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Radial multipliers on reduced free products of operator algebras

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Abstract

Let A_i be a family of unital C^*-algebras, respectively, of von Neumann algebras and $\phi : \mathbb{N}_0 \to \mathbb{C}$. We show that if a Hankel matrix related to ϕ is trace-class, then there exists a unique completely bounded map M_ϕ on the reduced free product of the A_i, which acts as a radial multiplier. Hereby we generalize a result of Wysoczański for Herz–Schur multipliers on reduced group C^*-algebras for free products of groups.

Keywords: Free probability; Operator algebras; Completely bounded multipliers

1. Introduction

Let $\mathscr{C}$ denote the set of functions ϕ on the non-negative integers $\mathbb{N}_0$ for which the matrix

$$h = \big(\phi(i+j) - \phi(i+j+1)\big)_{i,j \geqslant 0}$$

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is of trace-class. Let $G = *_{i\in I} G_i$ be the free product of discrete groups $(G_i)_{i\in I}$. In [13], J. Wysoczański proved that if $\phi \in \mathscr{C}$ and $\tilde{\phi} : G \to \mathbb{C}$ is defined by $\tilde{\phi}(e) = \phi(0)$ and $\tilde{\phi}(g_1 \cdots g_n) = \phi(n)$ for all $n > 0$ when $g_j \in G_{i_j} \setminus \{e\}$ and $i_1 \neq i_2 \neq \cdots \neq i_n$, then $\tilde{\phi}$ is a Herz–Schur multiplier on G and $\|\tilde{\phi}\|_{HS} \leqslant \|\phi\|_{\mathscr{C}}$, where $\|\cdot\|_{\mathscr{C}}$ is the norm on $\mathscr{C}$ defined in (2.2) below. In particular, there is a unique completely bounded map $M_\phi : C_r^*(G) \to C_r^*(G)$ such that $M_\phi(1) = \phi(0)1$ and

$$M_\phi\big(\lambda(g_1 \cdots g_n)\big) = \phi(n)\lambda(g_1 \cdots g_n)$$

when $g_j \in G_{i_j} \setminus \{e\}$ and $i_1 \neq i_2 \neq \cdots \neq i_n$ as above, and $\|M_\phi\|_{cb} \leqslant \|\phi\|_{\mathscr{C}}$. Furthermore, J. Wysoczański proved that $\|M_\phi\|_{cb} = \|\phi\|_{\mathscr{C}}$ in the cases when $|I| = \infty$ and $|G_i| = \infty$ for all $i \in I$. In the special case of $\phi_s(n) = s^n$ for $n \geqslant 0$ and $|s| < 1$ it follows that

$$\|M_{\phi_s}\|_{cb} \leqslant \|\phi_s\|_{\mathscr{C}} = \frac{|1-s|}{1-|s|}.$$

In this paper we will show that every function ϕ from $\mathscr{C}$ gives rise to radial multipliers M_ϕ on reduced free products of C^*-algebras and reduced free products of von Neumann algebras (cf. Theorem 2.2), satisfying $\|M_\phi\|_{cb} \leqslant \|\phi\|_{\mathscr{C}}$. Radial multipliers on general reduced free products of C^*-algebras were first considered by É. Ricard and Q. Xu in [11] and the weaker estimate $\|M_\phi\|_{cb} \leqslant |\phi(0)| + \sum_{n=1}^{\infty} 4n|\phi(n)|$ can be obtained from [11, Corollary 3.3].

The main result is proved in Section 5. In Section 6 we discuss a related set of functions $\mathscr{C}'$ (cf. Definition 6.3). It was used by T. Steenstrup, R. Szwarc and the first author in [6] to characterize radial multipliers on free groups $\mathbb{F}_n$ ($2 \leqslant n \leqslant \infty$). Moreover, C. Houdayer and É. Ricard used it in [7] to characterize multipliers on the free Araki–Woods factor $\Gamma(H_R, , U_t)''$ (cf. Section 6.3).

In Section 7 we obtain an integral representation of functions in the class $\mathscr{C}$ which together with N. Ozawa's result in [9], shows that for every hyperbolic group Γ, and every $\phi \in \mathscr{C}$, the function

$$\tilde{\phi}(x) = \phi\big(d(x,e)\big)$$

is a completely bounded Fourier multiplier on Γ (cf. Remark 7.6).

2. The main results

We start by defining the class $\mathscr{C}$, crucial in what follows.

Definition 2.1. Let $\mathscr{C}$ denote the set of functions $\phi : \mathbb{N}_0 \to \mathbb{C}$ for which the Hankel matrix $h = (\phi(i+j) - \phi(i+j+1))_{i,j\geqslant 0}$ is of trace-class.

If $\phi \in \mathscr{C}$, then $k = (\phi(i+j+1) - \phi(i+j+2))_{i,j\geqslant 0}$ is of trace-class, as well. Furthermore, we have

$$\sum_{n=0}^{\infty} \big|\phi(n) - \phi(n+1)\big| \leqslant \|h\|_1 + \|k\|_1 < \infty, \tag{2.1}$$

where $\|x\|_1 = Tr(|x|)$ is the trace-class norm for $x \in B(l^2(\mathbb{N}_0))$. This implies that $c = \lim_{n\to\infty} \phi(n)$ exists. For $\phi \in \mathscr{C}$ set

$$\|\phi\|_{\mathscr{C}} = \|h\|_1 + \|k\|_1 + |c|. \tag{2.2}$$

The main result of this paper is the following generalization of Wysoczański's result:

Theorem 2.2.

(1) *Let $\mathscr{A} = *_{i\in I}(\mathscr{A}_i, \omega_i)$ be the reduced free product of unital C^*-algebras $(\mathscr{A}_i)_{i\in I}$ with respect to states $(\omega_i)_{i\in I}$ for which the GNS-representation π_{ω_i} is faithful, for all $i \in I$.*
If $\phi \in \mathscr{C}$, then there is a unique linear completely bounded map $M_\phi : \mathscr{A} \to \mathscr{A}$ such that $M_\phi(1) = \phi(0)1$ and $M_\phi(a_1 a_2 \cdots a_n) = \phi(n) a_1 a_2 \cdots a_n$ whenever $a_j \in \mathring{\mathscr{A}}_{i_j} = \ker(\omega_{i_j})$ and $i_1 \neq i_2 \neq \cdots \neq i_n$. Moreover $\|M_\phi\|_{cb} \leqslant \|\phi\|_{\mathscr{C}}$.

(2) *Let $(\mathscr{M}, \omega) = \bar{*}_{i\in I}(\mathscr{M}_i, \omega_i)$ be the w^*-reduced free product of von Neumann algebras $(\mathscr{M}_i)_{i\in I}$ with respect to normal states $(\omega_i)_{i\in I}$ for which the GNS-representation π_{ω_i} is faithful, for all $i \in I$.*
If $\phi \in \mathscr{C}$, then there is a unique linear completely bounded normal map $M_\phi : \mathscr{M} \to \mathscr{M}$ such that $M_\phi(1) = \phi(0)1$ and $M_\phi(a_1 a_2 \cdots a_n) = \phi(n) a_1 a_2 \cdots a_n$ whenever $a_j \in \mathring{\mathscr{M}}_{i_j} = \ker(\omega_{i_j})$ and $i_1 \neq i_2 \neq \cdots \neq i_n$. Moreover $\|M_\phi\|_{cb} \leqslant \|\phi\|_{\mathscr{C}}$.

Remark 2.3. By J. Wysoczański's result in [13], the norm estimates in Theorem 2.2 are the best possible, as equality is attained if $|I| = \infty$ and $(A_i, \omega_i) = (C_r^*(G_i), \tau_i)$ for a family $(G_i)_{i\in I}$ of infinite discrete groups, where τ_i is the canonical trace on $C_r^*(G_i)$ coming from the left regular representation. In [13] Wysoczański also obtains explicit formulas for $\|M_\phi\|_{cb}$ in the case where $|I| < \infty$ and $|G_i| = \infty$ as well as in the case where $|I| < \infty$ and all the groups G_i are of the same finite order $|G_i|$. It would be interesting to know whether some of these results can be generalized to the case of arbitrary reduced free products of C^*-algebras.

We start by proving that the operator M_ϕ is unique, if it exists.

Lemma 2.4 *(Uniqueness in Theorem 2.2). The map M_ϕ is uniquely determined by the conditions in the hypothesis of Theorem 2.2.*

Proof. The algebra $\mathbb{C}1 + (\sum_{i\in I} \mathring{\mathscr{A}}_i) + (\sum_{i_1\neq i_2} \mathring{\mathscr{A}}_{i_1}\mathring{\mathscr{A}}_{i_2}) + \cdots$ is norm dense in $\mathscr{A}$ and, respectively, $\mathbb{C}1 + (\sum_{i\in I} \mathring{\mathscr{M}}_i) + (\sum_{i_1\neq i_2} \mathring{\mathscr{M}}_{i_1}\mathring{\mathscr{M}}_{i_2}) + \cdots$ is σ-weakly dense in $\mathscr{M}$. As M_ϕ is bounded, it is then uniquely defined on all of $\mathscr{A}$, respectively, on all of $\mathscr{M}$. □

Now to prove Theorem 2.2 we start by showing that it is enough to prove the result for the special case of the algebras $\mathscr{M} = B(H_i)$ equipped with ω_{Ω_i}, the vector state given by Ω_i, as this will imply the result for general C^*- and von Neumann-algebras. To do this we have to recall the definition of the Hilbert space free product.

Definition 2.5. Let $(H_i, \Omega_i)_{i\in I}$ be a family of Hilbert spaces with distinguished unit vectors and denote $\mathring{H}_i = \Omega_i^\perp$, for $i \in I$. Then as in [12] we define the *Hilbert space free product* by $H = *_{i\in I} H_i = \bigoplus_{n=0}^\infty H(n)$, where $H(n) = \bigoplus_{i_1\neq\cdots\neq i_n} \mathring{H}_{i_1} \otimes \cdots \otimes \mathring{H}_{i_n}$, for $n > 0$, and $H(0) = \mathbb{C}\Omega$. We will denote the projection from H to $H(n)$ by $P_n \in B(H)$, and let $Q_n = \sum_{k=n}^\infty P_k$.

Proposition 2.6. *If Theorem 2.2 part (2) holds for $(\mathscr{M}_i, \omega_i) = (B(H_i), \omega_{\Omega_i})$ for Hilbert spaces (H_i, Ω_i) and associated vector states ω_{Ω_i} then Theorem 2.2 holds in general.*

Proof. Assume Theorem 2.2 holds for $(B(H_i), \omega_{\Omega_i})$ for arbitrary H_i and Ω_i. Now let $\mathscr{A} = *_{i\in I}(\mathscr{A}_i, \omega_i)$, respectively, $(\mathscr{M}, \omega) = \bar{*}_{i\in I}(\mathscr{M}_i, \omega_i)$. Let $(H_i, \Omega_i) = (H_{\omega_i}, \xi_{\omega_i})$ be the Hilbert space and state coming from the GNS-representation of $\mathscr{A}_i$, respectively, $\mathscr{M}_i$, and let $(H, \Omega) = *_{i\in I}(H_i, \Omega_i)$ be their Hilbert space free product.

Now by [12, Definition 1.5.1], $\mathscr{A}_i$, respectively, $\mathscr{M}_i$ can be realized as subalgebras of $B(H)$ by the action defined as follows. If $a \in \mathscr{A}_i$, $\gamma_1 \otimes \cdots \otimes \gamma_n \in H$ with $\gamma_j \in \mathring{H}_{i_j} = \Omega_{i_j}^{\perp}$ and $i_1 \neq \cdots \neq i_n$ then

$$a(\gamma_1 \otimes \cdots \otimes \gamma_n) = a(\Omega_i) \otimes \gamma_1 \otimes \cdots \otimes \gamma_n$$

if $i \neq i_1$, and otherwise

$$a(\gamma_1 \otimes \cdots \otimes \gamma_n) = \big(a(\gamma_1) - \langle a(\gamma_1), \Omega_i\rangle \Omega_i\big) \otimes \gamma_2 \otimes \cdots \otimes \gamma_n + \langle a(\gamma_1), \Omega_i\rangle \gamma_2 \otimes \cdots \otimes \gamma_n.$$

Hence $M_\phi|_{\mathscr{A}}$ and $M_\phi|_{\mathscr{M}}$ can be obtained by restricting M_ϕ to the respective subalgebra of $B(H)$ and we then have $\|M_\phi|_{\mathscr{A}}\|_{cb} \leqslant \|M_\phi\|_{cb} \leqslant \|\phi\|_{\mathscr{C}}$, respectively, $\|M_\phi|_{\mathscr{M}}\|_{cb} \leqslant \|M_\phi\|_{cb} \leqslant \|\phi\|_{\mathscr{C}}$ which gives the desired general result, as the length of an operator in the free product is preserved when restricting to a subalgebra. □

We will prove the special case considered in Proposition 2.6 in the following sections.

3. Preliminaries

With notation as in Definition 2.5 we choose orthonormal bases $\mathring{\Gamma}_i$ for $\mathring{H}_i$, then $\Gamma_i = \mathring{\Gamma}_i \cup \{\Omega_i\}$ are bases for H_i. Put $\Lambda(0) = \{\Omega\}$ and $\Lambda(n) = \{\gamma_1 \otimes \cdots \otimes \gamma_n\colon\ \gamma_j \in \mathring{\Gamma}_{i_j}, i_1 \neq \cdots \neq i_n\}$ for all $n \geqslant 1$. Then $\Lambda(n)$ is an orthonormal basis for $H(n)$, for all $n \geqslant 0$ and $\Lambda = \bigcup_{n=0}^{\infty} \Lambda(n)$ is an orthonormal basis for H. Note that $\Lambda(1) = \bigcup_{i\in I} \mathring{\Gamma}_i$ considered as a subset of H.

Now we can define the basic operators in $B(H)$. Let $\gamma \in \Lambda(1)$. Let $L_\gamma, R_\gamma \in B(H)$ be the operators for which $L_\gamma \Omega = R_\gamma \Omega = \gamma$, and for $\chi = \chi_1 \otimes \cdots \otimes \chi_n \in \Lambda(n)$ where $\chi_j \in \mathring{\Gamma}_{i_j}$, $i_1 \neq \cdots \neq i_n$ and $\gamma \in \mathring{\Gamma}_i$ we have

$$L_\gamma(\chi) = \begin{cases} \gamma \otimes \chi & \text{if } i \neq i_1, \\ 0 & \text{if } i = i_1, \end{cases}$$

respectively,

$$R_\gamma(\chi) = \begin{cases} \chi \otimes \gamma & \text{if } i \neq i_n, \\ 0 & \text{if } i = i_n. \end{cases}$$

Note that L_γ and R_γ are well-defined partial isometries in $B(H)$. Moreover for all $\gamma \in \Lambda(1)$ and $n \geqslant 0$ we have $L_\gamma H(n) \subseteq H(n+1)$, respectively, $R_\gamma H(n) \subseteq H(n+1)$. Furthermore, for $\gamma \in \mathring{\Gamma}_i$, L_γ, R_γ are operators in $B(H_i) \subset B(H)$, when $B(H_i)$ is considered as a subalgebra of $B(H)$ as in the proof of Proposition 2.6. For $\gamma = \gamma_1 \otimes \cdots \otimes \gamma_n \in \Lambda(n)$ denote $L_\gamma = L_{\gamma_1} L_{\gamma_2} \cdots L_{\gamma_n}$, respectively, $R_\gamma = R_{\gamma_n} R_{\gamma_{n-1}} \cdots R_{\gamma_1}$, where we set $L_\Omega = R_\Omega = 1$.

Lemma 3.1. *Let $B(H_i)^\circ = \{a \in B(H_i)\colon \langle a\Omega_i, \Omega_i\rangle = 0\}$. Then the set* $\operatorname{span}(\{L_\gamma\colon \gamma \in \mathring{\Gamma}_i\} \cup \{L_\gamma^*\colon \gamma \in \mathring{\Gamma}_i\} \cup \{L_\gamma L_\delta^*\colon \gamma, \delta \in \mathring{\Gamma}_i\})$ *is σ-weakly dense in $B(H_i)^\circ$ considered as a subset of $B(H)$.*

Proof. Let $(e_{\gamma,\delta})_{\gamma,\delta\in\Gamma_i}$ be the matrix units of $B(H_i)$ corresponding to the basis Γ_i. Then $\operatorname{span}\{e_{\gamma,\delta}\colon (\gamma,\delta) \neq (\Omega_i,\Omega_i)\}$ is σ-weakly dense in $B(H_i)^\circ$. Moreover, by the natural embedding of $B(H_i)$ in $B(H)$ one gets for $\gamma, \delta \in \mathring{\Gamma}_i$ that $L_\gamma = e_{\gamma,\Omega_i}$, $L_\gamma^* = e_{\Omega_i,\gamma}$, and hence $L_\gamma L_\delta^* = e_{\gamma,\delta}$, which proves the lemma. □

Definition 3.2. Let $a = (a_i)_{i\geqslant 0} \in l^\infty(\mathbb{N}_0)$. Denote by D_a the bounded operator which is defined by $D_a(\xi) = a_n\xi$ for $\xi \in \Lambda(n)$, $n \geqslant 1$, respectively, $D_a(\Omega) = a_0\Omega$ and by linearity and continuity is extended to all of H.

Note that $D_a = \sum_{n=0}^\infty a_n P_n$ and $\|D_a\| = \|a\|_\infty$. Let S denote the standard shift on $l^\infty(\mathbb{N}_0)$, i.e., for $a = (a_i)_{i\geqslant 0} \in l^\infty(\mathbb{N}_0)$, let $S(a_0, a_1, a_2, \ldots) = (0, a_0, a_1, a_2, \ldots)$. To ease notation, consider separately the following two cases, which together contain all possible situations.

Definition 3.3. For $\xi \in \Lambda(k)$ and $\eta \in \Lambda(l)$, $k, l \geqslant 0$ we say that we are in

- *Case* 1: if $\xi = \Omega$ or $\eta = \Omega$ or $k, l \geqslant 1$ and $\xi = \xi_1 \otimes \cdots \otimes \xi_k$ and $\eta = \eta_1 \otimes \cdots \otimes \eta_l$ where $\xi_k \in \mathring{\Gamma}_i$, $\eta_l \in \mathring{\Gamma}_j$ and $i \neq j$, $i, j \in I$,

respectively,

- *Case* 2: if $k, l \geqslant 1$ and $\xi = \xi_1 \otimes \cdots \otimes \xi_k$ and $\eta = \eta_1 \otimes \cdots \otimes \eta_l$ where $\xi_k, \eta_l \in \mathring{\Gamma}_i$ for some $i \in I$.

4. Technical lemmas

Lemma 4.1. *(See [3, Theorem 1.3].) If $\|\sum_i u_i u_i^*\|, \|\sum_i v_i^* v_i\| < \infty$ for some $u_i, v_i \in B(H)$, then $\Phi(a) = \sum_i u_i a v_i$ defines a normal, completely bounded operator on $B(H)$ and $\|\Phi\|_{cb} \leqslant \|\sum_{i\in I} u_i u_i^*\|^{\frac{1}{2}} \|\sum_{i\in I} v_i^* v_i\|^{\frac{1}{2}}$.*

Definition 4.2. For $x, y \in l^2(\mathbb{N}_0)$ and $a \in B(H)$ set

$$\Phi_{x,y}^{(1)}(a) = \sum_{n=0}^\infty D_{(S^*)^n x} a D_{(S^*)^n y}^* + \sum_{n=1}^\infty D_{S^n x} \rho^n(a) D_{S^n y}^*,$$

respectively,

$$\Phi_{x,y}^{(2)}(a) = \sum_{n=0}^\infty D_{(S^*)^n x} a D_{(S^*)^n y}^* + \sum_{n=1}^\infty D_{S^n x} \rho^{n-1}\big(\epsilon(a)\big) D_{S^n y}^*$$

where $\rho(a) = \sum_{\gamma\in\Lambda(1)} R_\gamma a R_\gamma^*$ and $\epsilon(a) = \sum_{i\in I} q_i a q_i$ and q_i is the projection onto $\overline{\operatorname{span}}\{\xi \in \Lambda(n)\colon n \geqslant 1,\ \xi = \gamma_1 \otimes \cdots \otimes \gamma_n, \gamma_n \in \mathring{\Gamma}_i\}$ for $i \in I$.

We will show in Lemma 4.3 below that these maps are well-defined, normal and completely bounded on $B(H)$.

Lemma 4.3. *For $x, y \in l^2(\mathbb{N}_0)$, $\Phi^{(1)}_{x,y}$, $\Phi^{(2)}_{x,y}$ are well-defined, normal, completely bounded maps on $B(H)$, and $\|\Phi^{(1)}_{x,y}\|_{cb} \leqslant \|x\|_2\|y\|_2$, respectively, $\|\Phi^{(2)}_{x,y}\|_{cb} \leqslant \|x\|_2\|y\|_2$.*

Proof. Observe first, that for all $x \in l^2(\mathbb{N}_0)$,

$$\sum_{n=0}^{\infty} D_{(S^*)^n x} D^*_{(S^*)^n x} + \sum_{n=1}^{\infty} D_{S^n x} D^*_{S^n x} = \|x\|^2 1_{B(H)}$$

because each term in the sum is a constant positive operator on $H(m) = P_m H$, and for each fixed $m \in \mathbb{N}_0$ these constant operators add up to $\|x\|^2 1_{B(H(m))}$, because

$$\sum_{n=0}^{\infty} \bigl|x(m+n)\bigr|^2 + \sum_{n=1}^{m} \bigl|x(m-n)\bigr|^2 = \|x\|^2.$$

Observe next, that $D_{S^n x}$ vanishes on

$$H(0) \oplus H(1) \oplus \cdots \oplus H(n-1) = (1 - Q_n)(H)$$

where $Q_n = \sum_{k=n}^{\infty} P_k$ (cf. Definition 2.5). Hence also

$$\sum_{n=0}^{\infty} D_{(S^*)^n x} D^*_{(S^*)^n x} + \sum_{n=1}^{\infty} D_{S^n x} Q_n D^*_{S^n x} = \|x\|^2 1_{B(H)}. \tag{4.1}$$

Moreover ρ and ϵ are well-defined, normal, completely positive operators on $B(H)$, because

$$\sum_{\gamma \in \Lambda(1)} R_\gamma R^*_\gamma = Q_1$$

and $(q_i)_{i \in I}$ are pairwise orthogonal projections with sum $1 - P_0 = Q_1$.

In particular $\rho(1) = \epsilon(1) = Q_1$. More generally we have

$$\rho^n(a) = \sum_{\zeta \in \Lambda(n)} R_\zeta a R^*_\zeta, \quad a \in B(H),$$
$$\rho^{n-1}\bigl(\epsilon(a)\bigr) = \sum_{\zeta \in \Lambda(n-1)} R_\zeta p_i a p_i R^*_\zeta, \quad a \in B(H)$$

and

$$\rho^n(1) = Q_n,$$
$$\rho^{n-1}\bigl(\epsilon(1)\bigr) = \rho^{n-1}(Q_1) = Q_n.$$

Hence (4.1) implies that

$$\sum_{n=0}^{\infty} D_{(S^*)^n x} D^*_{(S^*)^n x} + \sum_{n=1}^{\infty} \sum_{\zeta \in \Lambda(n)} D_{S^n x} R_\zeta R^*_\zeta D^*_{S^n x} = \|x\|^2 1_{B(H)}$$

and

$$\sum_{n=0}^{\infty} D_{(S^*)^n x} D^*_{(S^*)^n x} + \sum_{n=1}^{\infty} \sum_{\zeta \in \Lambda(n-1)} \sum_{i \in I} D_{S^n x} R_\zeta q_i R^*_\zeta D^*_{S^n x} = \|x\|^2 1_{B(H)}.$$

These two formulas together with Lemma 4.1 show that $\Phi^{(i)}_{x,y}$ is a well-defined, normal, completely bounded operator on $B(H)$ and

$$\left\| \Phi^{(i)}_{x,y} \right\|_{cb} \leqslant \|x\|_2 \|y\|_2$$

for $i = 1, 2$ and for all $x, y \in l^2(\mathbb{N}_0)$. □

In the following lemmas we repeatedly will use the commutation relations $R_{\gamma_1} L_{\gamma_2} = L_{\gamma_2} R_{\gamma_1}$ for $\gamma_1, \gamma_2 \in \Lambda(1)$ and $Q_{n+k} L_\xi = L_\xi Q_n$, $Q_{n+k} R_\xi = R_\xi Q_n$ for $\xi \in \Lambda(k)$, as well as $\sum_{\zeta \in \Lambda(n)} R_\zeta R^*_\zeta = Q_n$ which can be easily verified.

Lemma 4.4. *Let $k, l \geqslant 0$. Then for every $\xi \in \Lambda(k)$ and $\eta \in \Lambda(l)$ we have for all $n \geqslant 0$, $\rho^n(L_\xi L^*_\eta) = L_\xi L^*_\eta Q_{l+n}$ and $\epsilon(L_\xi L^*_\eta) = \rho(L_\xi L^*_\eta)$ in Case 1, and, respectively, $\epsilon(L_\xi L^*_\eta) = L_\xi L^*_\eta$ in Case 2.*

Proof. For the first statement observe that

$$\begin{aligned} \rho^n\big(L_\xi L^*_\eta\big) &= \sum_{\zeta \in \Lambda(n)} R_\zeta L_\xi L^*_\eta R^*_\zeta \\ &= L_\xi \bigg(\sum_{\zeta \in \Lambda(n)} R_\zeta R^*_\zeta \bigg) L^*_\eta \\ &= L_\xi Q_n L^*_\eta \\ &= L_\xi L^*_\eta Q_{l+n}. \end{aligned}$$

For the second statement, let $\chi \in \Lambda(m)$. If $m > l$ then

$$\begin{aligned} \epsilon\big(L_\xi L^*_\eta\big)(\chi) &= \sum_{i \in I} q_i L_\xi L^*_\eta q_i(\chi) \\ &= L_\xi \sum_{i \in I} q_i q_i L^*_\eta(\chi) \\ &= L_\xi Q_1 L^*_\eta(\chi) \\ &= L_\xi L^*_\eta Q_{l+1}(\chi) \\ &= L_\xi L^*_\eta(\chi). \end{aligned}$$

While if $m = l$, $\chi = \eta$ and $\eta_l \in \mathring{\Gamma}_j$ for some $j \in I$ we have

$$\epsilon\big(L_\xi L_\eta^*\big)(\eta) = \sum_{i \in I} q_i L_\xi L_\eta^* q_i(\eta) = q_j L_\xi L_\eta^* q_j(\eta) = q_j L_\xi L_\eta^*(\eta) = q_j(\xi),$$

and this is equal to 0 in Case 1 (i.e., $\xi_k \notin \mathring{\Gamma}_j$), respectively, equal to ξ in Case 2 (i.e., $\xi_k \in \mathring{\Gamma}_j$).

Note that both sides vanish if $m = l$ and $\chi \neq \eta$, or $m < l$. □

Lemma 4.5. *Let $k, l \geqslant 0$. If $\xi \in \Lambda(k)$ and $\eta \in \Lambda(l)$ then*

$$\Phi_{x,y}^{(1)}\big(L_\xi L_\eta^*\big) = \left(\sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)}\right) L_\xi L_\eta^*$$

and, respectively,

$$\Phi_{x,y}^{(2)}\big(L_\xi L_\eta^*\big) = \begin{cases} \sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)} L_\xi L_\eta^* & \text{in Case } 1, \\ \sum_{t=0}^{\infty} x(k+t-1)\overline{y(l+t-1)} L_\xi L_\eta^* & \text{in Case } 2. \end{cases}$$

Proof. We prove this by showing that both sides act similarly on all simple tensors in H.

Indeed, let $m \geqslant 0$ and $\chi \in \Lambda(m)$ and let $n \geqslant 0$. If $\chi = \eta \otimes \zeta$, where $\zeta \in \Lambda(m-l)$ for some $l \geqslant 0$ we have for the common type of terms in $\Phi_{x,y}^{(1)}$ and $\Phi_{x,y}^{(2)}$ that

$$\begin{aligned} D_{(S^*)^n x} L_\xi L_\eta^* D_{(S^*)^n y}^*(\chi) &= \overline{y(m+n)} D_{(S^*)^n x} L_\xi L_\eta^*(\chi) \\ &= \overline{y(m+n)} D_{(S^*)^n x}(\xi \otimes \zeta) \\ &= x(k+m-l+n)\overline{y(m+n)} L_\xi L_\eta^*(\chi). \end{aligned} \tag{4.2}$$

Otherwise, if there is no ζ such that $\chi = \eta \otimes \zeta$, then both sides vanish, wherein we have used the convention that $x(p) = 0$ for $p < 0$.

For the other type of terms in $\Phi_{x,y}^{(1)}$, we get

$$\begin{aligned} D_{S^n x} \rho^n\big(L_\xi L_\eta^*\big) D_{S^n y}^*(\chi) &= \overline{y(m-n)} D_{S^n x} \rho^n\big(L_\xi L_\eta^*\big)(\chi) \\ &= \overline{y(m-n)} D_{S^n x} L_\xi L_\eta^* Q_{l+n}(\chi) \\ &= \overline{y(m-n)} D_{S^n x} Q_{l+n}(\xi \otimes \zeta) \\ &= x(k+m-l-n)\overline{y(m-n)} L_\xi L_\eta^* Q_{l+n}(\chi) \end{aligned} \tag{4.3}$$

where in the second equality we use Lemma 4.4 and the fact that both sides vanish if $n > m - l$.

We now estimate the other type of terms in $\Phi_{x,y}^{(2)}$. In Case 1 we similarly by use of Lemma 4.4 get

$$\begin{aligned} D_{S^n x} \rho^{n-1}\big(\epsilon\big(L_\xi L_\eta^*\big)\big) D_{S^n y}^*(\chi) &= D_{S^n x} \rho^n\big(L_\xi L_\eta^*\big) D_{S^n y}^*(\chi) \\ &= x(k+m-l-n)\overline{y(m-n)} L_\xi L_\eta^* Q_{l+n}(\chi) \end{aligned} \tag{4.4}$$

with both sides vanishing for $n > m - l$.

In Case 2 we get by Lemma 4.4

$$D_{S^n x}\rho^{n-1}\big(\epsilon\big(L_\xi L_\eta^*\big)\big)D_{S^n y}^*(\chi) = D_{S^n x}\rho^{n-1}\big(L_\xi L_\eta^*\big)D_{S^n y}^*(\chi)$$
$$= x(k+m-l-n)\overline{y(m-n)}L_\xi L_\eta^* Q_{l+n-1}(\chi) \tag{4.5}$$

with both sides vanishing for $n > m-l+1$.

Combining (4.2) and (4.3) we get

$$\begin{aligned}\Phi_{x,y}^{(1)}\big(L_\xi L_\eta^*\big)(\chi) &= \sum_{n=0}^{\infty} D_{(S^*)^n x} L_\xi L_\eta^* D_{(S^*)^n y}^* + \sum_{n=1}^{\infty} D_{S^n x}\rho^n\big(L_\xi L_\eta^*\big)D_{S^n y}^* \\ &= \sum_{n=0}^{\infty} x(k+m-l+n)\overline{y(m+n)}L_\xi L_\eta^*(\chi) \\ &\quad + \sum_{n=1}^{m-l} x(k+m-l-n)\overline{y(m-n)}L_\xi L_\eta^*(\chi) \\ &= \left(\sum_{n=l-m}^{\infty} x(k+m-l+n)\overline{y(m+n)}\right)L_\xi L_\eta^*(\chi) \\ &= \left(\sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)}\right)L_\xi L_\eta^*(\chi)\end{aligned}$$

as desired.

Similarly in Case 1, combining (4.2) and (4.4) we get

$$\Phi_{x,y}^{(2)}\big(L_\xi L_\eta^*\big)(\chi) = \left(\sum_{t=0}^{\infty} x(k+t)\overline{y(l+t)}\right)L_\xi L_\eta^*(\chi).$$

While in Case 2, combining (4.2) and (4.5) we get

$$\begin{aligned}\Phi_{x,y}^{(2)}\big(L_\xi L_\eta^*\big)(\chi) &= \sum_{n=0}^{\infty} x(k+m-l+n)\overline{y(m+n)}L_\xi L_\eta^*(\chi) \\ &\quad + \sum_{n=1}^{m-l+1} x(k+m-l-n)\overline{y(m-n)}L_\xi L_\eta^*(\chi) \\ &= \left(\sum_{t=0}^{\infty} x(k+t-1)\overline{y(l+t-1)}\right)L_\xi L_\eta^*(\chi).\end{aligned}$$

This completes the proof. □

We now establish some technical results concerning the maps $\phi \in \mathscr{C}$.

Lemma 4.6. *Let $\phi \in \mathscr{C}$ and let h, k and c be as in Definition 2.1. Put $\psi_1(n) = \sum_{i=0}^{\infty}(\phi(n+2i) - \phi(n+2i+1))$ and $\psi_2(n) = \psi_1(n+1)$, for $n \geqslant 0$. Then $\phi(n) = \psi_1(n) + \psi_2(n) + c$ for $n \geqslant 0$ and the entries $h_{i,j}$ and $k_{i,j}$ of h and k are given by $h_{i,j} = \psi_1(i+j) - \psi_1(i+j+2)$, respectively, $k_{i,j} = \psi_2(i+j) - \psi_2(i+j+2)$, for $i, j \geqslant 0$.*

Proof. By (2.1) we have

$$\Bigl|\lim_{n\to\infty} \psi_1(n)\Bigr| \leqslant \lim_{n\to\infty} \sum_{i=0}^{\infty} \bigl|\phi(n+2i) - \phi(n+2i+1)\bigr| = 0.$$

A similar statement holds for ψ_2 and therefore $\lim_{n\to\infty} \psi_1(n) = 0$ and $\lim_{n\to\infty} \psi_2(n) = 0$. Next, let $n \geqslant 0$ be fixed. Then simple computations give $\psi_1(n) + \psi_2(n) = \phi(n) - c$, and $\psi_1(n) - \psi_1(n+2) = \phi(n) - \phi(n+1)$, respectively, $\psi_2(n) - \psi_2(n+2) = \phi(n+1) - \phi(n+2)$. Using these equations, we get the desired formulas for $h_{i,j}$, respectively, $k_{i,j}$. □

Remark 4.7. Since h, k are trace-class, it is well known (cf. [6, p. 13]) that there exist $x_i, y_i, z_i, w_i \in l^2(\mathbb{N}_0)$ such that $h = \sum_{i=1}^{\infty} x_i \odot y_i$ and $\sum \|x_i\|_2 \|y_i\|_2 = \|h\|_1$, respectively, $k = \sum_{i=1}^{\infty} z_i \odot w_i$ and $\sum \|z_i\|_2 \|w_i\|_2 = \|k\|_1$. Here we use the notation $(u \odot v)(t) = \langle t, v\rangle u$, for $u, v, t \in l^2(\mathbb{N}_0)$.

Lemma 4.8. *For ψ_1 and ψ_2 as in Lemma 4.6, and x_i, y_i, z_i, and w_i as in Remark 4.7 we have $\psi_1(k+l) = \sum_{i=1}^{\infty}\sum_{t=0}^{\infty} x_i(k+t)\overline{y_i(l+t)}$ and $\psi_2(k+l) = \sum_{i=1}^{\infty}\sum_{t=0}^{\infty} z_i(k+t)\overline{w_i(l+t)}$.*

Proof. Let $k, l \geqslant 0$, then

$$\begin{aligned}
\psi_1(k+l) &= \sum_{t=0}^{\infty} \psi_1(k+l+2t) - \psi_1(k+l+2t+2) \\
&= \sum_{t=0}^{\infty} h_{k+t,l+t} \\
&= \sum_{t=0}^{\infty}\sum_{i=1}^{\infty} x_i(k+t)\overline{y_i(l+t)} \\
&= \sum_{i=1}^{\infty}\sum_{t=0}^{\infty} x_i(k+t)\overline{y_i(l+t)}
\end{aligned}$$

where the sums are absolutely convergent, and we use Lemma 4.6 for the first two equalities. A similar reasoning applies to ψ_2. □

5. Proof of the main result

As shown in Section 2, it is enough to prove the following lemma in order to obtain the result of the main theorem.

Proposition 5.1. *Let $(H,\Omega) = *_{i\in I}(H_i,\Omega_i)$ be the reduced free product of Hilbert spaces $(H_i)_{i\in I}$ with unit vector Ω_i and let $\omega_i(a) = \langle a\Omega_i,\Omega_i\rangle$ for $a \in B(H_i)$ where we realize $B(H_i)$ as subalgebras of $B(H)$ as in the proof of Proposition 2.6. Then for every $\phi \in \mathscr{C}$, there exists a linear, completely bounded, normal map M_ϕ on $B(H)$ such that $M_\phi(1) = \phi(0)1$ and $M_\phi(a_1a_2\cdots a_n) = \phi(n)a_1a_2\cdots a_n$ whenever $n \geqslant 1$, $i_1,\dots,i_n \in I$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ and $a_j \in B(H_{i_j})^\circ = \ker(\omega_{i_j})$. Moreover, $\|M_\phi\|_{cb} \leqslant \|\phi\|_{\mathscr{C}}$.*

The proof of Proposition 5.1 will be divided into a series of lemmas.

Lemma 5.2. *Let $T: B(H) \to B(H)$ be a bounded, linear, normal map, and let $\phi: \mathbb{N}_0 \to \mathbb{C}$. The following statements are equivalent:*

(a) *For all $n \geqslant 1$, $i_1,\dots,i_n \in I$ with $i_1 \neq i_2 \neq \cdots \neq i_n$ and $a_j \in B(H_{i_j})^\circ = \ker(\omega_{i_j})$, we have $T(1) = \phi(0)1$ and $T(a_1a_2\cdots a_n) = \phi(n)a_1a_2\cdots a_n$.*
(b) *For all $k,l \geqslant 0$ and $\xi \in \Lambda(k)$, $\eta \in \Lambda(l)$ we have*

$$T\big(L_\xi L_\eta^*\big) = \begin{cases} \phi(k+l)L_\xi L_\eta^* & \text{in Case } 1,\\ \phi(k+l-1)L_\xi L_\eta^* & \text{in Case } 2.\end{cases}$$

Proof. Assume (a) and let $k,l \geqslant 1$, $\xi \in \Lambda(k)$, $\eta \in \Lambda(l)$. Now by the definition of L_ξ we have $L_\xi L_\eta^* = L_{\xi_1}\cdots L_{\xi_k}L_{\eta_l}^*\cdots L_{\eta_1}^*$.

If we are in Case 1, there exist $i, j \in I$, $i \neq j$ such that $\xi_k \in \mathring{\Gamma}_i$ and $\eta_l \in \mathring{\Gamma}_j$. Hence all adjacent terms above are from different $B(H_i)^\circ$, hence $L_\xi L_\eta^*$ is of the form $a_1\cdots a_n$ in (a) with $n = k+l$.

On the other hand, if we are in Case 2, there exists $i \in I$ such that $\xi_k, \eta_l \in \mathring{\Gamma}_i$. In this case $L_{\xi_k}L_{\eta_l}^* \in B(H_i)^\circ$, hence $L_\xi L_\eta^*$ is of the form $a_1\cdots a_n$ in (a) with $n = k+l-1$. Applying (a) we get the conclusion of (b) for $k,l \geqslant 1$. If $k = 0$ or $l = 0$, e.g., $\xi = \Omega$ or $\eta = \Omega$ the result follows similarly by using $L_\Omega = 1$.

Assume (b). Using Kaplansky's density theorem [8, Theorem 5.3.5] and the fact that the product is jointly σ-strong continuous on bounded sets, by Lemma 3.1 it is enough to check that $T(1) = \phi(0)1$ and $T(a_1\cdots a_n) = \phi(n)a_1\cdots a_n$ whenever $n \geqslant 1$ and $a_j \in \{L_\gamma\colon \gamma \in \mathring{\Gamma}_{i_j}\} \cup \{L_\gamma^*\colon \gamma \in \mathring{\Gamma}_{i_j}\} \cup \{L_\gamma L_\delta^*\colon \gamma,\delta \in \mathring{\Gamma}_{i_j}\}$ where $i_1 \neq i_2 \neq \cdots \neq i_n$.

It is easy to check that $L_\gamma^* L_\delta = 0$ when $\gamma,\delta \in \Lambda(1)$, $\gamma \neq \delta$. In particular, $L_\gamma^* L_\delta = 0$ when $\gamma \in \mathring{\Gamma}_{i_j}$ and $\delta \in \mathring{\Gamma}_{i_{j+1}}$, since $i_j \neq i_{j+1}$. Hence $a_1\cdots a_n = 0$, unless $a_1\cdots a_n = L_{\gamma_1}\cdots L_{\gamma_k}L_{\delta_l}^* \cdots L_{\delta_1}^*$ for some $\gamma_j \in \mathring{\Gamma}_{i_j}$, $\delta_s \in \mathring{\Gamma}_{r_s}$, $i_j \neq i_{j+1}$, $r_s \neq r_{s+1}$ and $i_1,\dots,i_k,r_1,\dots,r_l \in I$.

If we are in Case 1, we have $i_k \neq r_l$. Hence neighboring elements on the right hand side are from different $B(H_i)^\circ$ and thus $n = k+l$. If we are in Case 2, we have $i_k = r_l$. Hence $L_{\gamma_k}L_{\delta_l}^* \in B(H_{i_k})^\circ$, thus $n = k+l-1$. Now (b) gives the result for $k \geqslant 1$ or $l \geqslant 1$. Moreover, the $k = l = 0$ case of (b) gives $T(1) = \phi(0)1$. $\square$

Next, we explicitly construct such a map T.

Definition 5.3. Let $\phi \in \mathscr{C}$. Define maps

$$T_1 = \sum_{i=1}^\infty \Phi^{(1)}_{x_i,y_i} \quad \text{and} \quad T_2 = \sum_{i=1}^\infty \Phi^{(2)}_{z_i,w_i}$$

where $\Phi^{(\cdot)}_{x,y}$ are as in Definition 4.2, ψ_1, ψ_2 as in Lemma 4.6, and x_i, y_i, z_i, w_i as in Remark 4.7. Moreover, define $T = T_1 + T_2 + c\,\mathrm{Id}$ where Id denotes the identity operator on $B(H)$, and $c = \lim_{n\to\infty} \phi(n)$.

First we prove that these maps are well-defined, normal, and completely bounded, afterwards we will prove that the maps exhibit the right behavior.

Lemma 5.4. *The maps T_1, T_2 and T are normal and completely bounded and $\|T\|_{cb} \leqslant \|\phi\|_{\mathscr{C}}$.*

Proof. Let $x_i, y_i \in l^2(\mathbb{N}_0)$ then we have by Lemma 4.3 that $\|\Phi^{(1)}_{x_i,y_i}\|_{cb} \leqslant \|x_i\|_2\|y_i\|_2$. Furthermore, since $T_1 = \sum_{i=1}^\infty \Phi^{(1)}_{x_i,y_i}$ we have by Remark 4.7

$$\|T_1\|_{cb} \leqslant \sum_{i=1}^\infty \big\|\Phi^{(1)}_{x_i,y_i}\big\|_{cb} \leqslant \sum_{i=1}^\infty \|x_i\|_2\|y_i\|_2 = \|h\|_1,$$

respectively,

$$\|T_2\|_{cb} \leqslant \sum_{i=1}^\infty \big\|\Phi^{(2)}_{z_i,w_i}\big\|_{cb} \leqslant \sum_{i=1}^\infty \|z_i\|_2\|w_i\|_2 = \|k\|_1.$$

Hence $\|T\|_{cb} \leqslant \|T_1\|_{cb} + \|T_2\|_{cb} + \|c\,\mathrm{Id}\|_{cb} \leqslant \|h\|_1 + \|k\|_1 + |c| = \|\phi\|_{\mathscr{C}}$ as desired. The normality of T_1, T_2 and T follows from the normality of $\Phi^{(1)}_{x,y}$ and $\Phi^{(2)}_{x,y}$. $\square$

Lemma 5.5. *For T_1, T_2 defined as above and for all $k, l \geqslant 0$, $\xi \in \Lambda(k)$, $\eta \in \Lambda(l)$ we have $T_1(L_\xi L_\eta^*) = \psi_1(k+l)L_\xi L_\eta^*$, respectively,*

$$T_2\big(L_\xi L_\eta^*\big) = \begin{cases} \psi_2(k+l)L_\xi L_\eta^* & \textit{in Case } 1, \\ \psi_2(k+l-2)L_\xi L_\eta^* & \textit{in Case } 2. \end{cases}$$

Proof. Let $k, l \geqslant 0$ and $\xi \in \Lambda(k)$, $\eta \in \Lambda(l)$. Now by Lemmas 4.5 and 4.8 we have

$$\begin{aligned} T_1\big(L_\xi L_\eta^*\big) &= \sum_{i=1}^\infty \Phi^{(1)}_{x_i,y_i}\big(L_\xi L_\eta^*\big) \\ &= \left(\sum_{i=1}^\infty \sum_{t=0}^\infty x_i(k+t)\overline{y_i(l+t)}\right)\big(L_\xi L_\eta^*\big) \\ &= \psi_1(k+l)L_\xi L_\eta^*. \end{aligned}$$

Furthermore, in Case 1 we have by Lemmas 4.5 and 4.8

$$T_2\big(L_\xi L_\eta^*\big) = \sum_{i=1}^\infty \Phi^{(2)}_{z_i,w_i}\big(L_\xi L_\eta^*\big)$$

$$= \left(\sum_{i=1}^{\infty} \sum_{t=0}^{\infty} z_i(k+t)\overline{w_i(l+t)} \right) \left(L_\xi L_\eta^* \right)$$
$$= \psi_2(k+l) L_\xi L_\eta^*,$$

respectively, in Case 2

$$\begin{aligned} T_2\left(L_\xi L_\eta^*\right) &= \sum_{i=1}^{\infty} \Phi^{(2)}_{z_i,w_i}\left(L_\xi L_\eta^*\right) \\ &= \left(\sum_{i=1}^{\infty} \sum_{t=0}^{\infty} z_i\left((k-1)+t\right)\overline{w_i\left((l-1)+t\right)} \right) \left(L_\xi L_\eta^* \right) \\ &= \psi_2(k+l-2) L_\xi L_\eta^*. \end{aligned}$$

This completes the proof. □

Lemma 5.6. *For T defined as above and for all $k, l \geqslant 0$ and $\xi \in \Lambda(k)$, $\eta \in \Lambda(l)$ we have*

$$T\left(L_\xi L_\eta^*\right) = \begin{cases} \phi(k+l) L_\xi L_\eta^* & \text{in Case } 1, \\ \phi(k+l-1) L_\xi L_\eta^* & \text{in Case } 2. \end{cases}$$

Note that by Lemma 5.2 this implies that $T(1) = \phi(1)1$ and that for $n \geqslant 1$, $T_\phi(a_1 a_2 \cdots a_n) = \phi(n) a_1 a_2 \cdots a_n$.

Proof. Assume we are in Case 1, then

$$\begin{aligned} T\left(L_\xi L_\eta^*\right) &= T_1\left(L_\xi L_\eta^*\right) + T_2\left(L_\xi L_\eta^*\right) + c L_\xi L_\eta^* \\ &= \left(\psi_1(k+l) + \psi_2(k+l) + c\right) L_\xi L_\eta^* \\ &= \phi(k+l) L_\xi L_\eta^*. \end{aligned}$$

Here we use the definition of T, then Lemma 5.5, and lastly Lemma 4.6. If we are in Case 2, we similarly get

$$\begin{aligned} T\left(L_\xi L_\eta^*\right) &= T_1\left(L_\xi L_\eta^*\right) + T_2\left(L_\xi L_\eta^*\right) + c L_\xi L_\eta^* \\ &= \left(\psi_1(k+l) + \psi_2(k+l-2) + c\right) L_\xi L_\eta^* \\ &= \left(\psi_2(k+l-1) + \psi_1(k+l-1) + c\right) L_\xi L_\eta^* \\ &= \phi(k+l-1) L_\xi L_\eta^*. \end{aligned}$$

Here we furthermore use $\psi_2(n) = \psi_1(n+1)$, for $n \geqslant 0$. □

Combining Lemmas 5.6 and 5.4 we obtain Proposition 5.1, and therefore an application of Proposition 2.6 yields the conclusion of Theorem 2.2.

6. Examples

6.1. The case $\phi_s(n) = s^n$

As a first example we will look at a simple ϕ where $\|\phi\|_{\mathscr{C}}$ can be calculated explicitly.

Corollary 6.1. *Let $\mathbb{D} = \{s \in \mathbb{C}\colon |s| < 1\}$ and $s \in \mathbb{D}$. Denote by ϕ_s the function $\phi_s(n) = s^n$. Then ϕ_s defines a radial multiplier M_{ϕ_s} on $\mathscr{A} = *_{i\in I}(\mathscr{A}_i, \omega_i)$, respectively, $(\mathscr{M}, \omega) = \bar{*}_{i\in I}(\mathscr{M}_i, \omega_i)$ as in Theorem 2.2. Moreover, $\|M_{\phi_s}\|_{cb} \leqslant |1-s|/(1-|s|)$.*

Proof. The conclusion follows from Theorem 2.2, once we show that ϕ belongs to $\mathscr{C}$ and that $\|\phi\|_{\mathscr{C}} = \|h\|_1 + \|k\|_1 + |c| = |1-s|/(1-|s|)$.

Observe first that $c = \lim_{n\to\infty}\phi(n) = \lim_{n\to\infty} s^n = 0$ as $|s| < 1$. Furthermore, $\phi(i+j+1) - \phi(i+j+2) = s(\phi(i+j) - \phi(i+j+1))$ so $k = s\cdot h$, hence $\|\phi\|_{\mathscr{C}} = (1+|s|)\|h\|_1$. Moreover $\phi(i+j) - \phi(i+j+1) = (1-s)s^{i+j}$ so $h = (1-s)m$, where m is the matrix $m_{i,j} = s^{i+j}$. This gives $\|\phi\|_{\mathscr{C}} = (1+|s|)\|h\|_1 = (1+|s|)|1-s|\|m\|_1$. Now $m = a \odot \bar{a}$, where $a = (s^k)_{k\geqslant 0} \in l^2(\mathbb{N}_0)$, hence $\|m\|_1 = \|a\|_2^2 = 1/(1-|s|^2)$. Combining these calculations we get $\|\phi\|_{\mathscr{C}} = |1-s|/(1-|s|)$, which proves the corollary. □

6.2. Wysoczański's theorem

As a second example, we will show that Wysoczański's result, apart from determining when equality holds, is a special case of Theorem 2.2.

Theorem 6.2. *(See [13, Theorem 6.1].) Let $G = *_{i\in I} G_i$ be the free product of a family of discrete groups, and let $g \in G$ be $g = g_1 g_2 \cdots g_n$ where $g_j \in G_{i_j}\setminus\{e\}$, $j_1, \ldots, j_n \in I$ and $j_1 \neq j_2 \neq \cdots \neq j_n$. If $\phi \in \mathscr{C}$ then $\tilde{\phi}(g) = \phi(n)$ is a Herz–Schur multiplier on G. Moreover $\|\tilde{\phi}\|_{HS} \leqslant \|\phi\|_{\mathscr{C}}$.*

Proof. Let $\phi \in \mathscr{C}$ and $g = g_1 \cdots g_n \in G$ as above. Now by [1, p. 301] and [4] we have $\|\tilde{\phi}\|_{HS} = \|\tilde{\phi}\|_{M_0A(G)} = \|\tilde{M}_{\tilde{\phi}}\|_{cb}$ where $\tilde{M}_{\tilde{\phi}}$ is the operator $\tilde{M}_{\tilde{\phi}}(\lambda(g)) = \tilde{\phi}(g)\lambda(g)$ for $g \in G$ and λ the left regular representation. By the definition of $\tilde{\phi}$ this is $\tilde{M}_{\tilde{\phi}}(\lambda(g)) = \phi(n)\lambda(g)$ and by the definition of $L(G)$ we have $\lambda(g) = \lambda(g_1)\lambda(g_2)\cdots\lambda(g_n)$. Hence $\tilde{M}_{\tilde{\phi}}(\lambda(g)) = M_\phi(\lambda(g))$, where M_ϕ is as defined in Theorem 2.2. Applying the theorem one obtains that $\tilde{\phi}$ is a Herz–Schur multiplier, and $\|\tilde{\phi}\|_{HS} \leqslant \|\phi\|_{\mathscr{C}}$. □

6.3. Relation to Houdayer and Ricard's results

Recently, C. Houdayer and É. Ricard in [7] proved results concerning radial multipliers on free Araki–Woods factors related to functions from a class $\mathscr{C}'$ quite similar to $\mathscr{C}$. Although their results apply to different objects than those considered in this paper, we will discuss in this section the issue of how their methods could be applied to prove Theorem 2.2.

We start by defining the class of functions $\mathscr{C}'$, mentioned above.

Definition 6.3. Let $\mathscr{C}'$ denote the set of functions $\phi : \mathbb{N}_0 \to \mathbb{C}$ for which the Hankel matrix $\hat{h} = (\phi(i+j) - \phi(i+j+2))_{i,j\geqslant 0}$ is of trace-class.

Observe that this implies the existence of $c_1, c_2 \in \mathbb{C}$ and a unique $\psi : \mathbb{N}_0 \to \mathbb{C}$ such that $\phi(n) = c_1 + (-1)^n c_2 + \psi(n)$ and $\lim_{n\to\infty} \psi(n) = 0$. For $\phi \in \mathscr{C}'$ put $\|\phi\|_{\mathscr{C}'} = |c_1| + |c_2| + \|\hat{h}\|_1$.

In [7] the following two results for functions in the class $\mathscr{C}'$ are proved, note the resemblance with Theorem 2.2. In what follows $\Gamma(H, U_t)''$ denotes the free Araki–Woods factor associated to a real Hilbert space H and a one parameter group of orthogonal transformations (U_t). (See [7, Sections 2.5 and 3.1] for more precise definitions.) As usual S denotes the standard shift on $l^2(\mathbb{N}_0)$.

Theorem 6.4. *(See [7, Proposition 3.3].) A function ϕ belongs to $\mathscr{C}'$ if and only if the operator γ defined by $\gamma(S^i(S^*)^j) = \phi(i+j)$ extends to a bounded map on $C^*(S)$. Moreover, $\|\gamma\|_{C^*(S)^*} = \|\phi\|_{\mathscr{C}'}$, and we say that γ is the* radial functional *associated with ϕ.*

Theorem 6.5. *(See [7, Theorem 3.5].) Let $\phi : \mathbb{N}_0 \to \mathbb{C}$. Then ϕ defines a completely bounded radial multiplier on $\Gamma(H, U_t)''$ if and only if the radial functional γ on $C^*(S)$ associated to ϕ is bounded. Moreover, $\|M_\phi\|_{cb} = \|\gamma\|_{C^*(S)^*}$.*

In Theorem 6.7 below we give a characterization of the set $\mathscr{C}$ similar to the characterization of $\mathscr{C}'$ in Theorem 6.4.

Lemma 6.6. *A function $\phi : \mathbb{N}_0 \to \mathbb{C}$ belongs to $\mathscr{C}$ if and only if $\tilde{\phi} : \mathbb{N}_0 \to \mathbb{C}$ defined by*

$$\tilde{\phi}(n) = \begin{cases} \phi(\frac{n}{2}), & n \text{ even}, \\ 0, & n \text{ odd} \end{cases}$$

belongs to $\mathscr{C}'$. Moreover $\|\phi\|_{\mathscr{C}} = \|\tilde{\phi}\|_{\mathscr{C}'}$.

Proof. Assume that $\phi \in \mathscr{C}$, and let h, k, c be as in Definition 2.1. Then h, k are trace-class Hankel matrices, $c \in \mathbb{C}$ and

$$\|\phi\|_{\mathscr{C}} = \|h\|_1 + \|k\|_1 + |c|.$$

Moreover, put $\hat{h} = (\tilde{\phi}(i+j) - \tilde{\phi}(i+j+2))_{i,j\geqslant 0}$. Then by Definition 6.3, $\tilde{\phi} \in \mathscr{C}'$ if and only if $\hat{h}$ is of trace-class and

$$\|\tilde{\phi}\|_{\mathscr{C}'} = \|\hat{h}\|_1 + |c_1| + |c_2|$$

where $c_1, c_2 \in \mathbb{C}$ are given by

$$c_1 + c_2 = \lim_{n\to\infty} \tilde{\phi}(2n), \qquad c_1 - c_2 = \lim_{n\to\infty} \tilde{\phi}(2n+1).$$

It is elementary to check, that $\hat{h} = h \oplus k$ with respect to the decomposition $l^2(\mathbb{N}_0) = l^2(\mathbb{N}_0)^{\text{even}} \oplus l^2(\mathbb{N}_0)^{\text{odd}}$ where $l^2(\mathbb{N}_0)^{\text{even}}$ (respectively, $l^2(\mathbb{N}_0)^{\text{odd}}$) consist of the vectors $(\xi(n))_{n\geqslant 0} \in l^2(\mathbb{N}_0)$, which are non-zero only for n even (respectively n odd). Moreover $c_1 = c_2 = \frac{1}{2}c$. Hence $\tilde{\phi} \in \mathscr{C}'$ and

$$\|\tilde{\phi}\|_{\mathscr{C}'} = \|h \oplus k\|_1 + |c| = \|h\|_1 + \|k\|_1 + |c| = \|\phi\|_{\mathscr{C}}.$$

Conversely if $\tilde{\phi} \in \mathscr{C}'$, then $\hat{h}$ is a trace-class operator of the form $\hat{h} = h' \oplus k'$ with respect to the above decomposition of $l^2(\mathbb{N}_0)$, and $c_1 = c_2$, and with h, k, c as in Definition 2.1, $h = h'$, $k = k'$ and $c = 2c_1 = 2c_2$. Hence $\phi \in \mathscr{C}$ and

$$\|\phi\|_{\mathscr{C}} = \|h\|_1 + \|k\|_1 + |c| = \|\hat{h}\|_1 + |c_1| + |c_2| = \|\tilde{\phi}\|_{\mathscr{C}'}. \qquad \square$$

Theorem 6.7.

(1) $C^*(S^2, SS^*) = \operatorname{span}\{S^k(S^*)^l\colon k, l \geqslant 0,\ k+l \text{ even}\}$.
(2) *A function $\phi : \mathbb{N}_0 \to \mathbb{C}$ is in the set $\mathscr{C}$ if and only if there exists an $\omega \in C^*(S^2, SS^*)^*$, such that*

$$\omega\big(S^k\big(S^*\big)^l\big) = \phi\left(\frac{k+l}{2}\right), \quad k, l \geqslant 0,\ k+l \text{ even}. \tag{6.1}$$

Moreover $\|\phi\|_{\mathscr{C}} = \|\omega\|$.

Proof. (1) This is obvious from the relation $S^*S = 1$.

(2) Assume that $\phi \in \mathscr{C}$. By Lemma 6.6

$$\tilde{\phi}(n) = \begin{cases} \phi(\frac{n}{2}), & n \text{ even}, \\ 0, & n \text{ odd} \end{cases}$$

is in $\mathscr{C}'$. Hence by Theorem 6.4 there exists $\tilde{\omega} \in C^*(S)^*$, such that

$$\tilde{\omega}\big(S^k\big(S^*\big)^l\big) = \tilde{\phi}(k+l) = \begin{cases} \phi(\frac{n}{2}), & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}$$

Hence the restriction ω of $\tilde{\omega}$ to $C^*(S^2, SS^*)$ satisfies (6.1), and

$$\|\omega\| \leqslant \|\tilde{\omega}\| = \|\tilde{\phi}\|_{\mathscr{C}'} = \|\phi\|_{\mathscr{C}}.$$

Let $\alpha : C^*(S) \to C^*(S)$ be the automorphism of order 2 given by $\alpha(S) = -S$, and let i be the identity map on $C^*(S)$. Then $\beta = \frac{1}{2}(i + \alpha)$ satisfies

$$\beta\big(S^k\big(S^*\big)^l\big) = \begin{cases} S^k(S^*)^l, & k+l \text{ even}, \\ 0, & k+l \text{ odd}. \end{cases}$$

Hence β maps $C^*(S)$ onto $C^*(S^2, SS^*)$ and $\|\beta\|_{cb} = 1$. Moreover, $\tilde{\omega} = \omega \circ \beta$. Hence $\|\tilde{\omega}\| \leqslant \|\omega\|$, so altogether we have shown that $\|\omega\| = \|\phi\|_{\mathscr{C}}$.

Conversely, if $\omega \in C^*(S^2.SS^*)^*$ and $\phi : \mathbb{N}_0 \to \mathbb{C}$ satisfies (6.1), then $\tilde{\omega} = \omega \circ \beta \in C^*(S)^*$, $\|\tilde{\omega}\| = \|\omega\|$, and

$$\tilde{\omega}\big(S^k\big(S^*\big)^l\big) = \begin{cases} \phi(\frac{k+l}{2}), & k+l \text{ even}, \\ 0, & k+l \text{ odd}. \end{cases}$$

Hence by Theorem 6.4, the function

$$\tilde{\phi}(n) = \begin{cases} \phi(\frac{n}{2}), & n \text{ even}, \\ 0, & n \text{ odd} \end{cases}$$

is in $\mathscr{C}'$ and $\|\tilde{\phi}\|_{\mathscr{C}'} = \|\tilde{\omega}\|$. This combined with Lemma 6.6 shows that $\phi \in \mathscr{C}$ and $\|\phi\|_{\mathscr{C}} = \|\tilde{\omega}\| = \|\omega\|$. $\square$

The proof by Houdayer and Ricard of Theorem 6.5 could be used to provide a much simpler proof of Theorem 2.2 if one could prove the existence of a unital $*$-homomorphism

$$\pi : C^*\big(L_\gamma : \gamma \in \Lambda(1)\big) \to C^*\big(L_\gamma : \gamma \in \Lambda(1)\big) \otimes C^*\big(S^2, SS^*\big)$$

such that

$$\pi\big(L_\xi L_\eta^*\big) = \begin{cases} L_\xi L_\eta^* \otimes S^{2k}(S^*)^{2l} & \text{in Case 1}, \\ L_\xi L_\eta^* \otimes S^{2k-1}(S^*)^{2l-1} & \text{in Case 2}. \end{cases} \tag{6.2}$$

If (6.2) holds, then for $\phi \in \mathscr{C}$ we let $\omega \in C^*(S^2, SS^*)^*$ denote the corresponding functional from Theorem 6.7. Then

$$\omega\big(S^k\big(S^*\big)^l\big) = \phi\bigg(\frac{k+l}{2}\bigg), \quad k, l \geqslant 0,\ k+l \text{ even}.$$

Put next $T = (id \otimes \omega) \circ \pi$, then T is completely bounded with

$$\|T\|_{cb} \leqslant \|\omega\| = \|\phi\|_{\mathscr{C}}$$

and

$$T\big(L_\xi L_\eta^*\big) = \begin{cases} \phi(k+l) L_\xi L_\eta^* & \text{in Case 1}, \\ \phi(k+l-1) L_\xi L_\eta^* & \text{in Case 2}. \end{cases}$$

This would give a much more direct proof of Lemma 5.6 and hence also of Theorem 2.2.

It is however not possible to construct such an isomorphism. Let for instance $|I| = 1$ and $\dim(H) = 2$ and let e_{ij} denote the matrix units with respect to the basis (Ω, γ) of H. Then we have $e_{01}e_{10} = e_{00}$, but $\Phi(e_{01})\Phi(e_{10}) = e_{00} \otimes S^*S \neq 1 \otimes 1 - e_{11} \otimes SS^* = \Phi(e_{00})$.

However, note that it would be sufficient if there existed a unital completely positive π satisfying (6.2). To find such an operator we can regard $l^2(\mathbb{N}_0) = l^2(\mathbb{N}_0)^{\text{even}} \oplus l^2(\mathbb{N}_0)^{\text{odd}}$. In this case S^2 on $l^2(\mathbb{N}_0)$ can be realized as $S \oplus S$ and SS^* on $l^2(\mathbb{N}_0)$ can be realized as $SS^* \oplus 1$. Then it would be enough to find unital completely positive operators π_1, π_2 such that

$$\pi_1\big(L_\xi L_\eta^*\big) = L_\xi L_\eta^* \otimes S^k\big(S^*\big)^l, \tag{6.3}$$

respectively,

$$\pi_2\big(L_\xi L_\eta^*\big) = \begin{cases} L_\xi L_\eta^* \otimes S^k (S^*)^l & \text{in Case 1,} \\ L_\xi L_\eta^* \otimes S^{k-1} (S^*)^{l-1} & \text{in Case 2.} \end{cases} \tag{6.4}$$

Since $\pi(1) = \pi(2) = 1$ we have $\|\pi_1\|_{cb}, \|\pi_2\|_{cb} \leqslant 1$ and $\pi = \pi_1 \oplus \pi_2$ is unital completely positive too. Hence $T = (\mathrm{Id} \otimes w) \circ \pi$ would be as desired.

Set $U_n = \sum_{i=0}^{\infty} P_{i+n} \otimes e_{i0}$, where e_{ij} are the matrix units in $B(l^2(\mathbb{N}_0))$, and use the convention $P_m = 0$ if $m < 0$. Now it can be shown that

$$\pi_1(x) = \sum_{n=-\infty}^{0} U_n (x \otimes 1) U_n^* + \sum_{n=1}^{\infty} U_n \big(\rho^n(x) \otimes 1\big) U_n^*,$$

respectively,

$$\pi_2(x) = \sum_{n=-\infty}^{0} U_n (x \otimes 1) U_n^* + \sum_{n=1}^{\infty} U_n \big(\rho^{n-1}\big(\epsilon(x)\big) \otimes 1\big) U_n^*$$

are unital completely positive and fulfill (6.3) and (6.4). The proof of this fact can be given by an argument quite similar to that given in Sections 4 and 5. We leave the details to the reader.

7. Integral representation of functions from $\mathscr{C}$

Let $\mathscr{C}'$ denote the set of functions $\phi : \mathbb{N}_0 \to \mathbb{C}$ from Definition 6.3. Based on Peller's characterization of Hankel matrices of trace class from [10, Theorem 1′], the following integral representation for functions in $\mathscr{C}'$ was proved in [6]. The set $\mathscr{C}'$ is not defined in [6], but the result follows from [6, Theorem 2.12 and Theorem 4.2].

Theorem 7.1. *Let $\psi : \mathbb{N}_0 \to \mathbb{C}$ be a function. Then the following are equivalent*:

(1) $\psi \in \mathscr{C}'$.
(2) *There exists a complex Borel measure μ on $\mathbb{D}$ and constants $c_+, c_- \in \mathbb{C}$, such that*

$$\psi(n) = c_+ + (-1)^n c_- + \int_{\mathbb{D}} s^n \, d\mu(s) < \infty \tag{7.1}$$

and

$$\int_{\mathbb{D}} \frac{|1 - s^2|}{1 - |s|^2} \, d|\mu|(s) < \infty.$$

Moreover, for $\phi \in \mathscr{C}'$, the measure μ in (7.1) can be chosen such that

$$|c_+| + |c_-| + \int_{\mathbb{D}} \frac{|1 - s^2|}{1 - |s|^2} \, d|\mu|(s) \leqslant \frac{8}{\pi} \|\psi\|_{\mathscr{C}'}.$$

We will prove next a similar characterization of functions in $\mathscr{C}$:

Theorem 7.2. *Let $\phi : \mathbb{N}_0 \to \mathbb{C}$ be a function. Then the following are equivalent*:

(1) $\phi \in \mathscr{C}$.
(2) *There exists a constant $c \in \mathbb{C}$ and a complex Borel measure ν on $\mathbb{D}$ such that*

$$\phi(n) = c + \int_{\mathbb{D}} s^n \, d\nu(s) \tag{7.2}$$

and

$$\int_{\mathbb{D}} \frac{|1-s|}{1-|s|} \, d|\nu|(s) < \infty.$$

Moreover, for $\phi \in \mathscr{C}$, the measure ν in (7.2) can be chosen such that

$$|c| + \int_{\mathbb{D}} \frac{|1-s|}{1-|s|} \, d|\nu|(s) \leqslant \frac{8}{\pi} \|\phi\|_{\mathscr{C}}.$$

Proof. (1) implies (2). Let $\phi \in \mathscr{C}$ and put

$$\tilde{\phi}(n) = \begin{cases} \phi(\frac{n}{2}) & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Then by Lemma 6.6, $\tilde{\phi} \in \mathscr{C}'$ and $\|\tilde{\phi}\|_{\mathscr{C}'} = \|\phi\|_{\mathscr{C}}$. From Theorem 7.1 there exists a complex measure μ on $\mathbb{D}$ and constants $c_+, c_- \in \mathbb{C}$ such that

$$\phi(n) = \tilde{\phi}(2n) = c_+ + c_- + \int_{\mathbb{D}} s^{2n} \, d\mu(s) < \infty$$

and

$$|c_+| + |c_-| + \int_{\mathbb{D}} \frac{|1-s^2|}{1-|s|^2} \, d|\mu|(s) \leqslant \frac{8}{\pi} \|\phi\|_{\mathscr{C}}.$$

Let ν be the range measure of μ by the map $s \mapsto s^2$ of $\mathbb{D}$ onto $\mathbb{D}$, and put $c = c_+ + c_-$. Then $|\nu|$ is less or equal to the range measure of $|\mu|$ by the map $s \mapsto s^2$. Hence

$$\phi(n) = c + \int_{\mathbb{D}} s^{2n} \, d\mu(s) = c + \int_{\mathbb{D}} s^n \, d\nu(s)$$

and

$$|c| + \int_{\mathbb{D}} \frac{|1-s|}{1-|s|}\, d|\nu|(s) \leqslant |c_+| + |c_-| + \int_{\mathbb{D}} \frac{|1-s^2|}{1-|s^2|}\, d|\mu|(s) \leqslant \frac{8}{\pi}\|\phi\|_{\mathscr{C}}.$$

This proves (1) implies (2) and the last statement in Theorem 7.2.

Conversely if (2) holds, the Hankel matrices h, k from Definition 2.1 have the entries

$$h_{ij} = \int_{\mathbb{D}} s^{i+j}(1-s)\, d\nu(s)$$

and

$$k_{ij} = \int_{\mathbb{D}} s^{i+j} s(1-s)\, d\nu(s).$$

By the proof of Corollary 6.1,

$$\left\| \left(s^{i+j}\right)_{i,j\geqslant 0} \right\|_1 = \frac{1}{1-|s|^2}, \quad s \in \mathbb{D}.$$

Hence

$$\|h\|_1 + \|k\|_1 \leqslant \int_{\mathbb{D}} \frac{|1-s| + |s(1-s)|}{1-|s|^2}\, d|\nu|(s) = \int_{\mathbb{D}} \frac{|1-s|}{1-|s|}\, d|\nu|(s) < \infty$$

which shows that $\phi \in \mathscr{C}$. □

In [9] N. Ozawa proved that if Γ is a discrete hyperbolic group (in the sense of M. Gromov [5]), then Γ is weakly amenable. The proof was obtained by showing that the metric $d : \Gamma \times \Gamma \to \mathbb{N}_0$ (w.r.t. the Cayley graph of Γ) satisfies three properties (1)–(3) listed in [9, Theorem 1].

As an application of Theorem 7.2, we will show below, that property (1) from [9] implies property (3) and hence is sufficient for Γ being weakly amenable. For the definition of weak amenability and of the constant $\Lambda(\Gamma)$ for a weakly amenable group Γ, we refer to [2, Section 12.3].

Recall that a metric on a discrete metric space (X, d) is called *proper* if the ball $B(x, r) = \{y \in X\colon d(x, y) < r\}$ is finite for all $x \in X$ and all $r > 0$.

Theorem 7.3. *Let Γ be a discrete countable group and let $d : \Gamma \times \Gamma \to \mathbb{N}_0$ be a proper left invariant metric. Put*

$$\phi_s(x) = s^{d(x,e)}, \quad s \in \mathbb{D},\ x \in \Gamma.$$

Assume that there exists a constant $C \geqslant 1$, such that $\phi_s \in M_0A(\Gamma)$ for all $s \in \mathbb{D}$ and

$$\|\phi_s\|_{M_0A(\Gamma)} \leqslant C\frac{|1-s|}{1-|s|}, \quad s \in \mathbb{D}. \tag{7.3}$$

Then Γ is weakly amenable with constant $\Lambda(\Gamma) \leqslant C$.

Remark 7.4. As in [6] we have used the notation $M_0A(\Gamma)$ for the set of completely bounded Fourier multipliers on Γ. Note that in [2, Section 12.3] the space $M_0A(\Gamma)$ is denoted $B_2(\Gamma)$.

We first prove

Lemma 7.5.

(1) *Put* $\chi_n(k) = \delta_{kn}$ *for* $n, k \geqslant 0$. *Then* $\chi_n \in \mathscr{C}$ *and*

$$\|\chi_n\|_{\mathscr{C}} \leqslant \max\{1, 4n\}, \quad n \geqslant 0.$$

(2) *For* $r \in (0,1)$ *and* $l \geqslant 0$, *put*

$$\phi_r(k) = r^k,$$
$$\phi_{r,n}(k) = \begin{cases} r^k, & 0 \leqslant k \leqslant n, \\ 0, & k > n, \end{cases}$$

then $\phi_r, \phi_{r,n} \in \mathscr{C}$, $\|\phi_r\|_{\mathscr{C}} = 1$ *and for fixed* $r \in (0,1)$

$$\lim_{n\to\infty} \|\phi_r - \phi_{r,n}\|_{\mathscr{C}} = 0.$$

Proof. From Definition 2.1 we have $\chi_n \in \mathscr{C}$, and $\|\chi_n\|_{\mathscr{C}} = \|H_n\|_1 + \|K_n\|_1$ where

$$H_n(i,j) = \chi_n(i+j) - \chi_n(i+j+1),$$
$$K_n(i,j) = \chi_n(i+j+1) - \chi_n(i+j+2).$$

If $H = (h_{ij})_{i,j=0}^{\infty}$ is a matrix of complex numbers for which $\sum_{i,j} |h_{ij}| < \infty$, then H is of trace-class and $\|H\|_1 \leqslant \sum_{i,j=0}^{\infty} |h_{ij}|$. Hence $\|\chi_n\|_{\mathscr{C}} \leqslant (2n+1) + (2n-1)$ for $n \geqslant 1$ and $\|\chi_0\|_{\mathscr{C}} \leqslant 1$ which proves (1). It follows from Corollary 6.1, that $\|\phi_r\|_{\mathscr{C}} = 1$, $0 < r < 1$. By (1),

$$\|\phi_r - \phi_{r,n}\|_{\mathscr{C}} = \left\| \sum_{k=n+1}^{\infty} r^k \chi_k \right\|_{\mathscr{C}} \leqslant \sum_{k=n+1}^{\infty} 4kr^k$$

which proves (2). □

Proof of Theorem 7.3. Let $\phi : \mathbb{N}_0 \to \mathbb{C}$ be a function from $\mathscr{C}$, and put

$$\tilde{\phi}(x) = \phi\big(d(x,e)\big), \quad x \in \Gamma.$$

Then by (7.3) and the integral representation of ϕ from Theorem 7.2 it follows that $\tilde{\phi}$ is a completely bounded Fourier multiplier on Γ and that

$$\|\tilde{\phi}\|_{M_0A(\Gamma)} \leqslant \frac{8C}{\pi} \|\phi\|_{\mathscr{C}}. \tag{7.4}$$

Let ϕ_r and $\phi_{r,n}$ be as in Lemma 7.5. Then by (7.3) $\|\tilde{\phi}_r\|_{M_0A(\Gamma)} \leqslant C$.
Moreover by (7.4) and Lemma 7.5

$$\lim_{n\to\infty} \|\tilde{\phi}_r - \tilde{\phi}_{r,n}\|_{M_0A(\Gamma)} = 0$$

for fixed $r \in (0,1)$. Put $r_k = 1 - 1/k$, $k \geqslant 1$ and choose for each $k \geqslant 2$ an $n_k \geqslant k$, such that

$$\|\tilde{\phi}_{r_k} - \tilde{\phi}_{r_k,n_k}\|_{M_0A(\Gamma)} \leqslant \frac{1}{k}.$$

Then $\psi_k = \tilde{\phi}_{r_k,n_k}$ form a sequence of functions on Γ, such that $\|\psi_k\|_{M_0A(\Gamma)} < C + 1/k$, and $\lim_{k\to\infty} \psi_k(x) = 1$ for all $x \in \Gamma$. Furthermore, the ψ_k are finitely supported as the metric is assumed to be proper. Hence Γ is weakly amenable and $\Lambda(\Gamma) \leqslant C$. □

Remark 7.6. By [9, Theorem 1] and the proof of Theorem 7.3 it follows that for every hyperbolic group Γ and every $\phi \in \mathscr{C}'$, the function

$$\tilde{\phi}(x) = \phi\big(d(x,e)\big), \quad x \in \Gamma,$$

is a completely bounded Fourier multiplier on Γ and $\|\tilde{\phi}\|_{M_0A(\Gamma)} \leqslant \frac{8C}{\pi}\|\phi\|_{\mathscr{C}'}$, where C is the constant in [9, Theorem 1 (1)].

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