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Global Attractivity in a Second-Order Nonlinear Difference Equation

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Consider the difference equation

$$x_{n+1} = x_n f(x_{n-1}), \qquad n = 0, 1, 2, ..., \tag{1}$$

where the function f satisfies the following conditions:

(i) $f \in \mathbf{C}[[0, \infty), (0, \infty)]$ and $f(u)$ is nonincreasing in u;

(ii) The equation $f(x) = 1$ has a unique positive solution;

(iii) If $\bar{x}$ denotes the unique positive solution of $f(x) = 1$, then

$$[xf(x) - \bar{x}](x - \bar{x}) > 0 \qquad \text{for} \quad x \neq \bar{x}.$$

Then $\bar{x}$ is a global attractor of all positive solutions of Eq. (1).

1. INTRODUCTION AND PRELIMINARIES

Our aim in this paper is to establish the following global attractivity result for a second order nonlinear difference equation.

THEOREM 1. *Consider the difference equation*

$$x_{n+1} = x_n f(x_{n-1}), \qquad n = 0, 1, 2, ..., \tag{1}$$

where the function f satisfies the following conditions:

(i) $f \in \mathbf{C}[[0, \infty), (0, \infty)]$ *and* $f(u)$ *is nonincreasing in* u;

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(ii) *The equation $f(x) = 1$ has a unique positive solution;*

(iii) *If $\bar{x}$ denotes the unique positive solution of $f(x) = 1$, then*

$$[xf(x) - \bar{x}](x - \bar{x}) > 0 \qquad \text{for} \quad x \neq \bar{x}. \tag{2}$$

Then $\bar{x}$ is a global attractor of all positive solutions of Eq. (1).

By a *solution* of Eq. (1) we mean a sequence $\{x_n\}$ which is defined for $n \geqslant -1$ and which satisfies Eq. (1) for $n \geqslant 0$. If a_{-1} and a_0 are two given nonnegative numbers, then Eq. (1) has a unique solution satisfying the initial conditions

$$x_{-1} = a_{-1} \qquad \text{and} \qquad x_0 = a_0. \tag{3}$$

If $a_{-1} \geqslant 0$ and $a_0 > 0$, then clearly, the solution of the initial value problem (1) and (3) is positive for $n \geqslant 0$. In this paper, we will only investigate solutions of Eq. (1) which are positive for $n \geqslant 0$. Such solutions will also be called *positive solutions.*

An immediate application of Theorem 1 is to the discrete delay logistic model

$$y_{n+1} = \frac{\alpha y_n}{1 + \beta y_{n-1}}, \qquad n = 0, 1, 2, ..., \tag{4}$$

where

$$\alpha \in (1, \infty) \qquad \text{and} \qquad \beta \in (0, \infty). \tag{5}$$

The more general equation

$$N_{t+1} = \frac{\alpha N_t}{1 + \beta N_{t-k}},$$

where α and β satisfy (5) and k is a nonnegative integer, was proposed by E. C. Pielou in her books [2, p. 22; 3, p. 79] as a discrete analog of the delay logistic equation

$$\dot{N}(t) = rN(t) \left[1 - \frac{N(t - \tau)}{P} \right].$$

This paper has been motivated by the results in [1] where it was shown that if (5) holds, then every positive solution of Eq. (4) converges to its positive equilibrium $(\alpha-1)/\beta$. The same result follows now as a corollary of Theorem 1. In this example,

$$f(x)=\frac{\alpha}{1+\beta x}$$

and the unique positive solution of $f(x)=1$ is

$$\bar{x}=(\alpha-1)/\beta.$$

Clearly, all hypotheses of Theorem 1 are satisfied and so we have the following corollary.

COROLLARY 1 [1]. *Assume that* (5) *holds and let* $\{y_n\}$ *be any solution of* Eq. (4) *with*

$$y_{-1}\geq 0 \qquad \textit{and} \qquad y_0>0.$$

Then

$$\lim_{n\to\infty} y_n=\frac{\alpha-1}{\beta}.$$

Another application of Theorem 1 is to the more general equation

$$y_{n+1}=\frac{\alpha y_n}{(1+a y_{n-1})^b+\beta y_{n-1}}, \qquad n=0, 1, 2, ... \tag{6}$$

which without delay in the denominator is the population model for annual plants which was derived in [4]. In Eq. (6) we assume that

$$\alpha\in(1,\infty), \qquad a\in[0,\infty), \qquad b\in[0,1], \qquad \text{and} \qquad \beta\in(0,\infty). \tag{7}$$

In this example,

$$f(x)=\frac{\alpha}{(1+ax)^b+\beta x}$$

and $\bar{x}$ is the unique positive solution of the equation

$$(1+ax)^b+\beta x=\alpha.$$

One can see that all the hypotheses of Theorem 1 are satisfied and so we have the following corollary.

COROLLARY 2. *Assume that* (7) *holds and let* $\{y_n\}$ *be any solution of Eq.* (6) *with*

$$y_{-1} \geqslant 0 \quad \textit{and} \quad y_0 > 0.$$

Then

$$\lim_{n \to \infty} y_n = \bar{x}.$$

2. PROOF OF THEOREM 1

Let $\{x_n\}$ be a positive solution of Eq. (1). We must prove that

$$\lim_{n \to \infty} x_n = \bar{x}.$$

There are only two cases to consider.

Case 1. For n sufficiently large,

$$\text{either } x_n \geqslant \bar{x} \tag{8}$$

$$\text{or } x_n \leqslant \bar{x}. \tag{9}$$

We will assume that (8) holds. The case where (9) holds is similar and will be omitted. Then for n sufficiently large, Eq. (1) yields

$$x_{n+1} \leqslant x_n f(\bar{x}) = x_n$$

and so $\{x_n\}$ decreases to a positive limit, $\lambda \geqslant \bar{x}$. By taking limits on both sides of Eq. (1) we find

$$f(\lambda) = 1$$

and in view of (3), $\lambda = \bar{x}$. The proof in this case is complete.

Case 2. The sequence $\{x_n\}$ is "strictly" oscillating about the positive equilibrium $\bar{x}$ in the sense that $x_n - \bar{x}$ obtains, for infinitely many values of n, positive values and also obtains, for infinitely many values of n, negative values. Then the sequence $\{x_n\}$ consists of a "string" of positive terms which are strictly below the positive equilibrium $\bar{x}$, followed by a string of terms which are greater than or equal to $\bar{x}$, followed by a string of positive terms strictly less than $\bar{x}$, etc. We will call these strings "negative semi-cycles" and "positive semicycles," respectively. In view of the hypothesis that $\{x_n\}$ is strictly oscillating about $\bar{x}$, it follows that a positive semicycle cannot have two consecutive terms equal to $\bar{x}$ (for otherwise, $x_n - \bar{x}$ would be identically equal to zero).

First we will prove that every semicycle contains at least three terms. We will give the proof for a positive semicycle. The proof for a negative semicycle is similar and will be omitted. To this end, let $x_{n-1} < \bar{x}$ be the last term in a negative semicycle. Then $x_n \geqslant \bar{x}$ is the first term of the following positive semicycle. We have

$$f(x_{n-1}) \geqslant f(\bar{x}) = 1$$

and so

$$x_{n+1} = x_n f(x_{n-1}) = x_n \geqslant \bar{x}. \tag{10}$$

Note that x_{n+1} must be strictly greater that $\bar{x}$. This follows from (10), if $x_n > \bar{x}$. And if $x_n = \bar{x}$, it follows from (10) and the additional observation that the sequence $\{x_n\}$, in this case, cannot have two consecutive terms equal to $\bar{x}$. Now from (1), (10), and (2) we see that

$$\begin{aligned} x_{n+2} &= x_{n+1} f(x_n) && (11) \\ &\geqslant x_n f(x_n) \\ &\geqslant \bar{x}. && (12) \end{aligned}$$

Also note that $x_{n+2} > \bar{x}$. This follows from (11) if $x_n = \bar{x}$. And if $x_n > \bar{x}$, the inequality in (12) is strict and so again, $x_{n+2} > \bar{x}$.

Next, observe that

$$x_{n+2} - x_{n+1} = x_{n+1}[f(x_n) - f(\bar{x})]$$

and so the following is true:

$$\begin{aligned} x_{n+2} &\geqslant x_{n+1} \quad && \text{if} \quad x_n < \bar{x} \\ x_{n+2} &= x_{n+1} \quad && \text{if} \quad x_n = \bar{x} \\ x_{n+2} &\leqslant x_{n+1} \quad && \text{if} \quad x_n > \bar{x}. \end{aligned}$$

This implies that the maximum in a positive semicycle and the minimum in a negative semicycle are equal to the value of the second term in the respective semicycle.

Now to complete the proof, consider the four consecutive semicycles,

$$C_{r-1}, C_r, C_{r+1}, C_{r+2}$$

with C_{r-1} being a negative semicycle, as

$$\begin{aligned} C_{r-1} &= \{x_{k+1}, x_{k+2}, ..., x_l\}, && \text{negative semicycle} \\ C_r &= \{x_{l+1}, x_{l+2}, ..., x_m\}, && \text{positive semicycle} \\ C_{r+1} &= \{x_{m+1}, x_{m+2}, ..., x_n\}, && \text{negative semicycle} \\ C_{r+2} &= \{x_{n+1}, x_{n+2}, ..., x_p\}, && \text{positive semicycle.} \end{aligned}$$

By using Eq. (1) twice, then (2), and by the decreasing nature of f we find

$$x_{l+2} = x_{l+1} f(x_l) = x_l f(x_l) f(x_{l-1}) < \bar{x} f(x_{l-1}) \leqslant \bar{x} f(x_{k+2}). \tag{13}$$

Similarly,

$$x_{m+2} > \bar{x} f(x_{l+2}). \tag{14}$$

By combining (13) and (14) and by using the decreasing nature of f we find

$$x_{m+2} > \bar{x} f(\bar{x} f(x_{k+2})). \tag{15}$$

In a similar way we obtain

$$x_{n+2} < \bar{x} f(\bar{x} f(x_{l+2})). \tag{16}$$

From (15), by applying (2), and then by using the decreasing nature of f, and then by applying (2) again we obtain

$$x_{m+2} > \bar{x} f\left(\frac{\bar{x}^2}{x_{k+2}}\right) > \bar{x} \frac{\bar{x}}{\bar{x}^2/x_{k+2}} = x_{k+2}. \tag{17}$$

In a similar way, (16) yields

$$x_{n+2} < \bar{x} f\left(\frac{\bar{x}^2}{x_{l+2}}\right) < \bar{x} \frac{\bar{x}}{\bar{x}^2/x_{l+2}} = x_{l+2}. \tag{18}$$

From (17) we see that the subsequence of $\{x_n\}$ which consists of the terms with the smallest values in the negative semicycles increases to a finite limit, which we denote by λ. Similarly, the subsequence of $\{x_n\}$ which consists of the terms with the largest values in the positive semicycles decreases to a finite limit, which we denote by L. Clearly, $0 < \lambda \leqslant \bar{x}$, $L \geqslant \bar{x}$ and from (15) and (16) it follows that

$$\lambda \geqslant \bar{x} f(\bar{x} f(\lambda)) \tag{19}$$

and

$$L \leqslant \bar{x} f(\bar{x} f(L)). \tag{20}$$

The proof will be complete, if we show that

$$\lambda = L = \bar{x}.$$

Suppose, for the sake of contradiction, that $0 < \lambda < \bar{x}$. The case where $L > \bar{x}$ is similar and will be omitted. Then, from (2),

$$\lambda f(\lambda) < \bar{x}$$

and so

$$\bar{x} f(\lambda) < \bar{x}^2/\lambda.$$

By using the decreasing nature of f we see that

$$f(\bar{x} f(\lambda)) \geqslant f(\bar{x}^2/\lambda). \tag{21}$$

As $\bar{x}^2/\lambda > \bar{x}$ it follows from (19), (21), and (2) that

$$\lambda \geqslant \bar{x} f(\bar{x} f(\lambda)) \geqslant \bar{x} f\left(\frac{\bar{x}^2}{\lambda}\right) > \bar{x}\,\frac{\bar{x}}{\bar{x}^2/\lambda} = \lambda.$$

This is a contradiction and the proof is complete.

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