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Existence of positive periodic solutions for higher-order ordinary differential equations☆

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ABSTRACT

This paper deals with the existence of positive periodic solutions for the nth-order ordinary differential equation

$$u^{(n)}(t) = f(t,\ u(t),\ u'(t),\ \ldots,\ u^{(n-1)}(t)),$$

where $n \geq 2$, $f : \mathbb{R} \times [0, \infty) \times \mathbb{R}^{n-1} \to \mathbb{R}$ is a continuous function and $f(t, x_0, x_1, \ldots, x_{n-1})$ is 2π-periodic in t. Some existence results of positive 2π-periodic solutions are obtained assuming f satisfies some superlinear or sublinear growth conditions on $x_0, x_1, \ldots, x_{n-1}$. The discussion is based on the fixed point index theory in cones.

1. Introduction

In this paper, we are concerned with the existence of positive 2π-periodic solutions for the nth-order ordinary differential equation

$$u^{(n)}(t) = f(t, u(t), u'(t), \ldots, u^{(n-1)}(t)), \tag{1}$$

where $n \geq 2$ is a positive integer, $f : \mathbb{R} \times [0, \infty) \times \mathbb{R}^{n-1} \to \mathbb{R}$ is a continuous function and $f(t, x_0, x_1, \ldots, x_{n-1})$ is 2π-periodic with respect to t.

The existence problem of periodic solutions is an important topic in qualitative analysis of ordinary differential equations. In recent years, the existence of positive periodic solutions of some ordinary differential equations with special forms has been researched by several authors; see [1–16]. For the second-order equation

$$u''(t) + a(t)u(t) = f(t, u(t)), \tag{2}$$

where $a \in C(\mathbb{R}, \mathbb{R}^+)$ is a ω-periodic function, the existence and multiplicity of positive periodic solutions are discussed by the authors of [1–6]. One of the well-known results is that if f satisfies the superlinear growth or sublinear condition, Eq. (2) has at least one positive periodic solution; see [1,3,4]. This result is concluded from Krasnoselskii's fixed point theorem of cone expansion or compression, and it is improved and extended by more precise theory of the fixed point index in cones; see [2,5,6]. In [7], Lui, Ge and Gui considered the periodic problem of the second-order equation

$$-(pu')'(t) + q(t)u(t) = f(t, u(t)),$$

where the coefficients $p, q \in C(\mathbb{R}, \mathbb{R}^+)$ are ω-periodic and $p(t) > 0$ for every $t \geq 0$. Using Leggett–Williams fixed point theorem in cones, they obtained the existence result of three positive periodic solutions. This work was recently extended

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by Anderson and Avery in [8] to the second-order equation

$$-(pu')'(t) - r(t)p(t)u'(t) + q(t)u(t) = f(t, u(t))$$

and the corresponding discrete version.

The existence of positive periodic solutions of some third-order equations has been discussed in [9–12]. In [10], Chu and Zhou considered the periodic boundary value problem for the third-order equation

$$u'''(t) + \rho^3 u(t) = f(t, u(t)),$$

where $\rho \in (0, \frac{1}{\sqrt{3}})$ is a constant and $f \in C([0, 2\pi] \times (0, \infty))$. Using Krasnoselskii's fixed point theorem in cones, they obtained the existence results of positive solutions. Their results extended the one obtained by the Schauder fixed point theorem in [9]. In [11], by Krasnoselskii's fixed point theorem in cones, Feng established some existence and multiplicity results of positive periodic solutions for the third-order equation

$$u'''(t) + \alpha u''(t) + \beta u'(t) = f(t, u(t)),$$

where α and β are positive constants and satisfy certain conditions. Recently, these works were extended to the more general third-order equations by Li in [12].

In [13] and [14], the author using the fixed point index theory in cones obtained some existence results of positive solutions for periodic boundary problems of the fourth-order equation

$$u^{(4)}(t) - \beta u''(t) + \alpha u(t) = f(t, u(t))$$

and the nth-order equation

$$u^{(n)}(t) + a_{n-1}u^{(n-1)}(t) + \cdots + a_0 u(t) = f(t, u(t)),$$

respectively. The existence of positive periodic solutions for some special higher-order ordinary differential equations was also considered in [15,16]. The existence of the other periodic solutions for the higher-order ordinary differential equations was discussed in [17–19]. In [17], Li considered the existence and uniqueness of the nth-order periodic boundary problem. He presented some new spectral conditions for the nonlinearity $f(t, u)$ to guarantee the existence and uniqueness. In [18], Li and Mu researched the existence of odd periodic solutions for a $2n$th-order ordinary differential equation. Their obtained some existence results of odd periodic solutions under the nonlinearity respectively satisfies linear, superlinear or sublinear growth conditions. In [19], Cabada developed a monotone method in the presence of lower and upper solutions for some higher-order periodic boundary value problems on time scales. In this paper, we are interested in the existence of positive periodic solutions for higher-order ordinary differential equations.

In the works on the existence of positive periodic solutions mentioned above, the newly discovered positivity of Green function of the corresponding linear periodic boundary value problems plays an important role. The positivity guarantees that the integral operators of the periodic problems are cone-preserving in the cone

$$P = \{u \in C[0, \omega] \mid u(t) \geq \sigma \|u\|, t \in [0, \omega]\} \tag{3}$$

in the Banach space $C[0, \omega]$, where σ is a positive constant. Hence the fixed point theorems of cone mapping can be applied to these periodic problems. However, all of these works are on the special equations whose nonlinearities contain no derivative terms, and few researchers consider the existence of positive periodic solutions for the general nth-order Eq. (1) that nonlinearity f contains the derivative terms $u'(t), u''(t), \ldots, u^{(n-1)}(t)$.

The purpose of this paper is to establish the existence results of positive periodic solutions to the general nth-order Eq. (1). For the periodic problem of Eq. (1), since the corresponding integral operator has no definition on cone P, the argument methods used in [1–16] are not applicable. We will use a completely different method to treat Eq. (1). Our main results will be given in Section 3. Some preliminaries to discuss Eq. (1) are presented in Section 2.

2. Preliminaries

Let $C_{2\pi}(\mathbb{R})$ denote the Banach space of all continuous 2π-periodic functions with norm $\|u\|_C = \max_{0\leq t\leq 2\pi} |u(t)|$. Generally, for $m \in \mathbb{N}$, we use $C^m_{2\pi}(\mathbb{R})$ to denote the Banach space of all mth-order continuous differentiable 2π-periodic functions with the norm $\|u\|_{C^m} = \sum_{k=0}^{m} \|u^{(k)}\|_C$. Let $C^+_{2\pi}(\mathbb{R})$ denote the cone of all nonnegative functions in $C_{2\pi}(\mathbb{R})$.

Let M_n be a positive constant given by

$$M_n = \begin{cases} \left(2\cos\left(\dfrac{\pi}{2n}\right)\right)^{-n}, & \text{if } n = 2k+1 \text{ for a } k \in \mathbb{N}, \\ \left(2\cos\left(\dfrac{\pi}{n}\right)\right)^{-n}, & \text{if } n = 4k \text{ for a } k \in \mathbb{N}, \\ 2^{-n}, & \text{if } n = 4k-2 \text{ for a } k \in \mathbb{N}. \end{cases} \tag{4}$$

By the maximum principle of the differential operator $L_n u = u^{(n)} + Mu$ in periodic boundary condition in [20] or [14] (See [20, Lemma 2.4], or [14, Lemma 3]), we have the following lemma:

Lemma 2.1. *Let $M \in (0, M_n)$ be a constant. Then the linear nth-order boundary value problem*

$$\begin{cases} u^{(n)}(t) + Mu(t) = 0, \quad t \in [0, 2\pi], \\ u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, \ldots, n-2, \\ u^{(n-1)}(0) - u^{(n-1)}(2\pi) = 1, \end{cases} \tag{5}$$

has a unique solution $U \in C^n[0, 2\pi]$. Moreover, $U(t) > 0$ for $t \in [0, 2\pi]$.

Let $M \in (0, M_n)$. For $h \in C_{2\pi}(\mathbb{R})$, we consider the existence of 2π-periodic solution of the linear nth-order differential equation

$$u^{(n)}(t) + Mu(t) = h(t), \quad t \in \mathbb{R}. \tag{6}$$

Hereinafter, we use $U(t)$ to denote the unique solution of the boundary value problem (5).

Lemma 2.2. *Let $M \in (0, M_n)$. Then for every $h \in C_{2\pi}(\mathbb{R})$, the linear Eq. (6) has a unique 2π-periodic solution $u(t)$ which is given by*

$$u(t) = \int_{t-2\pi}^{t} U(t-s)h(s)\mathrm{d}s := Sh(t), \quad t \in \mathbb{R}. \tag{7}$$

Moreover, $S : C_{2\pi}(\mathbb{R}) \to C_{2\pi}^{n-1}(\mathbb{R})$ is a completely continuous linear operator.

Proof. Making derivative for the expression (7) and using the boundary condition of $U(t)$, we obtain that

$$u^{(i)}(t) = \int_{t-2\pi}^{t} U^{(i)}(t-s)h(s)\mathrm{d}s, \quad i = 1, 2, \ldots, n-1, \tag{8}$$

and

$$\begin{aligned} u^{(n)}(t) &= (U^{(n-1)}(0) - U^{(n-1)}(2\pi))h(t) + \int_{t-2\pi}^{t} U^{(n)}(t-s)h(s)\mathrm{d}s \\ &= h(t) - M\int_{t-2\pi}^{t} U(t-s)h(s)\mathrm{d}s \\ &= h(t) - Mu(t). \end{aligned}$$

Therefore, $u(t)$ satisfies Eq. (6). Let $\tau = s + 2\pi$, from (7) and the periodicity of h it follows that

$$\begin{aligned} u(t) &= \int_{t}^{t+2\pi} U(t + 2\pi - \tau)h(\tau - 2\pi)\mathrm{d}\tau \\ &= \int_{t}^{t+2\pi} U(t + 2\pi - \tau)h(\tau)\mathrm{d}\tau = u(t + 2\pi). \end{aligned}$$

Hence, $u(t)$ is a 2π-periodic solution of Eq. (6). The existence implies that $u(t)$ is the unique 2π-periodic solution of Eq. (6).

Clearly, $S : C_{2\pi}(\mathbb{R}) \to C_{2\pi}^{n-1}(\mathbb{R})$ is completely continuous. □

Let $M \in (0, M_n)$. Then the solution of Eq. (5) $U(t) > 0$ for every $t \in [0, 2\pi]$. If $h \in C_{2\pi}^{+}(\mathbb{R})$ and $h(t) \not\equiv 0$, by (7) the 2π-periodic solution of Eq. (6) $u(t) > 0$ for every $t \in \mathbb{R}$, and it is called positive 2π-periodic solution. This positivity is important for our discussion. Define positive constants σ and $C_1, \ldots, C_{n-1}$ by

$$\sigma = \frac{\min_{t\in I} U(t)}{\max_{t\in I} U(t)}, \qquad C_i = \frac{\max_{t\in I} |U^{(i)}(t)|}{\min_{t\in I} U(t)} \quad (i = 1, \ldots, n-1), \tag{9}$$

where $I = [0, 2\pi]$. Choose a cone K in $C_{2\pi}^{n-1}(\mathbb{R})$ by

$$K = \{u \in C_{2\pi}^{n-1}(\mathbb{R}) \mid u(t) \geq \sigma \|u\|_C, |u^{(i)}(t)| \leq C_i u(t),\ i = 1, 2 \cdots, n-1,\ t \in \mathbb{R}\}. \tag{10}$$

We have the following lemma.

Lemma 2.3. *Let $M \in (0, M_n)$. Then for every $h \in C_{2\pi}^{+}(\mathbb{R})$, the 2π-periodic solution of Eq. (6) $u = Sh \in K$. Namely, $S(C_{2\pi}^{+}(\mathbb{R})) \subset K$.*

Proof. Let $h \in C_{2\pi}^{+}(\mathbb{R})$, $u = Sh$. For every $t \in \mathbb{R}$, from the expression (7) and the periodicity of h it follows that

$$u(t) = \int_{t-2\pi}^{t} U(t-s)h(s)\mathrm{d}s \le \max_{\tau\in I} U(\tau)\int_{t-2\pi}^{t} h(s)\mathrm{d}s = \max_{\tau\in I} U(\tau)\int_{0}^{2\pi} h(s)\mathrm{d}s,$$

and therefore,

$$\|u\|_C \le \max_{\tau\in I} U(\tau)\int_{0}^{2\pi} h(s)\mathrm{d}s.$$

Using (7) again, we obtain that

$$u(t) = \int_{t-2\pi}^{t} U(t-s)h(s)\mathrm{d}s \ge \min_{\tau\in I} U(\tau)\int_{t-2\pi}^{t} h(s)\mathrm{d}s = \min_{\tau\in I} U(\tau)\int_{0}^{2\pi} h(s)\mathrm{d}s \ge \sigma\|u\|_C.$$

By the expression (8) we obtain that

$$\begin{aligned}|u^{(i)}(t)| &\le \int_{t-2\pi}^{t} |U^{(i)}(t-s)|h(s)\mathrm{d}s \le \max_{\tau\in I} |U^{(i)}(\tau)|\int_{t-2\pi}^{t} h(s)\mathrm{d}s \\ &= \max_{\tau\in I} |U^{(i)}(\tau)|\int_{0}^{2\pi} h(s)\mathrm{d}s \\ &= C_i \min_{\tau\in I} U(\tau)\int_{0}^{2\pi} h(s)\mathrm{d}s \le C_i u(t), \quad i = 1, \dots, n-1.\end{aligned}$$

Hence, $u \in K$. □

Now we consider the nonlinear equation (1). Hereinafter, we assume that the nonlinearity f satisfies the following hypothesis:

(F0) There exists a positive constant $M \in (0, M_n)$ such that

$$f(t, x_0, x_1, \dots, x_{n-1}) + Mx_0 \ge 0$$

for every $x_0 \in [0, +\infty)$ and $t, x_1, \dots, x_{n-1} \in \mathbb{R}$.

Let

$$f_1(t, x_0, x_1, \dots, x_{n-1}) = f(t, x_0, x_1, \dots, x_{n-1}) + Mx_0,$$

then $f_1(t, x_0, x_1, \dots, x_{n-1}) \ge 0$ for $x \ge 0$ and $t, x_1, \dots, x_{n-1} \in \mathbb{R}$, and Eq. (1) is rewritten to

$$u^{(n)}(t) + Mu(t) = f_1(t, u(t), u'(t), \dots, u^{(n-1)}(t)), \quad t \in \mathbb{R}. \tag{11}$$

For every $u \in K$, set

$$F(u)(t) := f_1(t, u(t), u'(t), \dots, u^{(n-1)}(t)), \quad t \in \mathbb{R}. \tag{12}$$

Then $F : K \to C_{2\pi}^{+}(\mathbb{R})$ is a continuous mapping. We define the integral operator A acting on cone K by

$$Au(t) = \int_{t-2\pi}^{t} U(t-s)f_1(s, u(s), u'(s), \dots, u^{(n-1)}(s))\mathrm{d}s = (S \circ F)(t). \tag{13}$$

From Assumption (F0) and Lemma 2.3, we easily obtain that

Lemma 2.4. *$A(K) \subset K$, and $A : K \to K$ is completely continuous.*

By the definition of operator S, a positive 2π-periodic solution of Eq. (1) is equivalent to a nonzero fixed point of A. We will find the nonzero fixed point of A by using the fixed point index theory in cones.

We recall some concepts and conclusions on the fixed point index in cones in [21,22]. Let E be a Banach space and $K \subset E$ be a closed convex cone in E. Assume Ω is a bounded open subset of E with boundary $\partial\Omega$, and $K \cap \Omega \ne \emptyset$. Let $A : K \cap \overline{\Omega} \to K$ be a completely continuous mapping. If $Au \ne u$ for every $u \in K \cap \partial\Omega$, then the fixed point index $i(A, K \cap \Omega, K)$ is well defined. One important fact is that if $i(A, K \cap \Omega, K) \ne 0$, then A has a fixed point in $K \cap \Omega$. The following two lemmas are needed in our argument.

Lemma 2.5 ([22]). *Let Ω be a bounded open subset of E with $\theta \in \Omega$, and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega$ and $0 < \lambda \leq 1$, then $i(A, K \cap \Omega, K) = 1$.*

Lemma 2.6 ([22]). *Let Ω be a bounded open subset of E and $A : K \cap \overline{\Omega} \to K$ a completely continuous mapping. If there exists an $v \in K \setminus \{\theta\}$ such that $u - Au \neq \tau v$ for every $u \in K \cap \partial\Omega$ and $\tau \geq 0$, then $i(A, K \cap \Omega, K) = 0$.*

3. Main results

We consider the existence of positive 2π-periodic solutions of Eq. (1). Let $f \in C(\mathbb{R} \times [0, \infty) \times \mathbb{R}^{n-1})$ satisfy Assumption (F0) and $f(t, x_0, x_1, \ldots, x_{n-1})$ be 2π-periodic in t. Let $C_1, \ldots, C_{n-1}$ be the positive constants defined by (9) and $I = [0, 2\pi]$. To be convenient, we introduce the notations

$$f_0 = \liminf_{x_0 \to 0^+} \min_{t\in I, |x_i| \leq C_i x_0, i=1,\ldots,n-1} \frac{f(t, x_0, x_1, \ldots, x_{n-1})}{x_0},$$
$$f^0 = \limsup_{x_0 \to 0^+} \max_{t\in I, |x_i| \leq C_i x_0, i=1,\ldots,n-1} \frac{f(t, x_0, x_1, \ldots, x_{n-1})}{x_0},$$
$$f_\infty = \liminf_{x_0 \to +\infty} \min_{t\in I, |x_i| \leq C_i x_0, i=1,\ldots,n-1} \frac{f(t, x_0, x_1, \ldots, x_{n-1})}{x_0},$$
$$f^\infty = \limsup_{x_0 \to +\infty} \max_{t\in I, |x_i| \leq C_i x_0, i=1,\ldots,n-1} \frac{f(t, x_0, x_1, \ldots, x_{n-1})}{x_0}.$$

Our main results are as follows:

Theorem 3.1. *Suppose that $f : \mathbb{R} \times [0, \infty) \times \mathbb{R}^{n-1} \to \mathbb{R}$ is continuous and $f(t, x_0, x_1, \ldots, x_{n-1})$ is 2π-periodic in t. If f satisfies Assumption* (F0) *and the condition*

(F1) $f^0 < 0, \quad f_\infty > 0,$

then Eq. (1) *has at least one positive 2π-periodic solution.*

Theorem 3.2. *Suppose that $f : \mathbb{R} \times [0, \infty) \times \mathbb{R}^{n-1} \to \mathbb{R}$ is continuous and $f(t, x_0, x_1, \ldots, x_{n-1})$ is 2π-periodic in t. If f satisfies Assumption (F0) and the condition*

(F2) $f_0 > 0, \quad f^\infty < 0,$

then Eq. (1) *has at least one positive 2π-periodic solution.*

In Theorem 3.1, the condition (F1) allows that $f(t, x_0, x_1, \ldots, x_{n-1})$ may be superlinear in $x_0, x_1, \cdots, x_{n-1}$. For the application, see Example 3.1 below. In Theorem 3.2, the condition (F2) allows that $f(t, x_0, x_1, \ldots, x_{n-1})$ may be sublinear in $x_0, x_1, \cdots, x_{n-1}$; see Example 3.2 below.

Proof of Theorem 3.1. Choose working space $E = C_{2\pi}^{n-1}(\mathbb{R})$. Let $K \subset C_{2\pi}^{n-1}(\mathbb{R})$ be the closed convex cone in $C_{2\pi}^{n-1}(\mathbb{R})$ defined by (10) and $A : K \to K$ be the completely continuous operator defined by (13). Then the positive 2π-periodic solution of Eq. (1) is equivalent to nonzero fixed point of A. Let $0 < r < R < +\infty$ and set

$$\Omega_1 = \{u \in E \mid \|u\|_{C^{n-1}} < r\}, \qquad \Omega_2 = \{u \in E \mid \|u\|_{C^{n-1}} < R\}. \tag{14}$$

We show that the operator A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ when r is small enough and R large enough.

By the assumption of $f^0 < 0$ and the definition of f^0, there exist $\varepsilon \in (0, M)$ and $\delta > 0$, such that

$$f(t, x_0, x_1, \ldots, x_{n-1}) \leq -\varepsilon x_0, \quad \text{for } t \in I,\ |x_i| \leq C_i x_0, \quad i = 1, \ldots, n-1,\ 0 \leq x_0 \leq \delta. \tag{15}$$

Choose $r \in (0, \delta)$. We prove that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$, namely $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega_1$ and $0 < \lambda \leq 1$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $0 < \lambda_0 \leq 1$ such that $\lambda_0 Au_0 = u_0$, then by the definition of A and Lemma 2.2, $u_0 \in C_{2\pi}^n(\mathbb{R})$ satisfies the differential equation

$$u_0^{(n)}(t) + Mu_0(t) = \lambda_0 f_1(t, u_0(t), u_0'(t), \ldots, u_0^{(n-1)}(t)), \quad t \in \mathbb{R}. \tag{16}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, we have

$$\begin{gathered} |u_0^{(i)}(t)| \leq C_i u_0(t), \quad i = 1, \ldots, n-1, \\ 0 < \sigma\|u_0\|_C \leq u_0(t) \leq \|u_0\|_C \leq \|u_0\|_{C^{n-1}} = r < \delta, \quad t \in \mathbb{R}. \end{gathered} \tag{17}$$

Hence from (15) it follows that

$$f(t, u_0(t), u_0'(t), \ldots, u_0^{(n-1)}(t)) \leq -\varepsilon u_0(t), \quad t \in I.$$

By this inequality, (16) and the definition of f_1 we have

$$u_0^{(n)}(t) + Mu_0(t) \le \lambda_0(Mu_0(t) - \varepsilon u_0(t)) \le (M - \varepsilon)u_0(t), \quad t \in I.$$

Integrating both sides of this inequality on I and using the periodicity of u_0, we obtain that

$$M \int_0^{2\pi} u_0(t)\mathrm{d}t \le (M - \varepsilon) \int_0^{2\pi} u_0(t)\mathrm{d}t.$$

Since $\int_0^{2\pi} u_0(t)\mathrm{d}t \ge 2\pi\sigma\|u_0\|_C > 0$, it follows that $M \le M - \varepsilon$, which is a contradiction. Hence A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_1$. By Lemma 2.5 we have

$$i(A, K \cap \Omega_1, K) = 1, \tag{18}$$

On the other hand, since $f_\infty > 0$, by the definition of f_∞, there exist $\varepsilon_1 > 0$ and $H > 0$ such that

$$f(t, x_0, x_1, \ldots, x_{n-1}) \ge \varepsilon_1 x_0, \quad \text{for } t \in I,\ |x_i| \le C_i x_0, \quad i = 1, \ldots, n-1,\ x_0 \ge H. \tag{19}$$

Choose $R > \max\{\frac{1+C_1+\cdots+C_{n-1}}{\sigma}H,\ \delta\}$ and let $v(t) \equiv 1$. Clearly, $v \in K \setminus \{\theta\}$. We show that A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_2$, namely $u - Au \ne \tau v$ for every $u \in K \cap \partial\Omega_2$ and $\tau \ge 0$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $\tau_1 \ge 0$ such that $u_1 - Au_1 = \tau_1 v$, since $u_1 - \tau_1 v = Au_1$, by definition of A and Lemma 2.2, $u_1 \in C_{2\pi}^n(\mathbb{R})$ satisfies the differential equation

$$u_1^{(n)}(t) + M(u_1(t) - \tau_1) = f_1(t, u_1(t), u_1'(t), \ldots, u_1^{(n-1)}(t)), \quad t \in \mathbb{R}. \tag{20}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, we have

$$u_1(t) \ge \sigma\|u_1\|_C, \quad |u_1^{(i)}(t)| \le C_i u_1(t), \quad i = 1, \ldots, n-1,\ t \in \mathbb{R}. \tag{21}$$

By the latter inequalities of (21), we have

$$\|u_1^{(i)}\|_C \le C_i\|u_1\|_C, \quad i = 1, \ldots, n-1.$$

This implies that

$$\|u_1\|_{C^{n-1}} = \sum_{i=0}^{n-1} \|u^{(i)}\|_C \le (1 + C_1 + \cdots + C_{n-1})\|u_1\|_C.$$

Consequently,

$$\|u_1\|_C \ge \frac{1}{1 + C_1 + \cdots + C_{n-1}}\|u_1\|_{C^{n-1}}. \tag{22}$$

By (22) and the former inequality of (21), we have

$$\begin{aligned} u_1(t) \ge \sigma\|u_1\|_C &\ge \frac{\sigma}{1 + C_1 + \cdots + C_{n-1}}\|u_1\|_{C^{n-1}} \\ &= \frac{\sigma R}{1 + C_1 + \cdots + C_{n-1}} > H, \quad t \in I. \end{aligned}$$

From this, the latter inequalities of (21) and (19), it follows that

$$f(t, u_1(t), u_1'(t), \ldots, u_1^{(n-1)}(t)) \ge \varepsilon_1 u_1(t), \quad t \in I.$$

By this inequality, (20) and the definition of f_1, we have

$$u_1^{(n)}(t) + M(u_1(t) - \tau_1) \ge (M + \varepsilon_1)u_1(t), \quad t \in I.$$

Integrating this inequality on I and using the periodicity of u_1, we get that

$$M \int_0^{2\pi} u_1(t)\mathrm{d}t - 2\pi M\tau_1 \ge (M + \varepsilon_1) \int_0^{2\pi} u_1(t)\mathrm{d}t.$$

Since $\int_0^{2\pi} u_1(t)\mathrm{d}t \ge 2\pi\sigma\|u_1\|_C > 0$, form this inequality it follows that $M \ge M + \varepsilon_1$, which is a contradiction. This means that A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_2$. By Lemma 2.6,

$$i(A, K \cap \Omega_2, K) = 0, \tag{23}$$

Now by the additivity of fixed point index, (18) and (23) we have

$$i(A, K \cap (\Omega_2 \setminus \overline{\Omega}_1), K) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = -1.$$

Hence A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive 2π-periodic solution of Eq. (1). $\square$

Proof of Theorem 3.2. Let $\Omega_1, \Omega_2 \subset C^{n-1}_{2\pi}(\mathbb{R})$ be defined by (14). We prove that the operator A defined by (13) has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$ if r is small enough and R large enough.

By the assumption of $f_0 > 0$ and the definition of f_0, there exist $\varepsilon > 0$ and $\delta > 0$, such that

$$f(t, x_0, x_1, \dots, x_{n-1}) \geq \varepsilon x_0, \quad \text{for } t \in I, |x_i| \leq C_i x_0, \quad i = 1, \dots, n-1,\ 0 \leq x_0 \leq \delta. \tag{24}$$

Choose $r \in (0, \delta)$ and $v(t) \equiv 1$. We prove that A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_1$, namely $u - Au \neq \tau v$ for every $u \in K \cap \partial\Omega_1$ and $\tau \geq 0$. In fact, if there exist $u_0 \in K \cap \partial\Omega_1$ and $\tau_0 \geq 0$ such that $u_0 - Au_0 = \tau_0 v$, since $u_0 - \tau_0 v = Au_0$, by definition of A and Lemma 2.2, $u_0(t) \in C^n_{2\pi}(\mathbb{R})$ satisfies the differential equation

$$u_0{}^{(n)}(t) + M(u_0(t) - \tau_0) = f_1(t, u_0(t), u_0{}'(t), \dots, u_0{}^{(n-1)}(t)), \quad t \in \mathbb{R}. \tag{25}$$

Since $u_0 \in K \cap \partial\Omega_1$, by the definitions of K and Ω_1, u_0 satisfies (17). From (17) and (24) it follows that

$$f(t, u_0(t), u_0{}'(t), \dots, u_0{}^{(n-1)}(t)) \geq \varepsilon u_0(t), \quad t \in I.$$

By this inequality, (25) and the definition of f_1, we have

$$u_0{}^{(n)}(t) = f(t, u_0(t), u_0{}'(t), \dots, u_0{}^{(n-1)}(t)) + M\tau_0 \geq \varepsilon u_0(t), \quad t \in I.$$

Integrating this inequality on I and using the periodicity of u_0, we obtain that $\int_0^{2\pi} u_0(t)\mathrm{d}t \leq 0$. But by the definition of K, $\int_0^{2\pi} u_0(t)\mathrm{d}t \geq 2\pi\sigma\|u_0\|_C > 0$, this is a contradiction. Hence A satisfies the condition of Lemma 2.6 in $K \cap \partial\Omega_1$. By Lemma 2.6 we have

$$i(A, K \cap \Omega_1, K) = 0. \tag{26}$$

Since $f^\infty < 0$, by the definition of f^∞, there exist $\varepsilon_1 \in (0, M)$ and $H > 0$ such that

$$f(t, x_0, x_1, \dots, x_{n-1}) \leq -\varepsilon_1 x_0, \quad \text{for } t \in I,\ |x_i| \leq C_i x_0, \quad i = 1, \dots, n-1,\ x_0 \geq H. \tag{27}$$

Choosing $R > \max\{\frac{1+C_1+\cdots+C_{n-1}}{\sigma}H,\ \delta\}$, we show that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$, namely $\lambda Au \neq u$ for every $u \in K \cap \partial\Omega_2$ and $0 < \lambda \leq 1$. In fact, if there exist $u_1 \in K \cap \partial\Omega_2$ and $0 < \lambda_1 \leq 1$ such that $\lambda_1 Au_1 = u_1$, then by definition of A and Lemma 2.2, $u_1 \in C^n_{2\pi}(\mathbb{R})$ satisfies the differential equation

$$u_1{}^{(n)}(t) + Mu_1(t) = \lambda_1 f_1(t, u_1(t), u_1{}'(t), \dots, u_1{}^{(n-1)}(t)), \quad t \in \mathbb{R}. \tag{28}$$

Since $u_1 \in K \cap \partial\Omega_2$, by the definition of K, u_1 satisfies (21). From (21) we easily prove that u_1 satisfies (22). By the former inequality of (21) and (22), we have

$$\begin{aligned} u_1(t) \geq \sigma\|u_1\|_C &\geq \frac{\sigma}{1 + C_1 + \cdots + C_{n-1}}\|u_1\|_{C^{n-1}} \\ &= \frac{\sigma R}{1 + C_1 + \cdots + C_{n-1}} > H, \quad t \in I. \end{aligned}$$

From this, (21) and (27) it follows that

$$f(t, u_1(t), u_1{}'(t), \dots, u_1{}^{(n-1)}(t)) \leq -\varepsilon_1 u_1(t), \quad t \in I.$$

By this and (28), we have

$$u_1{}^{(n)}(t) + Mu_1(t) \leq \lambda_1(Mu_1(t) - \varepsilon_1 u_1(t)) \leq (M - \varepsilon_1)u_1(t), \quad t \in I.$$

Integrating this inequality on I and using the periodicity of u_1, we obtain that

$$M\int_0^{2\pi} u_1(t)\mathrm{d}t \leq (M - \varepsilon_1)\int_0^{2\pi} u_1(t)\mathrm{d}t.$$

Since $\int_0^{2\pi} u_1(t)\mathrm{d}t \geq 2\pi\sigma\|u_1\|_C > 0$, from this inequality it follows that $M \leq M - \varepsilon_1$, which is a contradiction. This means that A satisfies the condition of Lemma 2.5 in $K \cap \partial\Omega_2$. By Lemma 2.5,

$$i(A, K \cap \Omega_2, K) = 1. \tag{29}$$

Now, from (26) and (29) it follows that

$$i(A, K \cap (\Omega_2 \setminus \overline{\Omega}_1), K) = i(A, K \cap \Omega_2, K) - i(A, K \cap \Omega_1, K) = 1.$$

Hence A has a fixed point in $K \cap (\Omega_2 \setminus \overline{\Omega}_1)$, which is a positive 2π-periodic solution of Eq. (1). □

Example 1. Consider the superlinear nth-order ordinary differential equation

$$u^{(n)} = a(t)u + \sum_{i=0}^{n-1} b_i(t)(u^{(i)})^2, \quad t \in \mathbb{R}. \tag{30}$$

where $a(t), b_0(t), b_1(t), \dots, b_{n-1}(t) \in C_{2\pi}(\mathbb{R})$. Assume that

$$-M_n < a(t) < 0, \qquad b_i(t) > 0\ (i = 0, 1, \dots, n-1), \quad t \in \mathbb{R}. \tag{31}$$

Choose $M = -\min_{t\in I} a(t)$. It is easy to verify that the nonlinearity

$$f(t, x_0, x_1, \dots, x_{n-1}) = a(t)x + \sum_{i=0}^{n-1} b_i(t){x_i}^2$$

satisfies the conditions (F0) and (F1). Hence, by Theorem 3.1, Eq. (30) has at least one positive 2π-periodic solution.

Example 2. Consider the sublinear nth-order differential equation

$$u^{(n)} = a(t)u + \sum_{i=0}^{n-1} b_i(t)\sqrt[3]{u^{(i)}}, \quad t \in \mathbb{R}. \tag{32}$$

where $a(t),\ b_0(t),\ b_1(t), \dots, b_{n-1}(t) \in C_{2\pi}(\mathbb{R})$ and satisfy the Assumption (30). Let $M = -\min_{t\in I} a(t)$, then the nonlinearity

$$f(t, x_0, x_1, \dots, x_{n-1}) = a(t)x + \sum_{i=0}^{n-1} b_i(t)\sqrt[3]{x_i}$$

satisfies the Assumption (F0). It is easily to see that the condition (F2) holds. Hence, by Theorem 3.2, Eq. (32) has at least one positive 2π-periodic solution.

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