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Group Algebras of Some Torsion-free Groups

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1. RESULTS

In [3] the open question regarding the semi-simplicity of group algebras over fields of characteristic 0 was considered. A second open question is the following (see [7] and [9]):

If G is a torsion-free group and if F is any field, is the group algebra FG semi-simple?

If G is torsion-free nilpotent (or even locally nilpotent), then FG is semi-simple. For, by Lemma 1 of [3], we may suppose that G is finitely generated. For any prime p, let $\mathfrak{F}_p$ denote the class of finite p-groups. Then by [4], G is a residually-$\mathfrak{F}_p$ group. If we choose p different from the characteristic of F, then FG is semi-simple, by the Corollary to Theorem 2.3 of [11]. On the other hand, if G is only soluble, then G need not be residually finite even. For example, let H be the group of 3×3 unitriangular matrices over Q (the rational field) and suppose that $H = \langle a_0, a_1, a_2, \ldots \rangle$. Let

$$T = \langle t \rangle \cong C_\infty, \qquad \overline{W} = H \,\overline{\wr}\, T,$$

the complete wreath product of H and T. So $\overline{W}$ is torsion-free and soluble ($\overline{W}''' = 1$). Let u be the element of the base group of $\overline{W}$ defined by $u_{2^r} = a_r$, if $r \geqslant 0$, and $u_n = 1$ otherwise. Then let $G = \langle t, u \rangle$, $M = \langle u^G \rangle$. According to [6] M' is isomorphic to the direct product of a countable infinity of copies of H'; and H' is isomorphic to the additive group of Q. Thus since Q is not residually finite, neither is G. Nevertheless we will prove

THEOREM A. *Let G be torsion-free and a finite extension of a soluble group, and let F be any field. Then FG is semi-simple.*

Using Lemma 1 of [3], there is a corollary to Theorem A (analogous to Theorem C of [3]). In particular we have

THEOREM B. *Let G be torsion-free and locally soluble. Then FG is semi-simple, for any field F.*

A locally soluble group is an SN-group ([*10*]). If a group G has an abelian series $(\Lambda_\sigma, V_\sigma; \sigma \in \Omega)$ with torsion-free factors $\Lambda_\sigma / V_\sigma$ (forcing G to be torsion-free), then Bovdi ([*1*]) proves that FG is semi-simple for any field F. However there exist even polycyclic torsion-free groups which do not possess an abelian series with torsion-free factors ([*8*]). A variation of the proof of Theorem A of [*3*] gives an alternative proof of Bovdi's result:

THEOREM C. *Let G be an SN-group with an abelian series having torsion-free factors. Then FG is semi-simple, for any field F.*

2. PROOFS

2.1 *Proof of Theorem* C. By Lemma 2 of [*3*], we may suppose that F contains at least 3 elements.

Suppose, if possible, that G is an SN-group with an abelian series having torsion-free factors such that FG is not semi-simple. Then choose a nonzero element x belonging to the Jacobson radical $J(FG)$ of FG. Write $x = \sum_1^n \lambda_i g_i$, where $n \geqslant 1$, $0 \neq \lambda_i \in F$ $(1 \leqslant i \leqslant n)$, and $g_1, \ldots, g_n$ are distinct elements of G. Since this expression for x is unique, we may call n the *length* of x. Without loss of generality, suppose that $g_1 = 1$.

Let $H = \langle g_1, \ldots, g_n \rangle$. Then there exists a subgroup $K \lhd H$ such that $H/K \cong C_\infty$. Let $H = \langle K, t \rangle$. Then $g_i = k_i t^{m_i}$, $k_i \in K$, $1 \leqslant i \leqslant n$. So $m_1 = 0$, $k_1 = 1$. Let m be the g.c.d. of the absolute values of the nonzero integers among $m_1, \ldots, m_n$. Write $m_i = mn_i$, $1 \leqslant i \leqslant n$, $s = t^m$, $H_1 = \langle K, s \rangle$. Then $x = \sum_1^n \lambda_i k_i s^{n_i} \in J(FH_1)$, by Lemma 1 of [*3*], and $n_1 = 0$.

Each element of FH_1 is uniquely expressible in the form

$$\sum_{-\infty}^{\infty} \theta_i s^i,$$

where $\theta_i \in FK$ and all but a finite number of the θ_i's are 0. Choose $\mu \in F$ such that $0 \neq \mu \neq 1$. Then

$$\phi : \sum_{-\infty}^{\infty} \theta_i s^i \mapsto \sum_{-\infty}^{\infty} \mu^i \theta_i s^i$$

defines an automorphism of FH_1. Thus

$$x\phi = \sum_1^n \lambda_i \mu^{n_i} k_i s^{n_i} \in J(FH_1).$$

Therefore $y = x - x\phi \in J(FH_1)$. Also $y \neq 0$. For otherwise, $\mu^{n_i} = 1$, $1 \leqslant i \leqslant n$. Since the g.c.d. of the absolute values of the nonzero n_i's is 1, and since $\mu \neq 1$, it would follow that $n_i = 0$, $1 \leqslant i \leqslant n$; that is $g_i \in K$, $1 \leqslant i \leqslant n$, and so $H = K$, giving a contradiction. Finally, y has smaller length than x.

Since H_1 is a counter-example to the Theorem, it follows that there exists a counter-example, say M, and an element of length 1 in $J(FM)$. This is impossible, and so our contradiction establishes Theorem C.

2.2 *Proof of Theorem* A. Let G be torsion-free with a normal soluble subgroup N of finite index. We argue by induction on the derived length of N. Thus suppose, if possible, that the Theorem is false, and let G be a counter-example. Let N have a finite abelian series

$$1 = H_0 \leqslant H_1 \leqslant H_2 \leqslant \cdots \leqslant H_m = N, \tag{1}$$

where $m \geqslant 1$ and each $H_i \lhd G$. We show that we may suppose that $H_1 \leqslant \zeta_1(G)$, the centre of G.

Let T_1 be a transversal to H_1 in G containing the identity. Then each nonzero element $x \in J(FG)$ is uniquely expressible in the form

$$x = \sum_1^n \alpha_i g_i ,$$

where $n \geqslant 1$, $0 \neq \alpha_i \in FH_1$ $(1 \leqslant i \leqslant n)$, and the g_i's are distinct elements of T_1. Choose such an element x with n minimal. Without loss of generality we may suppose that $g_1 = 1$. Let $h \in H_1$. Then

$$x^h = h^{-1}xh = \sum_1^n \alpha_i g_i^h \in J(FG).$$

Thus $x - x^h = \sum_2^n \alpha_i(1 - [h, g_i^{-1}])\, g_i \in J(FG)$, and $\alpha_i(1 - [h, g_i^{-1}]) \in FH_1$, $2 \leqslant i \leqslant n$. Therefore, by choice of x, $x - x^h = 0$. Then for each i,

$$\alpha_i(1 - [h, g_i^{-1}]) = 0. \tag{2}$$

Now H_1, as a torsion-free abelian group, is ordered ([2]). Therefore FH_1 has no zero-divisors. Thus (2) implies that $hg_i = g_ih$, $1 \leqslant i \leqslant n$, all $h \in H_1$. Hence by Lemma 1 of [3], we may suppose that $G = \langle H_1, g_i\, ; 1 \leqslant i \leqslant n \rangle$; and then

$$H_1 \leqslant \zeta_1(G). \tag{3}$$

Suppose that $m = 1$. Then by a theorem due to Schur (see [5], Theorem 8.1), $G' = 1$. So G is torsion-free abelian, and then FG is semi-simple, by Theorem C. Thus we must have $m \geqslant 2$. Suppose that G is a counter-example, with m in (1) minimal, and satisfying (3). We show that there exists such a group with H_2 in (1) abelian. Then the Theorem will follow by our induction argument.

Let T_2 be a transversal to H_2 in G containing the identity. Then each nonzero element $x \in J(FG)$ is uniquely expressible in the form

$$x = \sum_1^n \gamma_j f_j ,$$

where $n \geqslant 1$, $0 \neq \gamma_j \in FH_2$ $(1 \leqslant j \leqslant n)$, and the f_j's are distinct elements of T_2. Let M be the set consisting of those elements x for which n is minimal and some f_j, say f_1, is 1. Let S be a transversal to H_1 in H_2 containing the identity. For each $x \in M$, write

$$\gamma_1 = \sum_1^l \alpha_i e_i ,$$

where $l \geqslant 1$, $0 \neq \alpha_i \in FH_1$ $(1 \leqslant i \leqslant l)$, and the e_i's are distinct elements of S. Now choose $x \in M$ with l minimal. Without loss of generality, suppose that $e_1 = 1$. Let $b \in H_2$. Then

$$x^b = \alpha_1 + \sum_2^l \alpha_i [b, e_i^{-1}]\, e_i + \sum_2^n \gamma_j^b [b, f_j^{-1}] f_j \in J(FG),$$

by (3). Also $\alpha_i[b, e_i^{-1}] \in FH_1$ $(2 \leqslant i \leqslant l)$ and $\gamma_j^b[b, f_j^{-1}] \in FH_2$ $(2 \leqslant j \leqslant n)$. Then $x - x^b \in J(FG)$; and by choice of x,

$$x - x^b = 0. \tag{4}$$

Thus, for each i, $\alpha_i(1 - [b, e_i^{-1}]) = 0$; and as above $be_i = e_i b$, all $b \in H_2$. For $2 \leqslant j \leqslant n$, let

$$\gamma_j = \sum_{k=1}^{l_j} \alpha_{jk} e_{jk} ,$$

where $l_j \geqslant 1$, $0 \neq \alpha_{jk} \in FH_1$, and the e_{jk}'s are distinct elements of S. Then from (4),

$$\gamma_j = (\gamma_j f_j)^b f_j^{-1} = \sum_{k=1}^{l_j} \alpha_{jk} (e_{jk} f_j)^b f_j^{-1} ,$$

$2 \leqslant j \leqslant n$. Thus the number of conjugates of each $e_{jk} f_j$ under transformation by elements $b \in H_2$ cannot exceed the length of γ_j. Now by Lemma 1 of [3], we may suppose that $G = \langle H_1, e_i, e_{jk} f_j ; 1 \leqslant i \leqslant l, 1 \leqslant k \leqslant l_j, 2 \leqslant j \leqslant n \rangle$

and (3) is still satisfied. Then $K = \zeta_1(G) \cap H_2$ has finite index in H_2. Therefore, again by Schur's theorem, H_2 is abelian.

This completes the proof of Theorem A.

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