# Inconstancy of finite and infinite sequences 

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#### Abstract

In order to study large variations or fluctuations of finite or infinite sequences (time series), we bring to light an 1868 paper of Crofton and the (Cauchy-)Crofton theorem. After surveying occurrences of this result in the literature, we introduce the inconstancy of a sequence and we show why it seems more pertinent than other criteria for measuring its variational complexity. We also compute the inconstancy of classical binary sequences including some automatic sequences and Sturmian sequences.


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The voyage of the best ship is a zig-zag line of a hundred tacks.
See the line from a sufficient distance, and it straightens itself to
the average tendency...(Ralph Waldo Emerson, Emerson Essays, 1899).

## 1. Introduction

How is it possible to define and to detect large variations or fluctuations of a sequence (with possible applications to the [discrete] time evolution of biological, financial, musical phenomena and so on). The usual approach is based on computing the distance of the associated piecewise affine function to the corresponding linear regression line, i.e., on computing the residual variance. But this quantity somehow describes total distance to "regularity", and says nothing about possibly large local fluctuations: for example, it may not discriminate between an exponentially growing function and a fractal-like "chaotic" (disordered) curve. In particular one should remember that dictionaries defining "fluctuation" use words with a similar meaning among which "wavering", "unsteadiness", "vacillation", "erraticness", "variability", etc.

We suggest here to bring to light - especially for applications to sequences - a paper of Crofton dated 1868 [20] (see also the papers of Cauchy [13,14] and the papers of Steinhaus [58] and of Dupain, Kamae and Mendès France [22]). Crofton studies the average number of intersection points of a curve with random straight lines. But this average number can be thought of as a measure of the fluctuations of the curve. Namely, for a straight line or a curve "looking like a straight line", this average number is equal to 1 , while it has a very large value for a "very complicated" curve. Following this idea, we propose a measure of large variations of a sequence and we compare it with the residual variance. Conversely, this measure

[^0]will allow us to decide whether a sequence is "more complicated" than another in cases where the visual aspect does not suffice to suggest an intuitive answer. We will also show that this measure can be applied to infinite sequences satisfying some technical condition (in particular certain automatic sequences as well as Sturmian sequence; see, e.g., [5]) to describe their "complexity".

As will be recalled, the ideas of Cauchy and Crofton were already used in various contexts: one of our purposes is to insist on their usefulness for measuring the complexity of discrete phenomena, as a compromise between measuring intensity, time and consecutive repetitions. These ideas will be applied in a subsequent paper (see [61]) to fluctuations of biological parameters, e.g., the weight, or the Quetelet index, ${ }^{1}$ often called the BMI (Body Mass Index; see, e.g., [52,8,55,54,53]) for children: are "large fluctuations" of the BMI risk factors for cardiovascular diseases in relation with the metabolic syndrome? This question was addressed with other tools in [60] (see also the references therein). We also aim to try to apply this measure of fluctuations to other questions, e.g., analyzing fluctuations of the stockmarket, and quantifying the "smoothness" of musical themes.

## 2. Defining the Inconstancy of a curve

A possible approach for describing large variations or large fluctuations of a curve is to "compare" it with a straight line. More precisely we can count the number of intersection points of random straight lines with the given curve: if this number is small on average, the curve behaves roughly as a straight line; if this number is large, the curve is "complicated". Is there an "easy" way to compute this number? The Cauchy-Crofton theorem answers the question.

### 2.1. The Cauchy-Crofton theorem

Consider a plane curve $\Gamma$. Let $\ell(\Gamma)$ denote the length of $\Gamma$ and let $\delta(\Gamma)$ denote the perimeter of the closed curve forming the edge of the convex hull of $\Gamma$. Let $\Omega(\Gamma)$ be the set of straight lines which intersect $\Gamma$. Any line can be defined as the set of $(x, y)$ such that $x \cos \theta+y \sin \theta-\rho=0$, where $\theta$ belongs to $[0, \pi)$ and $\rho$ is a positive real number. A straight line is therefore completely determined by $(\rho, \theta)$. Letting $\mu$ denote the Lebesgue measure on the set $\{(\rho, \theta), \rho \geq 0, \theta \in[0, \pi)\}$, the average number of intersection points between the curve $\Gamma$ and a line in $\Omega$ is defined by

$$
\mathcal{N}(\Gamma):=\int_{D \in \Omega(\Gamma)} \sharp(\Gamma \cap D) \frac{\mathrm{d} \rho \mathrm{~d} \theta}{\mu(\Omega(\Gamma))} .
$$

The following result can be found in [20, p. 184-185]; see also the papers of Cauchy [13,14].
Theorem 2.1 (Cauchy-Crofton). The average number of intersection points between the curve $\Gamma$ and the straight lines in $\Omega$ satisfies

$$
\mathcal{N}(\Gamma)=\frac{2 \ell(\Gamma)}{\delta(\Gamma)}
$$

Remark 2.2. In his paper, Crofton speaks of "Local or Geometrical Probability"; he writes about Probabilities, "The rigorous precision, as well as the extreme beauty of the methods and results... the subtlety and delicacy of the reasoning...", and he quotes Laplace: "ce calcul délicat". Crofton's result is explained in Steinhaus' paper [58]. It is presented in an illuminating way with several examples in the paper of Dupain, Kamae, and Mendès France [22]: these authors studied the notion of entropy of a curve and of temperature of a curve introduced by Mendès France in [38]. Note that the occurrence of the number 2 in the numerator can be understood by considering the case where $\Gamma$ is a segment: the average number of intersection points is equal to 1 , while the perimeter of the convex hull of the segment is twice the length of the segment (why twice? go back to the definition ["closed curve..."] or think of the case where the segment is replaced by a thin rectangle whose width tends to zero).

Remark 2.3. The reader will have noted that Crofton's approach has much to do with the famous Buffon needle problem [11, p. 100-104], also known as the Buffon-Laplace needle problem; see [31, p. 359-360]. The area of this type of result is known as "Integral Geometry". This terminology seems to have been introduced by Blaschke in his "Vorlesungen über Integralgeometrie" [9,10]. More recent references are the book of Santaló [56], and the forthcoming book of Langevin [33] (see also [32]). An interesting review of the books of Blaschke and of the 1936 edition of the book of Santaló is [45]. A nice exposition of the (proof of the) theorem of Cauchy-Crofton, where the curve is only supposed to be rectifiable, can be found in the paper of Ayari and Dubuc [6]. We also recommend for a first approach the texts of Mendès France [42] and of Teissier [59]. Note that the Crofton theorem is also (and more correctly) called the Cauchy-Crofton theorem in the literature.

[^1]Remark 2.4. Using the theorem of Cauchy-Crofton to define a measure of complexity of a curve was first suggested by Mendès France [36, page 92]; also see [37,38]. It was also proposed later, e.g., in [15] where the name "folding index" is used. Also note that the Crofton formulas in [20] are used frequently in many fields. These include complex motor behaviour in human movements [16] (also see [17,18]), study of human blood and transfusion [62], simulation of gravitational evolution [57], anisotropies of the secondary cosmic microwave background [25], grain size distribution analysis for polycrystalline thin films [19], image analysis of crystalline agglomerates [49], measurement of convolution in cotton fibers [28], all applications of LIS (Line-Intercept Sampling), e.g., to the statistical analysis of vegetation or wildlife, see for example [63] and the references therein (in particular [30] in the references below), spatial analysis of urban maps [24], in a discussion about examples of information processing coming from neurophysiology, cognitive psychology, and perception [48, pp. 11821185], and even relations between art and complexity [43] (also see [46,47]).

### 2.2. The inconstancy of a curve

The theorem of Cauchy-Crofton suggests the following definition.
Definition 2.5. Let $\Gamma$ be a plane curve of length $\ell(\Gamma)$ and such that the perimeter of its convex hull is equal to $\delta(\Gamma)$. The inconstancy of the curve $\Gamma$, denoted $\mathcal{I}(\Gamma)$, is defined by

$$
\mathcal{I}(\Gamma):=\frac{2 \ell(\Gamma)}{\delta(\Gamma)}
$$

Remark 2.6. The above definition and the Cauchy-Crofton theorem show that the inconstancy of the union of two curves is at most the sum of the inconstancies of these curves, that the inconstancy of a curve is equal to the inconstancy of its translated, rotated or homothetic curve, etc.

## 3. Comparison with other criteria

Other criteria for measuring fluctuations of a discrete curve can be found in the literature for real (e.g., biological) phenomena: qualitative classification with predetermined cut-off points, maximal values, residual variance, etc. (see, e.g., the discussion in [60, pp. 316-317] for weight fluctuations). By oversimplifying most of the various definitions, one could say that they aim to measure the "distance" between the considered curve and a straight line, but this distance can be computed globally or locally. We recall the definition of regression line, residual variance, and mean square error.
Definition 3.1. Let $\left(x_{i}, y_{i}\right)_{i=1,2, \ldots, n}$ be a family of $n \geq 3$ points. Their regression line is the straight line that minimizes the sum of squares of distances from the ( $x_{i}, y_{i}$ )'s to it. Letting $\bar{x}=\left(\sum_{1 \leq i \leq n} x_{i}\right) / n$ and $\bar{y}=\left(\sum_{1 \leq i \leq n} y_{i}\right) / n$ denote the averages of the $x_{i}^{\prime}$ 's and of the $y_{i}$ 's, the equation of the regression line is

$$
Y=\hat{a} X+\hat{b}, \quad \text { where } \hat{a}=\frac{\sum_{1 \leq i \leq n}\left(x_{i}-\bar{x}\right)\left(y_{i}-\bar{y}\right)}{\sum_{1 \leq i \leq n}\left(x_{i}-\bar{x}\right)^{2}} \text { is the correlation, and } \hat{b}=\bar{y}-\hat{a} \bar{x}
$$

The MSE (i.e., mean square error) and the RMSE (i.e., root mean square error) of the ( $x_{i}, y_{i}$ )'s are defined by

$$
M S E:=\frac{1}{n-2} \sum_{1 \leq i \leq n}\left(y_{i}-\hat{a} x_{i}-\hat{b}\right)^{2} \quad \text { and } \quad R M S E:=\sqrt{M S E}
$$

The mean square error is sometimes called residual variance.
We also introduce some notation.
Definition 3.2. Let $n$ be a positive integer. We define $\Gamma\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ to be the union of the $n$ segments $(0,0)-\left(1, a_{1}\right)$, $\left(1, a_{1}\right)-\left(2, a_{2}\right), \ldots\left(n-1, a_{n-1}\right)-\left(n, a_{n}\right)$. (Note that we have $(n+1)$ points, and that, without loss of generality, we suppose that the curve begins at the origin.)

### 3.1. Why is MSE not satisfactory to measure fluctuations?

In this section we show two curves having same length: one is "fluctuating", the other increases quickly, but their residual variances are both equal to 6; see Fig. 1. Note that when we say that the first curve is more "fluctuating" than the second one, it means for example that for a variation of weight or of BMI, the first curve is really fluctuating, while the second one just shows some (possibly quick) growth (also see Remark 3.6 and the beginning of Section 4.3).

### 3.2. Comparing MSE and inconstancy

Are residual variance and inconstancy of a curve comparable? We will prove that this is not the case, even for very simple curves, thanks to two easy lemmas.


Fig. 1. Same MSE.
Lemma 3.3. Let $\mathcal{R}\left(\Gamma\left(a_{1}, a_{2}\right)\right)$ be the residual variance of the curve $\Gamma\left(a_{1}, a_{2}\right)$. Then

$$
\mathcal{R}\left(\Gamma\left(a_{1}, a_{2}\right)\right)=\frac{\left(2 a_{1}-a_{2}\right)^{2}}{6}
$$

Proof. The computation is straightforward. The linear regression straight line is parallel to $(0,0)-\left(2, a_{2}\right)$, and it contains the center of gravity of the triangle $(0,0),\left(1, a_{1}\right),\left(2, a_{2}\right)$; or simply compute from Definition $3.1: \bar{x}=1, \bar{y}=\left(a_{1}+a_{2}\right) / 3$, $\hat{a}=a_{2} / 2$, and $\hat{b}=\left(2 a_{1}-a_{2}\right) / 6$, hence $\mathcal{R}\left(\Gamma\left(a_{1}, a_{2}\right)\right)=\left(2 a_{1}-a_{2}\right)^{2} / 6$.

Lemma 3.4. Let $\Gamma\left(a_{1}, a_{2}\right)$ be the curve defined as the union of the two straight line segments $(0,0)-\left(1, a_{1}\right)$ and $\left(1, a_{1}\right)-\left(2, a_{2}\right)$. Then, $\mathcal{I}\left(\Gamma\left(a_{1}, a_{2}\right)\right)$, the inconstancy of $\Gamma\left(a_{1}, a_{2}\right)$, is given by

$$
\mathcal{I}\left(\Gamma\left(a_{1}, a_{2}\right)\right)=\frac{2}{1+\frac{\sqrt{a_{2}^{2}+4}}{\sqrt{a_{1}^{2}+1}+\sqrt{\left(a_{2}-a_{1}\right)^{2}+1}}}
$$

Proof. The proof is again straightforward. The length of $\Gamma\left(a_{1}, a_{2}\right)$ and the perimeter of the convex hull of $\Gamma\left(a_{1}, a_{2}\right)$ are given respectively by

$$
\sqrt{a_{1}^{2}+1}+\sqrt{\left(a_{2}-a_{1}\right)^{2}+1} \text { and } \sqrt{a_{1}^{2}+1}+\sqrt{\left(a_{2}-a_{1}\right)^{2}+1}+\sqrt{a_{2}^{2}+4}
$$

We can now state the non-comparability of residual variance and inconstancy.
Proposition 3.5. Residual variance and inconstancy of a curve are not comparable. More precisely, there exist four curves $\Gamma_{i}$, $i=1,2,3,4$, (see Fig. 2 and the proof below) such that, if $\mathcal{R}\left(\Gamma_{i}\right)$ and $\mathcal{I}\left(\Gamma_{i}\right)$ are their residual variances and inconstancies, then the following inequalities hold:

$$
\begin{aligned}
& \mathcal{R}\left(\Gamma_{1}\right)<\mathcal{R}\left(\Gamma_{2}\right)<\mathcal{R}\left(\Gamma_{3}\right)<\mathcal{R}\left(\Gamma_{4}\right) \\
& \mathcal{I}\left(\Gamma_{4}\right)<\mathcal{I}\left(\Gamma_{2}\right)<\mathcal{I}\left(\Gamma_{1}\right)<\mathcal{I}\left(\Gamma_{3}\right) .
\end{aligned}
$$

Proof. Using Lemmas 3.3 and 3.4 above, we get the residual variances $\mathcal{R}\left(\Gamma_{i}\right)$ and inconstancies $\mathcal{I}\left(\Gamma_{i}\right)$ of the following curves $\Gamma_{i}$

$$
\begin{array}{lll}
\Gamma_{1}:=\Gamma(1,0) & \mathcal{R}\left(\Gamma_{1}\right)=\frac{2}{3} \approx 0.67 & \mathcal{I}\left(\Gamma_{1}\right)=\frac{2 \sqrt{2}}{1+\sqrt{2}} \approx 1.17 \\
\Gamma_{2}:=\Gamma(0,3) & \mathcal{R}\left(\Gamma_{2}\right)=\frac{3}{2} \approx 1.50 & \mathcal{I}\left(\Gamma_{2}\right)=\frac{2+2 \sqrt{10}}{1+\sqrt{10}+\sqrt{13}} \approx 1.07 \\
\Gamma_{3}:=\Gamma(2,0) & \mathcal{R}\left(\Gamma_{3}\right)=\frac{8}{3} \approx 2.67 & \mathcal{I}\left(\Gamma_{3}\right)=\frac{2 \sqrt{5}}{1+\sqrt{5}} \approx 1.38 \\
\Gamma_{4}:=\Gamma(0,5) & \mathcal{R}\left(\Gamma_{4}\right)=\frac{25}{6} \approx 4.17 & \mathcal{I}\left(\Gamma_{4}\right)=\frac{2+2 \sqrt{26}}{1+\sqrt{26}+\sqrt{29}} \approx 1.06
\end{array}
$$



Fig. 2. Comparing residual variance and inconstancy.


Fig. 3. Varying the intermediate value.
Remark 3.6. Comparing $\mathcal{I}\left(\Gamma_{2}\right)$ and $\mathcal{I}\left(\Gamma_{4}\right)$ shows again that "fluctuating" is not the same as "growing". More generally, with the notation of Lemma 3.4 above, looking at $\mathcal{I}(\Gamma(0, x))$, shows that $\mathcal{I}(\Gamma(0,0))=1=\lim _{x \rightarrow \infty} \mathcal{I}(\Gamma(0, x))$. When $x$ varies from 0 to $\infty$ the quantity $\mathcal{I}(\Gamma(0, x))$ increases from 1 to a small value $>1$ then it decreases back to 1 .
Remark 3.7. There are other quantities that also "measure" the fluctuations of a curve. For example, keeping the notations of Definition 3.1: the total variation is defined as the mean of $\left(y_{i}-\bar{y}\right)^{2}$, i.e., as $\left(\sum_{1 \leq i \leq n}\left(y_{i}-\bar{y}\right)^{2}\right) / n$; the maximal distance is defined as $\max _{1 \leq i \leq n}\left|y_{i}-\hat{a} x_{i}-\hat{b}\right|$. The reader can easily compute these quantities for the curve $\Gamma\left(a_{1}, a_{2}\right)$ and check that they are not comparable to the inconstancy of $\Gamma\left(a_{1}, a_{2}\right)$.

Total variation: $\frac{2\left(a_{1}^{2}+a_{2}^{2}-a_{1} a_{2}\right)}{9}$. Maximal distance: $\frac{\left|2 a_{1}-a_{2}\right|}{3}$.

## 4. Pertinence of the use of inconstancy: simple arguments

### 4.1. A single fluctuation

Taking again the example in the previous section of a curve consisting of two straight line segments, let us vary the value $a_{1}$, say $x:=a_{1}$, and fix $a_{2}=a$ (see Fig. 3). The inconstancy $\mathcal{I}(\Gamma(x, a))$ is thus given by

$$
\mathcal{I}(\Gamma(x, a))=\frac{2}{1+\frac{\sqrt{a^{2}+4}}{\sqrt{x^{2}+1}+\sqrt{(a-x)^{2}+1}}}
$$

This map $x \rightarrow \mathcal{I}(\Gamma(x, a))$ is increasing for $x \geq a / 2$, which is in agreement with what a "fluctuation" should be.
It is clear that $\mathcal{I}(\Gamma(x, a))=\mathcal{I}(\Gamma(a-x, a))$, which shows that the line $x=a / 2$ is a symmetry axis. In other words, "exchanging" the two segments, more precisely replacing $((0,0)-(1, x)),((1, x)-(2, a))$ by $((0,0)-(1, a-x)),((1, a-x)-$ $(2, a))$, does not change the inconstancy (see Fig. 4). Of course this is a necessary condition for a fluctuation criterion.


Fig. 4. Symmetry.


Fig. 5. Graph of $\mathcal{I}(\Gamma(x, a))$.


Fig. 6. Varying $a$ in the graph of $\mathcal{I}(\Gamma(x, a))$.

It is easy to show that $\mathcal{I}(\Gamma(a / 2, a))=1$ (no fluctuation) and $\lim _{x \rightarrow+\infty} \mathcal{I}(\Gamma(x, a))=2$ (when $x$ is large, the value of $x$ is not really important, the inconstancy is close to 2 ). We also have that $(\mathcal{I}(\Gamma(x, a)))^{\prime}=0$ if and only if $x=a / 2$. In particular, the graph of the function $\mathcal{I}(\Gamma(x, a))$ has the aspect shown in Figs. 5 and 6.

We note that the curve is "flat" in the neighborhood of $a / 2$, or even for $x \in(0, a)$; see Fig. 5 . This means that the inconstancy $(\mathcal{I}(\Gamma(x, a)))$, which is equal to 1 for $x=a / 2$, remains close to 1 when the two slopes of the curve have the


Fig. 7. Signs of slopes.
same sign, while it is larger when the signs of the slopes are opposite, which correctly describes what a fluctuation should be (the MSE does not have this property); see Fig. 7. Also the inconstancy $(\mathcal{I}(\Gamma(x, a)))$ tends quickly to 2 when $a$ is small; see Fig. 6.

### 4.2. General remarks

If we look more generally at the inconstancy of $\Gamma\left(a_{1}, a_{2}, \ldots, a_{n}\right)$, what will clearly matter for its size is the sequence of slopes: growth and signs of consecutive terms are crucial characteristics of the sequence, which corresponds to the intuitive idea of "fluctuation". Of course we always have the straightforward bounds

$$
1 \leq \mathcal{I}\left(\Gamma\left(a_{1}, a_{2}, \ldots, a_{n}\right)\right) \leq n
$$

(count the possible number of intersection points of $\Gamma\left(a_{1}, a_{2}, \ldots, a_{n}\right)$ with a random straight line and apply Theorem 2.1).
Conversely the inconstancy may be used to discriminate between curves, i.e., to decide whether a curve fluctuates more than another, when the "visual aspect" does not suffice to assert an intuitive answer. We give two examples.

### 4.3. Fluctuations of curves with four points

In Fig. 8 inconstancies permit to discriminate between "less fluctuating" and "more fluctuating" curves, though there is no visual evidence of which curve fluctuates more. It is interesting to note that the maximum of the function is not really taken into account, only the variations count (look, e.g., at the two examples with inconstancy 1.58 in Fig. 8).

### 4.4. A case where inconstancy does not discriminate

The lengths and inconstancies of the two curves $\Gamma(\sqrt{3}, \sqrt{3}, 0)$ and $\Gamma(2 \sqrt{6} / 5,4 \sqrt{6} / 5,0)$ (see Fig. 9) are the same.

## 5. Inconstancy of sequences

Inconstancy of (finite or infinite) sequences can be defined in a straightforward way from what precedes.
Definition 5.1. Let $\left(u_{n}\right)_{0 \leq n \leq N}$ be a finite sequence of real numbers, with $u_{0}=0$ say. Let $\Gamma_{n}$ be the union of the straight line segments $(0,0)-\left(1, u_{1}\right),\left(1, u_{1}\right)-\left(2, u_{2}\right), \ldots,\left(n-1, u_{n-1}\right)-\left(N, u_{N}\right)$, then the inconstancy of $\left(u_{n}\right)_{0 \leq n \leq N}$ is defined by

$$
\mathcal{I}\left(\left(u_{n}\right)_{0 \leq n \leq N}\right):=\mathcal{I}\left(\Gamma_{N}\right) .
$$

Let $\left(u_{n}\right)_{n \geq 0}$ be an infinite sequence of real numbers, with $u_{0}=0$ say. Then the inconstancy of $\left(u_{n}\right)_{n \geq 0}$ is defined by

$$
\mathcal{I}\left(\left(u_{n}\right)_{n \geq 0}\right):=\limsup _{N \rightarrow \infty} \mathcal{I}\left(\left(u_{n}\right)_{0 \leq n \leq N}\right) \quad \text { (or } \lim _{N \rightarrow \infty} \mathcal{I}\left(\left(u_{n}\right)_{0 \leq n \leq N}\right) \text { if the limit exists). }
$$

The inconstancy of an infinite sequence depends in particular of how long and frequently the sequence levels off: this is particularly clear for binary sequences as shown in Theorem 5.2 below.



Fig. 8. Inconstancy discriminates between fluctuations.


Fig. 9. Same inconstancy.

Theorem 5.2. - (i) Let $\left(u_{n}\right)_{0 \leq n \leq N}$ be a finite sequence taking two values 0 and $h>0$, with $u_{0}=0$. Let $\alpha \geq 1$ be the index such that $u_{0}=u_{1}=\cdots=u_{\alpha-1}=0$ and $u_{\alpha} \neq 0$. In other words $\alpha$ is the length of the longest initial string of 0's. Analogously let $\beta$ be the length of the longest final string of 0's. If $\beta=0$, let $\gamma \geq 0$ be the largest index such that $u_{\gamma}=0$. Let $\mathcal{N}_{00}, \mathcal{N}_{h h}, \mathcal{N}_{0 h}, \mathcal{N}_{h 0}$
be respectively the number of blocks of the form $00, h h, 0 h, h 0$ in the sequence. Then

$$
\mathcal{I}\left(\left(u_{n}\right)_{0 \leq n \leq N}\right)= \begin{cases}2 \frac{\mathcal{N}_{00}+\mathcal{N}_{h h}+\left(\sqrt{1+h^{2}}\right)\left(\mathcal{N}_{0 h}+\mathcal{N}_{h 0}\right)}{\sqrt{h^{2}+\alpha^{2}}+N-\alpha-\beta+\sqrt{h^{2}+\beta^{2}}+N} & \text { if } \beta>0 \\ 2 \frac{\mathcal{N}_{00}+\mathcal{N}_{h h}+\left(\sqrt{1+h^{2}}\right)\left(\mathcal{N}_{0 h}+\mathcal{N}_{h 0}\right)}{\sqrt{h^{2}+\alpha^{2}}+N-\alpha+\sqrt{h^{2}+(N-\gamma)^{2}}+\gamma} & \text { if } \beta=0 .\end{cases}
$$

- (ii) Let $\left(u_{n}\right)_{n \geq 0}$ be an infinite sequence taking two values 0 and $h>0$, with $u_{0}=0$. We make the assumption that the frequencies of occurrences of the blocks $00, h h, 0 h, h 0$ in the sequence exist and are respectively equal to $\mathcal{F}_{00}, \mathcal{F}_{h h}, \mathcal{F}_{0 h}, \mathcal{F}_{h 0}$. Then

$$
\mathcal{I}\left(\left(u_{n}\right)_{n \geq 0}\right)=\mathcal{F}_{00}+\mathcal{F}_{h h}+\left(\sqrt{1+h^{2}}\right)\left(\mathcal{F}_{0 h}+\mathcal{F}_{h 0}\right)=1+\left(\sqrt{1+h^{2}}-1\right)\left(\mathcal{F}_{0 h}+\mathcal{F}_{h 0}\right)
$$

Similarly let $\left(u_{n}\right)_{n \geq 0}$ be an infinite sequence taking only finitely many real values, and let $H$ be this set of values. We make the assumption that the frequencies of occurrences of all length-2 blocks $j j^{\prime}\left(j, j^{\prime} \in H\right)$ exist and are respectively equal to $\mathcal{F}_{j j^{\prime}}$. Then

$$
\begin{aligned}
\mathcal{I}\left(\left(u_{n}\right)_{n \geq 0}\right) & =\sum_{j \in H} \mathcal{F}_{j j}+\sum_{j, j^{\prime} \in H, j<j^{\prime}}\left(\sqrt{1+\left(j^{\prime}-j\right)^{2}}\right)\left(\mathcal{F}_{j j^{\prime}}+\mathcal{F}_{j^{\prime} j}\right) \\
& =1+\sum_{j, j^{\prime} \in H, j<j^{\prime}}\left(\sqrt{1+\left(j^{\prime}-j\right)^{2}}-1\right)\left(\mathcal{F}_{j j^{\prime}}+\mathcal{F}_{j^{\prime} j}\right) .
\end{aligned}
$$

Proof. First let $\left(u_{n}\right)_{0 \leq n \leq N}$ be a finite sequence taking two values 0 and $h>0$. Let $\alpha \geq 1$ be the index such that $u_{0}=u_{1}=\cdots=u_{\alpha-1}=0$ and $u_{\alpha} \neq 0$. In other words $\alpha$ is the length of the longest initial string of 0 's. Analogously let $\beta \geq 0$ be the length of the longest final string of 0 's. Finally, if $\beta=0$, let $\gamma \geq 0$ be the largest index such that $u_{\gamma}=0$. It is almost immediate that the convex hull of the curve $\Gamma_{N}$ consists of the four straight line segments $((0,0)-(\alpha, h)),((\alpha, h)-$ $(N-\beta, h)),((N-\beta, h)-(N, 0)),((0,0)-(N, 0))$ if $\beta>0$, and $((0,0)-(\alpha, h)),((\alpha, h)-(N, h)),((0,0)-(\gamma, 0)),((\gamma, 0)-$ $(N, h)$ ) if $\beta=0$ (there are only three segments if $\gamma=0$, which implies $\alpha=1$ ). Hence

$$
\delta\left(\Gamma_{N}\right)= \begin{cases}\sqrt{h^{2}+\alpha^{2}}+N-\alpha-\beta+\sqrt{h^{2}+\beta^{2}}+N & \text { if } \beta>0 \\ \sqrt{h^{2}+\alpha^{2}}+N-\alpha+\sqrt{h^{2}+(N-\gamma)^{2}}+\gamma & \text { if } \beta=0\end{cases}
$$

while the length of the curve is

$$
\ell\left(\Gamma_{N}\right)=\mathcal{N}_{00}+\mathcal{N}_{h h}+\left(\sqrt{1+h^{2}}\right)\left(\mathcal{N}_{0 h}+\mathcal{N}_{h 0}\right)
$$

This gives the first part of the theorem, namely

$$
\mathcal{I}\left(\left(u_{n}\right)_{0 \leq n \leq N}\right)= \begin{cases}2 \frac{\mathcal{N}_{00}+\mathcal{N}_{h h}+\left(\sqrt{1+h^{2}}\right)\left(\mathcal{N}_{0 h}+\mathcal{N}_{h 0}\right)}{\sqrt{h^{2}+\alpha^{2}}+N-\alpha-\beta+\sqrt{h^{2}+\beta^{2}}+N} & \text { if } \beta>0 \\ 2 \frac{\mathcal{N}_{00}+\mathcal{N}_{h h}+\left(\sqrt{1+h^{2}}\right)\left(\mathcal{N}_{0 h}+\mathcal{N}_{h 0}\right)}{\sqrt{h^{2}+\alpha^{2}}+N-\alpha+\sqrt{h^{2}+(N-\gamma)^{2}}+\gamma} & \text { if } \beta=0 .\end{cases}
$$

In order to prove the second part of the theorem, we will directly address the case of a sequence $\left(u_{n}\right)_{n \geq 0}$ taking any finite number of values (the proof is simpler than our original one, thanks to a remark of one of the referees). The length of the curve $\Gamma_{n}$ clearly is

$$
\sum_{j \in H} \mathcal{N}_{j j}+\sum_{j, j^{\prime} \in H, j<j^{\prime}} \sqrt{1+\left(j^{\prime}-j\right)^{2}}\left(\mathcal{N}_{i j^{\prime}}+\mathcal{N}_{j^{\prime} j}\right) .
$$

The perimeter of the convex hull of $\Gamma_{n}$ satisfies

$$
2 \sqrt{N^{2}+u_{N}^{2}} \leq \delta\left(\Gamma_{n}\right) \leq 2 N+M_{N}, \quad \text { where } M_{N}:=\max \left\{u_{n}, 0 \leq n \leq N\right\}
$$

(the inequality on the left is due to the fact that the perimeter is larger than twice the distance between $(0,0)$ and $\left(N, u_{N}\right)$; the right inequality comes from the fact that the length of the convex hull is less than the perimeter of the rectangle $(0,0)-$ $\left.\left(0, M_{N}\right)-\left(N, M_{N}\right)-(N, 0)\right)$. Since the sequence $\left(u_{n}\right)_{n \geq 0}$ takes only finitely many values, this shows that

$$
\delta\left(\gamma_{N}\right)=2 N+O(1)
$$

Hence

$$
\mathcal{I}\left(\left(u_{n}\right)_{n \geq 0}\right)=\sum_{j \in H} \mathcal{F}_{j j}+\sum_{j, j^{\prime} \in H, j<j^{\prime}} \sqrt{1+\left(j^{\prime}-j\right)^{2}}\left(\mathcal{F}_{j j^{\prime}}+\mathcal{F}_{j^{\prime} j}\right) .
$$

But

$$
\sum_{j \in H} \mathcal{F}_{j j}+\sum_{j, j^{\prime} \in H,, j<j^{\prime}}\left(\mathcal{F}_{j j^{\prime}}+\mathcal{F}_{j^{\prime} j}\right)=1
$$

hence the result.

## 6. Computing the inconstancy of classical sequences

Using Theorem 5.2 we see that the inconstancy of an infinite binary ( 0,1 )-sequence must belong to the interval $[1, \sqrt{2}]$. Bound 1 is reached if and only if $\mathcal{F}_{01}+\mathcal{F}_{10}=0$, i.e., $\mathcal{F}_{01}=\mathcal{F}_{10}=0$. Bound $\sqrt{2}$ is reached when $\mathcal{F}_{01}+\mathcal{F}_{10}=1$, i.e., $\mathcal{F}_{00}+\mathcal{F}_{11}=0$, i.e., $\mathcal{F}_{00}=\mathcal{F}_{11}=0$. Let us compute the inconstancy of some classical binary sequences.

### 6.1. Periodic sequences

The sequence $\left(0^{d} 1\right)^{\infty}=(00 \ldots 01)^{\infty}$ (periodic of period $(d+1)$, where the period pattern consists of $d$ symbols 0 followed by one symbol 1 ). It is easy to compute $\mathcal{F}_{00}=\frac{d-1}{d+1}, \mathcal{F}_{11}=0, \mathcal{F}_{01}=\mathcal{F}_{10}=\frac{1}{d+1}$. Hence

$$
\mathcal{I}\left(\left(0^{d} 1\right)^{\infty}\right)=\frac{d-1+2 \sqrt{2}}{d+1}
$$

In particular, $\mathcal{I}\left((01)^{\infty}\right)=\sqrt{2}=1.414 \ldots$ while $\mathcal{I}\left(\left(0^{d} 1\right)^{\infty}\right)$ tends to 1 when $d$ tends to infinity: this corresponds to the fact that the curve becomes more and more flat when $d$ increases. The case $d=1$ is somehow the worst case among periodic and nonperiodic binary sequences in terms of levelling off (or flatness).

### 6.2. Random sequences

A random sequence of 0 's and 1 's. For almost all binary sequences we have $\mathcal{F}_{00}=\mathcal{F}_{11}=\mathcal{F}_{01}=\mathcal{F}_{10}=\frac{1}{4}$. Hence if $\left(r_{n}\right)_{n \geq 0}$ is "a random sequence" of 0 's and 1 's, then

$$
\mathcal{I}\left(\left(r_{n}\right)_{n \geq 0}\right)=\frac{1+\sqrt{2}}{2}=1.207 \ldots
$$

### 6.3. Some automatic sequences

We first recall a few notions of combinatorics on words; see, e.g. [5]. A finite set is called an alphabet. Its elements are called letters. For an alphabet $A$, we let $A^{*}$ denote the free monoid spanned by $A$ and equipped with the concatenation. Elements of $A^{*}$ are called words on $A$; the length of the word $a_{1} a_{2} \ldots a_{n}$, with $a_{i} \in A$, is $n$. Homomorphisms of monoids are called morphisms. A morphism from $A^{*}$ to $B^{*}$ is determined by the images of the letters in $A$. It is called uniform if the images of all letters have the same length. The transition matrix of a morphism $\sigma: A^{*} \rightarrow B^{*}$ counts the number of times the letter $b_{j}$ in $B$ occurs in $\sigma\left(a_{i}\right)$. Finally a sequence is called automatic if it is the pointwise image of a fixed point of a nontrivial uniform morphism.

- Recall that the Thue-Morse sequence with values 0 and 1 can be defined as the fixed point beginning with 0 of the morphism $0 \rightarrow 01,1 \rightarrow 10$ (see, e.g., [4]): it is the most famous example of automatic sequences (see, e.g., [5]). The first few terms of the Thue-Morse sequence $\left(m_{n}\right)_{n \geq 0}$ are


## $011010011001011010 \ldots$

The frequencies of occurrences of blocks of length 2 are given by $\mathcal{F}_{00}=\mathcal{F}_{11}=\frac{1}{6}$ and $\mathcal{F}_{01}=\mathcal{F}_{10}=\frac{1}{3}$ : this is a classical exercise that involves the morphism on four letters defined by $a \rightarrow a b, b \rightarrow c a, c \rightarrow c d, d \rightarrow a c$. An alternative proof consists of noting that the sequence $\left(\left(m_{n}+m_{n+1}\right) \bmod 2\right)_{n \geq 0}$ is the period doubling sequence, i.e., the fixed point of the morphism $1 \rightarrow 10,0 \rightarrow 11$; the sum of frequencies of the blocks 01 and 10 in the Thue-Morse sequence is thus the frequency of 1 's in the period doubling sequence which is easily seen to be $2 / 3$ (look at the transition matrix of the morphism $1 \rightarrow 10,0 \rightarrow 11$ ). Hence

$$
\mathcal{I}\left(\left(m_{n}\right)_{n \geq 0}\right)=\frac{1+2 \sqrt{2}}{3}=1.276 \ldots
$$

Note that the "high" value of this inconstancy is related to the absence of long strings of 0's or of 1's: namely the Thue-Morse sequence does not contain the blocks 000 and 111.

- The Shapiro-Rudin sequence $\left(r_{n}\right)_{n \geq 0}$ with values 0 and 1 can be defined as the sequence of parities of the number of (possibly overlapping) 11 's in he binary expansions of the integers $0,1,2, \ldots, n \ldots$ (see, e.g., [5]). It is clear that the sum
of frequencies of occurrences of the blocks 01 and 10 is the frequency of occurrences of the letter 1 in the sequence $\left(r_{n}^{\prime}\right)_{n \geq 0}$ defined by $r_{n}^{\prime}:=\left(r_{n}+r_{n+1}\right) \bmod 2$. This last sequence is easily seen to be the pointwise image under the map $a \rightarrow 0$, $b \rightarrow 0, c \rightarrow 1, d \rightarrow 1$ of the infinite fixed point of the morphism $a \rightarrow a b, b \rightarrow c d, c \rightarrow a d, d \rightarrow c b$. (Hint: prove that both this pointwise image and the sequence $\left(r_{n}^{\prime}\right)_{n \geq 0}$ satisfy the recursive relations $r_{4 n}^{\prime}=0, r_{4 n+1}^{\prime}=r_{2 n}^{\prime}, r_{4 n+2}^{\prime}=1$, $r_{4 n+3}^{\prime}=1+r_{2 n+1}^{\prime} \bmod 2$, with $r_{0}^{\prime}=0$. From this it is straightforward that the frequency of occurrences of 1 in the sequence $\left(r_{n}^{\prime}\right)_{n \geq 0}$ is equal to $1 / 2$. Hence

$$
\mathcal{I}\left(\left(r_{n}\right)_{n \geq 0}\right)=\frac{1+\sqrt{2}}{2}=1.207 \ldots
$$

which is the same inconstancy as for a random sequence.

- The (regular) paperfolding sequence $\left(z_{n}\right)_{n \geq 0}$ with values 0 and 1 can be defined by $z_{4 n}=0, z_{4 n+1}=1, z_{2 n+1}=z_{n}$. Reasoning as for the Shapiro-Rudin sequence (left to the reader) leads to

$$
\mathcal{I}\left(\left(r_{n}\right)_{n \geq 0}\right)=\frac{1+\sqrt{2}}{2}=1.207 \ldots
$$

which is again the same inconstancy as for a random sequence.

### 6.4. Sturmian sequences

Recall that a Sturmian sequence can be defined as a (binary) sequence having exactly $n+1$ blocks of length $n$ for every integer $n \geq 1$ (see, e.g., [5,34]). In particular Sturmian sequences are not ultimately periodic, and the blocks 00 and 11 cannot both occur in a same Sturmian sequence. Since interchanging 0's and 1's in a Sturmian sequence gives a Sturmian sequence, we may suppose that no 11 occurs. But then the frequencies of occurrences of the blocks 01 and 10 in the sequence are both equal to the frequency of occurrence of 1 , hence to the slope of the Sturmian sequence (see, e.g., [5, Theorem 10.5.8, page 318]). Thus the inconstancy of a Sturmian sequence of slope $\alpha \in(0,1)$ without the block 11 in it (resp. of slope $1-\alpha \in(0,1)$ without the block 00 in it) is

$$
\mathcal{I}=1+2(\sqrt{2}-1) \alpha
$$

Recall that if the sequence does not contain the block 11 , then $\alpha$ belongs to $(0,1 / 2)$, hence as expected $\mathcal{I}$ belongs to (1, $\sqrt{2}$ ).
Remark 6.1. A possible application of inconstancy of infinite sequences can be to "predict" the $n$th term of a very long (or infinite) sequence knowing its first $n-1$ terms: if $n$ is large enough, $u_{n}$ "should" be close to a value minimizing the difference $\left|\mathcal{I}\left(\Gamma_{n}\right)-\mathcal{I}\left(\Gamma_{n-1}\right)\right|$.

Remark 6.2. A different way of defining the inconstancy of a binary sequence could be to interpret it as a sequence on the alphabet $\{\mathrm{L}(\mathrm{eft}), \mathrm{R}$ (ight) $\}$. Then to associate with this (LR) sequence a 2 D curve drawn on the lattice $\mathbb{Z}^{2}$, consisting of horizontal and vertical segments. The first segment is $(0,0)-(1,0)$; then for each value of the LR sequence we make a $\pm \pi / 2$ turn. The inconstancy of the sequence could be defined as the inconstancy of the curve obtained that way. The reader will have recognized curves studied, e.g., in [44], where paperfolding sequences enter the picture. This notion of inconstancy for sequences would thus be terminologically closer to the "folding index" of [15]. Since the choice of $\pm \pi / 2$ is arbitrary (another angle could have been chosen), it is not clear whether this definition is pertinent or if one should consider all possible angles, thus obtaining a set of inconstancies for any given sequence.

## 7. Algorithmic aspects

In order to compute the inconstancy $\mathcal{I}(\Gamma):=\frac{2 \ell(\Gamma)}{\delta(\Gamma)}$ of a piecewise affine curve $\Gamma$, the perimeter of the convex hull of $\Gamma$ is needed. Hence we need to construct the convex hull of a finite set consisting of, say, $n$ points. Several algorithms are available, their complexity is in $\mathcal{O}(n \log n)$ (see, e.g., the Graham scan studied in [27], the Jarvis march studied in [29]; see also, e.g., the papers [51,50]-in particular [50] gives an optimal real-time algorithm for planar convex hulls).

Implementations of these algorithms are classical in usual softwares: for example the command convhull in Maple (with the package Convex), the command ConvexHull in Mathematica, the command convex_hull in Scilab, or the command convhull (see also convhulln) in Matlab. Also note that Qhull computes convex hulls, Delaunay triangulations, Voronoi diagrams, halfspace intersections about a point, furthest-site Delaunay triangulations, and furthestsite Voronoi diagrams (see http://www.qhull.org/). Demonstrations of computations can be found on several sites; see e.g., http://www.piler.com/convexhull/.
http://www.cs.princeton.edu/courses/archive/fall08/cos226/demo/ah/GrahamScan.html.
http://www.cs.princeton.edu/courses/archive/fall08/cos226/demo/ah/ConvexHull.html.
http://www.cse.unsw.edu.au/~lambert/java/3d/hull.html.

## 8. Conclusion

Inspired by the theorem of Cauchy-Crofton, the inconstancy of a curve could be a way of detecting large fluctuations of a curve, different from (and hopefully better than) usual indexes such as the residual variance. We intend to test this idea in three domains: fluctuations of biological parameters [61], fluctuations of the stockmarket [3] and smoothness of musical themes [1]. Two other directions could be the following. First, a way of discriminating between models that describe a given phenomenon with the same error bound (e.g., prediction of electric load and consumption) could be to choose the model for which the difference between data and predictions has maximal inconstancy (when the inconstancy is close to 1 , this difference is "quasi-affine"; this means that there is a "quasi-affine" bias in the model that can/should be corrected a priori). Second, we alluded to fractal-like "chaotic" (disordered) curves in the introduction; coming across, e.g., the paper [12] we recall that measuring the "complexity" of geographic objects classically involves their fractal dimension and, e.g., their "length"; we could also think of looking at their inconstancy (typically how complicated a river can be, i.e., how far from straight it looks, can be measured by the number of intersection points with a random straight line). A natural question then occurs: to what extent fractal dimension and inconstancy are related? Or what can be said of the intersection with straight lines of a set with given fractal dimension? Such questions also make sense for (in)finite sequences, in particular in view of Remark 6.2. Of course the length of such curves is usually infinite while the length of the convex envelop is finite (think of the von Koch curve). What could be looked at for fractals obtained by "iteration" is the inconstancy at each finite step of the iteration: it is conceivable that the fractal dimension shows up, though this is not the case for the von Koch curve. Some ideas about these questions can be found, e.g., in [58,35,2,44,40,41], in particular in relation with the entropy of a curve, as discussed in several papers of Mendès France. We will conclude this paper with that notion of entropy for a plane curve. Let $p_{n}$ be the probability that a straight line cuts the plane curve $\Gamma$ in exactly $n$ points, then the theorem of Cauchy-Crofton says that

$$
\sum_{n \geq 1} n p_{n}=\frac{2 \ell(\Gamma)}{\delta(\Gamma)}
$$

It is natural to define the entropy of $\Gamma$ by

$$
H(\Gamma):=\sum_{n \geq 1} p_{n} \log \frac{1}{p_{n}}
$$

Now how large can this expression be? Define the set of sequences $\mathcal{P}$ by

$$
\mathcal{P}:=\left\{\left(p_{n}\right)_{n \geq 1} ; p_{n} \geq 0, \sum_{n \geq 1} p_{n}=1, \sum_{n \geq 1} n p_{n}=\frac{2 \ell(\Gamma)}{\delta(\Gamma)}\right\}, \quad \text { and let } \quad H_{\max }(\Gamma):=\max _{\mathcal{P}} H(\Gamma) .
$$

It can be proven (see [22] for details, also see [39]) that

$$
H_{\max }(\Gamma)=\log \frac{2 \ell(\Gamma)}{\delta(\Gamma)}+\frac{\beta}{e^{\beta}-1}
$$

where $\beta:=\log \frac{2 \ell(\Gamma)}{2 \ell(\Gamma)-\delta(\Gamma)}$ (the quantity $\beta$ can be seen as the inverse of the temperature of the curve).
A modified definition is thus proposed in [38], namely

$$
H(\Gamma):=\log \frac{2 \ell(\Gamma)}{\delta(\Gamma)}
$$

This definition was used in several papers (see, e.g., [26,21,7]). With our terminology, it reads, as noted by Mendès France, "the entropy is the logarithm of the inconstancy". The reader might think of comparing this statement with the classical Weber-Fechner law in psychophysics according to which "sensation is proportional to the logarithm of excitation" ([23]; see also http://psychclassics.yorku.ca/Fechner/).

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[^1]:    ${ }^{1}$ In [52] Quetelet asserts that weights vary like heights squared for adults but more like (heights) ${ }^{5 / 2}$ for children (see p. 52-53, and p. 61), while the "simplified" definition of the BMI is the ratio of the weight by the height squared.

