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On the Zeros of Some Generalized Hypergeometric Functions

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Let $a_1, \ldots, a_p, b_1, \ldots, b_p$ be real constants with $a_1, \ldots, a_p \neq 0, -1, -2, \ldots$ and $b_1, \ldots, b_p > 0$, and let ${}_pF_p(z) = {}_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z)$. It is shown that the following three conditions are equivalent to each other: (i) ${}_pF_p(z)$ has only a finite number of zeros, (ii) ${}_pF_p(z)$ has only real zeros, and (iii) the a_j's can be re-indexed so that $a_1 = b_1 + m_1, \ldots, a_p = b_p + m_p$ for some nonnegative integers $m_1, \ldots, m_p$.

Key Words: generalized hypergeometric function; multiplier sequence.

1. INTRODUCTION

This paper is concerned with the zeros of generalized hypergeometric functions, namely the functions of the form

$$ {}_pF_q(z) = {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; z) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{z^n}{n!}. \quad (1) $$

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Here $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ $(\neq 0, -1, -2, \ldots)$ are constant and $(a)_n = \Gamma(a+n)/\Gamma(a)$, that is,

$$(a)_0 = 1 \quad \text{and} \quad (a)_n = a(a+1)\cdots(a+n-1) \qquad (n = 1, 2, \ldots).$$

Observe that we must assume $b_1, \ldots, b_1 \neq 0, -1, -2, \ldots$ in order that the series may be well defined, and that the series reduces to a polynomial whenever some a_j is equal to 0 or a negative integer.

In 1929, Hille published a paper entitled *Note on some hypergeometric series of higher order* [3]. In this paper, he studied the zeros of the series (1) in the case where $p \leq q$ and the parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ are real. (Note that (1) defines an entire function if and only if $p \leq q$, provided $a_1, \ldots, a_p \neq 0, -1, -2, \ldots$.) In particular, he proved that (i) if $a \neq 0, -1, -2, \ldots$, the necessary and sufficient condition for ${}_1F_1(a; b; z)$ to have only a finite number of zeros is that $a - b$ is a non-negative integer, and that (ii) if $b_1, \ldots, b_p > 0$ and $m_1, \ldots, m_p$ are non-negative integers, then ${}_pF_p(b_1 + m_1, \ldots, b_p + m_p; b_1, \ldots, b_p; z)$ has real zeros only. He proved (ii) in the special case where $m_1 = \cdots = m_p$, but the same method yields the general case.

The purpose of this paper is to generalize Hille's results mentioned above. For convenience, we introduce the following three conditions on ${}_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z)$.

F. ${}_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z)$ *has only a finite number of zeros*.

R. ${}_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z)$ *has only real zeros*.

I. *The a_j's can be re-indexed so that $a_1 = b_1 + m_1, \ldots, a_p = b_p + m_p$ for some non-negative integers $m_1, \ldots, m_p$.*

What Hille has proved are (i) if $p = 1$ and $a_1 \neq 0, -1, -2, \ldots$, then **F** and **I** are equivalent, and (ii) if $b_1, \ldots, b_p > 0$, then **I** implies **R**. In this paper, we first consider the general case when the parameters $a_1, \ldots, a_p$ and $b_1, \ldots, b_q$ are allowed to have complex values, and prove that if $a_1, \ldots, a_p \neq 0, -1, -2, \ldots$, then **F** and **I** are equivalent (Theorem 1, Section 2). Next, in Section 3, we apply a method due to Barnes to show that ${}_pF_p(z)$ is a real entire function and $a_1, \ldots, a_p$ are real, then ${}_pF_p(z)$ has only a finite number of real zeros (Theorem 2). As a result of Theorem 1 and Theorem 2 we obtain Theorem 3 which states that if $a_1, \ldots, a_p$ $(\neq 0, -1, -2, \ldots)$ are real and $b_1, \ldots, b_p > 0$, then **F**, **R**, and **I** are equivalent to each other. Finally, we conclude this paper with some examples which show that the assumptions in our theorems cannot be weakened (Section 4).

2. NUMBER OF ZEROS

In this section, we will prove the following theorem.

THEOREM 1. *If* $a_1, \dots, a_p \neq 0, -1, -2, \dots$, *then* **F** *and* **I** *are equivalent.*

Proof. We first prove that **I** implies **F**. Let ϑ denote the differential operator $z(d/dz)$, so that for each convergent power series $\sum A_n z^n$ and a constant $a \in \mathbb{C}$ we have

$$\sum_{n=0}^{\infty} (a+n) A_n z^n = (a+\vartheta) \sum_{n=0}^{\infty} A_n z^n.$$

If $b \neq 0, -1, -2, \dots$ and m is a non-negative integer, then

$$\begin{aligned}
&{}_1F_1(b+m; b; z) \\
&\quad = \sum_{n=0}^{\infty} \frac{(b+m)_n}{(b)_n} \frac{z^n}{n!} \\
&\quad = \frac{1}{(b)_m} \sum_{n=0}^{\infty} (b+n)(b+1+n) \cdots (b+m-1+n) \frac{z^n}{n!} \\
&\quad = \frac{1}{(b)_m} (b+\vartheta)(b+1+\vartheta) \cdots (b+m-1+\vartheta) \sum_{n=0}^{\infty} \frac{z^n}{n!},
\end{aligned}$$

so that ${}_1F_1(b+m; b; z) = P(z)e^z$ for some polynomial $P(z)$ of degree m.

In general, if $b_1, \dots, b_p \neq 0, -1, -2, \dots$ and $m_1, \dots, m_p$ are non-negative integers, then we have

$$\begin{aligned}
&{}_pF_p(b_1+m_1, \dots, b_p+m_p; b_1, \dots, b_p; z) \\
&\quad = \left[\prod_{j=1}^{p} \frac{1}{(b_j)_{m_j}} \prod_{k=0}^{m_j-1} (b_j+k+\vartheta) \right] \sum_{n=0}^{\infty} \frac{z^n}{n!},
\end{aligned}$$

and hence ${}_pF_p(b_1+m_1, \dots, b_p+m_p; b_1, \dots, b_p; z) = P(z)e^z$ for some polynomial $P(z)$ of degree $m_1 + \cdots + m_p$. This proves that **I** implies **F**.

Conversely, suppose that $a_1, \dots, a_p \neq 0, -1, -2, \dots$ and that ${}_pF_p(z) = {}_pF_p(a_1, \dots, a_p; b_1, \dots, b_p; z)$ has only a finite number of zeros. Then ${}_pF_p(z)$ is an entire function of order 1 with finitely many zeros. Therefore ${}_pF_p(z)$ can be written in the form

$$ {}_pF_p(z) = (c_0 + c_1 z + \cdots + c_N z^N) e^{\alpha z}, \tag{2}$$

where $\alpha, c_0, \dots, c_N$ are constants with $\alpha, c_N \neq 0$. It is well known [1, p. 60] and easy to see that ${}_pF_p(z)$ satisfies the differential equation

$$\left[\frac{d}{dz}(\vartheta + b_1 - 1) \cdots (\vartheta + b_p - 1) - (\vartheta + a_1) \cdots (\vartheta + a_p)\right] {}_pF_p(z) = 0. \tag{3}$$

From (2) and (3), we obtain

$$(\alpha^{p+1} - \alpha^p) c_N z^{N+p} + \text{lower terms} = 0,$$

so that $\alpha = 0$ or 1. Since $\alpha \neq 0$, it follows that $\alpha = 1$, and hence (2) becomes

$$\sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_p)_n} \frac{z^n}{n!} = (c_0 + c_1 z + \cdots + c_N z^N) \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

Therefore we have

$$\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_p)_n} \frac{1}{n!} = \frac{c_0}{n!} + \frac{c_1}{(n-1)!} + \cdots + \frac{c_N}{(n-N)!} \qquad (n = N, N+1, \dots). \tag{4}$$

Let $f(t)$ be the polynomial defined by

$$f(t) = c_0 + c_1 t + c_2 t(t-1) + \cdots + c_N t(t-1) \cdots (t - N + 1).$$

Then (4) implies that

$$\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_p)_n} = f(n) \qquad (n = N, N+1, \dots),$$

and hence we have

$$\begin{aligned} \frac{(a_1 + n) \cdots (a_p + n)}{(b_1 + n) \cdots (b_p + n)} f(n) &= \frac{(a_1 + n) \cdots (a_p + n)}{(b_1 + n) \cdots (b_p + n)} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_p)_n} \\ &= \frac{(a_1)_{n+1} \cdots (a_p)_{n+1}}{(b_1)_{n+1} \cdots (b_p)_{n+1}} = f(n+1) \\ &\qquad (n = N, N+1, \dots), \end{aligned}$$

so that

$$(a_1+n)\cdots(a_p+n)f(n) = (b_1+n)\cdots(b_p+n)f(n+1)$$
$$(n = N, N+1, \dots). \quad (5)$$

Since $f(t)$ is a polynomial, Eq. (5) implies that

$$(a_1+t)\cdots(a_p+t)f(t) = (b_1+t)\cdots(b_p+t)f(t+1). \quad (6)$$

Now, using (6) and an induction, we will show that we can re-index $a_1, \dots, a_p$ so that $a_1 - b_1, \dots, a_p - b_p$ are non-negative integers. If $p + N = p + \deg f(t) = 1$, then we must have $p = 1$ and $N = 0$, so that our assertion is trivial.

Let $p + N > 1$ and assume the induction hypothesis. Since $b_1 + t$ is a factor of the right-hand side of (6), it is a factor of the left-hand side also. Therefore $b_1 + t$ divides $(a_1 + t)\cdots(a_p + t)$, or $b_1 + t$ divides $f(t)$.

If $b_1 + t$ divides $(a_1 + t)\cdots(a_p + t)$, then $b_1 = a_k$ for some k. Re-index $a_1, \dots, a_p$ so that $a_1 = b_1$. Then (6) implies that

$$(a_2+t)\cdots(a_p+t)f(t) = (b_2+t)\cdots(b_p+t)f(t+1).$$

Hence, by the induction hypothesis, we can re-index $a_2, \dots, a_p$ so that $a_2 - b_2, \dots, a_p - b_p$ are non-negative integers.

If $b_1 + t$ divides $f(t)$, then there is a polynomial $f_1(t)$ of degree $N - 1$ such that $f(t) = (b_1 + t)f_1(t)$. From (6), we obtain

$$\begin{aligned}(a_1+t)\cdots(a_p+t)f_1(t) &= (b_2+t)\cdots(b_p+t)f(t+1)\\ &= (b_2+t)\cdots(b_p+t)(b_1+t+1)f_1(t+1).\end{aligned}$$

Again, the induction hypothesis implies that we can re-index $a_1, \dots, a_p$ so that $a_1 - b_1 - 1, a_2 - b_2, \dots, a_p - b_p$ are non-negative integers. This proves our assertion. ∎

3. REALITY OF ZEROS

We start this section with an asymptotic equality due to Barnes [1]. Let $a_1, \dots, a_p$ and $b_1, \dots, b_p$ be complex constants and assume that $b_1, \dots, b_p \neq 0, -1, -2, \dots$. In [1, pp. 80–83], it is shown that for each $\epsilon > 0$ we have

$$\prod_{j=1}^{p} \frac{\Gamma(a_j)}{\Gamma(b_j)} {}_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z)$$

$$= e^z z^{\Sigma a_j - \Sigma b_j}\left(1 + O\left(\frac{1}{z}\right)\right) \qquad \left(|z| \to \infty, |\arg z| \le \frac{\pi}{2} - \epsilon\right). \quad (7)$$

Here arg is the principal branch of the argument. Hence ${}_pF_p(z)$ has only a finite number of zeros in the sector $|\arg z| \le \frac{\pi}{2} - \epsilon$ for each $\epsilon > 0$.

Next, we will show that if

$$\frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_p)_n} \in \mathbb{R} \qquad (n = 0, 1, 2, \ldots),$$

and if $a_1, \ldots, a_p$ are real, then ${}_pF_p(z)$ has only a finite number of negative real zeros. If some a_j is equal to zero or a negative integer, then ${}_pF_p(z)$ is a polynomial, and if the condition **I** holds, then ${}_pF_p(z)$ has only a finite number of zeros, by Theorem 1. Hence we may assume, without loss of generality, that $a_1, \ldots, a_p \ne 0, -1, -2, \ldots$ and that the condition **I** does not hold. Then the function $\Gamma(a_1 + s) \cdots \Gamma(a_p + s)/\Gamma(b_1 + s) \cdots \Gamma(b_p + s)$ has infinitely many poles all of which are different from $0, 1, 2, \ldots$. Let $\alpha_1, \alpha_2, \ldots$ denote the distinct poles of $\Gamma(a_1 + s) \cdots \Gamma(a_p + s)/\Gamma(b_1 + s) \cdots \Gamma(b_p + s)$. For each $k = 1, 2, \ldots$ let $R_k(z)$ denote the residue of the function

$$\frac{\Gamma(a_1 + s) \cdots \Gamma(a_p + s)}{\Gamma(b_1 + s) \cdots \Gamma(b_p + s)} \Gamma(-s)(-z)^s$$

at the pole $s = \alpha_k$. Here $(-z)^s$ is defined by

$$(-z)^s = \exp[s \log(-z)]$$

for $s \in \mathbb{C}$ and $z \ne 0$; log is the principal branch of the logarithm. Then it is easy to see that $R_k(z)/(-z)^{\alpha_k}$ is a polynomial of $\log(-z)$ for each $k = 1, 2, \ldots$. See [6, p. 288] also.

Let $A > \max\{|\operatorname{Im} a_j| : j = 1, \ldots, p\}$, let K be an arbitrary positive real number such that $\operatorname{Re} \alpha_k \ne -K$ for all $k = 1, 2, \ldots$, and let C_1 denote the contour composed of the half line $s = -t + Ai$, $-\infty < t \le K$, the line segment from $-K + Ai$ to $-K - Ai$, and the half line $s = t - Ai$, $-K \le t$

$< \infty$. Then we have

$$\frac{-1}{2\pi i}\int_{C_1}\frac{\Gamma(a_1+s)\cdots\Gamma(a_p+s)}{\Gamma(b_1+s)\cdots\Gamma(b_p+s)}\Gamma(-s)(-z)^s\,ds$$

$$=\frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_p)}{}_pF_p(z)-\sum_{-K<\alpha_k}R_k(z)$$

for $\operatorname{Re} z < 0$.

Let C_2 denote the line $s=-K-it$, $-\infty<t<\infty$, and let $\epsilon>0$ be arbitrary. If $|\arg(-z)| = |\operatorname{Im}\log(-z)| \le \frac{\pi}{2}-\epsilon$, then

$$\int_{C_1}\frac{\Gamma(a_1+s)\cdots\Gamma(a_p+s)}{\Gamma(b_1+s)\cdots\Gamma(b_p+s)}\Gamma(-s)(-z)^s\,ds$$

$$=\int_{C_2}\frac{\Gamma(a_1+s)\cdots\Gamma(a_p+s)}{\Gamma(b_1+s)\cdots\Gamma(b_p+s)}\Gamma(-s)(-z)^s\,ds.$$

Moreover, we have

$$\int_{C_2}\frac{\Gamma(a_1+s)\cdots\Gamma(a_p+s)}{\Gamma(b_1+s)\cdots\Gamma(b_p+s)}\Gamma(-s)(-z)^s\,ds = O\left(|z|^{-K}\right)$$

$$\left(|\arg(-z)|\le\frac{\pi}{2}-\epsilon,\ |z|\to\infty\right),$$

so that

$$\frac{\Gamma(a_1)\cdots\Gamma(a_p)}{\Gamma(b_1)\cdots\Gamma(b_p)}{}_pF_p(z)=\sum_{-K<\alpha_k}R_k(z)+O\left(|z|^{-K}\right)$$

$$\left(|\arg(-z)|\le\frac{\pi}{2}-\epsilon,\ |z|\to\infty\right). \quad (8)$$

From (8), it follows that if $a_1,\dots,a_p$ are real and if $R_k(z)\neq 0$ for some k and z, then for each $\epsilon>0$ the function ${}_pF_p(z)$ has only a finite number of zeros in the sector $|\arg(-z)|\le\frac{\pi}{2}-\epsilon$, because $R_1(z)/(-z)^{\alpha_1}$, $R_2(z)/(-z)^{\alpha_2},\dots$ are polynomials of $\log(-z)$.

Remark. It seems hardly true that $R_k(z)=0$ for all k and z, but the authors were unable to prove that this is not the case.

To proceed further, we need the following lemma whose proof is almost trivial.

LEMMA. *Let $f(s)$ be a function which has a pole of order $m \geq 1$ at $s = \alpha$, and let $c_1, \ldots, c_{m-1}$ be arbitrary constants. Then either $f(s)$ has nonzero residue at $s = \alpha$ or there is a positive integer $l \leq m - 1$ such that $(s - c_1) \cdots (s - c_l) f(s)$ has nonzero residue at $s = \alpha$.*

From this lemma, it follows that if $z \neq 0$, then for each $k = 1, 2, \ldots$ there is a nonnegative integer l such that the residue of

$$\frac{\Gamma(a_1 + s) \cdots \Gamma(a_p + s)}{\Gamma(b_1 + s) \cdots \Gamma(b_p + s)} \Gamma(-s + l)(-z)^s$$

at $s = \alpha_k$ is not equal to zero. As a consequence, there is a non-negative integer l such that if $z \neq 0$, then the function

$$\frac{\Gamma(a_1 + l + s) \cdots \Gamma(a_p + l + s)}{\Gamma(b_1 + l + s) \cdots \Gamma(b_p + l + s)} \Gamma(-s)(-z)^s$$

has a pole other than $0, 1, 2, \ldots$ at which the residue is not equal to zero. Since

$$\begin{aligned} &{}_pF_p^{(l)}(a_1, \ldots, a_p; b_1, \ldots, b_p; z) \\ &\quad = \frac{(a_1)_l \cdots (a_p)_l}{(b_1)_l \cdots (b_p)_l} {}_pF_p(a_1 + l, \ldots, a_p + l; b_1 + l, \ldots, b_p + l; z), \end{aligned}$$

we conclude that if $a_1, \ldots, a_p$ are real, then some derivative of ${}_pF_p(z)$ has only a finite number of zeros in the sector $|\arg(-z)| \leq \frac{\pi}{2} - \epsilon$ for each $\epsilon > 0$.

Now suppose that $a_1, \ldots, a_p$ are real and that ${}_pF_p(z)$ is a real entire function, that is, its Maclaurin coefficients are all real. Then the above argument implies that there is a non-negative integer l such that ${}_pF_p^{(l)}(z)$ has only a finite number of negative real zeros, and hence Rolle's theorem implies that ${}_pF_p(z)$ must have a finite number of negative real zeros. On the other hand, Eq. (7) implies that ${}_pF_p(z)$ has finitely many positive real zeros. Consequently, ${}_pF_p(z)$ has only a finite number of real zeros.

THEOREM 2. *Suppose that ${}_pF_p(z)$ is a real entire function and $a_1, \ldots, a_p$ are real. Then ${}_pF_p(z)$ has only a finite number of real zeros.*

COROLLARY. *If ${}_pF_p(z)$ is a real entire function and $a_1, \ldots, a_p$ are real, then* **R** *implies* **F**.

As mentioned in the Introduction, Hille proved that if $b_1, \ldots, b_p > 0$, then **I** implies **R**, but he gave only a sketch of the proof. For completeness, we will give a detailed proof of the result. Our proof is based on the Pólya–Schur theory of multiplier sequences.

A sequence $\{\gamma_n\}_{n=0}^{\infty}$ of real numbers is said to be a *multiplier sequence* (of the first kind) if it has the following property.

M. *If $a_0 + a_1 z + \cdots + a_d z^d$ is a real polynomial with real zeros only, then the polynomial*

$$\gamma_0 a_0 + \gamma_1 a_1 z + \cdots + \gamma_d a_d z^d$$

also has real zeros only.

It follows immediately from the definition that if $\{\gamma_n\}$ and $\{\delta_n\}$ are multiplier sequences, then their termwise product $\{\gamma_n \delta_n\}$ is also a multiplier sequence.

The multiplier sequences are completely characterized by the following theorem of Pólya and Schur [4].

THE PÓLYA–SCHUR THEOREM. *A sequence $\{\gamma_n\}$ of real numbers is a multiplier sequence if and only if the function*

$$\Phi(z) = \sum_{n=0}^{\infty} \gamma_n \frac{z^n}{n!}$$

can be uniformly approximated on compact sets in the complex plane by a sequence of real polynomials all of whose zeros are real and of the same sign.

Let b be a positive real number. Then the Pólya–Schur theorem implies that $\{b+n\}_{n=0}^{\infty}$ is a multiplier sequence, because

$$\sum_{n=0}^{\infty} (b+n)\frac{z^n}{n!} = (z+b)e^z = \lim_{k\to\infty} (z+b)\left(1+\frac{z}{k}\right)^k.$$

Since the class of multiplier sequences is closed under termwise multiplication, it follows that for each positive integer m the sequence

$$\{(b+n)(b+1+n)\cdots(b+m-1+n)\}_{n=0}^{\infty}$$

is a multiplier sequence. Hence

$$\left\{\frac{(b+m)_n}{(b)_n}\right\}_{n=0}^{\infty} = \left\{\frac{1}{(b)_m}(b+n)(b+1+n)\cdots(b+m-1+n)\right\}_{n=0}^{\infty}$$

is a multiplier sequence for each positive integer m.

Now suppose that $b_1, \ldots, b_p$ are positive real numbers and $m_1, \ldots, m_p$ are nonnegative integers. Then it is clear that the sequence

$$\left\{ \frac{(b_1 + m_1)_n \cdots (b_p + m_p)_n}{(b_1)_n \cdots (b_p)_n} \right\}_{n=0}^{\infty}$$

is a multiplier sequence. Hence, by the Pólya–Schur theorem, the function

$$\begin{aligned} &{}_pF_p(b_1 + m_1, \ldots, b_p + m_p; b_1, \ldots, b_p; z) \\ &\quad = \sum_{n=0}^{\infty} \frac{(b_1 + m_1)_n \cdots (b_p + m_p)_n}{(b_1)_n \cdots (b_p)_n} \frac{z^n}{n!} \end{aligned}$$

can be uniformly approximated on compact sets in the complex plane by a sequence of real polynomials all of whose zeros are real and of the same sign; in particular, it has only real zeros. This completes the proof.

THEOREM 3. *If* $a_1, \ldots, a_p$ $(\neq 0, -1, -2, \ldots)$ *are real and* $b_1, \ldots, b_p > 0$, *then* **F**, **R**, *and* **I** *are equivalent*.

Proof. We have just shown that if $b_1, \ldots, b_p > 0$, then **I** implies **R**. Now, our theorem is an immediate consequence of Theorem 1 and the corollary to Theorem 2. ∎

4. EXAMPLES

In this section, we exhibit some examples which show that we cannot weaken the assumptions of the theorems in this paper without altering their conclusions. First of all, we remark that we must assume the condition that $a_1, \ldots, a_p \neq 0, -1, -2, \ldots$ in Theorem 1; otherwise the function ${}_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z)$ reduces to a polynomial, so that the implication $\mathbf{F} \Rightarrow \mathbf{I}$ does not hold.

Theorem 2 states that if ${}_pF_p(z)$ is a real entire function and $a_1, \ldots, a_p$ are real, then ${}_pF_p(a_1, \ldots, a_p; b_1, \ldots, b_p; z)$ has only a finite number of real zeros. The following example, however, shows that this is not the case in general.

EXAMPLE 1. The real entire function $f(z) = {}_2F_2(i, -i; 1, 1; z)$ has infinitely many real zeros. This may be shown as follows. According to (8) of

Section 3 or the asymptotic equality given in [1, Sect. 9], we have

$$\frac{2\pi}{e^{\pi} - e^{-\pi}} f(-t) = \frac{\Gamma(i)\Gamma(-2i)}{\Gamma(1-i)^2} t^{-i}(1 + O(t^{-1})) + \frac{\Gamma(-i)\Gamma(2i)}{\Gamma(1+i)^2} t^{i}(1 + O(t^{-1})),$$

as $t \to \infty$ with $t > 0$. Therefore there exist real constants A and B, with $A > 0$, such that

$$f(-t) = A\cos(B + \log t) + O(t^{-1}) \qquad (t \to \infty, t > 0),$$

and this proves our assertion.

Next, we consider the assumptions of Theorem 3, namely $a_1, \dots, a_p \neq 0, -1, -2, \dots$ and $b_1, \dots, b_p > 0$. The following example is about the condition $a_1, \dots, a_p \neq 0, -1, -2, \dots$.

EXAMPLE 2. Let $b > 0$. Then

$$\phi(x) = \frac{\Gamma(b)}{\Gamma(b+x)}$$

is a real entire function of order 1 and has negative real zeros only. Hence $\{(b)_n^{-1}\}_{n=0}^{\infty} = \{\phi(n)\}_{n=0}^{\infty}$ is a complex zero decreasing sequence by [2, Theorem 1.4]. In particular, it is a multiplier sequence. Let m be a positive integer. If we apply the multiplier sequence $\{(b)_n^{-1}\}_{n=0}^{\infty}$ to the polynomial $(1-z)^m$, then we obtain ${}_1F_1(-m; b; z)$, because

$${}_1F_1(-m; b; z) = \sum_{n=0}^{m} \frac{(-m)_n}{(b)_n} \frac{z^n}{n!} = \sum_{n=0}^{m} \frac{1}{(b)_n} \binom{m}{n} (-z)^n.$$

Therefore ${}_1F_1(-m; b; z)$ has real zeros only for each $m = 1, 2, \dots$ and $b > 0$. Consequently, we must assume $a_1, \dots, a_p \neq 0, -1, -2, \dots$ in order to obtain the implication $\mathbf{R} \Rightarrow \mathbf{I}$ in Theorem 3.

Our last example is concerned with the condition $b_1, \dots, b_p > 0$.

EXAMPLE 3. Let $b < 0$ and $b \neq -1, -2, \dots$. If m is a positive integer greater than $-b$, then ${}_1F_1(b+m; b; z)$ has at least $2[(1-b)/2]$ nonreal zeros.

Proof. Let m be a positive integer greater than $-b$. Then $\{(b+m)_n^{-1}\}_{n=0}^{\infty}$ is a complex zero decreasing sequence. On the other hand, the proof of Theorem 1 shows that ${}_1F_1(b+m; b; z) = P(z)e^z$ for some real polynomial $P(z)$ of degree m. Hence the number of nonreal zeros of

$$ {}_1F_1(b+m;b;z) = \sum_{n=0}^{\infty} \frac{(b+m)_n}{(b)_n} \frac{z^n}{n!} $$

is at least that of

$$ {}_0F_1(b;z) = \sum_{n=0}^{\infty} \frac{1}{(b)_n} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{(b+m)_n} \frac{(b+m)_n}{(b)_n} \frac{z^n}{n!}. $$

Now our assertion follows from a well-known theorem of Hurwitz [5, 15.27] which states that if $b < 0$ and $b \neq -1, -2, \ldots$, then ${}_0F_1(b;z)$ has exactly $2[(1-b)/2]$ nonreal zeros. ∎

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