Discrete Mathematics 22 (1978) 49–55.

THE CONNECTED RAMSEY NUMBER

David P. SUMNER

Department of Mathematics and Computer Science, University of South Carolina, Columbia, SC 29208, U.S.A.

Received 27 February 1976
Revised 14 October 1976

A graph G is totally connected if both G and $\bar{G}$ (its complement) are connected. The connected Ramsey number $r_c(F, H)$ is the smallest integer $k \geq 4$ so that if G is a totally connected graph of order k then either $F \subset G$ or $H \subset \bar{G}$. We show that if neither of F nor H contains a bridge, then $r_c(F, H) = r(F, H)$, the usual generalized Ramsey number of F and H. We compute $r_c(P_n, P_m)$, the connected Ramsey number for paths.

1. Introduction

In recent years there has been a flurry of activity involving the concept of the generalized Ramsey numbers. Let F and H be graphs (all our graphs will be finite and without loops or multiple edges). Then the Ramsey number $r(F, H)$ is the smallest integer n such that for any graph G of order at least n, either $F \subset G$ or $H \subset \bar{G}$. Here $F \subset G$ is meant to imply that F is a subgraph of G and $\bar{G}$ denotes the complement of G. An alternative point of view is that $r(F, H)$ is the smallest integer n such that if a graph G having order at least n is edge-colored with two colors, say red and blue, then the resulting coloring must contain either a red copy of F or a blue copy of H. For a good survey of the subject see Burr [2] and also Harary [5]. In his paper Burr mentions that many of the lower bounds for the generalized Ramsey numbers are obtained via certain canonical types of colorings (see Fig. 1).

In each case the coloring consists of two monochromatic complete graphs with all the edges between them having the same color. Certain more general canonical colorings (in which more than just two monochromatic complete graphs are used) also occur. When observing these colorings one is struck by the fact that in each case one of the colors induces a disconnected graph. Thus it seems plausable that if the restriction that each of G and $\bar{G}$ be a connected graph is placed on our graphs, then we may not need to require $r(F, H)$ vertices in G in order to insure that $F \subset G$ or $H \subset \bar{G}$. This turns out to often be the case. In order to investigate this situation, we define the *connected Ramsey number*.

Definition. (i) Let G be a graph. Then G is *totally connected* if each of G and $\bar{G}$ is a connected graph. (Note that every totally connected graph has order at least four).

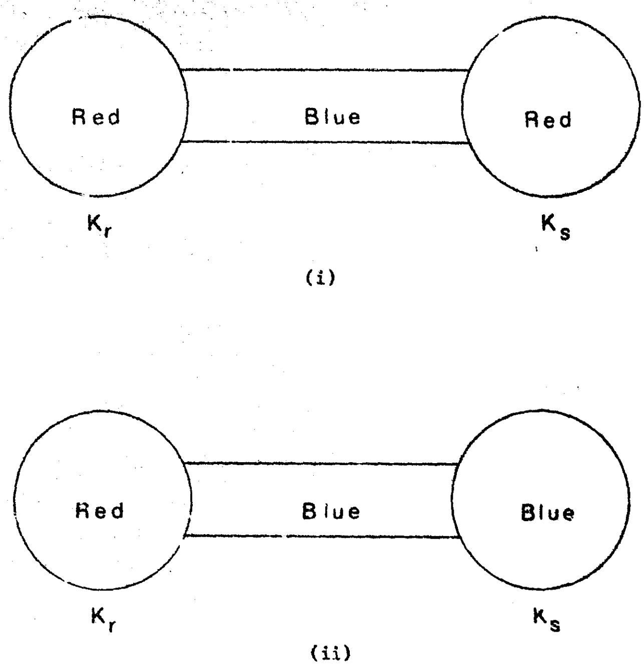

Fig. 1.

(ii) For two graphs F and H, the *connected Ramsey number* $r_c(F,H)$ is the smallest integer $k \geq 4$ so that if G is a totally connected graph of order at least k then either $F \subset G$ or $H \subset \bar{G}$.

It is our purpose in this paper to initiate an investigation of this number. First we note that for any two graphs F and H, $r_c(F,H)$ is well-defined and in fact, except in those trivial cases where $r(F,H) \leq 3$, we have $r_c(F,H) \leq r(F,H)$.

It is clear that we could conceiveably define many similar such Ramsey numbers. In fact, let P be any graph theoretic property. Then we may define the ***P-Ramsey number*** $r_P(F,H)$ of the graphs F and H to be the smallest integer n such that if G is a graph with at least n vertices and having property P, then either $F \subset G$ or $H \subset \bar{G}$. However for many properties this number would be unwieldly (at best) to work with. In particular, suppose that we have demonstrated that for every graph G having n vertices and property P, either $F \subset G$ or $H \subset \bar{G}$. Can we conclude that $r_P(F,H) \leq n$? Not necessarily. For there could exist a graph on $n+1$ vertices having property P but not having an n element induced subgraph with property P. Hence in order that there will be a decent chance of establishing the upper bounds on $r_P(F,H)$ in the usual way, we must require that the ***P-nucleus***, $G^{(P)} = \{v \mid G - v$ has property $P\}$ be non-empty for every graph G having property P.

For totally connected graphs this obstruction in establishing upper bounds is avoided via:

Theorem 1.1. [Bosák, Rosa, Znám]. *If G is a totally connected graph on at least five vertices, then there exists $v \in V(G)$ such that $G - v$ is also totally connected.*

A proof of this theorem appears at the end of [1].

Our next proposition, a simple consequence of the previous theorem, is a useful tool which occurs originally in Foulis [4]. A short proof not using Theorem 1.1 may be found in Sumner [6].

Theorem 1.2. *If G is a totally connected graph, then each of G and $\bar{G}$ contains a path on four vertices (i.e. P_4) as an induced subgraph.*

Note that since P_4 is a self-complementary graph, any four vertices that induce P_4 in G also induce P_4 in $\bar{G}$.

2. Blocks

We show in Section 3 that $r_c(F, H)$ is often smaller than $r(F, H)$. However, our next theorem shows that at least one of F and H must contain a bridge in order for this to happen.

Theorem 2.1. *Let F and H be graphs of order at least four and with edge connectivity at least two. Then $r_c(F, H) = r(F, H)$.*

Proof. Since we certainly have $r_c(F, H) \leq r(F, H)$, we need only establish the reverse inequality. Suppose then that G is a graph of order $r_c(F, H)$ but that $F \not\subset G$ and $H \not\subset \bar{G}$. Then G cannot be totally connected and so we may choose such a graph G so that whichever of G and $\bar{G}$ is not connected will have as few components as possible. With no loss of generality, we assume that it is G and not $\bar{G}$ that is disconnected.

Suppose that G consists of an isolated vertex and a complete graph, i.e. $G = K_1 \cup K_n$ ($n \geq 3$). Let a be the isolated vertex and let b and c be vertices in $G - a$. Then form the graph $G^* = G + ab - bc$. Then G^* is totally connected. Yet if $F \subset G^*$, then since $F \not\subset G$, F must contain the edge ab. But that is impossible since ab would be a bridge in F. Also, since $\overline{G^*}$ is a tree, $H \not\subset \bar{G}^*$. Hence we may suppose that G is not the disjoint union of a complete graph and an isolated vertex.

Now choose two vertices u and v as follows: If G has an isolated vertex let such a vertex be u and let v be any other vertex not adjacent to all the vertices in $G - u - v$. If G has no isolated vertices, then let A and B be any two components of G and choose $u \in A$ and $v \in B$. Now form the graph $G^* = G + uv$. Then if G^* is connected, it is totally connected and otherwise it has fewer components than G. In either case we must have $F \subset G^*$ or $H \subset \overline{G^*}$. However, if $F \subset G^*$, then F must contain the edge uv. But this is impossible since uv would be a bridge in F. Also if $H \subset \overline{G^*} = \bar{G} - uv$, then $H \subset \bar{G}$ which cannot be. Hence in any event we have a contradiction and the theorem follows.

We remark that as a consequence of Theorem 1.2, $r_c(P_4, H) = 4$ for any graph H

on at least four vertices. Thus if one of the graphs involved contains a bridge then the theorem need not hold.

3. Paths

In this section we will determine the connected Ramsey number for paths. We will let P_n and C_n denote a path and a cycle, respectively, on n vertices. Also for a graph G, if S is a subset of the vertices of G we will write $\langle S\rangle$ for the subgraph induced by S. If u and v are vertices of G we will write $u \sim v$ to indicate that u and v are adjacent in G and $u\Delta v$ otherwise; i.e. $u\Delta v$ means u and v are adjacent in $\bar{G}$.

Definition. Let n be a positive integer. Then n^* is the largest even integer smaller than n, i.e.

$$n^* = \begin{cases} n-1 & \text{if } n \text{ is odd,} \\ n-2 & \text{if } n \text{ is even.} \end{cases}$$

For the proof of Theorem 3.1 it is useful to keep in mind that $[(m-3)/2] = \frac{1}{2}m^* - 1$.

The next lemma appears in [3].

Lemma 3.1. *If n and m are positive even integers, then*
(i) $r(C_4, C_4) = 6$,
(ii) *If* $n \geqslant m \geqslant 4$ *and* $(n,m) \neq (4,4)$, *then* $r(C_n, C_m) = n + \frac{1}{2}m - 1$.

Lemma 3.2. (i) $r_c(P_6, P_6) = 6$,
(ii) $r_c(P_6, P_5) = 6$,
(iii) $r_c(P_5, P_5) = 5$.

Proof. We will verify (i). The proofs of (ii) and (iii) are similar. Clearly $r_c(P_6, P_6) \geqslant 6$. Let G be a totally connected graph on six vertices. Then by Theorem 1.2, there exist vertices a, b, c and d in G such that $\{a, b, c, d\}$ induces a path in both G and $\bar{G}$. Say $a \sim b$, $b \sim c$ and $c \sim d$. Let x and y be the remaining vertices of G. Then with no loss of generality we may assume that $x \sim y$. Now if $x \sim a$, then $yxabcd$ is a P_6 in G. Hence we may assume that $x\Delta a$ and similarly that $x\Delta d$, $y\Delta a$ and $y\Delta d$. Now if $x\Delta b$, then $caydxb$ is a P_6 in $\bar{G}$. Thus we may assume that $x \sim b$. If $y\Delta c$, then $bdxayc$ is a P_6 in $\bar{G}$. Hence we may suppose that $y \sim c$. But then $abxycd$ is a P_6 in G. Thus in any event either $P_6 \subset G$ or $P_6 \subset \bar{G}$. Hence $r_c(P_6, P_6) \leqslant 6$.

Theorem 3.1. (i) $r_c(P_n, P_4) = 4$ *for* $n \geqslant 4$.
(ii) *For n and m integers with* $n \geqslant m \geqslant 5$, $r_c(P_n, P_m) = n + [(m-3)/2] - 1$.

Proof. We have already noted that (i) is an immediate consequence of Theorem 1.2.

To prove (ii), let G be a totally connected graph of order $n+[(m-3)/2]-1$. If $n \leqslant 6$, the result is by Lemma 3.2. Hence we will assume that $n \geqslant 7$. Thus $(n^*, m^*) \neq (4,4)$ and so by Lemma 3.1

$$r(C_{n^*}, C_{m^*}) = n^* + \tfrac{1}{2}m^* - 1$$
$$\leqslant n + \tfrac{1}{2}m^* - 2 = n + [(m-3)/2] - 1.$$

Hence either $C_{n^*} \subset G$ or $C_{m^*} \subset \bar{G}$. Suppose that $C_{m^*} \subset \bar{G}$ and let C be a cycle of order m^* in $\bar{G}$. If m is odd then $C_{m-1} \subset \bar{G}$ and so since $\bar{G}$ is connected, some vertex v not in C is adjacent in $\bar{G}$ to some vertex of C and hence we have $P_m \subset \bar{G}$. Thus we can suppose that m is even and hence $m^* = m-2$. Also by (i) we may assume that $m > 4$ and so $m \geqslant 6$. Let $S = V(G) - C$. If $\langle S \rangle$ (in $\bar{G}$) contains a non-trivial component, then since $\bar{G}$ is connected, there exist vertices $a, b \in S$ with $a\Delta b$ and $b\Delta x$ for some vertex x in C. Hence $P_m \subset \bar{G}$. Thus we may assume that S is a set of independent vertices of $\bar{G}$. Now let $A = \{v \in C \mid v\Delta w$ for some $w \in S\}$. If $|A| > (m-2)/2$, then there would be two consecutive elements of C in A. But then either these two vertices are adjacent to distinct vertices in S in which case we have at once that $P_m \subset \bar{G}$ or else they are adjacent to the same element of S in which case $C_{m-1} \subset \bar{G}$ and so, since $\bar{G}$ is connected, we have in this case too that $P_m \subset \bar{G}$. Thus we may suppose that $|A| \leqslant (m-2)/2$. Hence there exists $B \subset C$ such that $|B| = (m-2)/2$ and no element of B is adjacent in $\bar{G}$ to any element of S. Thus we have partitioned $V(G)$ into the sets S, B and $D = C - B$, where, in G, $\langle S \rangle$ is complete, $|B| = |D| = (m-2)/2$ and every element of B is adjacent to every element of S. Thus since $|B \cup S| = n-2$ and $|S| \geqslant |B|$, then knowing that S is complete we can conclude that $C_{n-2} \subset \langle B \cup S \rangle$ (in G). Now if D had a non-trivial component, it would follow from the connectedness of G that $P_n \subset G$. Thus we may suppose that D is an independent set in G.

Now suppose that no vertex in S is adjacent in G to any vertex of D. Then as before since $|S| \geqslant |D| = (m-2)/2 \geqslant 2$, $\langle D \cup S \rangle$ (in $\bar{G}$) contains a copy of C_{m-2} in which every vertex of D is included. Thus since $\bar{G}$ is connected, if $\langle B \rangle$ (in $\bar{G}$) had a non-trivial component, we would have $P_m \subset \bar{G}$. Hence we may assume that B is independent in $\bar{G}$ and thus complete in G. Since G is connected and no vertex of D is adjacent to any vertex in $S \cup D$, if d is any vertex of D then we must have $d \sim b$ for some $b \in B$. But then since $\bar{G}$ is connected, there is some $x \in D$ with $b\Delta x$. But once again, there is some $y \in B$ with $y \sim x$. Thus since $\langle B \cup S \rangle$ (in G) is complete of order $n-2$, $\langle B \cup S \cup \{x, d\} \rangle$ (in G) contains P_n.

Suppose on the other hand that there is some vertex $u \in S$ adjacent in G to some vertex $v \in D$. Then there must exist some $w \in D$ with $u\Delta w$. Now if w is adjacent in G to any vertex of B, then it is easy to see that we have $P_n \subset \langle B \cup S \cup \{v, w\} \rangle$ (in G). Hence we may assume that $w \sim y$ for some $y \in S$ and that no vertex of D is adjacent in G to any vertex of B. Now if $\langle B \rangle$ (in G) had even a single edge, then it would follow that $\langle B \cup S \cup \{v, w\} \rangle$ (in G) contains P_n (the existence of an edge in B is needed in case $|B| = |S|$). Thus we can assume that $\langle B \rangle$ (in G) is independent.

But then $\langle B \cup D \rangle$ (in $\bar{G}$) is K_{m-2} and so since $u \not\sim w$ and y is adjacent in $\bar{G}$ to some element of $D - \{w\}$, we have $P_m \subset \langle B \cup D \cup \{u, y\}\rangle$ (in $\bar{G}$).

Hence the assumption that $C_{m^*} \subset \bar{G}$ has led to the conclusion that either $P_m \subset \bar{G}$ or $P_n \subset G$. Now suppose that $C_{n^*} \subset G$. If n were odd then, just as before, since G is connected we would have $P_n \subset G$. Thus we may suppose that n is even and so $n^* = n - 2$. Now we may argue in an exactly similar way as in the previous case to show that $V(G)$ may be partitioned into sets B, D and S so that $|B| = |D| = (n-2)/2$ and in $\bar{G}$ every element of B is adjacent to every element of S. But $|B| \geq |S| = \frac{1}{2}m^*$ and so $\langle B \cup S \rangle$ (in $\bar{G}$) contains C_{m^*}. But then we are in the previous case.

Thus we have $r_c(P_n, P_m) \leq n + [(m-3)/2] - 1$.

To obtain the reverse inequality consider the graph obtained from $K_{n-2} - e$ by adjoining to one end of the edge e, $[(m-3)/2]$ endpoints (see Fig. 2).

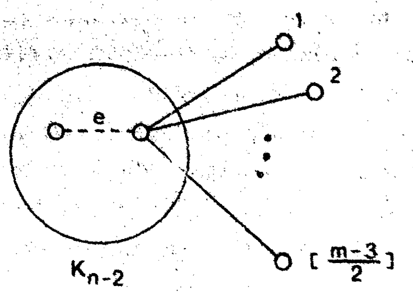

Fig. 2.

The graph so obtained has order $n + [(m-3)/2] - 2$ and does not contain P_n and its complement does not contain P_m. Thus $r_c(P_n, P_m) > n + [(m-3)/2] - 2$ and the proof of our theorem is complete.

It is interesting to observe the close relationship between the usual Ramsey numbers for paths and the connected Ramsey numbers. We recast Theorem 3.1 in a different way in order to delineate this relationship.

Corollary 3.1. *Let n and m be positive integers with $n \geq m \geq 5$. Then*

$$r_c(P_n, P_m) = \begin{cases} r(P_n, P_m) - 1 & \text{if } m \text{ is odd,} \\ r(P_n, P_m) - 2 & \text{if } m \text{ is even.} \end{cases}$$

4. Concluding remarks and questions

This paper has dealt with the number $r_c(F, H)$. It would seem worthwhile also to investigate the more general k-connected Ramsey number $r_k(F, H)$. $r_k(F, H)$ is the smallest integer n so that every graph G having $|G| \geq n$ and such that G is totally k-connected (i.e. both G and $\bar{G}$ are k-connected) will have $F \subset G$ or $H \subset \bar{G}$. Of course there may also be many other numbers $r_p(F, H)$ whose study may prove worthwhile to the overall investigation of the generalized Ramsey numbers. Finally we mention the problem of determining the other connected Ramsey numbers for pairs of graphs at least one of which contains a bridge. In particular $r_c(P_n, C_m)$ needs to be determined.

Acknowledgement

The author would like to thank the referee for suggesting an improvement in the presentation of the definition of the connected Ramsey number.

References

[1] J. Bosák, A. Rosa, S. Znám, On decomposition of complete graphs into factors with given diameters, in: Theory of Graphs (Academic Press, New York, 1969) 37–56.
[2] Stefan A. Burr, Generalized Ramsey theory for graphs — A survey, in: Graphs and Combinatorics (Springer-Verlag, Berlin 1974) 52–75.
[3] R.D. Faudree, R.H. Schelp, All Ramsey numbers for cycles in graphs, Discrete Math. 8 (1974) 313–329.
[4] D.J. Foulis, Empirical logic, Xeroxed course notes, Univeristy of Massachusetts, Amherst, MA (1969–1970).
[5] F. Harary, A survey of generalized Ramsey theory, in: Graphs and Combinatorics (Springer-Verlag, Berlin, 1974), 10–17.
[6] D.P. Sumner, Dacey Graphs, J. Austr. Math. Soc. 4 (1974) 492–502.