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Nonsymmetric configurations with natural index

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Abstract

The existence of nonsymmetric configurations (v_r, b_k) is discussed here for the first time as a general problem. By using results about resolvable and near-resolvable Steiner systems as well as difference triangle sets, the existence of all configurations with $k=3$ is proved. For $k \geqslant 4$ many infinite series of configurations with natural index are constructed, i.e. configurations where the number of blocks b is a multiple of the number of points v.

1. Introduction and notation

Definition 1.1. A *configuration* (v_r, b_k) is a finite incidence structure with the following properties:

(1) There are v points and b lines.

(2) There are k points on each line and r lines through each point.

(3) Two different lines intersect each other at most once and consequently two different points are connected by a line at most once.

Remark 1.2. If $v=b$ (and hence $r=k$) the configuration is called *symmetric* and denoted by v_k.

Configurations (v_r, b_k) with $k=2$ are r-regular graphs on v vertices. Since these graphs are investigated in graph theory it will be assumed that $k \geqslant 3$ for the rest of this paper.

Configurations have already been defined in the last century. There are also important results, especially about symmetric configurations, obtained more than 100 years ago. For a general report on the history of configurations, see [3]. The mathematical results which have been obtained about symmetric configurations are described in [2]. The Italian abbreviation cfz. will be used in many cases for the long

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word configuration since Italian mathematicians were very active in the research of configurations in the early days.

Remark 1.3. Of course, the dual of a configuration is also a configuration. It is convenient only to investigate configurations with $b \geqslant v$ and $r \geqslant k$ and to consider configurations without these properties as duals of configurations.

The following conditions are necessary for the existence of a configuration.

Lemma 1.4. *If there exists a configuration* (v_r, b_k), *the following conditions hold*:
(1) $v \leqslant b$ *and* $k \leqslant r$,
(2) $vr = bk$,
(3) $v \geqslant r(k-1)+1$.

Proof. The first condition holds by convention (cf. the remark above). The second condition is proved by counting the incident point-line pairs in two ways. Concerning the third condition consider all the points on the lines through a fixed point. These are $r(k-1)+1$ different points since two different lines intersect in at most one point. □

Remark 1.5. Since for given r, k, and suitable v there is at most one possible parameter b, only 3 of the 4 parameters are independent. That is the reason why there is a second suitable set of only 3 parameters for a configuration. Both parameter sets will be used in this paper.

Definition 1.6. The *order m* of a configuration (v_r, b_k) is defined as $m = k-1$, analogous to the order of a finite projective plane.

The *deficiency d* of a configuration (v_r, b_k) is $d = v - r(k-1) - 1$ and measures the 'distance' of the configuration from a Steiner system. Each point is not connected to exactly d other points of the configuration.

The *index t* of a configuration (v_r, b_k) is defined as $t = r/k$ (or $t = b/v$). If k divides r the configuration is said to have natural index. If $t = 1$ the configuration is symmetric.

In order to change from the parameters m, d, t to the parameters v, r, b, k the following formulas are needed.

Lemma 1.7. *A configuration with order m, deficiency d and index t is a cfz.* (v_r, b_k), *where* $k = m+1$, $r = t(m+1)$, $v = d + tm(m+1) + 1$, *and* $b = t(d + tm(m+1) + 1)$.

Closely related to a configuration is its configuration graph.

Definition 1.8. The configuration graph of a configuration (v_r, b_k) has the v points as vertices, and two vertices are connected by an edge if they are not collinear in the configuration.

The following lemma is obvious.

Lemma 1.9. *The configuration graph of a cfz.* (v_r, b_k) *is a d-regular graph on v vertices where d is the deficiency of the configuration.*

In this paper the research on symmetric configurations (cf. [2]) is extended to nonsymmetric configurations. The main aspect of this paper is to obtain many existence results similar to those in [2, Section 4]. A further paper [4] will especially cover nonsymmetric configurations with small deficiencies.

Similar to [2] the most important construction of configurations with natural index is done by using a generalization of Golomb rulers, the so-called difference triangle sets. A further method uses the existence of resolvable and near-resolvable Steiner systems for the construction of configurations with rational index. These methods are described in Section 2, which also contains a brief report about the construction methods used in [4].

In Section 3 the existence results are collected mainly, but not only, for configurations with natural index. The necessary existence conditions for configurations with $k=3$ are proved to be sufficient. Partial results are obtained for configurations with $k=4$ and $k=5$. Further it is shown that for nearly all configurations with natural index the existence conditions are sufficient.

The results from design theory which are needed are described very briefly. For further details, see [1, 6].

2. Construction methods

2.1. Configurations from resolvable Steiner systems

A Steiner system $S(2, k, v)$ is a configuration (v_r, b_k) with deficiency 0. The deletion of a parallel class of lines yields a configuration (v_{r-1}, b'_k) with $b' = v(r-1)/k$. Such an operation is possible as long as the configuration has a set of v/k pairwise-disjoint lines. If the original Steiner system is resolvable this procedure can be continued as long as $r \geqslant k$ (and even further). The following theorem is very useful for the construction of many configurations.

Theorem 2.1. *Let v be a multiple of k. There is a configuration* (v_r, b_k) *if there exists a resolvable Steiner system* $S(2, k, v)$.

The existence problem for resolvable Steiner systems $S(2, k, v)$ has been solved only partially. It is one of the oldest problems of design theory and was initiated in Kirkman's famous school-girl problem of 1850 [7] for the system $S(2, 3, 15)$ in the following way:

> Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.

Necessary conditions for the existence of a resolvable Steiner system $S(2,k,v)$ have been shown to be sufficient for $k=3$ by Ray-Chaudhuri and Wilson [10] in 1971 and for $k=4$ by Hanani et al. [5] in 1972.

Theorem 2.2 (Hanani et al. [5]). *There is a resolvable Steiner system* $S(2,3,v)$ *iff* $v\equiv 3 \bmod(6)$. *There is a resolvable Steiner system* $S(2,4,v)$ *iff* $v\equiv 4 \bmod(12)$.

As an example, a resolvable Steiner system $S(2,3,15)$ (exactly 4 out of the 80 systems have a resolution) yields the following configurations: $(15_6, 30_3)$, $(15_5, 25_3)$, $(15_4, 20_3)$, and (15_3).

The existence problem for resolvable Steiner systems $S(2,k,v)$ for $k\geqslant 5$ is still open. Very recently, the Chinese mathematicians Zhu and Du (see [6] for the exact reference) have proved that there is a resolvable Steiner system $S(2,5,v)$ for all $v\equiv 5 \bmod (20)$ and $v\geqslant 7865$. There are 113 undecided cases for smaller v, the smallest of which is $v=45$.

2.2. *Configurations from nearly resolvable Steiner systems*

A Steiner system $S(2,3,v)$ with $v\equiv 1 \bmod(6)$ cannot be resolvable since 3 does not divide v. However, if one point and all the lines containing this point (this construction is discussed in [4]) are removed, the remaining structure is a cfz. $((v-1)_{(r-1)}, (b-r)_3)$ with $v-1$ points. This configuration can be resolvable, i.e. partitioned into parallel classes.

Such configurations have been investigated in terms of *nearly Kirkman triple systems* in a series of papers by A. Kotzig, A. Rosa, R.D. Baker, R.M. Wilson, A.E. Brouwer, R. Rees, and D.R. Stinson. The result can be summarized in the following theorem (see [11] for further details).

Theorem 2.3. *There is a nearly Kirkman triple system* $NKTS(v)$ iff $v\equiv 0 \bmod(6)$, $v\geqslant 18$.

This implies the existence of a resolvable configuration (v_r, b_3), $v\equiv 0 \bmod(6)$, $r=(v-2)/2$, $b=v(v-2)/6$ for all $v\geqslant 18$.

By using the following theorem many configurations with $k=3$ can be constructed.

Theorem 2.4. *There is a configuration* (v_r, b_3) *with* $v\equiv 0 \bmod(6)$ *if there is a resolvable configuration* $(v_{r'}, b'_3)$ *with* $r\leqslant r'=(v-2)/2$, *and* $b\leqslant b'=v(v-2)/6$.

The following recent result by Hao [13] settles the existence problem for $k=4$ almost completely.

Theorem 2.5. *There is a nearly Kirkman system* $NKS(2,4,v)$ *if* $v\equiv 0 \bmod(12)$, $v\geqslant 24$ *and* $v\notin E$, *where* E *is a set of* 16 *values of* v *for which the existence problem is still open*:

$$E=\{84,120,132,180,216,264,312,324,372,456,552,648,660,804,852,888\}$$

It was shown by Kramer et al. [9] that there is a Steiner system $S(2,4,25)$ which has a near-resolution [9], that is, there is a resolvable cfz. $(24_7, 42_4)$ which implies the existence of some configurations with $v=24$, $k=4$.

2.3. *Difference triangle sets*

In [2] many symmetric configurations are constructed by means of Golomb rulers. The most recent report on Golomb rulers which also contains earlier references is given by Shearer [12]. There are only a few perfect Golomb rulers.

The combinatorial structures which are analogous to perfect Golomb rulers are the *perfect systems of difference sets* (PSDS). However, in most cases PSDS do not exist. This is the reason that scientists who want to apply these structures, for example, in radioastronomy, are looking for optimal solutions which are close to a PSDS in the same sense as Golomb rulers are close to perfect Golomb rulers. Results in this area have been obtained mainly by nonmathematicians and have been published in technical journals. A survey of the recent results on *difference triangle sets* (DTS) is contained in [8] and is summarized below.

The following is an example of an optimal (3, 3)-DTS.

$$\begin{array}{ccccc}
2 & & 5 & & 6 \\
& 7 & & 11 & \\
& & 13 & &
\end{array}
\qquad
\begin{array}{ccccc}
1 & & 9 & & 8 \\
& 10 & & 17 & \\
& & 18 & &
\end{array}
\qquad
\begin{array}{ccccc}
3 & & 12 & & 4 \\
& 15 & & 16 & \\
& & 19 & &
\end{array}$$

The notation is such that the first rows are the base rows of the triangles and the ith rows contain all the sums of i consecutive numbers of the base row. The required property for an (I, J)-DTS is that all the $IJ(J+1)/2$ elements in the I triangles of size J (i.e. the base row has J elements) are different. If these elements are the numbers from 1 to $IJ(J+1)/2$ then this is a PSDS. An (I, J)-DTS is called optimal if its largest entry is as small as possible in the set of all (I, J)-DTS. Let us denote the largest entry as the length of a DTS.

Since there is no (3, 3)-DTS of length 18 (this would be a PSDS), the above example is an optimal (3, 3)-DTS.

The following results are from [8]. Let $M(I, J)$ be the length of an optimal (I, J)-DTS.

Lemma 2.6. $M(I,2)=3I$ *for* $I\equiv 0,1 \bmod(4)$ *and*

$M(I,2)=3I+1$ *for* $I\equiv 2,3 \bmod(4)$.

$M(I,3)=6I$ *for* $I=1,4,\ldots,15$ *and* $I=18$ *and*

$M(I,3)=6I+1$ *for* $I=2,3$.

$M(I,4)=10I$ *for* $I=6,8,10,$

$$M(I,4)=10I+1 \quad \textit{for } I=1,4,5,7 \textit{ and}$$

$$M(I,4)=10I+2 \quad \textit{for } I=2,3.$$

For other values of $M(I,4)$ only relatively weak upper bounds are known.

Lemma 2.7. *The following are all known values for $M(I,J)$, $J \geqslant 5$:*

$$M(1,5)=17, \quad M(2,5)=34, \quad M(3,5)=49, \quad M(4,5)=65, \quad M(5,5)=83,$$

$$M(1,6)=25, \quad M(2,6)=51, \quad M(3,6)=77, \quad M(1,7)=34, \quad M(1,8)=44,$$

$$M(1,9)=55, \quad M(1,10)=72, \quad M(1,11)=85, \quad M(1,12)=106,$$

$$M(1,13)=127, \quad M(1,14)=151, \quad M(1,15)=177.$$

2.4. *Configurations from DTS*

The existence of DTS and optimal DTS can be used to construct configurations with natural index in the following way.

Theorem 2.8. *If there is an (I,J)-DTS of length L there is a configuration (v_r, b_k)* for $k=J+1$, $r=Ik$ *and* $v \geqslant 2L+1$. *Here* $b=Iv$.

Proof. Take as points of the I base lines 0 and the first entry of each row in the I difference triangles. Use a cyclic automorphism of order v to construct $Iv=b$ blocks. Since $v \geqslant 2L+1$, no two blocks of the configuration intersect in more than 1 point.

Remark 2.9. The smaller L is, the larger is the set of configurations which can be constructed by this method. This is the reason why it is interesting to use an optimal DTS. In any case, however, there remains only a finite number of small configurations which cannot be constructed by the above method.

Corollary 2.10. *For every pair of parameters (k,r) with k dividing r, there is only a finite number of nonexisting configurations (v_r, b_k).*

2.5. *Further construction methods*

There are other methods for the construction of nonsymmetric configurations which are described in [4] and which we briefly mention here.

The deletion of one or several parallel classes from a Steiner system or configuration yields again a configuration (see Section 2.1). The same is true if

(1) a point and all lines through this point are deleted from a Steiner system,

(2) k collinear points and all lines containing at least one of these points are deleted from a Steiner system,

(3) a set of pairwise nonconnected points is deleted from a configuration.

For further details and examples, see [4].

3. General results for $k=3$ and $k=4$

3.1. Configurations with $k=3$

The existence question of configurations with block size 3 is completely settled by the following theorem.

Theorem 3.1. *There is a configuration with $k=3$ iff the necessary conditions hold, i.e. $v \geqslant 2r+1$ and $vr=3b$.*

Proof. Since $vr=3b$ holds, v or r is a multiple of 3. $r \geqslant k=3$ implies $v \geqslant 2r+1 \geqslant 7$.

If $v \equiv 3 \bmod (6)$, take a resolvable Steiner triple system with v points (cf. Section 2.1) and delete the required number of parallel classes.

If $v \equiv 0 \bmod (6)$, take a resolvable configuration (v_r, b_3) with $d=1$ and delete the required number of parallel classes. This is possible for $v \geqslant 18$ (cf. Section 2.2). For $v=12$ there is no resolvable cfz. $(12_5, 20_3)$. There are exactly 5 of these configurations [4]. However, a configuration $(12_4, 16_3)$ can be constructed from cfz. $(12_5, 20_3)$ no. 1 by deleting a parallel class. There are exactly 229 configurations 12_3 (cf. [2]).

If r is a multiple of 3, that is the index t is natural, a cyclic configuration can be constructed by means of a difference triangle set (compare Lemma 2.5) if the deficiency is greater than 1. ($M(I,2)=3I$ or $3I+1$ for every I, i.e. there is an (I,J)-DTS of length $3I+1$. Hence, for $v \geqslant 6I+3$ the configuration exists.) Configurations with deficiency 0 are Steiner systems which exist. If the configuration has deficiency 1 it can be obtained by deletion of a point in a Steiner system $S(2,3,v+1)$ (cf. Section 2.5).

There remains the problem of determining the exact number of configurations (v_r, b_3) for all admissible parameters. This has been done for several small values. See [4] for the exact numbers of configurations (v_r, b_3), including symmetric configurations and Steiner systems.

3.2. Configurations with $k=4$

The situation for $k=4$ is much more complicated than for $k=3$ and the question whether the necessary existence conditions are also sufficient is still open. Until now no nonexistence results for $k=4$ are known but there are many undecided cases. The following description surveys the situation and gives partial results. Further results are contained in [4].

3.2.1. $v \equiv 4 \bmod (12)$

The only result which is totally analogous to the case $k=3$ is the following.

Theorem 3.2. *There is a cfz. (v_r, b_4) for every $v \equiv 4 \bmod (12)$, $v \geqslant 3r+1$ and $vr=4b$.*

Proof. Take a resolvable Steiner system $S(2,4,v)$ (cf. Section 2.1) and delete the required number of parallel classes. □

3.2.2. $v \equiv 0 \bmod (12)$

The existence of Steiner systems $S(2,4,v)$ for all $v \equiv 1 \bmod (12)$ yields the following result (see [4], cf. Section 2.5).

Theorem 3.3. *There is a cfz.* (v_r, b_4) *for every* $v \equiv 0 \bmod (12)$, $v = 3r + 3$ (*i.e.* $d = 2$), *and* $vr = 4b$.

The existence results of Shen Hao on the near-resolvability of Steiner systems $S(2,4,v)$, $v \equiv 1 \bmod (12)$, in all except 16 cases imply the following lemma.

Lemma 3.4. *There is a configuration* (v_r, b_4) *for every* $v \equiv 0 \bmod (12)$, $v \geqslant 3r + 1$, $vr = 4b$ *if* $v \notin E$ (cf. Section 2.2).

Proof. The existence of a near-resolvable $S(2,4,v)$ implies the existence of these configurations (see Section 2.2). □

3.2.3. $v \equiv 8 \bmod (12)$

A configuration $(20_6, 30_4)$ has been constructed in [4]. There exists a cyclic configuration $(20_5, 25_4)$ which can be constructed by using $0,1,3,7$ as base line $\bmod(20)$ and the 5 lines $0+i, 5+i, 10+i, 15+i$ for $i = 0, \ldots, 4$. There is a cyclic cfz. 20_4 with base block $0,1,4,6$.

3.2.4. $r \equiv 0 \bmod (4)$

If r is a multiple of 4 the existence of $(I,3)$-DTS can be used. The following result is incomplete since the results on DTS are only partially known.

Theorem 3.5. *There is a configuration* (v_r, b_4) *with* $r = 4t$, $v \geqslant 3r + 1$, $vr = 4b$ *for all* $1 \leqslant t \leqslant 15$, *except possibly* $t = 3$, $v = 38$.

Proof. For $t = 1$ and $t = 4, \ldots, 15$ we have $M(I,3) = 6I$. This implies the existence of all configurations (v_r, b_k) with $r = 4t$ and $v \geqslant 12t + 1 = 3r + 1$. For $t = 2$ and $t = 3$ all configurations can be constructed from $(2,3)$-DTS and $(3,3)$-DTS for $v \geqslant 27$ and $v \geqslant 39$, respectively. □

The configurations $(25_8, 50_4)$ and $(37_{12}, 111_4)$ are Steiner systems and exist. The configuration $(26_8, 52_4)$ has been constructed in [4]).

Example 3.6. As an illustration for the case $t=15$ in the theorem above the base lines for a cfz. $(182_{60}, 2730_4)$ given here are a consequence of the (15, 3)-DTS in [8]:

0, 1, 67, 90	0, 2, 64, 88	0, 3, 61, 87	0, 4, 63, 85
0, 5, 54, 83	0, 6, 51, 76	0, 7, 48, 79	0, 8, 42, 77
0, 9, 52, 82	0, 10, 28, 75	0, 11, 38, 57	0, 12, 44, 80
0, 13, 33, 50	0, 14, 53, 74	0, 15, 55, 71	

Remark 3.7. The existence of a cfz. $(181_{60}, 2715_4)$ which is a Steiner system $S(2, 4, 181)$ has been known already as well as the existence of $S(2, 4, 193)$ or cfz. $(193_{64}, 3088_4)$. Hence, apart from the cfz. $(38_{12}, 114_4)$ the smallest unknown configuration with $k=4$ and natural index is cfz. $(194_{64}, 3104_4)$.

3.2.5. $v \equiv r \equiv 2 \bmod (4)$

The smallest example is a cfz. $(22_6, 33_4)$. Such a configuration was constructed by A. Hartman (personal communication) and is shown below.

Example 3.8. The following 3 base lines define a cfz. $(22_6, 33_4)$:

$$\{0_1, 1_0, 2_0, 3_1\},\ \{0_0, 0_1, 4_1, 5_1\},\ \{0_0, 2_0, 5_0, 8_1\},$$

where an automorphism of order 11 acts by adding 1 mod(11). The indices are fixed.

In fact, these base lines produce all configurations with $r=6$, $k=4$ other than the cfz. $(20_6, 30_4)$.

Lemma 3.9. *There is a configuration* (v_6, b_4) *for all* $v \geqslant 20$, v *even*, $b = 3v/2$.

Proof. Develop the base lines of the example under an automorphism of order $v/2$ as above. This yields the differences $+1, -1, +2, -2, +3, -3, +5, -5$ for points with index 0, the differences $+1, -1, +3, -3, +4, -4, +5, -5$ for points with index 1 and the mixed differences $0, +1, -1, +2, -2, +3, +4, +5, +6, +8$ for points a_1 and b_0. For $v/2 \geqslant 11$ no 2 points are connected by more than 1 line.

For $v=20$ a configuration is constructed in [4]. □

3.3. Some general remarks

For $k=4$ it is open whether an existence theorem holds as in the case of $k=3$. There may be a set of parameters for which no configuration exists although the necessary condition holds. There is no such hope for greater k. At least for $k=5, \ldots, 15$ and for many other k it is known that there is no cfz. with $t=1$ and $d=1$ or $d=2$ (cf. [2, Table 4]).

3.4. Configurations with $k=5$

If $k=5$ the results are more incomplete than for $k=4$. The following result from [3] is proved by deletion of a point from the suitable Steiner system. Since all possible Steiner systems $S(2,5,v)$ exist, the following theorem holds.

Theorem 3.10. *There exists a configuration (v_r, b_5) with $vr=5b$ for all $v\equiv 0 \bmod (20)$, if $v=4r+4$.*

The existence of a configuration (v_r, b_5) implies that v or r is a multiple of 5.

If v is a multiple of 5, a configuration can be constructed by using a resolvable Steiner system ($v\equiv 5 \bmod (20)$) or a near-resolvable Steiner system ($v\equiv 0 \bmod (20)$). Since the knowledge about these structures is far from complete, I do not want to treat this case in detail. It should be mentioned that the result of Zhu and Du implies the following theorem for configurations.

Theorem 3.11. *There is a cfz.(v_r, b_5) if $v\equiv 5 \bmod (20)$, $v\geqslant 4r+1$, $vr=5b$, and $v\geqslant 7865$.*

The proof is obvious and analogous to that of Theorem 3.2. This theorem shows the existence of, for example, a configuration $(7865_{1965}, 3090945_5)$ which is constructed by deleting a parallel class from the Steiner system $S(2,5,7865)$ (or a configuration $(7865_{1966}, 3098810_5)$).

There remain two further cases ($v\equiv 10$ or $15 \bmod (20)$) which shall not be discussed here.

However, a few results are added for configurations with natural index, i.e. if r is a multiple of 5.

Theorem 3.12. *There exists a configuration (v_r, b_5) with $r=5t$, $v\geqslant 4r+1$ and $vr=5b$ for all $1\leqslant t\leqslant 10$, except $t=1$, $v=22$ (there is no cfz. 22_5!, see [2]), and possibly except $(t,v)=(2,42)$, $(2,43)$, $(3,62)$, $(3,63)$, $(4,82)$, $(5,102)$, $(7,142)$, $(9,182)$, $(9,183)$, $(9,185)$, $(9,186)$, $(9,187)$, $(9,188)$, $(9,189)$, $(9,190)$, $(9,191)$, $(9,192)$.*

Proof. Compare the values of $M(I,4)$ in Lemma 2.6. For $I=9$ there is a (9,4)-DTS of length 96 (may be this is not optimal). Together with the values of Lemma 2.6 and the results on Steiner systems $S(2,5,v)$, the existence is proved as in Theorem 3.5.

3.5. Configurations with $k\geqslant 6$ and natural index

For $k\geqslant 6$ the results about the existence of the Steiner systems $S(2,k,v)$ are too sporadic to be used here for general theorems. Configurations with natural index can be constructed from difference triangle sets. Only a few optimal DTS are known. In other cases the best known DTS (see [8]) is used to obtain the corresponding result. It does not make much sense to collect these results in theorems as $k \leqslant 5$.

However, there is the following important theorem which is easy to prove but has as consequence the fact that there are 'not too many' open problems left on the existence question for configuration with natural index.

Theorem 3.13. *For given k and r with $r = tk$ there is a $v_0(k,t)$ such that there is a cfz. (v_r, b_k) for all $v \geqslant v_0$ for which the necessary conditions hold.*

The proof is obvious by using Remark 2.9 and Corollary 2.10.

3.6. Prospects

There are several reasons why configurations should be investigated further. First they are one of the oldest structures of combinatorics. They are also very interesting structures with many properties which do not occur in Steiner systems where all points are connected by a line.

It would be worth considering the existence problem of configurations together with the existence problem of Steiner systems with $\lambda = 1$.

For $k = 3$ all configurations exist.

For $k = 4$ it is still open whether all configurations exist, but at least all symmetric configurations ($t = 1$) and all Steiner systems ($d = 0$) exist.

For $k = 5$ there is even a symmetric configuration 22_5 which fulfils the necessary conditions but does not exist. All Steiner systems, however, still exist.

For $k = 6$ not all possible Steiner systems exist. There is no $S(2,6,36)$, and affine plane of order 6. This may be a special case. There are also other $S(2,6,v)$ the existence of which is in doubt. For example, neither the existence nor the nonexistence of a $S(2,6,46)$ has been proved. However, all symmetric Steiner systems with $\lambda = 1$ (i.e. the projective plane of order 5) exist.

For $k = 7$ there is a nonexisting symmetric Steiner system, the projective plane of order 6.

The further investigation of configurations will perhaps clarify the observations described in this last part of the paper.

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