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Enhanced indexation based on second-order stochastic dominance

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Abstract

Second order Stochastic Dominance (SSD) has a well recognised importance in portfolio selection, since it provides a natural interpretation of the theory of risk-averse investor behaviour. Recently, SSD-based models of portfolio choice have been proposed; these assume that a reference distribution is available and a portfolio is constructed, whose return distribution dominates the reference distribution with respect to SSD. We present an empirical study which analyses the effectiveness of such strategies in the context of enhanced indexation. Several datasets, drawn from FTSE 100, SP 500 and Nikkei 225 are investigated through portfolio rebalancing and backtesting. Three main conclusions are drawn. First, the portfolios chosen by the SSD based models consistently outperformed the indices and the traditional index trackers. Secondly, the SSD based models do not require imposition of cardinality constraints since naturally a small number of stocks are selected. Thus, they do not present the computational difficulty normally associated with index tracking models. Finally, the SSD based models are robust with respect to small changes in the scenario set and little or no rebalancing is necessary.

In this paper we present a unified framework which incorporates (a) SSD, (b) downside risk (Conditional Value-at-Risk) minimisation and (c) enhanced indexation.

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1. Introduction

Second order Stochastic Dominance (SSD) has a well recognised importance in financial portfolio selection, due to its connection to the theory of risk-averse investor behaviour and tail risk minimisation. However, until recently, stochastic dominance was considered only as a theoretical tool and not as an active portfolio strategy, because the models applying this concept were regarded as intractable or at least very demanding from a computational point of view. Computationally tractable and scalable portfolio optimisation models which apply the concept of SSD were proposed recently (Dentcheva and Ruszczyński, 2006; Roman et al., 2006; Fabian et al., 2011). These portfolio optimisation models assume that a benchmark, that is, a desirable “reference” distribution is available; a portfolio is then “actively” constructed, whose return distribution dominates the reference distribution with respect to SSD.

Index tracking models are commonly referred to as “passive” asset allocation strategies. They also assume that a reference distribution (that of a financial index) is available. A portfolio is constructed with the aim of replicating, or tracking, the financial index. Traditionally, this is done by minimising the tracking error: the standard deviation of the differences between the portfolio and index returns (Roll, 1992). Other methods have been proposed (for a review of these methods, see for example Beasley et al., 2003; Canakgoz and Beasley, 2008). The “passive” portfolio strategy of index tracking is based on the well established “Efficient Market Hypothesis” (Fama, 1970) which implies that financial indices achieve the best returns over time.

A common problem with index tracking models is raised by their computational difficulty; this is due to implementing regulatory or trading constraints, e.g. cardinality constraints that limit the number of stocks in the chosen portfolio. It is known that most index tracking models naturally select a very large number of stocks in the composition of the portfolio. Cardinality constraints overcome this problem, but they require introduction of binary variables and thus the resulting model becomes more difficult to solve. Most of the literature in the field is concerned with overcoming this computational difficulty; see for example Beasley, Beasley et al. (2003), Canakgoz and Beasley (2008).

Enhanced indexation models are related to index tracking, in the sense that they also consider the return distribution of an index as a reference. They however aim to outperform the index by generating “excess” return (dIBartolomeo, 2000; Scowcroft and Sefton,
Enhanced indexation is a very new area of research and there is no generally accepted portfolio construction method in this field (Canakgoz and Beasley, 2008). The same computational issues as in index tracking are encountered.

Although the idea of enhanced indexation was formulated as early as 2000, the (few) enhanced indexation methods were proposed later in the research community (a review in Canakgoz and Beasley, 2008). Moreover, these methods are mostly concentrated on overcoming the computational difficulty raised by restricting the cardinality of the portfolios – not on answering the question if they do attain their stated purpose, i.e. obtain return in excess of the index.

From a theoretical perspective, enhanced indexation calls for further justification. The Efficient Market Hypothesis (EMH) is based on the key assumption that security prices fully reflect all available information – see Elton and Gruber (1995) also Lo (2005) for an insightful review of this topic. However, this hypothesis has been continuously challenged; the mere fact that academicians and practitioners commonly use “active” (i.e. non-index tracking) strategies is an indication for this. An attempt to reconcile the advocates and opponents of the EMH is the “adaptive market hypothesis” (Lo, 2005). Here, the idea is that the market “adapts” to the information received and is generally efficient but there are periods of time when it is not – and thus, these periods can be used by investors to make profit in excess of the market index. This would justly, from a theoretical point of view, the quest for techniques that seek to obtain excess return as compared to financial indices. Enhanced indexation aims to discover and exploit market inefficiencies.

There are very few empirical studies comparing performance of enhanced index funds with that of their proxy indices; a review is given in Krause (2009), see also Ahmed and Nanda (2005). However, no indication is given on the methods used for constructing the index funds. These studies mostly come to the conclusion that, although overall the universe of enhanced funds does not seem to outperform the market, there are situations when outperformance does occur and persists for some periods of time; this seems to be in line with the adaptive market hypothesis. Another conclusion is that there may be specific types of funds that do add value; however, as stated before, there is no indication on how to construct these types of funds.

In this paper, we analyse the effectiveness of our previously proposed SSD-based portfolio models (Roman et al., 2006; Fabian et al., 2011) as enhanced indexation strategies applied to three markets: FTSE 100, Nikkei 225 and SP 500. We also investigate aspects related to the practical application of these portfolio models: cardinality control and rebalancing.

The motivation and contribution of this work are as follows. We aim to show that strategies stemming from risk-averse models of economic behaviour (SSD) can be used as a criterion to dominate (and thereby enhance) a financial index. The resulting active portfolio models (Roman et al., 2006; Fabian et al., 2011) are a “passive” portfolio selection strategy. Their aim is to track a financial index’s performance over an extended period of time.

It has been recently shown that very large SSD-based models can be solved in seconds, using solution methods which apply the cutting plane approach, as proposed by Fabian et al. (2011). However, imposing additional constraints that add trading realism (for example cardinality constraints, which require additional binary variables) could increase dramatically the computational time. We empirically show that SSD-based models naturally select a small number of stocks in the composition of the portfolio, thus the cardinality constraints may be eliminated.

Another aspect of interest in real-life portfolio trading is the amount of rebalancing needed; how and when does the current portfolio change when new information comes into place. Within the stochastic optimisation paradigm, this aspect is in connection with the stability of the portfolio optimisation model to changes in the input data. In this paper, we investigate the changes in the solution portfolios over time triggered by new data on the assets.

The rest of the paper is organised as follows. In Section 2 we introduce index tracking and enhanced indexation. In Section 3 we discuss how Second Order Stochastic Dominance (SSD) is used as a choice criterion in portfolio selection. In Section 4 we formulate the proposed models for enhanced indexation based on SSD. The numerical experiments are presented in Section 5. Three data-sets, drawn from FTSE 100, Nikkei 225 and SP 500 are used for backtesting the proposed models in a rebalancing frame. Conclusions are presented in Section 6.

2. Index tracking and enhanced indexation

Let \( n \) denote the number of the assets in which we may invest at the beginning of a fixed time period. A portfolio \( x = (x_1, \ldots, x_n)^T \) represents the proportions of initial capital invested in the different assets. Let the random vector \( R = (R_1, \ldots, R_g)^T \) denote the returns of the assets at the end of the investment period. The return of the portfolio \( x \) is denoted by \( R_x = R^T x \), a random variable.

Let \( X \subset \mathbb{R}^n \) denote the set of the feasible portfolios. We assume that \( X \) is a convex polyhedron; for example, in the simplest case,

\[
X = \{(x_1, \ldots, x_n) | \sum_{j=1}^{n} x_j = 1, \; x_j \geq 0, \; \forall j \in \{1, \ldots, n\}\}
\]

It is usual to assume that the future returns of the assets are discrete random variables with a finite number of outcomes, obtained by scenario generation or finite sampling of historical data (this is also the assumption used throughout this paper). Consider \( S \) scenarios and \( p_i \) the probability of scenario \( i, \; i \in \{1, \ldots, S\} \); \( \sum_{i=1}^{S} p_i = 1 \). Let \( r_{ij} \) be the return of asset \( j \) under scenario \( i, \; i \in \{1, \ldots, S\}, \; j \in \{1, \ldots, n\} \). Thus, the random variable representing the return of asset \( j \) is finitely distributed over \( \{r_{1j}, \ldots, r_{Sj}\} \) with probabilities \( p_1, \ldots, p_S \). The random variable \( R_x \) representing the return of portfolio \( x = (x_1, \ldots, x_n) \) is finitely distributed over \( \{r_{1x}, \ldots, r_{Sx}\} \), where \( r_{ix} = x_1 r_{i1} + \ldots + x_n r_{in}, \forall i \in \{1, \ldots, S\} \).

The primary problem in “active” portfolio selection is how to find a portfolio \( x \) such that its return \( R_x \) is “maximised”. (Since \( R_x \) is a random variable, this requires further clarification. There are various models of choice under risk that specify a preference relation among random returns. A portfolio \( x \) is then chosen such that its return \( R_x \) is non-dominated with respect to the preference relation considered. We resume this discussion in Section 4). Index tracking models are a somewhat special category; they are a “passive” portfolio selection strategy. Their aim is to track a financial index’s return as close as possible, thus, to “minimise” the difference between \( R_x \) and the (known) return distribution \( R_x \).
of the financial index. The rationale behind it is the belief (advocated by many financial experts) that a financial index routinely beats any portfolio actively created from its component assets.

Traditionally, index tracking is done by minimising the volatility of the tracking error: the sum of the squared deviations of returns on the replicating portfolio from the index (Roll, 1992). Other approaches suggest the use of absolute deviations instead of the square deviations, which leads to a linear program (LP) instead of a QP (Clarke et al., 1994; Rudolf et al., 1999). There are also approaches where only the downside deviations from the index’ returns are considered (Rudolf et al., 1999).

One drawback of these methods is that the number of the stocks included in the portfolio is very large (as an example, in our numerical experiments, described in Section 5, the number of stocks selected in a tracking portfolio is at least half of the total number of stocks available); this makes the solution impractical in many cases, particularly if transaction costs are taken into consideration. For this reason, cardinality constraints (limiting the number of stocks in the composition) are normally imposed. It is known that cardinality constraints require the introduction of binary variables (one binary variable for each asset in the basket) and this hugely increases the computational difficulty of these models.

A lot of the research in the area of passive portfolio selection has been thus concentrated on modelling or solution techniques meant to handle the computational difficulty associated to index tracking – for a review, see Canakgoz and Beasley (2008), Beasley et al. (2003).

Enhanced indexation is a very new area of research – for a review, see Canakgoz and Beasley (2008). As in index tracking, the return distribution of a financial index is available and has to be “tracked” but with the intention of seeking excess return. There is no generally accepted method in this area. Usually, the same computational problems as in index tracking are encountered: cardinality constraints (thus, introduction of binary variables) have to be introduced, resulting in computational difficulty.

3. Second order stochastic dominance

As stated in Section 2, the problem in “active” portfolio selection is how to find a portfolio $x$ such that its return at the end of the investment period $R_x$ is “maximised”. Since portfolio returns are random variables, models that specify a preference relation among random returns are required. A portfolio $x$ is then chosen such that its return $R_x$ is non-dominated with respect to the preference relation considered – this is done via an optimisation model.

For portfolio selection, mean-risk models have been by far the most popular. They describe and compare random variables using two statistics: the expected value (mean) and a risk value. Various risk measures have been proposed in the literature, see for example Markowitz (1952), Fishburn (1977), Ogryczak and Ruszczyński (1999, 2001), Rockafellar and Uryasev (2000, 2002). Mean-risk models are convenient from a computational point of view and have an intuitive appeal, but their approach is somewhat oversimplified.

Expected utility theory (von Neumann and Morgenstern, 1947) compare random returns by comparing their expected utilities (larger value preferred). However, the expected utility values depend on the utility function that is used; the choice of a specific utility function is somewhat subjective.

Stochastic Dominance (SD) has been recognised as a sounder model of choice, as it exploits “the three p’s: price, probability and preference” (Lo, 1999). It is closely connected to the expected utility theory, but it eliminates the need to explicitly specify a utility function (see Whitmore and Findlay, 1978 for a detailed description of stochastic dominance relations, Kroll and Levy, 1980 for a review). With stochastic dominance, random variables are compared by pointwise comparison of functions constructed from their distribution functions. There are progressively stronger assumptions about the form of utility functions used in investment, which lead to first, second and higher orders of SD. For example, First order Stochastic Dominance (FSD) is connected to “non-satiation” behaviour. A random return is preferred to another with respect to FSD relation if its expected utility is higher, for any increasing utility function. This is a strong condition and thus many random returns cannot be ordered with respect to FSD.

In portfolio selection, Second order Stochastic Dominance (SSD) is particularly important, due to its relation to risk-averse behaviour, as explained below.

Let $R$ and $R'$ denote two random returns. Second-order stochastic dominance is defined by the following equivalent criteria:

(a) $E(U(R)) \geq E(U(R'))$ holds for any increasing and concave (integrable) utility function $U$.

(b) $E[(t - R_x)] \leq E[(t - R'_x)]$ holds for each $t \in \mathbb{R}$.

(c) $\text{Tail}_a(R) \succeq \text{Tail}_a(R')$ holds for each $0 < a < 1$, where $\text{Tail}_a(R)$ denotes the unconditional expectation of the least $a \times 100\%$ of the outcomes of $R$.

For the equivalence of (a) and (b) see for example Whitmore and Findlay (1978). The equivalence of (b) and (c) is shown in Ogryczak and Ruszczyński (2002).

If the relations above hold, we say that $R$ dominates $R'$ with respect to SSD; we use the notation $R \succeq_{SSD} R'$.

A portfolio $x$ is said to dominate (or be preferred to) another portfolio $y$ with respect to SSD if $R_x \succeq_{SSD} R_y$, where $R_x$ and $R_y$ are the (random) returns of portfolios $x$ and $y$ respectively. A similar notation is used: $x \succ_{SSD} y$.

A portfolio $x$ is said to be non-dominated (or efficient) with respect to SSD if there is no other feasible portfolio $y$ such that $y \succ_{SSD} x$.

It is known that increasing and concave utility functions express the preference of risk-averse investors, which is the observed economic behaviour. This underlines the importance of choosing SSD efficient solutions. Unfortunately, the SSD relation is expressed as a continuum of constraints in the form (b) above. This makes SSD-based portfolio models very difficult from a computational point of view. Only recently such models have been proposed in the literature (Dentcheva and Ruszczyński, 2003; Dentcheva and Ruszczyński, 2006; Roman et al., 2006; Fabian et al., 2011).

Dentcheva and Ruszczyński (2003), Dentcheva and Ruszczyński (2006) consider a benchmark return; a portfolio is then constructed, such that its return dominates the benchmark with respect to SSD (in addition, a functional of the portfolio’s return is optimised). Roman et al. (2006), Fabian et al. (2011) propose models whose solutions are SSD efficient portfolios. In addition, these portfolios have return distributions that comes uniformly close to given benchmark distributions (e.g. those of historical indices).

In the following section, we present the models proposed by Roman et al. (2006), Fabian et al. (2011) from an enhanced indexation perspective.

4. Enhanced Indexation based on SSD

Roman et al. (2006), Fabian et al. (2011) consider the case of S equally probable scenarios; under this assumption, the SD relations greatly simplify, as explained below.

Denote by $R_i$ the return of the financial index considered as a benchmark; this is a random variable with a known distribution, with $S$ equally probable outcomes (provided, for example, from historical observations). Its ordered outcomes are denoted by $R_i^{(1)} \leq \ldots \leq R_i^{(S)}$. Denote by $R$ a random return and with ordered
outcomes $R^{(1)} \leq \ldots \leq R^{(S)}$. The first and second order SD relations can be expressed as follows:

(a) $R$ dominates $R_s$ with respect to FSD (notation: $R \succeq_{FSD} R_s$) if and only if: $R^{(i)} \geq R_s^{(i)}$, $i = 1, \ldots, S$.

(b) $R$ dominates $R_s$ with respect to SSD (notation: $R \succeq_{SSD} R_s$) if and only if: $\sum_{i=1}^{S} R^{(i)} \geq \sum_{i=1}^{S} R_s^{(i)}$, $i = 1, \ldots, S$.

Following Fabian et al. (2011), given $x(0 < x \leq 1)$, we denote by $\text{Tail}_R(x)$ the unconditional expectation of the least $x \times 100\%$ outcomes of the random variable $R$. Thus, $\text{Tail}_R(x) = \sum_{i=1}^{S} R^{(i)}$.

The SSD relation can be expressed in relation to Conditional Value-at-Risk (CVaR) at $S$ different confidence levels. The CVaR of a random return $R$ at confidence level $x \in (0, 1)$ is the mathematical transcription of the concept “mean of losses” in the worst $x \times 100\%$ of cases (Acerbi and Tasche, 2002), where the loss is relative to zero payoffs. A formal definition of CVaR is given for example in Rockafellar and Uryasev (2000, 2002).

It follows easily that, in the case of equi-probable scenarios, the CVaR-optimization formula of Rockafellar and Uryasev (2000, 2002) can be re-formulated to:

$$\text{Tail}_R(x) = \frac{1}{x} \min \sum_{i=1}^{S} R^{(i)}_R$$

such that $\mathcal{J} \subset \{1, \ldots, S\}$, $|\mathcal{J}| = i$. Using (6), the achievement-maximisation problem (5) can be re-formulated to:

$$\max \, \mathcal{J}$$

such that $\mathcal{J} \subset \{1, \ldots, S\}$, $|\mathcal{J}| = i$.

(7) Usually provides the portfolio that improves the most on the worst outcomes of the chosen portfolio and of the index return $S$. The above SSD equivalence can be further written as:

$$\text{CVaR}_R(x) \leq \text{CVaR}_{R_s}(x) \text{ holding for } i = 1, \ldots, S$$

Thus, for finding the SSD efficient portfolios, a multi-objective approach is proposed, in which the $S$ objective functions (to be minimised) can be written as CVaR at confidence levels $x$ in $1 \ldots S$.

The specific SSD efficient solution that comes closest, in a uniform sense, to the reference distribution $R_k$ is chosen by using the reference-point method (Wierzbički, 1983); this transforms the multi-objective formulation into a single-objective optimization problem.

We shortly describe this approach, applied to the multi-objective model (2).

We use the following notation:

$$\mathcal{J} = (\tau_1, \ldots, \tau_S) := (\text{Tail}_1(R), \ldots, \text{Tail}_S(R))$$

The reference point method introduces a concave “achievement” function $\mathcal{I}_{\mathcal{J}}$ whose arguments are the components of the objective in (2). The simplest achievement function is

$$\mathcal{I}_{\mathcal{J}}(\tau_1, \ldots, \tau_S) := \min_{1 \leq i \leq S} (\tau_i - \bar{\tau}_i) = \min_{1 \leq i \leq S} (\text{Tail}_i(R^X) - \text{Tail}_i(R))$$

(3) Thus, the “achievement” function considers the worst difference between the tails of the resulting portfolio return and the tails of the index. A term $\varepsilon \sum_{i=1}^{S} (\tau_i - \bar{\tau}_i)$ with a small positive $\varepsilon$ is usually added to ensure Pareto-efficiency of the optimal solution, as described in Roman et al. (2006). The differences between the tails of the resulting portfolio return and the tails of the index at confidence levels $x$, $i = 1, \ldots, S$, are called “partial achievements”.

The single objective optimization problem basically maximises the worst “partial achievement” over $x \in X$:

$$\max \, \mathcal{I}_{\mathcal{J}}(\text{Tail}_1(R^X), \ldots, \text{Tail}_S(R^X))$$

such that $x \in X$.

Denoting by $\mathcal{J} = \min_{1 \leq i \leq S} (\text{Tail}_i(R^X) - \bar{\tau}_i)$ the worst partial achievement, the above problem is written as:

$$\max \, \mathcal{J}$$

such that $\mathcal{J} \subset \mathbb{R}$, $x \in X$.

To compute the quantities $\text{Tail}_i(R^X)$, Roman et al. (2006) used the CVaR-optimization formula of Rockafellar and Uryasev (2000, 2002). This approach requires introduction of a large number of additional variables. The result is a LP model of very large size, if the number of scenarios $S$ is high. (There are more than $S^2$ variables and constraints. The number of assets $n$ poses much less difficulty, since the number of constraints/variables grows only linearly with $n$. No binary variables are required). Only models of relatively small sizes (up to 500 scenarios) could be solved with this approach.

In Fabian et al. (2011), a cutting-plane approach is used for computing the quantities $\text{Tail}_i(R^X)$; this is based on the cutting plane representation of CVaR proposed by Künzi-Bay and Mayer (2006).

In the case of equally probable scenarios, a very intuitive cutting-plane representation for $\text{Tail}_i(R^X)$ follows:

$$\text{Tail}_i(R^X) = \frac{1}{S} \min \sum_{j \in \mathcal{J}} R^{(j)}_i$$

such that $\mathcal{J} \subset \{1, \ldots, S\}$, $|\mathcal{J}| = i$.

(6)

No additional variables are introduced in the above formulation. Theoretically an astronomical number of cuts are required, but in practice only a few of them are needed. Fabian et al. (2011) propose a cutting-plane solution method for solving the above problem and show that the computational time is dramatically decreased: problems with tens of thousands of scenarios are solved within seconds.

The above model (7) is based on comparison of “unscaled” tails $\text{Tail}_i$, $i = 1, \ldots, S$ (of the chosen portfolio and of the index return distributions). A similar model, based on the comparison of “scaled” tails (or, equivalently, CVaR’s at confidence levels $x$, $i = 1, \ldots, S$) is proposed by Fabian et al. (2011). Applying similarly the reference point method to (1) and using cutting-plane representations of CVaRs, the following model results:

$$\max \, \mathcal{J}$$

such that $\mathcal{J} \subset \mathbb{R}$, $x \in X$.

where $i = 1, \ldots, S$.
A natural question is whether one provides better solutions than the other. Fabian et al. (2011) show that the model (8) presents advantages over the “unscaled” model (7) from a theoretical point of view: it can be formulated as a risk minimisation model, considering a convex risk measure. In addition, it may present advantages from a practical point of view: the (in-sample) return distributions of the portfolios chosen with (8) are somewhat “shifted to the right” as compared to those obtained with (7), indicating overall higher returns - except for a small portion in the left tail. That is, under the most unfavourable scenarios, the unscaled model (7) could provide better solutions, i.e. leading to a less dramatic loss.

The models (8) and (7) described above are never infeasible, but always provide solutions that are SSD efficient – irrespective of the benchmark chosen by the user.

The benchmark reference distribution however plays an important role in choosing the solution. Our interest is the case when the benchmark distribution is that of a financial index. As far as we are aware, all numerical experiments checking the SSD efficiency of financial indexes’ distributions (Roman et al., 2006; Fabian et al., 2011; Fabian et al., 2011; Dentcheva and Ruszczyński, 2003, 2006) led to the same conclusion: that the indices were dominated with respect to SSD.

(In our models (8) and (7), it is easy to check this: a positive optimum indicates a reference distribution that is SSD dominated – please see Roman et al., 2006; Fabian et al., 2011 for more details).

Thus, a portfolio is chosen that improves on the index’s distribution, in the sense that it dominates it with respect to SSD. More precisely, the tails of the index are increased (or, equivalently, the CVaRs are decreased) until SSD efficiency is obtained.

For this reason, the models (8) and (7) can be viewed as enhanced indexation models.

**Remark 2.** Both models (8) and (7) are based on CVaR minimisation (at different confidence levels). Empirical studies show that portfolios obtained by CVaR minimisation have less stocks in the composition than portfolios based on variance minimisation. This aspect is resumed in Section 5.2, where the cardinality of portfolios chosen by SSD-models is investigated.

Roman et al. (2006), Fabian et al. (2011) conducted numerical tests in order to compare the return distributions of the chosen portfolios with those of the reference distributions. The chosen portfolios had clearly better return distributions than those of the corresponding financial indices (higher mean, lower variance, higher skewness, better left tail, etc.).

All these results were obtained in-sample, in a single period framework. Of more practical importance is however the actual, realised performance of these portfolios measured over time, as well as the easiness in applying these models. The realised performance of the portfolios is measured by the amount of excess return over the corresponding index. The easiness in applying the models is assessed by the number of stocks in the composition of the chosen portfolios and the amount of rebalancing needed.

(We earlier explained that models (8) and (7) do not pose computational problems and are fast to solve even for very large datasets. This is based on the assumption that no cardinality constraints are introduced. However, if the number of stocks selected in the portfolios is large, cardinality constraints have to be introduced and the same computational problems as in tracking error minimisation would be encountered.)

These aspects are investigated in the next section.

### 5. Computational study

#### 5.1 Scope of the study, dataset and implementation issues

We consider the two SSD-based models (8) and (7) in a rebalancing frame; for the purpose of comparison, we also consider the classical tracking error minimisation model of Roll (1992). We investigate the following aspects:

- the effectiveness in obtaining excess return on the financial index considered;
- the number of stocks in the composition of the chosen portfolios;
- the amount of rebalancing needed.

The data used in this analysis is drawn from 3 financial indices: FTSE 100, Nikkei 225 and SP 500. We considered a (rolling) one day investment period. We used daily past historical returns of the component stocks and of the corresponding indices as scenarios for the returns of the day following the portfolio decision. The historical data covers the period 02/04/09–22/12/11. We used the period 02/04/09–31/05/11 (564 working days) as the first in-sample data set; the return of the portfolio obtained with this data set was evaluated (backtested) on the following working day data (01/06/11). We used a rolling window approach, moving forward 1 day, always having 564 in-sample scenarios. For each market and each optimisation model, we performed 147 optimisations; the portfolios obtained were evaluated on the next working day’s historical returns. Thus, for each market and each optimisation model we obtained 147 ex-post (out-of sample) daily portfolio returns, over the period 01/06/11–22/12/11.

The FTSE 100 dataset has 97 component stocks, in addition to the FTSE100 index; these are the companies with full set of prices between 02/04/09–31/05/11. On a similar basis, we considered 222 stocks for the Nikkei 225 dataset and 494 stocks for the SP 500 dataset.

Following Fabian et al. (2011), the models were implemented using the AMPL modelling system (Fourer et al., 1989) and the AMPL Component Library (2005), integrated with C functions; the models were solved using the FortMP solver (Ellison et al., 1999). In all instances, the computational time was very small (a few seconds).

#### 5.2 Test results

For all the three markets, the portfolios chosen by the SSD-based models had an overall better performance than the corresponding indices. The tracking error minimisers (obtained via the model proposed by Roll (1992)) mimicked nearly identically the indices’ performance. This is best underlined by computing the ex-post compounded returns over the period 01/06/11–22/12/11 (147 working days).

Fig. 1 considers the case of FTSE 100 market. The index and the index trackers are generally “at loss”, while by implementing the portfolios obtained with the SSD unscaled model (7) there is overall a small gain. The portfolios obtained with the SSD scaled model (8) are those that result in the biggest gain – cumulating to nearly 40% towards the end of the backtesting period.

Fig. 2 considers the case of Nikkei 225 market. The index and the index trackers are consistently “at loss” (a cumulated loss of around 10% at the end of the backtesting period). The SSD unscaled strategy is outperformed by the index only on a few time periods at the beginning of the backtesting period, but overall it performs considerably better, resulting in a cumulated profit of at around 10% in the last half of the backtesting period. The best strategy is
again given by the SSD scaled model (8), which at the end of the backtesting period has a cumulated profit of approximately 50%.

A similar situation is in the case of the SP 500 market (Fig. 3). The index and the tracker are generally at loss, while the SSD unscaled portfolios perform consistently better, having a cumulated gain of approximately 10% over the last half of the backtesting period. Once again, the SSD scaled strategy gives the best results, clearly outperforming the other strategies and resulting in cumulated gains of 60%.

We next look into the number of stocks in the composition of the efficient portfolios and the amount of rebalancing needed at each time period.

In the case of FTSE 100 market, the unscaled model (7) resulted in portfolios with cardinalities between 5 and 11. The largest cardinality was obtained at the beginning of the backtesting period, when the chosen portfolios comprised 10 or 11 stocks. Table 1 displays the weights of the portfolios chosen during the first 10 time periods of the backtesting period, 01–14/06/2011 (only week days are considered).

The lowest cardinality was obtained for the end of the backtesting period (5 stocks). During the last half of the backtesting period, there were large periods of time when no rebalancing was needed, e.g. during time periods 58–82 (19/08/2011–22/09/2011) and 83–147 (23/09/2011–22/12/2011); please see Tables 2 and 3.

In the case of FTSE 100 market, the scaled model (8) resulted in portfolios with cardinalities between 7 and 12. Similarly with the unscaled model, the cardinality decreases towards the end of the rebalancing period. Unlike the case of the unscaled model, here there is rebalancing at each day during the backtested period – the composition of the efficient portfolios differs however only marginally from one day to the next. Table 4 presents the composition of the first 10 portfolios chosen with the scaled model (8), backtested during 01/06/2011–14/06/2011 (10 working days).

For the other two markets, the situation is similar.

With Nikkei 225, the portfolios selected with the SSD unscaled model (7) have between 6 and 10 stocks in the composition, while the “SSD scaled” portfolios are slightly more diversified: between 8 and 14 stocks in their composition.

In the case of S&P 500, the “SSD unscaled” portfolios have between 6 and 14 stocks in the composition, while the SSD scaled portfolios have between 7 and 28 stocks in the composition. As with the FTSE 100 data, there are periods of time when no rebalancing is needed for the “SSD unscaled” strategy, e.g. the optimal portfolio remains the same over 12/09/2011–22/12/2011.

The full set of results is very large – the interested readers can obtain this on request.

To conclude, in case of all three markets, the SSD-based portfolios have a small number of stocks in the composition (usually, much less than one tenth of the available stocks), which makes the imposition of cardinality constraints unnecessary. This behaviour is due to the CVaR-minimisation nature of the SSD-based models, as explained in Section 5. The composition of the optimal portfolios is stable with respect to moving the window frame for the in-sample data, particularly in the case of SSD unscaled model.
In comparison, the tracking errors minimisers contain nearly all of the available stocks in their composition making necessary the imposition of cardinality constraints.

6. Summary and conclusions

Two portfolio selection models that use Second-order Stochastic Dominance (SSD) as a choice criterion are presented from an enhanced indexation perspective. Both models consider a financial index’s return distribution; they produce portfolios whose return distributions improves on the index’s until SSD efficiency is attained. One model is based on comparison of “tails” (i.e., sums of ordered outcomes) between the chosen portfolio’s return distribution and the index’s return distribution; we refer to this model as the “unscaled” model. The other model uses comparison of “scaled tails” (i.e., averages of ordered outcomes), or, equivalently, comparison of CVaRs at different confidence levels. We refer to this model as the “scaled” model.

We have tested the effectiveness of these two models as enhanced indexation strategies, using three datasets: FTSE 100 (97 stocks), Nikkei 225 (222 stocks) and SP 500 (491 stocks). We have used the last half of 2011 (01/06/11–22/12/11) as a backtesting period, in a daily rebalancing frame: for each model and each market we have computed 147 ex-post compounded returns. These are “realised” returns: portfolio strategies are implemented and then evaluated on the next time period using real data. We have made a comparison with the indices’ performance and also with the performance of the index tracker portfolios obtained with Roll’s (1992) model.

Three conclusions are drawn.

First, the SSD-based models consistently outperform the corresponding indices, in the sense that higher returns are obtained over
most of the backtesting period. This aspect is emphasised by computing their compounded returns. All the three indices are generally at loss over the backtesting period, with the index trackers mimicking nearly perfectly their movements. In contrast, the portfolios obtained with the SSD models lead to overall profits. In particular, portfolios obtained via the SSD scaled model have a very good backtesting performance, consistently outperforming the corresponding indices (also the SSD unscaled portfolios) by a substantial amount. For all three markets, the SSD scaled strategy results in a compounded gain of 40% or above, while the indices have a compounded loss around 10%.

These results are somewhat not in line with studies like that of Ahmed and Nanda (2005) or Krause (2009), in which the authors conclude that overperformance of (enhanced) index funds does not often happen and is not consistent. We underline that these previous studies considered all enhanced index funds available at the time, with no indication of how they were constructed. In contrast, we here describe new strategies for obtaining enhanced funds, then compare their realised performance against that of the index.

The superlative out-of-sample performance of the SSD scaled model supports the original motivation of this model. Fabian et al. (2011) proposed the scaled model as an approach that improved the original model of Roman et al. (2006) from a theoretical as well as a practical point of view.

Secondly, the imposition of cardinality constraints seems to be unnecessary in the two SSD-based models. Due to their CVaR-minimisation nature, these models naturally select a much lower number of stocks than the established index tracking models. Usually, the SSD-based models select around one tenth of the available stocks or less. In comparison, the tracking error minimisers select almost all the available stocks and thus the cardinality has to be explicitly limited. Not imposing cardinality constraints has big advantages from a computational point of view and from a performance point of view.

Finally, the amount of necessary rebalancing in the SSD-based models is low, since the models are stable with the introduction of new scenarios, representing new information on the market. We rebalance the portfolios by moving the in-sample data window one day ahead. In particular, the unscaled model has a remarkable behaviour from this point of view, the chosen portfolios remaining the same (no rebalancing needed) over considerable periods of time.

These observations lead us to conclude that the SSD-based approach is a good strategy for portfolio construction; it provides consistently good returns, it is easy to implement due low cardinality and requires minimal rebalancing due to stable composition. We find it interesting that such consistent and clear outperformance of indices has been achieved using a remarkably small number of stocks, with no drastic rebalancing over time.

As future work in this area, we propose to evaluate the performance of the SSD-based models, compared to that of the indices, for a backtesting period in a bull market, that is, a market with rising prices. We also propose to formally investigate the stability of the SSD based models with respect to changes in the scenario set.

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