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Exchange Lemmas for Singular Perturbation Problems with Certain Turning Points

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In this work, singular perturbation problems with a certain type of turning point are studied from a geometric point of view. We first describe the delay

[illegible] and the center manifold theory for general invariant manifolds developed by Chow, Liu, and Yi, we extend the well-known exchange lemma, first formulated by Jones and Kopell for problems with normally hyperbolic slow manifolds, to problems with this type of turning point. Applications to singular boundary value problems with turning points are discussed.

1. INTRODUCTION

In this work, we extend the geometric singular perturbation theory to problems with a certain type of turning point.

Consider singularly perturbed ordinary differential equations of the form

$$\begin{aligned} \varepsilon \dot{x} &= F(x, y; \varepsilon), \\ \dot{y} &= G(x, y; \varepsilon), \end{aligned} \tag{1}$$

where $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$, F and G are smooth in their arguments, and $\dot{} = \frac{d}{d\tau}$ denotes the derivative with respect to the slow time τ. In terms of the fast time scale $t = \frac{\tau}{\varepsilon}$, the system becomes

$$\begin{aligned} x' &= F(x, y; \varepsilon), \\ y' &= \varepsilon G(x, y; \varepsilon), \end{aligned} \tag{2}$$

where $' = \frac{d}{dt}$ denotes the derivative with respect to the fast time t.

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While these two systems are equivalent for $\varepsilon \neq 0$, the different time scales give rise to two limiting systems. Letting $\varepsilon \to 0$ in (1), we obtain

$$\begin{aligned} 0 &= F(x, y; 0), \\ \dot{y} &= G(x, y; 0). \end{aligned} \tag{3}$$

Assume that there exists a function $x = H(y)$ for $y \in D$, a domain in $\mathbb{R}^n$, such that $F(H(y), y; 0) = 0$; that is, $S_0 = \{(x, y) : x = H(y)\}$ is a part of the slow manifold. The system (3) is then defined on S_0 and its dynamic is determined by the second equation only. On the other hand, letting $\varepsilon \to 0$ in (2) results in the system

$$\begin{aligned} x' &= F(x, y; 0), \\ y' &= 0, \end{aligned} \tag{4}$$

which has S_0 as a set of equilibria.

Roughly speaking, the two scalings single out the roles of the two components of the vector field. The dynamic in the vicinity of the slow manifold plays a role similar to that of Morse sets in Morse–Smale systems. A natural question is how the reduced flow (3) on the slow manifold affects the one in its neighborhood for $\varepsilon \neq 0$.

As a set of equilibria of the system (4), the linearization on S_0 is given by

$$\begin{pmatrix} F_x(p; 0) & F_y(p; 0) \\ 0 & 0 \end{pmatrix}$$

for $p \in S_0$. If the real part $\Re(\lambda(p)) \neq 0$ for all eigenvalues $\lambda(p)$ of $F_x(p; 0)$ and all $p \in S_0$, then the slow manifold S_0 is normally hyperbolic. The normally hyperbolic theory (see [7, 11]) implies the persistence of the slow manifold as well as the stable and unstable foliations. As a consequence, the dynamic in the vicinity of the slow manifold is completely determined by the one on the slow manifold. The latter statement is made precise, for practical reasons, through the exchange lemmas first formulated by Jones and Kopell in [16]. A special situation is treated early by Deng in [5]. The geometric singular perturbation theory (see [5, 8, 16, 21, 29, 30]) has been developed based on the normal hyperbolicity of the slow manifold together with other geometric properties, for example, the transversality of stable and unstable manifolds. It has been successfully applied to the study of heteroclinic, homoclinic, and periodic solutions and to singular boundary value problems.

With the presence of *turning points* where, for some eigenvalues $\lambda(p)$, the real part $\Re(\lambda(p))$ changes sign, the slow manifold fails to be normally hyperbolic in the vicinity of the turning points. Nevertheless, the theory has been employed in the study of some turning points. Kopell [17, 18] applied the theory to singularly perturbed turning-point problems and boundary layer problems exhibiting resonance. Milik and Szmolyan [24] studied the relaxation solutions of a chemical oscillator with turning points. In this work we identify a type of turning point and extend the geometric singular perturbation theory to problems with such turning points; more precisely, we establish the exchange lemmas for singular perturbation problems for this type of turning point. The purpose of the exchange lemmas is to describe the change of the smooth configuration of an invariant manifold as it passes a neighborhood of the slow manifold, which is evidently one of the most important ingredients of the geometric singular perturbation theory. In the case that the slow manifold is normally hyperbolic, Jones and Kopell [16] establish the exchange lemma. Its proof depends heavily on a Fenichel coordinates system (see [15, 16, 30]). The existence of such a coordinate system in the vicinity of a normally hyperbolic slow manifold follows from the normally hyperbolic theory. When turning points are relevant, the theory as well as the method developed in [7, 11] do not apply anymore. Instead, we use the center manifold theory for general invariant manifolds recently established by Chow *et al.* in [3] to achieve a similar coordinate syste—the Fenichel-type coordinate system. It turns out that the turning points affect the changes of configuration in three major ways, and three exchange lemmas are obtained accordingly.

The types of turning points involved possess the following property: The presence of the turning points will not destroy the slow manifold (due to special structures of the system, see hypotheses (H1) in Section 3). Its local properties were studied by Pontryagin and his school. The so-called *delay of stability loss* is the main feature of the turning points. Singular perturbations with such turning points arise in many important applications; for example, the class of Lotka–Volterra-type equations, a model problem of Howes and Parter [13, 19], a perturbation of the epidemic model of SIR type studied in [9], and traveling wave problems of a class of reaction–diffusion systems with nonlinearity depending on gradient.

The paper is organized as follows. In Section 2, we describe the interesting local properties of the turning points and discuss the effect on initial value problems. In Section 3, a Fenichel-type coordinate system is obtained in a neighborhood of the slow manifold. This is accomplished by an application of the center manifold theory [3] with a series of changes of coordinates. We then establish the exchange lemmas in Section 4. The formulation and the proof of the exchange lemmas follow those of Jones and co-workers [14–16, 30]. Section 5 is devoted to some general applications to singular boundary value problems. Finally, in the appendix, we justify a result on invariant foliations used in Section 3.

2. LOCAL PROPERTIES

Let us recall some important local properties of turning points (see [22, 25]). To illustrate the idea in a simple manner, we start with the simplest situation. Consider the singularly perturbed system

$$\begin{aligned} \varepsilon\dot{x} &= xf(x, y; \varepsilon), \\ \dot{y} &= g(x, y; \varepsilon) \end{aligned} \tag{5}$$

where $(x, y) \in \mathbb{R}^2$. Assume that the functions f and g are smooth in their arguments, $g(0, y; 0) > 0$, $f(0, 0; 0) = 0$ and $f(0, y; 0)\, y > 0$ for $y \neq 0$. Thus, $S_0 = \{x = 0\}$ is a branch of the slow manifold $\{xf(x, y; 0) = 0\}$ and, most importantly, it is invariant even for $\varepsilon \neq 0$. The linearization along S_0 of the corresponding fast system at $\varepsilon = 0$ is

$$\begin{pmatrix} f(0, y; 0) & 0 \\ 0 & 0 \end{pmatrix}.$$

The condition on f implies that $(0, 0)$ is a turning point and, on two sides of the turning point, the stability of the slow manifold changes; that is,

$S_- = \{(0, y) : y < 0\}$ is stable and $S_+ = \{(0, y) : y > 0\}$ is unstable.

If we consider an initial value problem with the initial data $x(0) = x_0 \neq 0$ and $y(0) = y_0 < 0$, then the solution immediately approaches S_- first to form an initial layer and then follows the slow manifold to give the outer solution. When it passes the turning point $(0, 0)$, S_+ repels the solution. The question is, up to what point is the outer solution valid, or, at what point does the solution begin to leave the slow manifold? Intuitively, the problem is a competition of the speed of the flow along the slow manifold and the rate of the repelling of the slow manifold. The effect of the type of turning points in system (5) was understood by Pontryagin back in the 1960s (see [25] and also [6, 26, 27]). We will give a description of the phenomenon as follows.

Let us define a map $P_0 : S_- \to S_+$ by $P_0(0, y) = (0, y_0) = (0, y \cdot \tau_0)$, where $\tau_0 > 0$ is determined by

$$\int_0^{\tau_0} f(0, y \cdot s; 0)\, ds = 0, \tag{6}$$

where $y \cdot \tau$ denotes the solution on S_0 at $\varepsilon = 0$. Or equivalently,

$$\int_y^{y_0} \frac{f(0, \xi; 0)}{g(0, \xi; 0)}\, d\xi = 0. \tag{7}$$

The assumptions on f and g imply that P_0 is a well-defined one-to-one map and $P_0(0, y) \to (0, 0)$ as $y \to 0$.

DEFINITION 2.1. The map P_0 will be called the *pairing map* and the pair of points $(0, y)$ and $P_0(0, y)$ will be called a *turning pair*.

The main result below reveals the delay of the stability loss.

THEOREM 2.2. *Fix $\delta \neq 0$ and $K > 0$. Choose Poincaré sections as*

$$I^\delta := \{(\delta, y) : y \in (-K, 0)\}; \qquad II^\delta := \{(\delta, y) : y \in (0, \infty)\}$$

and define the Poincaré map $P_\varepsilon \colon I^\delta \to II^\delta$ by $P_\varepsilon(\delta, y) = (\phi_1^\varepsilon(\tau_\varepsilon; \delta, y), \phi_2^\varepsilon(\tau_\varepsilon; \delta, y))$ at the time $\tau_\varepsilon > 0$ such that $\phi_1^\varepsilon(\tau_\varepsilon; \delta, y) = \delta$, where $(\phi_1^\varepsilon(\tau; x, y), \phi_2^\varepsilon(\tau; x, y))$ is the flow defined by the system (5) *corresponding to $\varepsilon \neq 0$ with the initial point (x, y). Then $P_\varepsilon \to P_0$ in the C^r-norm as $\varepsilon \to 0$ if we identify I^δ with I^0 and II^δ with II^0 in the obvious way.*

Proof. See [22, 25]. ∎

As an immediate consequence, we have (see Fig. 1)

COROLLARY 2.3. *Consider the initial value problem $x(0) = x_0 \neq 0$ and $y(0) = y_0 < 0$ for the system* (5). *The outer expansion is valid until $P_0(0, y_0) = (0, y_1)$ in the sense that* (i) *for any $0 < y < y_1$ the solution is exponentially in ε close to the slow manifold up to* $(0, y)$ *for small* ε; (ii) *for any $y > y_1$ the solution leaves the vicinity of the slow manifold before* $(0, y)$.

Proof. See [22]. ∎

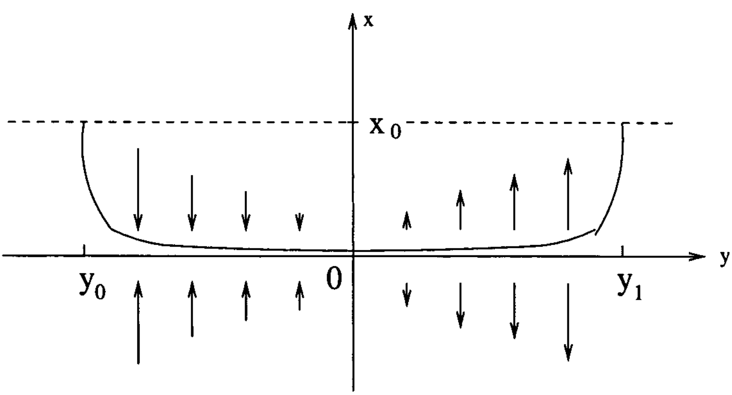

FIG. 1. The delay phenomena.

EXAMPLE 2.4. An application of Corollary 2.3 explains a phenomenon observed by O'Malley in [28] from a simple linear system. The system is

$$\varepsilon \dot{x} = (\tau - 1)\, x. \tag{8}$$

The slow manifold is given by $x = 0$ and the exact solution of an initial value problem is given by $x(\tau; \varepsilon) = e^{-(1-\tau/2)\,\tau/\varepsilon} x(0)$. It can be readily seen that the outer solution expansion is valid up to $\tau = 2$. If we try to solve the initial value problem $x(0) \neq 0$ using the traditional asymptotic expansion procedure, we obtain, for the inner problem and in terms of the fast time t, the solution $x(t; \varepsilon) = e^{-t} x(0) + 1/2 t^2 e^{-t} x(0) + \cdots$. In particular, $x(t; \varepsilon) \to 0$ as $t \to \infty$. For the outer solution, we then seek the solution with $x(0) = 0$ on the slow manifold. It turns out that $x(\tau; \varepsilon) = 0$ for all τ is a solution. There are two questions which arise here. One comes from the fact that the outer solution is actually valid beyond $\tau = 1$, the turning point; the other is that it is not valid after $\tau = 2$, the symmetric point of the initial point with respect to the turning point. It is known that the complicated phenomenon was caused by the presence of the turning point $\tau = 1$.

If we augment $\dot{\tau} = 1$ to the equation and replace τ with $y + 1$, we have

$$\begin{aligned} \varepsilon \dot{x} &= xy, \\ \dot{y} &= 1. \end{aligned} \tag{9}$$

In terms of system (5), we have $f(x, y; \varepsilon) = y$ and $g(x, y; \varepsilon) = 1$, and hence $(0, 0)$ is a turning point and $(0, -1)$ and $(0, 1)$ form a turning pair. The initial value problem $x(0) \neq 0$ for system (8) is equivalent to the initial value problem $x(0) \neq 0$ and $y(0) = -1$ for system (9). Corollary 2.3 asserts that the outer solution expansion is valid up to $y = 1$ or $\tau = 2$.

We now state the result for higher dimensional slow manifolds and one fast direction. Thus, we consider

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y; \varepsilon)\, x, \\ \dot{y} &= g(x, y; \varepsilon), \end{aligned} \tag{10}$$

where $x \in \mathbb{R}$ and $y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. Assume that the functions f and g are smooth in their arguments,

$$f(0, y; 0) = \begin{cases} <0, & \text{if } \ y_1 < 0 \\ =0, & \text{if } \ y_1 = 0 \\ >0, & \text{if } \ y_1 > 0 \end{cases}$$

and $(0, g(0, y; 0))$ is transversal to $\{y_1 = 0\}$ on $\{x = 0\}$ pointing from $\{y_1 < 0\}$ to $\{y_1 > 0\}$. Thus, $S_0 := \{x = 0\}$ is a branch of the slow manifold and is also invariant for $\varepsilon \neq 0$. The set $L := \{x = 0, y_1 = 0\}$, consisting of turning points, is a hypersurface of S_0. The two sides of L on S_0 will be denoted by

$$S_- := \{(0, y) : y_1 < 0\} \qquad \text{and} \qquad S_+ := \{(0, y) : y_1 > 0\}.$$

On S_0, the slow flow is given by

$$\dot{y} = g(0, y; 0).$$

Let $y \cdot \tau$ denote the solution with the initial condition y. Define the pairing map $P_0 \colon S_- \to S_+$ by $P_0(0, y^0) = (0, y^0 \cdot T) = (0, y^1)$, where $T > 0$ is determined by

$$\int_0^T f(0, y^0 \cdot \tau; 0)\, d\tau = 0, \tag{11}$$

or equivalently, y^1 is determined by the line integral condition

$$\int_{\mathrm{Or}(y^0, y^1)} \frac{f(0, y; 0)}{|g(0, y; 0)|^2} \langle g(0, y; 0), dy \rangle = 0, \tag{12}$$

where $\mathrm{Or}(y^0, y^1)$ denotes the curve formed by the solution $y^0 \cdot \tau$ from y^0 to y^1. In fact, if we parameterize the curve $\mathrm{Or}(y^0, y^1)$ by the solution $y^0 \cdot \tau$, then the line integral equation (12) reduces to the equation (11). An advantage of the second formula is that one can evaluate the line integral using any parameterization of the curve instead of the one given by the solution; for example, if the slow flow is a two dimensional Hamiltonian flow, then the orbits are given by the level curves of the Hamiltonian and they usually have natural parameterization while the solutions are not representable in general.

THEOREM 2.5. *Fix $\delta \neq 0$ and $K > 0$. Choose Poincaré sections as*

$$I^\delta := \{(\delta, y) : y_1 \in (-K, 0)\}; \qquad II^\delta := \{(\delta, y) : y_1 \in (0, \infty)\}$$

and define the Poincaré map $P_\varepsilon \colon I^\delta \to II^\delta$ by $P_\varepsilon(\delta, y) = (\phi_1^\varepsilon(\tau_\varepsilon; \delta, y), \phi_2^\varepsilon(\tau_\varepsilon; \delta, y))$ at the time $\tau_\varepsilon > 0$ such that $\phi_1^\varepsilon(\tau_\varepsilon; \delta, y) = \delta$, where $(\phi_1^\varepsilon(\tau; x, y), \phi_2^\varepsilon(\tau; x, y))$ is the flow defined by the system (10) corresponding to $\varepsilon \neq 0$ with the initial point (x, y). Then $P_\varepsilon \to P_0$ in the C^r-norm as $\varepsilon \to 0$ if we identify I^δ with I^0 and II^δ with II^0 in the obvious way.

3. FENICHEL-TYPE COORDINATES

As shown in [16, 30], in establishing the exchange lemmas a Fenichel coordinate system in a neighborhood of the normally hyperbolic slow manifold is crucial for the analysis. There, the existence of a Fenichel coordinate system follows from the normally hyperbolic theory. Although the slow manifold of the singularly perturbed system we consider here is not normally hyperbolic, we are able to construct a similar coordinate system (the Fenichel-type coordinate system) by taking into the account the properties of the turning points. As far as for the geometric theory, the normally hyperbolic theory is replaced with the center manifold theorem for invariant manifolds developed in [3].

The hypotheses on the system (2) are as follows.

(H1) (Persistence). The slow manifold

$$S_0 := \{(x, y) : x = H(y),\ y \in D\},$$

where D is a domain in $\mathbb{R}^n$ and $H\colon D \to \mathbb{R}^m$ is a function, persists; that is, there is a smooth function $H\colon D \times (-\varepsilon_0, \varepsilon_0) \to \mathbb{R}^m$ such that

$$S_\varepsilon := \{(x, y) : x = H(y; \varepsilon),\ y \in D\}$$

is invariant under system (2) for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ and $H(y; 0) = H(y)$.

(H2) (Turning Surface). There is a smooth hypersurface L of S_0 which divides S_0 into two parts

$$S_- := \{(x, y) : x = H(y),\ y \in D_-\} \qquad \text{and}$$

$$S_+ := \{(x, y) : x = H(y),\ y \in D_+\},$$

such that, for some $\alpha_0 < 0 < \beta_0$, the eigenvalues of $F_x(p; 0)$ for $p = (H(y), y) \in S_0$, denoted by

$$\alpha_i(y) \quad \text{for } i = 1, ..., l, \qquad \beta_j(y) \text{ for } j = 1, ..., k, \qquad \text{and} \qquad \lambda_0(y),$$

satisfy $\Re(\alpha_i(y)) < \alpha_0 < \lambda_0(y) < \beta_0 < \Re(\beta_j(y))$, and $\lambda_0(y)$ changes sign at $p = (H(y), y) \in L$, and $\lambda_0(y) < 0$ for $y \in D_-$ and $\lambda_0(y) > 0$ for $y \in D_+$.

Clearly, we have $k + l + 1 = m$. Let $B = \{y \in D : (H(y), y) \in L\}$ be the projection of L to D. Thus, $B = \partial D_- \cap \partial D_+$.

(H3) (Transversality). On D, the reduced vector field $G(H(y), y; 0)$ is transversal to B, and G points from D_- to D_+.

LEMMA 3.1. *If the hypotheses* (H1)–(H3) *are satisfied by system* (2), *then there exists a coordinate system in a neighborhood of L such that, for* $\varepsilon \neq 0$ *small, the system can be written as, in the fast time scale t,*

$$
\begin{aligned}
u' &= U(u, v, w, y; \varepsilon)\, u, \\
v' &= V(u, v, w, y; \varepsilon)\, v, \\
w' &= \lambda_0(w, y; \varepsilon)\, w + \Lambda(u, v, w, y; \varepsilon)(u, v), \\
y' &= \varepsilon(h(y; \varepsilon) + a(w, y; \varepsilon)\, w + H(u, v, w, y; \varepsilon)(u, v)),
\end{aligned}
\tag{13}
$$

or, in the slow time scale τ,

$$
\begin{aligned}
\varepsilon\dot{u} &= U(u, v, w, y; \varepsilon)\, u, \\
\varepsilon\dot{v} &= V(u, v, w, y; \varepsilon)\, v, \\
\varepsilon\dot{w} &= \lambda_0(w, y; \varepsilon)\, w + \Lambda(u, v, w, y; \varepsilon)(u, v), \\
\dot{y} &= h(y; \varepsilon) + a(w, y; \varepsilon)\, w + H(u, v, w, y; \varepsilon)(u, v),
\end{aligned}
\tag{14}
$$

where $u \in \mathbb{R}^k$; $v \in \mathbb{R}^l$; $w \in \mathbb{R}$; $L = \{y_1 = 0\}$; $S_0 = \{u = v = w = 0\}$; $h(0, y_2, ..., y_n; 0)$ *transverses to L on* S_0; $U(0, 0, 0, y; \varepsilon)$ *and* $V(0, 0, 0, y; \varepsilon)$ *are in the Jordan canonical forms*; *and* $U(0, 0, 0, y; 0)$ *and* $V(0, 0, 0, y; 0)$ *have eigenvalues* $\beta_j(y)$ *for* $j = 1, ..., k$ *and* $\alpha_i(y)$ *for* $i = 1, ..., l$ *respectively*; $\lambda_0(0, y; 0) = \lambda_0(y)$; $\Lambda(u, v, w, y; \varepsilon)$ *and* $H(u, v, w, y; \varepsilon)$ *are bilinear forms*; *and* $a(w, y; \varepsilon)$ *is a vector in* $\mathbb{R}^n$ *and, for any fixed* $\delta_0 > 0$ *small,* $a(w, y; \varepsilon) = 0$ *for* $|y_1| \geqslant \delta_0$.

Proof. This will be accomplished through several changes of variables together with applications of invariant manifold theory.

Step 1. The first change of variables is made to put S_ε and L as regions on coordinate subspaces. By the hypothesis (H1), we have that

$$S_\varepsilon := \{(x, y) : x = H(y; \varepsilon),\ y \in D\}$$

is invariant. We may also assume, locally around the interested turning points, that L is given by

$$L = \{(x, y) : x = H(y),\ y_1 = \phi(y_2, ..., y_n)\}.$$

Set

$$
\begin{aligned}
&X = x - H(y; \varepsilon), \\
&Y_1 = y_1 - \phi(\bar{y}), \qquad \bar{Y} := (Y_2, ..., Y_n) = \bar{y} := (y_2, ..., y_n),
\end{aligned}
$$

and note that the inverse of this change of variables is given by

$$x = X + H(Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon), \qquad y_1 = Y_1 + \phi(\bar{Y}), \qquad \bar{y} = \bar{Y}.$$

In the new coordinates, the system (2) becomes

$$\begin{aligned} X' &= F(X + H(Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon), Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon) \\ &\quad - \varepsilon DH(Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon)\, G(X + H(Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon), Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon), \\ Y' &= \varepsilon G(X + H(Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon), Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon). \end{aligned}$$

The invariance of $x = H(y; \varepsilon)$ is equivalent to

$$F(H(y; \varepsilon), y; \varepsilon) = \varepsilon DH(y; \varepsilon)\, G(H(y; \varepsilon), y; \varepsilon),$$

or

$$\begin{aligned} &F(H(Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon), Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon) \\ &\quad = \varepsilon DH(Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon)\, G(H(Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon), Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon). \end{aligned}$$

Hence, there exists $\mathscr{F}(X, Y; \varepsilon) : \mathbb{R}^m \to \mathbb{R}^m$ such that

$$\begin{aligned} &F(X + H, Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon) - \varepsilon DHG(X + H, Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon) \\ &\quad = \mathscr{F}(X, Y; \varepsilon)\, X. \end{aligned}$$

We then end up with

$$\begin{aligned} X' &= \mathscr{F}(X, Y; \varepsilon)\, X \\ Y' &= \varepsilon g(X, Y; \varepsilon), \end{aligned} \tag{15}$$

where $g(X, Y; \varepsilon) = G(X + H, Y_1 + \phi(\bar{Y}), \bar{Y}; \varepsilon)$. In this new coordinates system, $S_0 = \{X = 0\}$ and $L = \{X = 0, Y_1 = 0\}$.

Step 2. Next, we make use of the existence of center, stable, and unstable manifolds of S_0 to split X into three components.

Note that $\mathscr{F}(0, Y; 0) = F_x(H, Y_1 + \phi(\bar{Y}), \bar{Y}; 0)$ has eigenvalues α_i, λ_0, and β_j. Therefore, the standard invariant manifold theory and the center manifold theory developed in [3] imply that there exist stable, center, and unstable manifolds $W^{\mathrm{s}}_\varepsilon$, $W^{\mathrm{c}}_\varepsilon$, and $W^{\mathrm{u}}_\varepsilon$ which, at $\varepsilon = 0$, are tangent to the eigenspaces corresponding to α_i, λ_0, and β_j, respectively. Furthermore, $W^{\mathrm{s}}_\varepsilon$ and $W^{\mathrm{u}}_\varepsilon$ are invariantly foliated. Following the argument of Fenichel [8] (see also [15, 16]), there exists a local coordinate system $(\bar{u}, \bar{v}, \bar{w}, \bar{y})$ so that system (15) becomes

$$\bar{u}' = \bar{U}(\bar{u}, \bar{v}, \bar{w}, y; \varepsilon)\, \bar{u},$$
$$\bar{v}' = \bar{V}(\bar{u}, \bar{v}, \bar{w}, y; \varepsilon)\, \bar{v},$$
$$\bar{w}' = \bar{\lambda}_0(\bar{u}, \bar{v}, \bar{w}, y; \varepsilon)\, \bar{w} + \bar{A}(\bar{u}, \bar{v}, \bar{w}, y; \varepsilon)(\bar{u}, \bar{v}),$$
$$y' = \varepsilon(\bar{h}(y; \varepsilon) + \bar{a}(\bar{u}, \bar{v}, \bar{w}, y; \varepsilon)\, \bar{w} + \bar{H}(\bar{u}, \bar{v}, \bar{w}, y; \varepsilon)(\bar{u}, \bar{v}))$$

with $\bar{u} \in \mathbb{R}^k$; $\bar{v} \in \mathbb{R}^l$; $\bar{w} \in \mathbb{R}$; $\bar{U}(0, 0, 0, y; 0)$ has eigenvalues β_j for $j = 1, ..., k$; $\bar{V}(0, 0, 0, y; 0)$ has eigenvalues α_i for $i = 1, ..., l$; and $\bar{\lambda}_0(0, 0, 0, y; 0) = \lambda_0(y)$.

Step 3. A further technical but crucial point is that there are locally invariant foliations of the center-stable and center-unstable manifolds $W_\varepsilon^{\text{cs}}$ and $W_\varepsilon^{\text{cu}}$ over the center manifold W_ε^{c}. See Lemma 6.1 in the appendix for a construction. Making use of these foliations, there exists a local coordinate system (u, v, w, y) such that the system can be further reduced to

$$u' = U(u, v, w, y; \varepsilon)\, u,$$
$$v' = V(u, v, w, y; \varepsilon)\, v,$$
$$w' = \lambda_0(w, y; \varepsilon)\, w + A(u, v, w, y; \varepsilon)(u, v),$$
$$y' = \varepsilon(h(y; \varepsilon) + a(w, y; \varepsilon)\, w + H(u, v, w, y; \varepsilon)(u, v)).$$

The assertion that $a(w, y; \varepsilon) = 0$ for $|y_1| \geqslant \delta_0$ follows from the existence of stable and unstable foliations of the corresponding part of the center manifold.

Step 4. Finally, let $u = P(y; \varepsilon)\, \bar{u}$ and $v = Q(y; \varepsilon)\, \bar{v}$, where $P(y; \varepsilon)$ and $Q(y; \varepsilon)$ are matrices putting $U(0, 0, 0, y; \varepsilon)$ and $V(0, 0, 0, y; \varepsilon)$ into their Jordan canonical forms, respectively. The resulting system has the desired form. ▌

The coordinate system established in Lemma 3.1 will be referred to as the *Fenichel-type coordinates system.*

4. EXCHANGE LEMMAS

Having the Fenichel-type coordinate system, we will extend the exchange lemma to the cases where the slow manifold possesses the type of turning points discussed above.

Let $\mathscr{B} = \{(u, v, w, y) : |u|, |v|, |w| \leqslant \Delta,\ y \in D = D_- \cup D_+\}$ be a neighborhood of the slow manifold, where Δ is small enough so that the Fenichel-type coordinate system is valid in $\mathscr{B}$. For $\varepsilon > 0$, let Γ_ε be an orbit entering $\mathscr{B}$ at $q_\varepsilon^0 = (u_\varepsilon^0, v_\varepsilon^0, w_\varepsilon^0, y_\varepsilon^0)$ and exiting at $q_\varepsilon^1 = (u_\varepsilon^1, v_\varepsilon^1, w_\varepsilon^1, y_\varepsilon^1)$ later on. Let M_ε be a $(k + \sigma)$-dimensional invariant manifold containing Γ_ε, where k is the dimension of the unstable fiber of the slow manifold as in (H2) and $\sigma \geqslant 1$ (see Remarks 4.1 and 4.2 for the explanation of this requirement on the dimension). The objective of the exchange lemmas is to describe the C^1 configuration of M_ε at q_ε^1 in terms of that of M_ε at q_ε^0. Different from the results for normally hyperbolic slow manifolds in [14, 16, 30], etc., the phenomena depend significantly on the positions of y_0^0 and y_0^1 relative to the pairing map P_0. Corresponding to the different relative positions, the w component of the orbit Γ_ε will be treated as *stable*, *center*, *and unstable* ones and the three exchange lemmas, Theorem 4.4, Theorem 4.7, and Theorem 4.10 will be obtained.

4.1. *Evolution of Forms along a Solution*

To track the C^1 configuration of M_ε along the orbit Γ_ε, we apply the treatment in [14, 16] of using differential forms. The idea is that the tangent space of the $(k+\sigma)$-dimensional manifold M_ε at a point on Γ_ε can be coordinated by the evaluations—the so-called Plücker coordinates—of $(k+\sigma)$-forms on the tangent space. The value of a form on the tangent space measures the $(k+\sigma)$-volume of the projection of the tangent space to the space corresponding to the form. In this subsection, we will derive the differential equation that the $(k+\sigma)$-forms satisfy along a solution of system (14).

Let $(u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau))$ be a solution of system (14) in $\mathscr{B}$. For $\sigma \geqslant 1$, all $(k+\sigma)$-forms form a vector space. We will use the following base consisting of three types of basic forms: type I, type II, and type III.

For any integers p and q with $p \leqslant q$, let $\Sigma(p, q)$ denote the set of all monotone and one-to-one maps π: $(1, ..., p) \to (1, ..., q)$.

The type-I basic forms are

$$\xi_\pi := du_1 \wedge \cdots \wedge du_k \wedge dy_{\pi(1)} \wedge \cdots \wedge dy_{\pi(\sigma)}$$

where $\pi \in \Sigma(\sigma, n)$. Thus, there are $C(n, \sigma)$ many such basic type-I $(k+\sigma)$-forms, where $C(n, \sigma)$ is the combination number of choosing σ elements from n elements.

The type-II basic forms are

$$\eta_\pi := du_1 \wedge \cdots \wedge du_k \wedge dw \wedge dy_{\pi(1)} \wedge \cdots \wedge dy_{\pi(\sigma-1)},$$

where $\pi \in \Sigma(\sigma-1, n)$, and there are $C(n, \sigma-1)$ many such basic forms.

The type-III basic forms are further divided into $k+1$ subgroups: $III_0, ..., III_k$. The type-III_i group consists of

$$\rho_\pi := du_{\pi(1)} \wedge \cdots \wedge du_{\pi(i)} \wedge dz_{i+1} \wedge \cdots \wedge dz_{k+\sigma}$$

where $\pi \in \Sigma(i, k)$ and z_j ranges over v_i, w, and y_i. In the case that $i = k$, at least one of the z_j's belongs to $\{v_\alpha\}$.

Taking the differential on the slow system (14), we get

$$
\begin{aligned}
\varepsilon d\dot{u} &= U\, du + (D_z U\, dz)\, u, \\
\varepsilon d\dot{v} &= V\, dv + (D_z V\, dz)\, v, \\
\varepsilon d\dot{w} &= \lambda_0\, dw + (D_w \lambda_0\, dw)\, w + (D_y \lambda_0\, dy)\, w \\
&\qquad + \Lambda \cdot (du, v) + \Lambda \cdot (u, dv) + (D_z \Lambda\, dz)(u, v), \\
d\dot{y} &= D_y h\, dy + (a + w\, D_w a)\, dw + w\, D_y a\, dy \\
&\qquad + H \cdot (du, v) + H \cdot (u, dv) + (D_z H\, dz)(u, v),
\end{aligned} \tag{16}
$$

where z runs over u, v, w, and y. To illustrate the derivation of the differential equations for the $(k+\sigma)$-forms, let us derive the differential equation for a type-I form with the assumption that the solution followed is on the slow manifold where $u = v = w = 0$. In the computation, the following notations will be used

$$
d^k u := du_1 \wedge \cdots \wedge du_k; \qquad d^\sigma y_\pi := dy_{\pi(1)} \wedge \cdots \wedge dy_{\pi(\sigma)}.
$$

For any $\pi_i \in \Sigma(\sigma, n)$, the corresponding type-I form is

$$
\xi_{\pi_i} = d^k u \wedge d^\sigma y_{\pi_i}.
$$

Taking the derivative with respect to τ and using the relations (16),

$$
\begin{aligned}
\varepsilon \dot{\xi}_{\pi_i} &= \sum_\alpha \varepsilon\, du_1 \wedge \cdots \wedge d\dot{u}_\alpha \wedge \cdots \wedge du_k \wedge d^\sigma y_{\pi_i} \\
&\quad + \sum_\gamma \varepsilon d^k u \wedge dy_{\pi_i(1)} \wedge \cdots \wedge d\dot{y}_{\pi_i(\gamma)} \wedge \cdots \wedge dy_{\pi_i(\sigma)} \\
&= \sum_{\alpha, \beta} du_1 \wedge \cdots \wedge (U_{\alpha\beta} du_\beta) \wedge \cdots \wedge du_k \wedge d^\sigma y_{\pi_i} \\
&\quad + \sum_{\gamma, p} \varepsilon d^k u \wedge dy_{\pi_i(1)} \wedge \cdots \wedge D_{y_p} h_{\pi_i(\gamma)} dy_p \wedge \cdots \wedge dy_{\pi_i(\sigma)} \\
&\quad + \sum_\gamma \varepsilon d^k u \wedge dy_{\pi_i(1)} \wedge \cdots \wedge a_{\pi_i(\gamma)} dw \wedge \cdots \wedge dy_{\pi_i(\sigma)} \\
&= \operatorname{tr} U \xi_{\pi_i} + \varepsilon \sum_{\pi_j} A_{\pi_i \pi_j} \xi_{\pi_j} + \sum_\gamma \varepsilon a_{\pi_i(\gamma)} \eta_{\pi_i(\gamma)},
\end{aligned}
$$

where the summation in the second term above is over all π_j's such that the image of π_j differs from that of π_i by exactly one element, say $\pi_i(\gamma_1) \notin \operatorname{Im} \pi_j$ and $\pi_j(\gamma_2) \notin \operatorname{Im} \pi_i$, and $A_{\pi_i \pi_j} = (-1)^{|\gamma_2 - \gamma_1|} D_{y_{\pi_j(\gamma_2)}} h_{\pi_i(\gamma_1)}$, and

$$\eta_{\pi_i(\gamma)} = d^k u \wedge dy_{\pi_i(1)} \wedge \cdots \wedge dy_{\pi_i(\gamma-1)} \wedge dw \wedge dy_{\pi_i(\gamma+1)} \wedge \cdots \wedge dy_{\pi_i(\sigma)}.$$

In general, along a solution $(u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau))$ in $\mathscr{B}$, if we set $\xi = (\xi_{\pi_1}, ..., \xi_{\pi_c})^\tau$ with $\pi_i \in \Sigma(\sigma, n)$, where $c = C(n, \sigma)$, $\eta = (\eta_{\pi_1}, ..., \eta_{\pi_d})^\tau$ with $\pi_i \in \Sigma(\sigma - 1, n)$, where $d = C(n, \sigma - 1)$ and $\rho = (\rho_\pi : \pi \in \Sigma(i, k), i = 0, ..., k)$ is the vector formed by all the type-III forms, then a computation yields

Lemma 4.1. *The forms ξ, η, and ρ satisfy the linear nonautonomous system*

$$\begin{aligned}
\varepsilon\dot{\xi} &= (\operatorname{tr} U + \phi_1)\, \xi + \varepsilon(A^{\mathrm{I}} + \theta_1^{\mathrm{I}})\, \xi + \varepsilon\theta_2 \eta + \theta_3 \rho, \\
\varepsilon\dot{\eta} &= (\operatorname{tr} U + \lambda_0 + \phi_2)\, \eta + \varepsilon(A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\, \eta + \theta_4 \xi + \theta_5 \rho, \\
\varepsilon\dot{\rho} &= (\operatorname{tr} U + B + \lambda_0 + \phi_3)\, \rho + \theta_6 \xi + \theta_7 \eta
\end{aligned} \tag{17}$$

where the argument of the coefficients of ξ, η, and ρ in the above system is $(u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau); \varepsilon)$, and where A^{I} and A^{II} are matrices formed from $D_y h(y_\varepsilon(\tau))$; B is in the Jordan canonical form with entries on the main diagonal less than or equal to $-\gamma_0$ with $\gamma_0 = \min\{-\alpha_0, \beta_0\}$; ϕ_i's are $\mathbb{R}$-valued functions; and there exists a constant $K > 0$ independent of ε such that

$$|\phi_1| \leqslant K\,|u|, \qquad |\phi_2| \leqslant K(|u| + |w| + \varepsilon), \qquad |\phi_3| \leqslant K(|u| + |v| + |w|),$$

$$|\theta_1^{\mathrm{I,\,II}}| \leqslant K(|u|\ |v| + |D_y a|\ |w|),$$

$$|\theta_2| \leqslant K(|u|\ |v| + |a| + |D_w a|\ |w|), \qquad |\theta_3| \leqslant K\,|u|,$$

$$|\theta_4| \leqslant K(|u|\ |v| + |w|), \qquad |\theta_5| \leqslant K\,|u|,$$

$$|\theta_6| \leqslant K\,|v|, \qquad |\theta_7| \leqslant K\,|v|. \qquad \blacksquare$$

4.2. *The First Exchange Lemma*

Let us denote the stable (resp., unstable, center-stable, center-unstable, and center) manifold of S_0 by $W^{\mathrm{s}}(S_0)$ (resp. $W^{\mathrm{u}}(S_0)$, $W^{\mathrm{cs}}(S_0)$, $W^{\mathrm{cs}}(S_0)$, and $W^{\mathrm{c}}(S_0)$) in $\mathscr{B}$. For example, $W^{\mathrm{s}}(S_0) = \{u = 0, w = 0\}$, $W^{\mathrm{cs}}(S_0) = \{u = 0\}$, and $W^{\mathrm{c}}(S_0) = \{u = 0, v = 0\}$.

Let $\phi_\varepsilon^\tau(q_\varepsilon^0) = (u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau))$ denote the solution of system (14) with the initial condition $q_\varepsilon^0 = (u_\varepsilon(0), v_\varepsilon(0), w_\varepsilon(0), y_\varepsilon(0))$. To shorten the notation, let $\lambda_0^\varepsilon(\tau) := \lambda_0(u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau); \varepsilon)$. The slow flow on S_0 at $\varepsilon = 0$ will be denoted by $y \cdot \tau$ for $(0, y) \in S_0$. On the center manifold $W^{\mathrm{c}}(S_0)$ the system is the same as that discussed in Section 2. We denote the pairing map by P_0. Also, for $y^1, y^2 \in S_0$, we will use the notation $y^1 \prec y^2$ (resp. $y^1 \succ y^2$) if $y^1 = y^0 \cdot \tau_1$, $y^2 = y^0 \cdot \tau_2$, and $\tau_1 < \tau_2$ (resp. $\tau_1 > \tau_2$).

For a set N, let $\omega(N)$ (resp. $\alpha(N)$) denote the ω-limit (resp. α-limit) set of N for the system (13) at $\varepsilon = 0$.

The letters K and C will denote positive constants whose values may change with each occurrence but will not depend on ε.

Let $1 \leqslant \sigma \leqslant n$ be an integer and let M_ε be a $(k+\sigma)$-dimensional invariant manifold of system (13) which is smooth in ε (including $\varepsilon = 0$).

We impose the following assumptions.

(A1) M_0 intersects $W^{cs}(S_-)$ transversally at $q_0^0 \in \partial \mathscr{B}$.

Remark 4.1. If we denote $N_0 = M_0 \cap W^{cs}(S_-)$, then, as a consequence of (A1), $\dim N_0 = \sigma$. Since M_0 is invariant, we have $\sigma \geqslant 1$.

(A2) The set $\omega(N_0) \cap \mathscr{B}$ is a $(\sigma - 1)$-dimensional submanifold of S_-.

(A3) $(0, h(y; 0)) \notin T_{(0,\, y)}\omega(N_0)$ for $(0, y) \in \omega(N_0)$.

Remark 4.2. If $\dim \omega(N_0) = 0$, or equivalently, $\sigma = 1$, then (A3) is automatic. Also, (A3) requires that $\dim \omega(N_0) \leqslant n - 1$; that is $\sigma \leqslant n$. Together with Remark 4.1, this explains the above requirement of $1 \leqslant \sigma \leqslant n$.

Suppose that the singular orbit on S_0 followed by M_ε has end points $y^0 \in S_-$ and $y^1 \in S_+$. The First Exchange Lemma deals with the case that $y^1 \prec P_0(y^0)$. In view of the Exchange Lemma of [16, 30], etc., the w-component can be regarded as a *stable* one in this case. Thus, the exchange lemma obtained in this subsection is a generalization of that in [16, 30].

The next lemma is a C^0 version of the Exchange Lemma. It estimates the location of the orbit $\phi_\varepsilon^\tau(q_\varepsilon^0)$ relative to its singular limit on the slow manifold.

LEMMA 4.2. *Assume* (H1)–(H3) *and* (A1)–(A3). *If M_ε enters $\mathscr{B}$ through the point q_ε^0 which depends on ε smoothly and exits $\mathscr{B}$ through the point $q_\varepsilon^1 = \phi_\varepsilon^{\tau_\varepsilon}(q_\varepsilon^0)$ on the face $|u| = \Delta$ with $\tau_\varepsilon \to \tau_0 > 0$ and $\alpha(q_0^1) \prec P_0(\omega(q_0^0))$, then there exist constants $K > 0$ and $C > 0$ such that, for $\Delta > 0$ small, $\varepsilon > 0$ small, and $\tau \in [0, \tau_\varepsilon]$,*

$$|u_\varepsilon(\tau)| \leqslant K e^{-\beta_0(\tau_\varepsilon - \tau)/\varepsilon}, \qquad |v_\varepsilon(\tau)| \leqslant K e^{\alpha_0 \tau/\varepsilon}, \qquad |w_\varepsilon(\tau)| \leqslant K e^{-C\tau/\varepsilon},$$

and

$$|y_\varepsilon(\tau) - y_0 \cdot \tau| \leqslant K(\varepsilon + |y_\varepsilon(0) - y_0|) \leqslant K\varepsilon.$$

Proof. The estimates for $|u_\varepsilon(\tau)|$ and $|v_\varepsilon(\tau)|$ follow from the results of Proposition 3.1 in [16]. We now estimate $|w_\varepsilon(\tau)|$ and $|y_\varepsilon(\tau) - y_0 \cdot \tau|$.

Choose Δ small enough such that $\alpha_0 < \lambda_- \leqslant \lambda_0(w, y; \varepsilon) \leqslant \lambda_+ < \beta_0$ for $(u, v, w, y) \in \mathscr{B}$. Recall that $\lambda_0^\varepsilon(\tau) := \lambda_0(u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau); \varepsilon)$.

First we show that if

$$\int_0^\tau \lambda_0^\varepsilon(s)\, ds \leqslant -r_0 \tau$$

for some constant $r_0 > 0$ and $\tau \in [0, \tau_\varepsilon]$, then the estimates in the lemma hold. To see this, note that

$$w_\varepsilon(\tau) = e^{(1/\varepsilon) \int_0^\tau \lambda_0^\varepsilon(s)\, ds} w_\varepsilon(0) + \int_0^\tau e^{(1/\varepsilon) \int_s^\tau \lambda_0^\varepsilon(t)\, dt} \Lambda \cdot (u_\varepsilon(s), v_\varepsilon(s))\, ds$$

and

$$|u_\varepsilon(\tau)|\, |v_\varepsilon(\tau)| \leqslant K e^{-\beta_0(\tau_\varepsilon - \tau)/\varepsilon} e^{\alpha_0 \tau/\varepsilon} \leqslant K e^{((\alpha_0 + \beta_0)\, \tau - \beta_0 \tau_\varepsilon)/\varepsilon}.$$

Therefore,

$$\begin{aligned} |w_\varepsilon(\tau)| &\leqslant e^{-r_0 \tau/\varepsilon}\, |w_\varepsilon(0)| + K \int_0^\tau e^{(\lambda_+(\tau - s) + (\alpha_0 + \beta_0)\, s - \beta_0 \tau_\varepsilon)/\varepsilon}\, ds \\ &\leqslant \Delta e^{-r_0 \tau/\varepsilon} + K\varepsilon(e^{\alpha_0 \tau_\varepsilon/\varepsilon} + e^{-(\beta_0 - \lambda_+)\, \tau_\varepsilon/\varepsilon}), \end{aligned}$$

which implies the estimate of $|w_\varepsilon(\tau)|$ in the lemma. Using the above estimates for $|u_\varepsilon(\tau)|\, |v_\varepsilon(\tau)|$ and $|w_\varepsilon(\tau)|$, and

$$y_\varepsilon(\tau) = y_\varepsilon(0) + \int_0^\tau (h(y_\varepsilon(s); \varepsilon) + a w_\varepsilon(s) + H \cdot (u_\varepsilon(s), v_\varepsilon(s)))\, ds,$$

we have

$$\begin{aligned} |y_\varepsilon(\tau) - y_0 \cdot \tau| &\leqslant |y_\varepsilon(0) - y_0| + \int_0^\tau |Dh|\, |y_\varepsilon(s) - y_0 \cdot s|\, ds \\ &\quad + |a| \int_0^\tau |w_\varepsilon(s)|\, ds + |H| \int_0^\tau |u_\varepsilon(s)|\, |v_\varepsilon(s)|\, ds \\ &\leqslant |y_\varepsilon(0) - y_0| + K\varepsilon + \int_0^\tau K\, |y_\varepsilon(s) - y_0 \cdot s|\, ds. \end{aligned}$$

By an application of Gronwall's inequality, we get

$$|y_\varepsilon(\tau) - y_0 \cdot \tau| \leqslant K(\varepsilon + |y_\varepsilon(0) - y_0|) \leqslant K\varepsilon.$$

Thus, to complete the proof, it remains to show the following claim.

CLAIM. There exists a constant $r_0 > 0$ such that

$$\int_0^\tau \lambda_0^\varepsilon(s)\, ds \leqslant -r_0 \tau, \tag{18}$$

for $\tau \in [0, \tau_\varepsilon]$ and ε small.

Proof of Claim. Since $\alpha(q_0^1) \prec P_0(\omega(q_0^0))$ and $\tau_\varepsilon \to \tau_0$, $\int_0^\tau \lambda_0(y_0 \cdot s)\, ds < 0$ for $0 < \tau \leqslant \tau_\varepsilon$. Also,

$$\lim_{\tau \to 0} \frac{1}{\tau} \int_0^\tau \lambda_0(y_0 \cdot s)\, ds = \lambda_0(y_0) < 0.$$

Thus, there exists a constant $r_0 > 0$ such that, for $\tau \in [0, \tau_\varepsilon]$,

$$\frac{1}{\tau} \int_0^\tau \lambda_0(y_0 \cdot s)\, ds \leqslant -2r_0 \qquad \text{or} \qquad \int_0^\tau \lambda_0(y \cdot s)\, ds \leqslant -2r_0 \tau.$$

We may assume that

$$r_0 < -\lambda_0(w_\varepsilon(0), y_\varepsilon(0); \varepsilon). \tag{19}$$

Let T be the supermum of the τ's satisfying the inequality (18) for a fixed ε. Then, $T > 0$ due to the inequality (19). We will show that $T = \tau_\varepsilon$. Suppose, on the contrary, that $T < \tau_\varepsilon$. Note that

$$w_\varepsilon(\tau) = e^{(1/\varepsilon) \int_T^\tau \lambda_0^\varepsilon(s)\, ds} w_\varepsilon(T) + \int_T^\tau e^{(1/\varepsilon) \int_s^\tau \lambda_0^\varepsilon(t)\, dt} \Lambda \cdot (u_\varepsilon(s), v_\varepsilon(s))\, ds$$

and

$$|u_\varepsilon(\tau)|\ |v_\varepsilon(\tau)| \leqslant K e^{((\alpha_0 + \beta_0)\tau - \beta_0 \tau_\varepsilon)/\varepsilon}.$$

Therefore, for $\tau \leqslant T + \min\{r_0 T/2\lambda_+, (\beta_0 - \lambda_+)\tau_\varepsilon/2\lambda_+\}$,

$$\begin{aligned}
|w_\varepsilon(\tau)| &\leqslant e^{(1/\varepsilon) \int_T^\tau \lambda_0^\varepsilon(s)\, ds} |w_\varepsilon(T)| + K \int_T^\tau e^{(1/\varepsilon) \int_s^\tau \lambda_0^\varepsilon(t)\, dt} e^{((\alpha_0 + \beta_0)s - \beta_0 \tau_\varepsilon)/\varepsilon}\, ds \\
&\leqslant e^{(1/\varepsilon) \int_T^\tau \lambda_0^\varepsilon(s)\, ds} |w_\varepsilon(T)| + K \int_T^\tau e^{(\lambda_+(\tau - s) + (\alpha_0 + \beta_0)s - \beta_0 \tau_\varepsilon)/\varepsilon}\, ds \\
&\leqslant K e^{-r_0 T/2\varepsilon} + K\varepsilon e^{-(\beta_0 - \lambda_+)\tau_\varepsilon/2\varepsilon},
\end{aligned}$$

and, using an argument the same as that given earlier,

$$|y_\varepsilon(\tau) - y_0 \cdot \tau| \leqslant K\varepsilon.$$

We then have, for $\tau \leqslant T + \min\{r_0 T/2\lambda_+, (\beta_0 - \lambda_+) \tau_\varepsilon/2\lambda_+\}$ and ε small,

$$\int_0^\tau \lambda_0^\varepsilon(s)\, ds \leqslant \int_0^\tau \lambda_0(y_0 \cdot s)\, ds + |D_w \lambda_0| \int_0^\tau |w_\varepsilon(s)|\, ds$$

$$+ |D_y \lambda_0| \int_0^\tau |y_\varepsilon(s) - y_0 \cdot s|\, ds + |D_\varepsilon \lambda_0| \int_0^\tau \varepsilon\, ds$$

$$\leqslant -2r_0 \tau + K\varepsilon\tau \leqslant -r_0 \tau.$$

This contradicts to $T < \tau_\varepsilon$. The proof is then completed. ∎

Next, we prove a technical lemma which is the crucial estimate for a proof of the exchange lemmas.

LEMMA 4.3. *Consider the linear nonautonomous system*

$$\dot{\xi} = (A^{\mathrm{I}} + \theta_1^{\mathrm{I}})\, \xi + \theta_2 \eta + \frac{\theta_3}{\varepsilon}\, \rho,$$

$$\dot{\eta} = \frac{\lambda_0 + \phi_2 - \phi_1}{\varepsilon}\, \eta + (A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\, \eta + \frac{\theta_4}{\varepsilon}\, \xi + \frac{\theta_5}{\varepsilon}\, \rho,$$

$$\dot{\rho} = \frac{B + \lambda_0 + \phi_3 - \phi_1}{\varepsilon}\, \rho + \frac{\theta_6}{\varepsilon}\, \xi + \frac{\theta_7}{\varepsilon}\, \eta$$

where the coefficients A^{I}, A^{II}, B, ϕ_i, *and* θ_i *are as in* (17). *Assume the hypotheses in Lemma* 4.2. *Then there exist constants* $K > 0$ *and* $C > 0$ *such that, for* $\varepsilon > 0$ *small and* $\tau \in [0, \tau_\varepsilon]$,

$$|\xi(\tau)| \leqslant K(|\xi(0)| + \varepsilon\, |\eta(0)|) + K e^{-C\tau_\varepsilon/\varepsilon}\, |\rho(0)|,$$

$$|\eta(\tau)| \leqslant K e^{-C\tau/\varepsilon} \left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon} \right) + K e^{-C\tau_\varepsilon/\varepsilon}\, |\rho(0)|,$$

and

$$|\rho(\tau)| \leqslant K e^{-C\tau/\varepsilon} (|\rho(0)| + |\xi(0)| + |\eta(0)|).$$

Proof. The estimate is complicated but very basic. It uses a variation of the constant formula and Gronwall's inequality.

Let $\Phi_1(\tau)$, $\Phi_2(\tau)$, and $\Phi_3(\tau)$ be the principal fundamental matrix solutions at $\tau = 0$ of the systems with system matrices

$$A^{\mathrm{I}} + \theta_1^{\mathrm{I}}, \qquad \frac{\lambda_0 + \phi_2 - \phi_1}{\varepsilon} + A^{\mathrm{II}} + \theta_1^{\mathrm{II}}, \qquad \frac{B + \lambda_0 + \phi_3 - \phi_1}{\varepsilon},$$

respectively. Then, we have

$$|\Phi_1(\tau)\,\Phi_1^{-1}(s)| \leqslant K, \qquad |\Phi_2(\tau)\,\Phi_2^{-1}(s)| \leqslant Ke^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}$$

and

$$|\Phi_3(\tau)\,\Phi_3^{-1}(s)| \leqslant Ke^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}e^{-\gamma_0(r-s)/\varepsilon}$$

for some constant $K>0$, for $\tau > s$ in $[0, \tau_\varepsilon]$ and $\gamma_0 < \min\{-\alpha_0, \beta_0\}$.

By the variation of the constant formula and the estimates given above, we have

$$|\xi(\tau)| \leqslant K\,|\xi(0)| + K\int_0^\tau \left(|\theta_2|\,|\eta(s)| + \frac{|\theta_3|}{\varepsilon}\,|\rho(s)|\right) ds, \tag{20}$$

$$\begin{aligned}|\eta(\tau)| \leqslant{}& Ke^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds}\,|\eta(0)|\\ &+K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\left(\frac{|\theta_4|}{\varepsilon}\,|\xi(s)| + \frac{|\theta_5|}{\varepsilon}\,|\rho(s)|\right) ds,\end{aligned} \tag{21}$$

and

$$\begin{aligned}|\rho(\tau)| \leqslant{}& Ke^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds}e^{-\gamma_0\tau/\varepsilon}\,|\rho(0)|\\ &+K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}e^{-\gamma_0(r-s)/\varepsilon}\left(\frac{|\theta_6|}{\varepsilon}\,|\xi(s)| + \frac{|\theta_7|}{\varepsilon}\,|\eta(s)|\right) ds.\end{aligned} \tag{22}$$

Substituting the estimate in (20) into (22),

$$\begin{aligned}|\rho(\tau)| \leqslant{}& Ke^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds}e^{-\gamma_0\tau/\varepsilon}\,|\rho(0)|\\ &+K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}e^{-\gamma_0(r-s)/\varepsilon}\,\frac{|\theta_6|}{\varepsilon}\,|\xi(0)|\,ds\\ &+K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}e^{-\gamma_0(r-s)/\varepsilon}\,\frac{|\theta_6|}{\varepsilon}\int_0^s |\theta_2|\,|\eta(t)|\,dt\,ds\\ &+K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}e^{-\gamma_0(r-s)/\varepsilon}\,\frac{|\theta_6|}{\varepsilon}\int_0^s \frac{|\theta_3|}{\varepsilon}\,|\rho(t)|\,dt\,ds\\ &+K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}e^{-\gamma_0(r-s)/\varepsilon}\,\frac{|\theta_7|}{\varepsilon}\,|\eta(s)|\,ds.\end{aligned}$$

Recalling that $|\theta_6| \leqslant K\,|v_\varepsilon| \leqslant K e^{\alpha_0 \tau/\varepsilon}$ and $\lambda_0^\varepsilon(\tau) > \alpha_0$, there exists a constant $0 < C \leqslant \min\{\lambda_0 - \alpha_0, \gamma_0\}$ such that

$$\int_0^\tau e^{-(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} e^{-\gamma_0(r-s)/\varepsilon} \frac{|\theta_6|}{\varepsilon}$$

$$\leqslant \frac{K}{\varepsilon} \int_0^\tau e^{-(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(t)\,dt} e^{-\gamma_0(r-s)/\varepsilon} e^{\alpha_0 s/\varepsilon}\,ds \leqslant K e^{-C\tau/\varepsilon}.$$

Therefore,

$$|\rho(\tau)| \leqslant K e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} e^{-C\tau/\varepsilon}\left(|\rho(0)| + |\xi(0)| + \int_0^\tau |\theta_2|\,|\eta(s)|\,ds\right)$$

$$+ K e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} \int_0^\tau e^{-(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(t)\,dt} e^{-\gamma_0(r-s)/\varepsilon} \frac{|\theta_7|}{\varepsilon}\,|\eta(s)|\,ds$$

$$+ K e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} e^{-C\tau/\varepsilon} \int_0^\tau \frac{|\theta_3|}{\varepsilon}\,|\rho(s)|\,ds.$$

Dividing $e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} e^{-C\tau/\varepsilon}$ on both sides, applying Gronwall's inequality, and using the estimates $|\theta_3| \leqslant K\,|u_\varepsilon| \leqslant K e^{-\beta_0(\tau_\varepsilon - \tau)/\varepsilon}$ and

$$\int_0^\tau \frac{e^{-\beta_0(\tau_\varepsilon - s)/\varepsilon}}{\varepsilon} e^{(1/\varepsilon)\int_0^s \lambda_0^\varepsilon(t)\,dt} e^{-Cs/\varepsilon}\,ds \leqslant K,$$

one has

$$|\rho(\tau)| \leqslant K e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} e^{-C\tau/\varepsilon}\left(|\rho(0)| + |\xi(0)| + \int_0^\tau |\theta_2|\,|\eta(s)|\,ds\right)$$
$$+ K e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} \int_0^\tau e^{-(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} e^{-\gamma_0(r-s)/\varepsilon} \frac{|\theta_7|}{\varepsilon}\,|\eta(s)|\,ds. \tag{23}$$

Substituting the estimates (20) and (23) into (21),

$$|\eta(\tau)| \leqslant K e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} |\eta(0)| + K \int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt} \frac{|\theta_4|}{\varepsilon}\,|\xi(0)|\,ds$$

$$+ K \int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt} \frac{|\theta_4|}{\varepsilon} \int_0^s \left(|\theta_2|\,|\eta(t)| + \frac{|\theta_3|}{\varepsilon}\,|\rho(t)|\right) dt\,ds$$

$$+ K \int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt} \frac{|\theta_5|}{\varepsilon}\,|\rho(s)|\,ds$$

$$
\begin{aligned}
&\leqslant K e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds}|\eta(0)| + K\,|\xi(0)| \int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\,\frac{|\theta_4|}{\varepsilon}\,ds \\
&\quad + K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\,\frac{|\theta_4|}{\varepsilon}\,ds \int_0^\tau |\theta_2|\,|\eta(s)|\,ds \\
&\quad + K(|\rho(0)| + |\xi(0)|)\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\,\frac{|\theta_4|}{\varepsilon}\,ds \\
&\quad \times \int_0^\tau \frac{|\theta_3|}{\varepsilon}\,e^{(1/\varepsilon)\int_0^s \lambda_0^\varepsilon(t)\,dt} e^{-Cs/\varepsilon}\,ds \\
&\quad + K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\,\frac{|\theta_4|}{\varepsilon}\,ds \int_0^\tau \frac{|\theta_3|}{\varepsilon}\,e^{(1/\varepsilon)\int_0^s \lambda_0^\varepsilon(t)\,dt} e^{-Cs/\varepsilon}\,ds \\
&\quad \times \int_0^\tau |\theta_2|\,|\eta(s)|\,ds \\
&\quad + K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\,\frac{|\theta_4|}{\varepsilon}\,ds \int_0^\tau \frac{|\theta_3|}{\varepsilon} e^{(1/\varepsilon)\int_0^s \lambda_0^\varepsilon(t)\,dt} e^{-\gamma_0 s/\varepsilon}\,ds \\
&\quad\quad \times \int_0^\tau e^{-(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(t)\,dt} e^{\gamma_0 s/\varepsilon}\,\frac{|\theta_7|}{\varepsilon}\,|\eta(s)|\,ds \\
&\quad + K(|\rho(0)| + |\xi(0)|)\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\,\frac{|\theta_5|}{\varepsilon}\,e^{(1/\varepsilon)\int_0^s \lambda_0^\varepsilon(t)\,dt} e^{-Cs/\varepsilon}\,ds \\
&\quad + K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\,\frac{|\theta_5|}{\varepsilon}\,e^{(1/\varepsilon)\int_0^s \lambda_0^\varepsilon(t)\,dt} e^{-Cs/\varepsilon}\,ds \int_0^\tau |\theta_2|\,|\eta(s)|\,ds \\
&\quad + K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\,\frac{|\theta_5|}{\varepsilon}\,e^{(1/\varepsilon)\int_0^s \lambda_0^\varepsilon(t)\,dt} e^{-\gamma_0 s/\varepsilon}\,ds \\
&\quad\quad \times \int_0^\tau e^{-(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} e^{\gamma_0 s/\varepsilon}\,\frac{|\theta_7|}{\varepsilon}\,|\eta(s)|\,ds.
\end{aligned}
$$

Note that $|\theta_4| \leqslant K(|u_\varepsilon|\,|v_\varepsilon| + |w_\varepsilon|)$; $|\theta_3|, |\theta_5| \leqslant K\,|u_\varepsilon|$; $|\theta_2| \leqslant K$; and $|\theta_7| \leqslant K\,|v_\varepsilon|$. From the estimates in Lemma 4.2, we have

$$
K\int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt}\,\frac{|\theta_4|}{\varepsilon}\,ds \leqslant \frac{K}{\varepsilon}\,e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds},
$$

and

$$K \int_0^\tau \frac{|\theta_i|}{\varepsilon} e^{-\gamma_0 s/\varepsilon} ds \leqslant K e^{-\gamma_0 \tau_\varepsilon/\varepsilon} \qquad \text{for} \quad i=3, 5.$$

Therefore,

$$\begin{aligned} |\eta(\tau)| \leqslant & K e^{(1/\varepsilon) \int_0^\tau \lambda_0^\varepsilon(s)\, ds} \left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon} + e^{-C\tau_\varepsilon/\varepsilon} |\rho(0)| \right) \\ & + \frac{K}{\varepsilon} e^{(1/\varepsilon) \int_0^\tau \lambda_0^\varepsilon(s)\, ds} \int_0^\tau |\eta(s)|\, ds \\ & + K e^{(1/\varepsilon) \int_0^\tau \lambda_0^\varepsilon(s)\, ds} e^{-\gamma_0 \tau_\varepsilon/\varepsilon} \int_0^\tau e^{-(1/\varepsilon) \int_0^s \lambda_0^\varepsilon(s)\, dt} e^{\gamma_0 s/\varepsilon} \frac{|v_\varepsilon|}{\varepsilon} |\eta(s)|\, ds. \end{aligned}$$

An application of Gronwall's inequality yields

$$|\eta(\tau)| \leqslant K e^{(1/\varepsilon) \int_0^\tau \lambda_0^\varepsilon(s)\, ds} \left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon} + e^{-C\tau_\varepsilon/\varepsilon} |\rho(0)| \right). \tag{24}$$

Substituting the estimate (24) into (23), we have

$$|\rho(\tau)| \leqslant K e^{(1/\varepsilon) \int_0^\tau \lambda_0^\varepsilon(s)\, ds} e^{-C\tau/\varepsilon} (|\rho(0)| + |\xi(0)| + |\eta(0)|). \tag{25}$$

Substituting the estimates (24) and (25) into (20) yields

$$|\xi(\tau)| \leqslant K(|\xi(0)| + \varepsilon\, |\eta(0)|) + K e^{-C\tau_\varepsilon/\varepsilon} |\rho(0)|. \tag{26}$$

The conclusion of the lemma then follows from the fact that $\int_0^\tau \lambda_0^\varepsilon(s)\, ds \leqslant -C\tau$ for some $C > 0$ and $\tau \in [0, \tau_\varepsilon]$ as in the proof of Lemma 4.2. ∎

Remark 4.3. Because we only require K to be independent of ε, the estimates (26), (24), and (25) hold true for $\tau \in [0, \tau_\varepsilon]$ as long as the property $\int_0^{\tau_\varepsilon} \lambda_0^\varepsilon(s)\, ds \leqslant K\varepsilon$ holds true.

We are now ready to obtain the corresponding C^1 Exchange Lemma (see Fig. 2 for an illustration).

THEOREM 4.4. *Assume* (H1)–(H3) *and* (A1)–(A3). *If M_ε enters $\mathscr{B}$ through the point q_ε^0 which depends on ε smoothly and exits $\mathscr{B}$ through the point $q_\varepsilon^1 = \phi_\varepsilon^{\tau_\varepsilon}(q_\varepsilon^0)$ on the face $|u| = \Delta$ with $\tau_\varepsilon \to \tau_0$ and $\alpha(q_0^1) \prec P_0(\omega(q_0^0))$, then, for some $\delta > 0$ small independent of ε, M_ε is C^1 $O(\varepsilon)$-close at q_ε^1 to the manifold*

$$W^{\mathrm{u}}(S_0)|_{\omega(N_0)\cdot(\tau_0-\delta,\, \tau_0+\delta)} = \{(u, 0, 0, y) : y \in \omega(N_0)\cdot(\tau_0 - \delta, \tau_0 + \delta)\}.$$

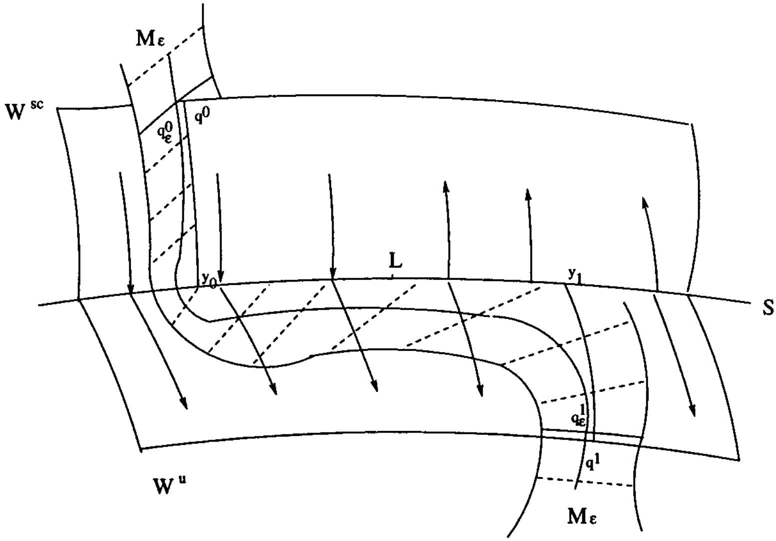

FIG. 2. The First Exchange Lemma.

Proof. Let $\xi(\tau)$, $\eta(\tau)$, and $\rho(\tau)$ be the types I, II and III forms corresponding to the tangent space of M_ε along the solution $(u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau))$ and let $\xi_0(\tau)$ denote the corresponding vector of type-I $(k+\sigma)$-forms at $\varepsilon = 0$ of

$$T_{(0,\,0,\,0,\,y^0\cdot\tau)}\,W^{\mathrm{u}}(S_0)|_{\omega(N_0)\cdot(\tau-\delta,\,\tau+\delta)}.$$

(The corresponding η_0 and ρ_0 are identical zeros.)

The statement of the theorem is equivalent to the estimate

$$|\eta(\tau_\varepsilon)| + |\rho(\tau_\varepsilon)| + |\xi(\tau_\varepsilon) - \xi_0(\tau_\varepsilon)| \leqslant K\varepsilon.$$

Recall the differential equation (17) that the forms satisfied along the solution $(u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau))$ are

$$\varepsilon\dot{\xi} = (\mathrm{tr}\ U + \phi_1)\,\xi + \varepsilon(A^I + \theta_1^{\mathrm{I}})\,\xi + \varepsilon\theta_2\eta + \theta_3\rho,$$

$$\varepsilon\dot{\eta} = (\mathrm{tr}\ U + \lambda_0 + \phi_2)\,\eta + \varepsilon(A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\,\eta + \theta_4\xi + \theta_5\rho,$$

$$\varepsilon\dot{\rho} = (\mathrm{tr}\ U + B + \lambda_0 + \phi_3)\,\rho + \theta_6\xi + \theta_7\eta.$$

By multiplying ξ, η, and ρ by the integrating factor $(1/|\xi(0)|)\, e^{-(1/\varepsilon)\int_0^\tau (\mathrm{tr}\ U + \phi_1)}$ and abusing the notations ξ, η, and ρ for the resulting forms, we get

$$\dot{\xi} = (A^{\mathrm{I}} + \theta_1^{\mathrm{I}})\,\xi + \theta_2 \eta + \frac{\theta_3}{\varepsilon}\,\rho,$$

$$\dot{\eta} = \frac{\lambda_0 + \phi_2 - \phi_1}{\varepsilon}\,\eta + (A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\,\eta + \frac{\theta_4}{\varepsilon}\,\xi + \frac{\theta_5}{\varepsilon}\,\rho, \tag{27}$$

$$\dot{\rho} = \frac{B + \lambda_0 + \phi_3 - \phi_1}{\varepsilon}\,\rho + \frac{\theta_6}{\varepsilon}\,\xi + \frac{\theta_7}{\varepsilon}\,\eta.$$

Note that the factor $1/|\xi(0)|$ normalizes the initial conditions and that after the normalization we have

$$|\xi(0)| = 1, \qquad |\eta(0)| \leqslant \frac{K}{\varepsilon}, \qquad \text{and} \qquad |\rho(0)| \leqslant \frac{K}{\varepsilon}$$

for some constant K. Applying Lemma 4.3 to system (27), we have that, for $\tau \leqslant \tau_\varepsilon$,

$$|\xi(\tau)| \leqslant K(|\xi(0)| + \varepsilon\,|\eta(0)| + |\rho(0)|\,e^{-C\tau_\varepsilon/\varepsilon}),$$

$$|\eta(\tau)| \leqslant K\,|\eta(0)|\,e^{-C\tau/\varepsilon} + \frac{K\,|\xi(0)|}{\varepsilon}\,e^{-C\tau/\varepsilon} + K\,|\rho(0)|\,e^{-C\tau_\varepsilon/\varepsilon},$$

and

$$|\rho(\tau)| \leqslant K(|\rho(0)| + \frac{|\xi(0)|}{\varepsilon} + |\eta(0)|)\,e^{-C\tau/\varepsilon}.$$

In particular, $|\eta(\tau_\varepsilon)| + |\rho(\tau_\varepsilon)| \leqslant K\varepsilon$. Next, we show that $|\xi(\tau_\varepsilon) - \xi_0(\tau_\varepsilon)| \leqslant K\varepsilon$. The equation that ξ_0 satisfied is

$$\dot{\xi}_0(\tau) = A^{\mathrm{I}}(y^0 \cdot \tau)\,\xi_0(\tau)$$

and that of ξ is

$$\dot{\xi}(\tau) = (A^{\mathrm{I}}(y_\varepsilon(\tau))) + \theta_1^{\mathrm{I}})\,\xi(\tau) + \theta_2 \eta(\tau) + \frac{\theta_3}{\varepsilon}\,\rho(\tau).$$

Therefore,

$$\begin{aligned}
|\xi(\tau) - \xi_0(\tau)| \leqslant{} & |\xi(0) - \xi_0(0)| + \int_0^\tau (|D_y A^{\mathrm{I}}|\,|y_\varepsilon(s) - y^0 \cdot s| + |\theta_1^{\mathrm{I}}|)\,|\xi(s)|\,ds \\
& + \int_0^\tau |A^{\mathrm{I}}|\,|\xi(s) - \xi_0(s)|\,ds + \int_0^\tau |\theta_2|\,|\eta(s)|\,ds \\
& + \int_0^\tau \frac{|\theta_3|}{\varepsilon}\,|\rho(s)|\,ds.
\end{aligned}$$

Note that, for some $\tau_*, \tau^* \in (0, \tau_\varepsilon)$ independent of ε, $a(w_\varepsilon(\tau), y_\varepsilon(\tau); \varepsilon) = 0$ for $\tau \notin [\tau_*, \tau^*]$. Thus, for $\tau \in [0, \tau_\varepsilon]$,

$$\int_0^\tau |\theta_2|\,|\eta(s)|\,ds \leqslant K \int_{\tau_*}^{\tau_*} |\eta(s)|\,ds$$
$$\leqslant K(\varepsilon\,|\eta(0)| + |\xi(0)|)\,e^{-C\tau_*/\varepsilon} + Ke^{-C\tau_\varepsilon/\varepsilon}\,|\rho(0)|.$$

Also, since

$$|\theta_1^{\mathrm{I}}| \leqslant K(|u_\varepsilon|\,|v_\varepsilon| + |D_y a|\,|w_\varepsilon|) \leqslant Ke^{-C\tau/\varepsilon} \qquad \text{and}$$
$$|\theta_3| \leqslant K\,|u_\varepsilon| \leqslant Ke^{-\beta_0(\tau_\varepsilon - \tau)/\varepsilon},$$

one can check easily that

$$\int_0^\tau |\theta_1^{\mathrm{I}}|\,|\xi(s)|\,ds \leqslant K\varepsilon(|\xi(0)| + \varepsilon\,|\eta(0)| + e^{-C\tau_\varepsilon/\varepsilon}\,|\rho(0)|) \leqslant K\varepsilon,$$

and

$$\int_0^\tau \frac{|\theta_3|}{\varepsilon}\,|\rho(s)|\,ds \leqslant K\left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon} + |\rho(0)|\right) e^{-C\tau_\varepsilon/\varepsilon} \leqslant K\varepsilon.$$

Therefore,

$$|\xi(\tau) - \xi_0(\tau)| \leqslant |\xi(0) - \xi_0(0)| + K\varepsilon + \int_0^\tau |A^{\mathrm{I}}|\,|\xi(s) - \xi_0(s)|\,ds.$$

An application of Gronwall's inequality then yields

$$|\xi(\tau) - \xi_0(\tau)| \leqslant K(\varepsilon + |\xi(0) - \xi_0(0)|).$$

Note that $\xi(0)$ is the vector of type-I $(k+\sigma)$ forms of $T_{\hat{q}_\varepsilon^0} W^{\mathrm{u}}(S_0)|_{\omega(N_0)\cdot(-\delta,\delta)}$. We have that $|\xi(0) - \xi_0(0)|$ is $O(\varepsilon)$ small. The proof is complete. ∎

4.3. *The Second Exchange Lemma*

The Second Exchange Lemma corresponds to the case that M_ε enters and exits $\mathscr{B}$ at points close to $W^{\mathrm{c}}(S_0)$ for which the delay of stability loss phenomenon in Theorem 2.2 dominates the evolution of the configuration. The stability of the w-component balances along this passage and the w-component can be viewed as a true *center* direction.

LEMMA 4.5. *Assume* (H1)–(H3) *and* (A1)–(A3) *and* $N_0 \cap \{|w| = \Delta\} \neq \varnothing$. *If* M_ε *enters* $\mathscr{B}$ *through the point* $q_\varepsilon^0 \in \{|w| = \Delta\}$ *which depends on* ε *smoothly and exits* $\mathscr{B}$ *through the point* $q_\varepsilon^1 = \phi_\varepsilon^{\tau_\varepsilon}(q_\varepsilon^0)$ *on the face* $|w| = \Delta$ *with*

$\tau_\varepsilon \to \tau_0 > 0$ *and* $\alpha(q_0^1) = P_0(\omega(q_0^0))$, *then there exists* $K > 0$ *such that, for* $\Delta > 0$ *small*, $\varepsilon > 0$ *small, and* $\tau \in [0, \tau_\varepsilon]$,

$$|u_\varepsilon(\tau)| \leqslant Ke^{-\beta_0(\tau_\varepsilon - \tau)/\varepsilon}, \qquad |v_\varepsilon(\tau)| \leqslant Ke^{\alpha_0 \tau/\varepsilon}, \qquad |w_\varepsilon(\tau)| \leqslant Ke^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\, ds},$$

and

$$|y_\varepsilon(\tau) - y_0 \cdot \tau| \leqslant K(\varepsilon + |y_\varepsilon(0) - y_0|).$$

Proof. This can be proved in a manner similarly to that of Lemma 4.2. The only change one may need is to break the proof into two steps. First, use the same proof to show the estimates for the first "half" trajectory where the inequality (18) holds and then to reverse the time and apply the same method for the second "half" trajectory. The details will be omitted. ▮

LEMMA 4.6. *Consider the linear system*

$$\dot{\xi} = (A^{\mathrm{I}} + \theta_1^{\mathrm{I}})\, \xi + \theta_2 \eta + \frac{\theta_3}{\varepsilon}\, \rho,$$

$$\dot{\eta} = \frac{\lambda_0 + \phi_2 - \phi_1}{\varepsilon}\, \eta + (A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\, \eta + \frac{\theta_4}{\varepsilon}\, \xi + \frac{\theta_5}{\varepsilon}\, \rho,$$

$$\dot{\rho} = \frac{B + \lambda_0 + \phi_3 - \phi_1}{\varepsilon}\, \rho + \frac{\theta_6}{\varepsilon}\, \xi + \frac{\theta_7}{\varepsilon}\, \eta,$$

where the coefficients A^{I}, A^{II}, B, ϕ_i, *and* θ_i *are as in* (17). *Assume the hypotheses in Lemma* 4.5. *Then there exist constants* $K > 0$ *and* $C > 0$ *such that, for* $\varepsilon > 0$ *small and for* $\tau \in [0, \tau_\varepsilon]$,

$$|\xi(\tau)| \leqslant K(|\xi(0)| + \varepsilon\, |\eta(0)|) + Ke^{-C\tau_\varepsilon/\varepsilon}\, |\rho(0)|,$$

$$|\eta(\tau)| \leqslant Ke^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\, ds} \left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon} \right) + Ke^{-C\tau_\varepsilon/\varepsilon}\, |\rho(0)|,$$

and

$$|\rho(\tau)| \leqslant Ke^{-C\tau/\varepsilon}(|\rho(0)| + |\xi(0)| + |\eta(0)|).$$

Proof. From Remark 4.3 and displays (24), and (25), and (26), it suffices to show that

$$\int_0^{\tau_\varepsilon} \lambda_0^\varepsilon(s)\, ds \leqslant K\varepsilon.$$

Recall that

$$w_\varepsilon(\tau) = e^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds} w_\varepsilon(0) + \int_0^\tau e^{(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\,dt} \Lambda \cdot (u_\varepsilon(s), v_\varepsilon(s))\,ds$$

and

$$|u_\varepsilon(\tau)|\,|v_\varepsilon(\tau)| \leqslant K e^{-\beta_0(\tau_\varepsilon - \tau)/\varepsilon} e^{\alpha_0 \tau/\varepsilon} \leqslant K e^{((\alpha_0 + \beta_0)\,\tau - \beta_0 \tau_\varepsilon)/\varepsilon}.$$

Thus,

$$|w_\varepsilon(\tau_\varepsilon)| \geqslant e^{(1/\varepsilon)\int_0^{\tau_\varepsilon} \lambda_0^\varepsilon(s)\,ds}\,|w_\varepsilon(0)| - K\varepsilon(e^{\alpha_0 \tau_\varepsilon/\varepsilon} + e^{-(\beta_0 - \lambda_+)\,\tau_\varepsilon/\varepsilon}).$$

From this and $|w_\varepsilon(\tau_\varepsilon)| = |w_\varepsilon(0)| = \Delta$ we have

$$e^{(1/\varepsilon)\int_0^{\tau_\varepsilon} \lambda_0^\varepsilon(s)\,ds} \leqslant 2$$

or

$$\int_0^{\tau_\varepsilon} \lambda_0^\varepsilon(s)\,ds \leqslant K\varepsilon.$$

This completes the proof.

We now prove the corresponding C^1 exchange lemma (see Fig. 3).

THEOREM 4.7. *Assume* (H1)–(H3), (A1)–(A3), *and that* $N_0 \cap \{|w| = \Delta\} \neq \varnothing$. *If* M_ε *enters* $\mathscr{B}$ *through the point* $q_\varepsilon^0 \in \{|w| = \Delta\}$ *which depends on* ε *smoothly and exits* $\mathscr{B}$ *through the point* $q_\varepsilon^1 = \phi_\varepsilon^{\tau_\varepsilon}(q_\varepsilon^0)$ *on the face* $|w| = \Delta$ *with*

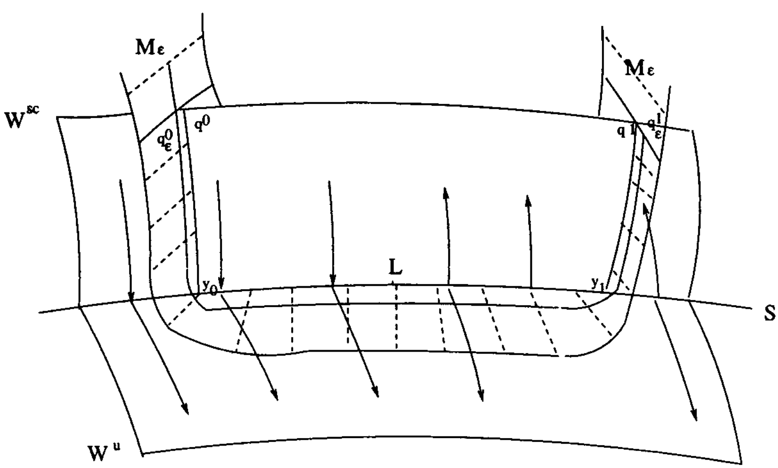

FIG. 3. The Second Exchange Lemma.

$\tau_\varepsilon \to \tau_0 > 0$ and $\alpha(q_0^1) = P_0(\omega(q_0^0))$, then M_ε is C^1 $O(\varepsilon)$-close at q_ε^1 to the manifold

$$W^{cs}(S_0)|_{P_0(\omega(N_0))} = \{(u, 0, w, y) : y \in P_0(\omega(N_0))\}.$$

Proof. The strategy for the proof of Theorem 4.4 can be modified for a proof of this theorem.

We decompose the tangent space of M_ε at q_ε^0 into the direct sum of the vector field direction and its orthogonal complement. The evolution of the former one is nothing but the vector field along the solution. For the latter, the evolution can be traced by the evaluations of all $(k+\sigma-1)$-forms.

Along the solution $(u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau))$ on M_ε, let $\xi(\tau)$, $\eta(\tau)$, and $\rho(\tau)$ be the vectors of the basic types I, II, and III $(k+\sigma-1)$-forms corresponding to the compliment of the tangent space of M_ε to the vector field, respectively (in the case that $\sigma = 1$ there is no η—the type-II $(k+\sigma-1)$-forms). Let $\xi_0(\tau)$ be the vector of type-I $(k+\sigma-1)$-forms at $\varepsilon = 0$ corresponding to the tangent space

$$T_{(0,\,0,\,0,\,y^0\cdot\tau)} W^{u}(S_0)|_{\omega(N_0)\cdot\tau}\,.$$

The corresponding η_0 and ρ_0 are identically zeros.

The statement of the theorem is then equivalent to the estimate

$$|\rho(\tau_\varepsilon| + |\xi(\tau_\varepsilon) - \xi_0(\tau_\varepsilon)| \leqslant K\varepsilon.$$

To obtain the estimate, we recall the equation that the forms satisfied

$$\varepsilon\dot{\xi} = (\mathrm{tr}\ U + \phi_1)\ \xi + \varepsilon(A^{\mathrm{I}} + \theta_1^{\mathrm{I}})\ \xi + \varepsilon\theta_2\eta + \theta_3\rho,$$

$$\varepsilon\dot{\eta} = (\mathrm{tr}\ U + \lambda_0 + \phi_2)\ \eta + \varepsilon(A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\ \eta + \theta_4\xi + \theta_5\rho,$$

$$\varepsilon\dot{\rho} = (\mathrm{tr}\ U + B + \lambda_0 + \phi_3)\ \rho + \theta_6\xi + \theta_7\eta.$$

Again, using the integrating factor $(1/|\xi(0)|)\, e^{-(1/\varepsilon)\int_0^\tau (\mathrm{tr}\ U + \phi_1)}$, we get

$$\dot{\xi} = (A^{\mathrm{I}} + \theta_1^{\mathrm{I}})\ \xi + \theta_2\eta + \frac{\theta_3}{\varepsilon}\ \rho,$$

$$\dot{\eta} = \frac{\lambda_0 + \phi_2 - \phi_1}{\varepsilon}\ \eta + (A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\ \eta + \frac{\theta_4}{\varepsilon}\ \xi + \frac{\theta_5}{\varepsilon}\ \rho,$$

$$\dot{\rho} = \frac{B + \lambda_0 + \phi_3 - \phi_1}{\varepsilon}\ \rho + \frac{\theta_6}{\varepsilon}\ \xi + \frac{\theta_7}{\varepsilon}\ \eta. \tag{28}$$

Due to the transversality condition (A1) and the normalization by the integrating factor, we have

$$|\xi(0)| = 1, \qquad |\eta(0)| \leqslant K, \qquad \text{and} \qquad |\rho(0)| \leqslant K$$

for some constant K. Applying Lemma 4.6 to system (28), we have that, for $\tau \leqslant \tau_\varepsilon$,

$$|\xi(\tau)| \leqslant K(|\xi(0)| + \varepsilon\,|\eta(0)|) + Ke^{-C\tau_\varepsilon/\varepsilon}\,|\rho(0)|,$$

$$|\eta(\tau)| \leqslant Ke^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\,ds}\left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon}\right) + K\,|\rho(0)|\,e^{-C\tau_\varepsilon/\varepsilon},$$

and

$$|\rho(\tau)| \leqslant Ke^{-C\tau/\varepsilon}(|\rho(0)| + |\xi(0)| + |\eta(0)|).$$

In particular, $|\rho(\tau_\varepsilon)| \leqslant K\varepsilon$. It remains to show that $|\xi(\tau_\varepsilon) - \xi_0(\tau_\varepsilon)| \leqslant K\varepsilon$.

The equation that ξ_0 satisfied is

$$\dot{\xi}_0(\tau) = A^{\mathrm{I}}(y^0 \cdot \tau)\,\xi_0(\tau)$$

and that of ξ is

$$\dot{\xi}(\tau) = (A^{\mathrm{I}}(y_\varepsilon(\tau)) + \theta_1^{\mathrm{I}})\,\xi(\tau) + \theta_2\eta(\tau) + \frac{\theta_3}{\varepsilon}\,\rho(\tau).$$

As in the proof of Lemma 4.4, we have

$$\begin{aligned}\int_0^\tau |\theta_2|\,|\eta(s)|\,ds &\leqslant K\int_{\tau_*}^{\tau^*} |\eta(s)|\,ds\\ &\leqslant K\varepsilon e^{-C\tau_*/\varepsilon}\left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon}\right) + Ke^{-C\tau_\varepsilon/\varepsilon}\,|\rho(0)| \leqslant K\varepsilon,\end{aligned}$$

$$\int_0^\tau |\theta_1^{\mathrm{I}}|\,|\xi(s)|\,ds \leqslant K\varepsilon(|\xi(0)| + \varepsilon\,|\eta(0)| + e^{-C\tau_\varepsilon/\varepsilon}\,|\rho(0)|) \leqslant K\varepsilon,$$

and

$$\int_0^\tau \frac{|\theta_3|}{\varepsilon}\,|\rho(s)|\,ds \leqslant K\left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon} + |\rho(0)|\right)e^{-C\tau_\varepsilon/\varepsilon} \leqslant K\varepsilon.$$

Hence,

$$|\xi(\tau)-\xi_0(\tau)| \leqslant |\xi(0)-\xi_0(0)| + \int_0^\tau (|D_y A^{\mathrm{I}}|\,|y_\varepsilon(s)-y^0\cdot s| + |\theta_1^{\mathrm{I}}|)\,|\xi(s)|\,ds$$

$$+\int_0^\tau |A^{\mathrm{I}}|\,|\xi(s)-\xi_0(s)|\,ds$$

$$+\int_0^\tau |\theta_2|\,|\eta(s)|\,ds + \int_0^\tau \frac{|\theta_3|}{\varepsilon}\,|\rho(s)|\,ds$$

$$\leqslant |\xi(0)-\xi_0(0)| + K\varepsilon + \int_0^\tau |A^{\mathrm{I}}|\,|\xi(s)-\xi_0(s)|\,ds.$$

An application of Gronwall's inequality then yields

$$|\xi(\tau)-\xi_0(\tau)| \leqslant K(\varepsilon + |\xi(0)-\xi_0(0)|).$$

Note that $\xi(0)$ is the vector of type-I $(k+\sigma-1)$-forms of $T_{\hat{q}_\varepsilon^0} W^{\mathrm{u}}(S_0)|_{\omega(N_0)}$. We have that $|\xi(0)-\xi_0(0)|$ is $O(\varepsilon)$ small and thus $|\xi(\tau)-\xi_0(\tau)| \leqslant K\varepsilon$. The proof is complete. ∎

4.4. *The Third Exchange Lemma*

Suppose that the singular orbit on S_0 followed by M_ε has endpoints $y^0 \in S_-$ and $y^1 \in S_+$. The last exchange lemma deals with the case that $y^1 \succ P_0(y^0)$ and the w-component is naturally treated as an *unstable* one. The lemma can again be viewed as a generalization of that of [16] and [30]. The assumptions (A1)–(A3) need to be replaced.

Let $2 \leqslant \sigma \leqslant n+1$ and let M_ε be a $(k+\sigma)$-dimensional invariant manifold of system (13) for small ε which is smooth in ε.

We change the assumptions (A1)–(A3) to the following.

(A1′) M_0 intersects $W^{\mathrm{s}}(S_-)$ transversally at $q_0^0 \in \partial\mathscr{B}$.

Remark 4.4. Let us denote that $N_0 = M_0 \cap W^{\mathrm{s}}(S_-)$. As a consequence of assumption (A1′), we now have $\dim N_0 = \sigma - 1$. Since M_0 is invariant, we need $\sigma \geqslant 2$.

(A2′) The set $\omega(N_0) \cap \mathscr{B}$ is a $(\sigma-2)$-dimensional submanifold of S_-.

(A3) $(0, h(y; 0)) \notin T_{(0,\,y)}\omega(N_0)$ for $(0, y) \in \omega(N_0)$.

Remark 4.5. If $\dim \omega(N_0) = 0$, or equivalently, $\sigma = 2$, then (A3′) is automatic. Also, (A3′) requires that $\dim \omega(N_0) \leqslant n-1$; that is, $\sigma \leqslant n+1$.

Lemma 4.8. *Assume* (H1)–(H3) *and* (A1′)–(A3′). *If* M_ε *enters* $\mathscr{B}$ *through the point* q_ε^0 *which depends on* ε *smoothly and exits* $\mathscr{B}$ *through the* *point* $q^1_\varepsilon = \phi^{\tau_\varepsilon}_\varepsilon(q^0_\varepsilon)$ *on the face* $|u| = \Delta$ *or* $|w| = \Delta$ *with* $\tau_\varepsilon \to \tau_0 > 0$ *and* $\alpha(q^1_0) \succ P_0(\omega(q^0_0))$, *then there exist constants* $K > 0$ *and* $C > 0$ *such that, for* $\Delta > 0$ *small,* $\varepsilon > 0$ *small, and* $\tau \in [0, \tau_\varepsilon]$,

$$|u_\varepsilon(\tau)| \leqslant Ke^{-\beta_0(\tau_\varepsilon - \tau)\varepsilon}, \qquad |v_\varepsilon(\tau)| \leqslant Ke^{\alpha_0 \tau/\varepsilon}, \qquad |w_\varepsilon(\tau)| \leqslant Ke^{-C(\tau_\varepsilon - \tau)/\varepsilon},$$

and

$$|y_\varepsilon(\tau) - y_0 \cdot \tau| \leqslant K(\varepsilon + |y_\varepsilon(0) - y_0|).$$

Proof. The estimates for $|u_\varepsilon(\tau)|$ and $|v_\varepsilon(\tau)|$ follow again from the results of Proposition 3.1 in [16]. For $|w_\varepsilon(\tau)|$ and $|y_\varepsilon(\tau) - y_0 \cdot \tau|$, one can apply the same proof as that of Lemma 4.2 after reversing the time. ∎

Next, we prove the corresponding technical lemma.

LEMMA 4.9. *Consider the linear system*

$$\dot{\xi} = -\frac{\lambda_0 + \phi_2 - \phi_1}{\varepsilon}\,\xi + (A^{\mathrm{I}} + \theta_1^{\mathrm{I}})\,\xi + \theta_2 \eta + \frac{\theta_3}{\varepsilon}\,\rho,$$

$$\dot{\eta} = (A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\,\eta + \frac{\theta_4}{\varepsilon}\,\xi + \frac{\theta_5}{\varepsilon}\,\rho,$$

$$\dot{\rho} = \frac{B + \phi_3 - \phi_2}{\varepsilon}\,\rho + \frac{\theta_6}{\varepsilon}\,\xi + \frac{\theta_7}{\varepsilon}\,\eta,$$

where the coefficients A^{I}, A^{II}, B, ϕ_i, *and* θ_i *are as in* (17). *Assume the hypotheses in Lemma* 4.8. *Then there exist constants* $K > 0$ *and* $C > 0$ *such that, for* $\varepsilon > 0$ *small and for* $\tau \in [0, \tau_1]$ *where* τ_1 *is determined by* $\int_0^{\tau_1} \lambda_0^\varepsilon(s)\, ds = 0$,

$$|\xi(\tau)| \leqslant K(|\xi(0)| + \varepsilon\,|\eta(0)|) + Ke^{-C\tau_\varepsilon/\varepsilon}\,|\rho(0)|,$$

$$|\eta(\tau)| \leqslant Ke^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\, ds}\left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon}\right) + Ke^{-C\tau_\varepsilon/\varepsilon}\,|\rho(0)|,$$

and

$$|\rho(\tau)| \leqslant Ke^{-C\tau/\varepsilon}(|\rho(0)| + |\xi(0)| + |\eta(0)|);$$

and, for $\tau \in [\tau_1, \tau_\varepsilon]$ *and for all small* $\varepsilon > 0$,

$$|\xi(\tau)| \leqslant Ke^{-C(\tau - \tau_1)/\varepsilon}\,(|\xi(0)| + \varepsilon\,|\eta(0)|) + Ke^{-C\tau_\varepsilon/\varepsilon}\,|\rho(0)|,$$

$$|\eta(\tau)| \leqslant K\left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon}\right) + Ke^{-C\tau_\varepsilon/\varepsilon}\,|\rho(0)|,$$

and

$$|\rho(\tau)| \leqslant Ke^{-C\tau/\varepsilon}\left(|\rho(0)| + \frac{|\xi(0)|}{\varepsilon} + |\eta(0)|\right).$$

Proof. We divide the estimate into two steps. First of all, let τ_1 be the slow time such that $\int_0^{\tau_1} \lambda_0^\varepsilon(s)\, ds = 0$. Using the result from Lemma 4.6, we have, for $\tau \in [0, \tau_1]$,

$$\begin{aligned}
&|\xi(\tau)| \leqslant K(|\xi(0)| + \varepsilon\, |\eta(0)|) + Ke^{-C\tau_\varepsilon/\varepsilon}\, |\rho(0)|,\\
&|\eta(\tau)| \leqslant Ke^{(1/\varepsilon)\int_0^\tau \lambda_0^\varepsilon(s)\, ds}\left(|\eta(0)| + \frac{|\xi(0)|}{\varepsilon}\right) + Ke^{-(C\tau_\varepsilon/\varepsilon)}\, |\rho(0)|,\\
&|\rho(\tau)| \leqslant Ke^{-C\tau/\varepsilon}(|\rho(0)| + |\xi(0)| + |\eta(0)|).
\end{aligned} \tag{29}$$

Next, let $\Phi_1(\tau)$, $\Phi_2(\tau)$, and $\Phi_3(\tau)$ be the principal fundamental matrix solutions at $\tau = \tau_1$ of the systems with system matrices

$$-\frac{\lambda_0 + \phi_2 - \phi_1}{\varepsilon} + A^{\mathrm{I}} + \theta_1^{\mathrm{I}}, \qquad A^{\mathrm{II}} + \theta_1^{\mathrm{II}}, \qquad \frac{B + \phi_3 - \phi_2}{\varepsilon},$$

respectively. Then we have

$$|\Phi_1(\tau)\, \Phi_1^{-1}(s)| \leqslant Ke^{-(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\, dt}, \qquad |\Phi_2(\tau)\, \Phi_2^{-1}(s)| \leqslant K,$$

and

$$|\Phi_3(\tau)\, \Phi_3^{-1}(s)| \leqslant Ke^{-\gamma_0(\tau - s)/\varepsilon},$$

for some constant $K > 0$, for $\tau > s$ in $[\tau_1, \tau_\varepsilon]$, and $\gamma_0 < \min\{-\alpha_0, \beta_0\}$.

By the variation of the constant formula and noting that $\int_s^\tau \lambda_0^\varepsilon(t)\, dt \geqslant C(\tau - s)$ for some $C > 0$ and $\tau \geqslant s$ in $[\tau_1, \tau_\varepsilon]$, we have, for $\tau \in [\tau_1, \tau_\varepsilon]$,

$$\begin{aligned}
|\xi(\tau)| &\leqslant Ke^{-(1/\varepsilon)\int_{\tau_1}^\tau \lambda_0^\varepsilon(s)\, ds}\, |\xi(\tau_1)|\\
&\quad + K\int_{\tau_1}^\tau e^{-(1/\varepsilon)\int_s^\tau \lambda_0^\varepsilon(t)\, dt}\left(|\theta_2|\, |\eta(s)| + \frac{|\theta_3|}{\varepsilon}\, |\rho(s)|\right) ds\\
&\leqslant Ke^{-C(\tau - \tau_1)/\varepsilon}\, |\xi(\tau_1)| + K\int_{\tau_1}^\tau e^{-C(\tau - s)/\varepsilon}\left(|\theta_2|\, |\eta(s)| + \frac{|\theta_3|}{\varepsilon}\, |\rho(s)|\right) ds,
\end{aligned} \tag{30}$$

$$|\eta(\tau)| \leqslant K\, |\eta(\tau_1)| + K\int_{\tau_1}^\tau \left(\frac{|\theta_4|}{\varepsilon}\, |\xi(s)| + \frac{|\theta_5|}{\varepsilon}\, |\rho(s)|\right) ds, \tag{31}$$

and

$$|\rho(\tau)| \leqslant Ke^{-\gamma_0(\tau-\tau_1)/\varepsilon} |\rho(\tau_1)|$$

$$+K \int_{\tau_1}^{\tau} e^{-\gamma_0(\tau-s)/\varepsilon} \left(\frac{|\theta_6|}{\varepsilon} |\xi(s)| + \frac{|\theta_7|}{\varepsilon} |\eta(s)| \right) ds. \tag{32}$$

Substituting the estimate (31) into (32),

$$|\rho(\tau)| \leqslant Ke^{-\gamma_0(\tau-\tau_1)/\varepsilon} |\rho(\tau_1)| + K \int_{\tau_1}^{\tau} e^{-\gamma_0(\tau-s)/\varepsilon} \frac{|\theta_6|}{\varepsilon} |\xi(s)| \, ds$$

$$+ K \int_{\tau_1}^{\tau} e^{-\gamma_0(\tau-s)/\varepsilon} \frac{|\theta_7|}{\varepsilon} |\eta(0)| \, ds$$

$$+ K \int_{\tau_1}^{\tau} e^{-\gamma_0(\tau-s)/\varepsilon} \frac{|\theta_7|}{\varepsilon} \int_{\tau_1}^{s} \frac{|\theta_4|}{\varepsilon} |\xi(t)| \, dt \, ds$$

$$+ K \int_{\tau_1}^{\tau} e^{-\gamma_0(\tau-s)/\varepsilon} \frac{|\theta_7|}{\varepsilon} \int_{\tau_1}^{s} \frac{|\theta_5|}{\varepsilon} |\rho(t)| \, dt \, ds.$$

Since $|\theta_7| \leqslant K \, |v_\varepsilon| \leqslant Ke^{\alpha_0 \tau/\varepsilon}$, one has

$$\int_{\tau_1}^{\tau} e^{-\gamma_0(\tau-s)/\varepsilon} \frac{|\theta_7|}{\varepsilon} \, ds \leqslant \frac{K}{\varepsilon} \int_{\tau_1}^{\tau} e^{-\gamma_0(\tau-s)/\varepsilon} e^{\alpha_0 s/\varepsilon} \, ds \leqslant Ke^{-\gamma_0 \tau/\varepsilon}.$$

Therefore,

$$|\rho(\tau)| \leqslant Ke^{-\gamma_0(\tau-\tau_1)/\varepsilon} (|\rho(\tau_1)| + e^{-\gamma_0 \tau_1/\varepsilon} |\eta(\tau_1)|)$$

$$+ Ke^{-\gamma_0 \tau/\varepsilon} \int_{\tau_1}^{\tau} e^{\gamma_0 s/\varepsilon} \frac{|\theta_6|}{\varepsilon} |\xi(s)| \, ds$$

$$+ Ke^{-\gamma_0 \tau/\varepsilon} \int_{\tau_1}^{\tau} \frac{|\theta_4|}{\varepsilon} |\xi(s)| \, ds$$

$$+ Ke^{-\gamma_0 \tau/\varepsilon} \int_{\tau_1}^{\tau} \frac{|\theta_5|}{\varepsilon} |\rho(s)| \, ds.$$

An application of Gronwall's inequality together with the estimate that

$$|\theta_5| \leqslant K \, |u_\varepsilon| \leqslant Ke^{-\beta_0(\tau_\varepsilon-\tau)/\varepsilon}$$

yields

$$|\rho(\tau)| \leqslant Ke^{-\gamma_0(\tau-\tau_1)/\varepsilon}(|\rho(\tau_1)| + e^{-\gamma_0\tau_1/\varepsilon}\,|\eta(\tau_1)|)$$

$$+ Ke^{-\gamma_0\tau/\varepsilon} \int_{\tau_1}^{\tau} e^{\gamma_0 s/\varepsilon} \frac{|\theta_6|}{\varepsilon}\,|\xi(s)|\,ds + Ke^{-\gamma_0\tau/\varepsilon} \int_{\tau_1}^{\tau} \frac{|\theta_4|}{\varepsilon}\,|\xi(s)|\,ds. \quad (33)$$

Substituting the estimates (31) and (33) into (30),

$$|\xi(\tau)| \leqslant Ke^{-C(\tau-\tau_1)/\varepsilon}\,|\xi(\tau_1)| + K\int_{\tau_1}^{\tau} e^{-C(\tau-s)/\varepsilon}\,|\theta_2|\,|\eta(\tau_1)|\,ds$$

$$+ K\int_{\tau_1}^{\tau} e^{-C(\tau-s)/\varepsilon}\,|\theta_2| \int_{\tau_1}^{s} \frac{|\theta_4|}{\varepsilon}\,|\xi(t)|\,dt\,ds$$

$$+ K\int_{\tau_1}^{\tau} e^{-C(\tau-s)/\varepsilon}\,|\theta_2| \int_{\tau_1}^{s} \frac{|\theta_5|}{\varepsilon}$$

$$\times e^{-\gamma_0(t-\tau_1)/\varepsilon}(|\rho(\tau_1)| + e^{-\gamma_0\tau_1/\varepsilon}\,|\eta(\tau_1)|)\,dt\,ds$$

$$+ K\int_{\tau_1}^{\tau} e^{-C(\tau-s)/\varepsilon}\,|\theta_2| \int_{\tau_1}^{s} \frac{|\theta_5|}{\varepsilon}\,e^{-\gamma_0 t/\varepsilon}\,dt\,ds \int_{\tau_1}^{\tau} e^{\gamma_0 s/\varepsilon}\frac{|\theta_6|}{\varepsilon}\,|\xi(s)|\,ds$$

$$+ K\int_{\tau_1}^{\tau} e^{-C(\tau-s)/\varepsilon}\,|\theta_2| \int_{\tau_1}^{s} \frac{|\theta_5|}{\varepsilon}\,e^{-\gamma_0 t/\varepsilon}\,dt\,ds \int_{\tau_1}^{\tau} \frac{|\theta_4|}{\varepsilon}\,|\xi(s)|\,ds$$

$$+ K\int_{\tau_1}^{\tau} e^{-C(\tau-s)/\varepsilon}\,\frac{|\theta_3|}{\varepsilon}\,e^{-\gamma_0(s-\tau_1)/\varepsilon}(|\rho(\tau_1)| + e^{-\gamma_0\tau_1/\varepsilon}\,|\eta(\tau_1)|)\,ds$$

$$+ K\int_{\tau_1}^{\tau} e^{-C(\tau-s)/\varepsilon}\,\frac{|\theta_3|}{\varepsilon} \int_{\tau_1}^{s} e^{-\gamma_0(s-t)/\varepsilon}\,\frac{|\theta_6|}{\varepsilon}\,|\xi(t)|\,dt\,ds$$

$$+ K\int_{\tau_1}^{\tau} e^{-C(\tau-s)/\varepsilon}\,\frac{|\theta_3|}{\varepsilon}\,e^{-\gamma_0 s/\varepsilon} \int_{\tau_1}^{s} \frac{|\theta_4|}{\varepsilon}\,|\xi(t)|\,dt\,ds.$$

In view of the choice of τ_1 and $a=0$ for $|y_1| \geqslant \delta_0$, we have, for $\tau \geqslant \tau_1$,

$$|\theta_2| \leqslant K(|u_\varepsilon|\,|v_\varepsilon| + |a| + |D_w aw|) \leqslant K\,|u_\varepsilon|\,|v_\varepsilon| \leqslant Ke^{-\gamma_0\tau_\varepsilon/\varepsilon},$$

and thus

$$\int_{\tau_1}^{\tau} e^{-C(\tau-s)/\varepsilon}\,|\theta_2| \leqslant K\varepsilon e^{-\gamma_0\tau_\varepsilon/\varepsilon}.$$

Therefore,

$$|\xi(\tau)| \leqslant Ke^{-C(\tau-\tau_1)/\varepsilon}\, |\xi(\tau_1)| + Ke^{-C\tau_\varepsilon/\varepsilon}(|\eta(\tau_1)| + |\rho(\tau_1)|)$$
$$+ K \int_{\tau_1}^{\tau} \frac{|\theta_4|}{\varepsilon}\, |\xi(s)|\, ds + Ke^{-\gamma_0\tau_\varepsilon/\varepsilon} \int_{\tau_1}^{\tau} e^{\gamma_0 s/\varepsilon}\, \frac{|\theta_6|}{\varepsilon}\, |\xi(s)|\, ds.$$

Applying Gronwall's inequality and using

$$|\theta_4| \leqslant K(|u_\varepsilon|\, |v_\varepsilon| + |w_\varepsilon|) \leqslant Ke^{-C(\tau_\varepsilon-\tau)/\varepsilon}, \qquad |\theta_6| \leqslant K\, |v_\varepsilon| \leqslant Ke^{\alpha_0\tau/\varepsilon},$$

we have

$$|\xi(\tau)| \leqslant Ke^{-C(\tau-\tau_1)/\varepsilon}\, |\xi(\tau_1)| + Ke^{-C\tau_\varepsilon/\varepsilon}(|\eta(\tau_1)| + |\rho(\tau_1)|). \tag{34}$$

Substituting (34) into (33),

$$|\rho(\tau)| \leqslant Ke^{-\gamma_0(\tau-\tau_1)/\varepsilon}\, |\rho(\tau_1)| + Ke^{-\gamma_0\tau/\varepsilon}(|\xi(\tau_1)| + |\eta(\tau_1)|). \tag{35}$$

Last, substituting (34) and (35) into (31),

$$|\eta(\tau)| \leqslant K\, |\eta(\tau_1)| + Ke^{-C\tau_\varepsilon/\varepsilon}(|\xi(\tau_1)| + |\rho(\tau_1)|).$$

Combining this with the estimate (29), we can then conclude the lemma. ∎

We are now ready to obtain the corresponding C^1 Exchange Lemma (Fig. 4).

THEOREM 4.10. *Assume* (H1)–(H3) *and* (A1′)–(A3′). *If* M_ε *enters* $\mathscr{B}$ *through the point* q_ε^0 *which depends on* ε *smoothly and exits* $\mathscr{B}$ *through the point* $q_\varepsilon^1 = \phi_\varepsilon^{\tau_\varepsilon}(q_\varepsilon^0)$ *on the face* $|u| = \Delta$ *or* $|w| = \Delta$ *with* $\tau_\varepsilon \to \tau_0$ *and* $\alpha(q_0^1) \succ P_0(\omega(q_0^0))$, *then for some* $\delta > 0$ *small independent of* ε, M_ε *is* C^1 $O(\varepsilon)$*-close at* q_ε^1 *to the manifold*

$$W^{\mathrm{cu}}(S_0)|_{\omega(N_0)\cdot(\tau_0-\delta,\, \tau_0+\delta)} = \{(u, 0, w, y) : y \in \omega(N_0)\cdot(\tau_0-\delta, \tau_0+\delta)\}.$$

Proof. Let $\xi(\tau)$, $\eta(\tau)$, and $\rho(\tau)$ again denote the vectors of the types I, II, and III $(k+\sigma)$-forms corresponding to the tangent space of M_ε along the solution $(u_\varepsilon(\tau), v_\varepsilon(\tau), w_\varepsilon(\tau), y_\varepsilon(\tau))$. Let $\eta_0(\tau)$ denote the vector of the type-II forms at $\varepsilon = 0$ corresponding to the tangent space

$$T_{(0,0,0,y^0\cdot\tau)}\, W^{\mathrm{cu}}(\omega(N_0)\cdot(\tau-\delta, \tau+\delta)).$$

The corresponding ξ_0 and ρ_0 are identical zeros.

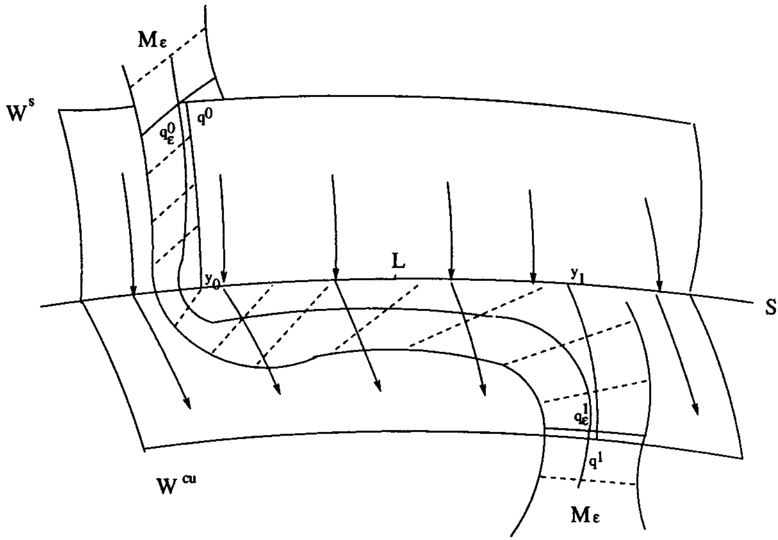

FIG. 4. The Third Exchange Lemma.

Differing from the first two exchange lemmas, the statement of the theorem is now equivalent to the estimate

$$|\xi(\tau_\varepsilon)| + |\rho(\tau_\varepsilon)| + |\eta(\tau_\varepsilon) - \eta_0(\tau_\varepsilon)| \leqslant K\varepsilon.$$

Starting with the differential equation for the $(k+\sigma)$-forms,

$$\varepsilon\dot{\xi} = (\mathrm{tr}\ U + \phi_1)\ \xi + \varepsilon(A^{\mathrm{I}} + \theta_1^{\mathrm{I}})\ \xi + \varepsilon\theta_2\eta + \theta_3\rho,$$

$$\varepsilon\dot{\eta} = (\mathrm{tr}\ U + \lambda_0 + \phi_2)\ \eta + \varepsilon(A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\ \eta + \theta_4\xi + \theta_5\rho,$$

$$\varepsilon\dot{\rho} = (\mathrm{tr}\ U + B + \lambda_0 + \phi_3)\ \rho + \theta_6\xi + \theta_7\eta,$$

and using the integrating factor $(1/|\eta(0)|)\ e^{-(1/\varepsilon)\int_0^\zeta(\mathrm{tr}\ U + \lambda_0 + \phi_2)}$, we get

$$\dot{\xi} = -\frac{\lambda_0 + \phi_2 - \phi_1}{\varepsilon}\ \xi + (A^{\mathrm{I}} + \theta_1^{\mathrm{I}})\ \xi + \theta_2\eta + \frac{\theta_3}{\varepsilon}\ \rho,$$

$$\dot{\eta} = (A^{\mathrm{II}} + \theta_1^{\mathrm{II}})\ \eta + \frac{\theta_4}{\varepsilon}\ \xi + \frac{\theta_5}{\varepsilon}\ \rho,$$

$$\dot{\rho} = \frac{B + \phi_3 - \phi_2}{\varepsilon}\ \rho + \frac{\theta_6}{\varepsilon}\ \xi + \frac{\theta_7}{\varepsilon}\ \eta. \tag{36}$$

The normalization gives rise to

$$|\xi(0)| \leqslant K\varepsilon, \qquad |\eta(0)| = 1, \qquad \text{and} \qquad |\rho(0)| \leqslant \frac{K}{\varepsilon}$$

for some constant K. Applying Lemma 4.9 to the system (3.6), we have that, for $0 \leqslant \tau \leqslant \tau_\varepsilon$,

$$|\xi(\tau)| \leqslant Ke^{-C\tau/\varepsilon}\, |\xi(0)| + K\varepsilon\, |\eta(0)| + |\rho(0)|\, e^{-C\tau_\varepsilon/\varepsilon},$$
$$|\eta(\tau)| \leqslant K\, |\eta(0)| + e^{-C\tau_\varepsilon/\varepsilon}(|\xi(0)| + |\rho(0)|),$$

and

$$|\rho(\tau)| \leqslant Ke^{-\gamma_0\tau/\varepsilon}(|\rho(0)| + |\eta(0)| + |\xi(0)|).$$

In particular, $|\xi(\tau_\varepsilon)| + |\rho(\tau_\varepsilon)| \leqslant K\varepsilon$. Next, we show that $|\eta(\tau_\varepsilon) - \eta_0(\tau_\varepsilon)|$ is $O(\varepsilon)$ small. The equation that η_0 satisfies is

$$\dot{\eta}_0(\tau) = A^{\mathrm{II}}(y^0 \cdot \tau)\, \eta_0(\tau),$$

and that η satisfies is

$$\dot{\eta}(\tau) = (A^{\mathrm{II}}(y_\varepsilon(\tau)) + \theta_1^{\mathrm{II}})\, \eta(\tau) + \frac{\theta_4}{\varepsilon}\, \xi + \frac{\theta_5}{\varepsilon}\, \rho.$$

Using the equations of η and η_0 and the estimates for θ_i, ξ, η, and ρ,

$$\begin{aligned}
|\eta(\tau) - \eta_0(\tau)| \leqslant{}& |\eta(0) - \eta_0(0)| + \int_0^\tau (|D_y A^{\mathrm{II}}|\, |y_\varepsilon(s) - y^0 \cdot s| + |\theta_1^{\mathrm{II}}|)\, |\eta(s)|\, ds \\
&+ K \int_0^\tau |\eta(s) - \eta_0(s)|\, ds + \int_0^\tau \frac{|\theta_4|}{\varepsilon}\, |\xi(s)|\, ds + \int_0^\tau \frac{|\theta_5|}{\varepsilon}\, |\rho(s)|\, ds \\
\leqslant{}& |\eta(0) - \eta_0(0)| + K\varepsilon\, |\eta(0)| + Ke^{-C\tau_0/\varepsilon}\, |\rho(0)| + K\, |\xi(0)| \\
&+ K \int_0^\tau |\eta(s) - \eta_0(s)|\, ds \\
\leqslant{}& |\eta(0) - \eta_0(0)| + K\varepsilon + K \int_0^\tau |\eta(s) - \eta_0(s)|\, ds.
\end{aligned}$$

An application of Gronwall's inequality then yields

$$|\eta(\tau) - \eta_0(\tau)| \leqslant K(\varepsilon + |\eta(0) - \eta_0(0)|).$$

Note that $\eta(0)$ is the vector of type-II $(k+\sigma)$-forms of $T_{\hat{q}_\varepsilon^0} W^{\mathrm{cu}}(\omega(N_0) \cdot (-\delta, \delta))$. We have that $|\eta(0) - \eta_0(0)|$ is $O(\varepsilon)$ small. The proof is complete. ∎

5. APPLICATION TO BOUNDARY VALUE PROBLEMS

In this section we apply the exchange lemmas to singular boundary value problems with turning points.

Consider the singularly perturbed system

$$\begin{aligned} \varepsilon \dot{u} &= f(u, v, \tau; \varepsilon), \\ \dot{v} &= g(u, v, \tau; \varepsilon) \end{aligned} \tag{37}$$

where $(u, v) \in \mathbb{R}^m \times \mathbb{R}^n$ with the boundary conditions

$$(u(0), v(0)) \in D^0 \qquad \text{and} \qquad (u(1), v(1)) \in D^1 \tag{38}$$

where D^0 and D^1 are two submanifolds of $\mathbb{R}^{m+n}$ of dimension d_0 and d_1, respectively.

The singularly perturbed boundary value problem has been one of the most important problems in singular perturbation theory due to its fundamental role in applications and its rich structures. A great amount of effort has been devoted to its study and a fruitful theory has been developed. The solution has the following general features. Part of the solution lies in the vicinity of the slow manifold governed by the dynamics of the slow variables v, which is far from meeting the boundary conditions in general; the correction around the boundary gives rise to the so-called *boundary layers* of the solution, which move rapidly from D^0 (D^1) toward the slow manifold in the forward (backward) time direction along the stable (unstable) manifold of the slow manifold—this is reflected by the dynamics of the fast variables u; there might be internal (transition) layers or others. From the classical asymptotic theory, one tries to find the asymptotic expansions of each part and to match them to form a global solution. From the geometric or dynamical system point of view, one tries to find the limiting singular solution formed piecewise by solutions of the limiting slow and fast systems and examine the possibility of lifting the singular solution to a true solution for $\varepsilon > 0$. Put another way, the question is, under what conditions is the singular solution shadowed by a true solution? With respect to the latter consideration, the problem has been well studied in e.g., [10, 21, 30], for the case where the relevant portion of the slow manifold is normally hyperbolic. In particular, exchange lemmas have been developed toward and successfully applied to the existence of the solution and the qualitative structure of its problem. Roughly speaking, in this case, a reasonable singular solution will be shadowed by a true solution [21, 30]. This is not true anymore for problems with turning points—the situation is so complicated that a general theory seems far beyond our reach (see [1, 4, 17, 18, 23, 31] and the references therein). For the problems with the type of turning points discussed in the previous sections, the answer depends significantly on the position of the limiting solution on the slow manifold relative to the pairing map as indicated in Theorem 2.2. We demonstrate the results of existence and non-existence for the extrema situations below. To start, we will convert the boundary value problem (37) and (38) to a connecting problem as, for example, in [30].

Regarding the time τ as a new variable, say $w=\tau$, and augmenting the equation $\dot{w}=1$ to the system (37), the above boundary value problem can be rephrased as the connecting problem: the existence of a solution $(u(\tau), v(\tau), w(\tau)) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}$ connects $D^0 \times \{0\}$ to $D^1 \times \{1\}$. The connecting problems thus obtained are special in that the time spent from $D^0 \times \{0\}$ to $D^1 \times \{1\}$ will always be one. In the follows, we will study the general connecting problem.

Consider the singularly perturbed system

$$\begin{aligned} \varepsilon \dot{x} &= f(x, y; \varepsilon), \\ \dot{y} &= g(x, y; \varepsilon), \end{aligned} \tag{39}$$

where $(x, y) \in \mathbb{R}^m \times \mathbb{R}^n$ with the connecting problem

$$(x(0), y(0)) \in D^L \qquad \text{and} \qquad (x(\tau_\varepsilon), y(\tau_\varepsilon)) \in D^R \tag{40}$$

for some $\tau_\varepsilon > 0$ and where D^L and D^R are two submanifolds of $\mathbb{R}^{m+n}$ of dimension d_L and d_R, respectively.

We assume that the hypotheses (H1)–(H3) are satisfied by system (39), and hence a Fenichel-type coordinate system as in Lemma 3.1 exists with k unstable, l stable, and one sign changing eigenvalues. As before, we denote the relative part of the slow manifold by S_0, the pairing map on the slow manifold by P_0, and $\mathcal{B}$ denotes a neighborhood of the slow manifold where the Fenichel-type coordinate system is valid. In the description, we use $y \cdot \tau$ for the flow on the slow manifold at $\varepsilon = 0$ and ϕ_ε^τ for the flow with $\varepsilon > 0$ small.

Let M_ε^L and M_ε^R denote the traces of D^L and D^R under the flow (39) or the equivalent fast system (so that M_0^L and M_0^R are also well-defined), respectively. As in [21, 30], the well-poseness of the boundary value problem requires that $d_L + d_R + 1 = m + n$. Since $\dim M_0^L = d_L + 1 = k + \sigma$, conditions (A1)–(A3) imply $1 \leqslant \sigma \leqslant n$, and thus $k \leqslant d_L \leqslant k + n - 1$ and $l + 1 \leqslant d_R \leqslant l + n$. Similarly, under conditions (A1′)–(A3′), we have $k + 1 \leqslant d_L \leqslant k + n$ and $l \leqslant d_R \leqslant l + n - 1$. As pointed out before, the result depends on the positions of the singular solution on the slow manifold, which can be characterized by the pairing map into three cases. Each case is studied using the corresponding exchange lemma. Thus, Theorems 5.1, 5.3, and 5.5 for the existence and their corollaries for the non-existence below are applications of the exchange lemmas and Theorems 4.4, 4.7, and 7.10.

THEOREM 5.1. *Assume*

(a) *The hypotheses* (H1)–(H3) *are satisfied by system* (39).

(b) $(f(x, y; 0), 0) \notin T_{(x, y)} D^L$ *and* $(f(x, y; 0), 0) \notin T_{(x, y)} D^R$.

(c) (A1)–(A3) *hold for* M_0^L *and* $N_0^L = M_0^L \cap W^{cs}(S_0)$.

(d) M_0^R *intersects* $W^u(S_+)$ *transversally, the set* $\alpha(N_0^R)$ *is a* $(\dim N_0^R - 1)$*-dimensional submanifold of* S_+ *where* $N_0^R = M_0^R \cap W^u(S_0)$.

(e) *On* S_0, *the sets* $\alpha(N_0^R) \cdot \mathbb{R}_-$ *and* $\omega(N_0^L) \cdot \mathbb{R}_+$ *intersect transversally. Thus, there exist* $(0, y^L) \in \omega(N_0^L)$, $(0, y^R) \in \alpha(N_0^R)$, *and* τ_0 *such that* $y^L \cdot \tau_0 = y^R$.

If $(0, y^R) \prec P_0(0, y^L)$, *then, for* $\varepsilon > 0$ *small, there exists a* τ_ε *which is* $O(\varepsilon)$ *close to* τ_0 *such that the connecting problem* (39) *and* (40) *has a locally unique solution.*

Proof. Condition (b) simply implies that M_0^L and M_0^R are smooth manifolds of $d_L + 1$ and $d_R + 1$ dimensional, respectively. The assumptions (c) and (d) imply that

$$\dim N_0^L = \dim M_0^L + \dim W^{cs}(S_-) - (m+n) = d_L - k + 1$$

and

$$\dim N_0^R = \dim M_0^R + \dim W^u(S_+) - (m+n) = d_R - l,$$

and thus $\dim \omega(N_0^L) = d_L - k$, $\dim \alpha(N_0^R) = d_R - l - 1$. The assumption (e) implies that

$$\dim[(\alpha(N_0^R) \cdot \mathbb{R}_-) \cap (\omega(N_0^L) \cdot \mathbb{R}_+)] = (d_L - k + 1) + (d_R - l) - n = 1.$$

Thus, locally, the trajectory on the slow manifold from $(0, y^L)$ to $(0, y^R)$ is the intersection $(\alpha(N_0^R) \cdot \mathbb{R}_-) \cap (\omega(N_0^L) \cdot \mathbb{R}_+)$. We now have a singular solution $\Gamma := \Gamma_L \cup \Gamma_0 \cup \Gamma_R$ to the boundary value problem, where

$$\Gamma_L := \{\gamma_L(t) : \gamma_L(0) \in D^L, \ \ \gamma_L(t) \to (0, y^L) \text{ as } t \to +\infty\};$$

$$\Gamma_0 := \{\gamma_0(\tau) : \gamma_0(0) = (0, y^L), \ \ \gamma_0(\tau_0) = (0, y^R)\};$$

$$\Gamma_R := \{\gamma_R(t) : \gamma_R(t) \to (0, y^R) \text{ as } t \to -\infty, \ \ \gamma_R(0) \in D^R\}.$$

Since γ_L and γ_R are fast "jump" solutions, we use the fast time t to parameterize the solutions. The singular solution γ_0 on the slow manifold is then naturally parameterized by the slow time τ.

Let $q^L=(u^L, v^L, w^L, y^L)=\Gamma_L\cap\partial\mathscr{B}$ and $q^R=(u^R, v^R, w^R, y^R)=\Gamma_R\cap\partial\mathscr{B}$. Since $q^L\in N_0^L$ and $q^R\in N_0^R$, $u^L=v^R=w^R=0$. If $(0, y^R)\prec P_0(0, y^L)$, then there exists $q_\varepsilon^L\in M_\varepsilon^L\cap\partial\mathscr{B}$ which is $O(\varepsilon)$ close to q^L such that $\phi_\varepsilon^\tau(q_\varepsilon^L)=q_\varepsilon^R$ is $O(\varepsilon)$ close to q^R and τ being $O(\varepsilon)$ close to τ_0. The conditions of the First Exchange Lemma (Theorem 4.4) are satisfied for M_ε^L entering $\mathscr{B}$ at q_ε^L and exiting $\mathscr{B}$ at q_ε^R. Applying the theorem, we conclude that, at q_ε^R, M_ε^L is C^1 $O(\varepsilon)$ close to $W^{\mathrm{u}}(S_0)|_{(\omega(N_0^L)\cdot(\tau_0-\delta,\,\tau_0+\delta))}$ for some $\delta>0$. Condition (d) and Proposition 1 in [30] then imply that M_ε^R intersects M_ε^L transversally which gives rise to a solution connecting D^L to D^R. Note that the time spent by γ_L and γ_R are $O(\varepsilon)$ close to zero in the time scale of τ. Thus, there exists a locally unique solution to the connecting problem with τ_ε $O(\varepsilon)$ close to τ_0. ∎

The following non-existence result is probably very surprising compared to the results in [30], for example.

COROLLARY 5.2. *Assume* (a)–(e) *in Theorem* 5.1. *Let Γ be the singular orbit constructed in the proof of Theorem* 5.1. *Suppose* $\gamma_L(0)=(0, v^L(0), w^L(0), y^L)\notin W^{\mathrm{s}}(S_0)$; *that is,* $w^L(0)\neq 0$. *If* $(0, y^R)\succ P_0(0, y^L)$, *then there is no solution for the connecting problem* (39) *and* (40) *in a neighborhood of Γ; that is, the singular orbit Γ is not shadowed by any true solution.*

Proof. If $w^L(0)\neq 0$, then Corollary 2.3 and Lemma 4.2 imply that, for any q_ε^0 close to $\gamma_L(0)$, the solution $\phi_\varepsilon^\tau(q_\varepsilon^0)$ with the initial condition q_ε^0 leaves $\mathscr{B}$ either before or near the hypersurface $\{y=P_0(y^L)\}$. Therefore, if $(0, y^R)\succ P_0(0, y^L)$, then M_ε^L will not intersect Γ_R and the conclusion of the corollary follows. ∎

THEOREM 5.3. *Assume:*

(a) *The hypotheses* (H1)–(H3) *are satisfied by system* (39).

(b) $(f(x, y; 0), 0)\notin T_{(x,y)}D^L$ *and* $(f(x, y; 0), 0)\notin T_{(x,y)}D^R$.

(c) (A1)–(A3) *hold for* M_0^L, $N_0^L=M_0^L\cap W^{\mathrm{cs}}(S_0)$, *and* $N_0^L\not\subset W^{\mathrm{s}}(S_0)$.

(d) M_0^R *intersects* $W^{\mathrm{cu}}(S_+)$ *transversally,*

$$N_0^R=M_0^R\cap W^{\mathrm{cu}}(S_0)\not\subset W^{\mathrm{u}}(S_0),$$

and $\alpha(N_0^R)$ *is a* $(\dim N_0^R-1)$*-dimensional submanifold of* S_+.

(e) *On* S_0, *the sets* $\alpha(N_0^R\setminus W^{\mathrm{u}}(S_0))$ *and* $P_0(\omega(N_0^L\setminus W^{\mathrm{s}}(S_0)))$ *intersect transversally. Thus, there exist* $(0, y^L)\in\omega(N_0^L\setminus W^{\mathrm{s}}(S_0))$, $(0, y^R)\in\alpha(N_0^R\setminus W^{\mathrm{u}}(S_0))$, τ_0, *such that* $y^L\cdot\tau_0=y^R$ *and* $P_0(0, y^L)=(0, y^R)$.

Then, for $\varepsilon \neq 0$ small, there exists τ_ε which is $O(\varepsilon)$ close to τ_0 such that the connecting problem (39) *and* (40) *has a locally unique solution.*

Proof. Condition (b) again implies that M_0^L and M_0^R are smooth manifolds of (d_L+1) and (d_R+1) dimension, respectively. The assumptions (c) and (d) imply that

$$\dim N_0^L = \dim M_0^L + \dim W^{cs}(S_-) - (m+n) = d_L - k + 1$$

and

$$\dim N_0^R = \dim M_0^R + \dim W^{cu}(S_+) - (m+n) = d_R - l + 1,$$

and thus $\dim \omega(N_0^L) = d_L - k$, $\dim \alpha(N_0^R) = d_R - l$. The assumption (e) then implies that

$$\dim[\alpha(N_0^R) \cap P_0(\omega(N_0^L))] = (d_L - k) + (d_R - l) - n = 0.$$

Thus, locally, $(0, y^L)$ is the unique point in $\omega(N_0^L)$ such that $P_0(0, y^L) \in \alpha(N_0^R)$. In view of Theorem 2.5, there exists a locally unique singular solution $\Gamma := \Gamma_L \cup \Gamma_0 \cup \Gamma_R$, where

$$\Gamma_L := \{\gamma_L(t) : \gamma_L(0) \in D^L,\ \ \gamma_L(t) \to (0, y^L) \text{ as } t \to +\infty\};$$
$$\Gamma_0 := \{\gamma_0(\tau) : \gamma_0(0) = (0, y^L),\ \ \gamma_0(\tau_0) = (0, y^R)\};$$
$$\Gamma_R := \{\gamma_R(t) : \gamma_R(t) \to (0, y^R) \text{ as } t \to -\infty,\ \ \gamma_R(0) \in D^R\}.$$

Let $q^L = (u^L, v^L, w^L, y^L) = \Gamma_L \cap \partial\mathscr{B}$ and $q^R = (u^R, v^R, w^R, y^R) = \Gamma_R \cap \partial\mathscr{B}$. Since $(0, y^L) \in \omega(N_0^L \setminus W^s(S_0))$ and $(0, y^R) \in \alpha(N_0^R \setminus W^u(S_0))$, we have $u^L = 0$, $w^L \neq 0$, $v^R = 0$, and $w^R \neq 0$. Since $(0, y^R) = P_0(0, y^L)$, Theorem 2.5 implies that there exists $q_\varepsilon^L \in M_\varepsilon^L \cap \partial\mathscr{B}$ which is $O(\varepsilon)$ close to q^L such that $\phi^\tau(q_\varepsilon^L) = q_\varepsilon^R$ is $O(\varepsilon)$ close to q_ε^R and τ is $O(\varepsilon)$ close to τ_0. An application of the Second exchange lemma (Theorem 4.7) to M_ε^L concludes that, at q_ε^R, it is C^1 $O(\varepsilon)$ close to $W^{cs}(P_0(\omega(N_0^L)))$. Condition (d) and Proposition 1 in [30] implies that M_ε^R intersects M_ε^L transversally. Since the time spent by γ_L and γ_R are $O(\varepsilon)$ close to zero in the time scale of τ, there exists a locally unique solution to the connecting problem with τ_ε $O(\varepsilon)$ close to τ_0. ∎

Similarly, we have the following non-existence result.

Corollary 5.4. *Assume* (a)–(e) *in Theorem* 5.3. *Let Γ be the singular orbit constructed in the proof of Theorem* 5.3 *and let* $\gamma_L(0) = (0, v^L(0), w^L(0), y^L)$ *be the point as in the construction of Γ_L. If* $(0, y^R) \neq P_0(0, y^L)$ *but close, then there is no solution for the connecting problem* (39) *and* (40) *in a neighborhood of Γ; that is, the singular orbit Γ is not shadowed by any true solution.*

Proof. Note that it is already assumed that $w^L(0) \neq 0$. Corollary 2.3 and Lemma 4.5 then imply that, for any q_ε^0 close to $\gamma_L(0)$, the solution $\phi_\varepsilon^\tau(q_\varepsilon^0)$ with the initial condition q_ε^0 leaves $\mathscr{B}$ either before the hypersurface $\{y = P_0(y^L)\}$ along $W^u(S_0)$ or near $\{y = P_0(y^L)\}$. Note also that $\Gamma_R \cap W^u(S_0) = \emptyset$. Therefore, if $(0, y^R) \neq P_0(0, y^L)$, then M_ε^L will not intersect Γ_R. This completes the proof. ∎

THEOREM 5.5. *Assume*

(a) *The hypotheses* (H1)–(H3) *are satisfied by system* (39).

(b) $(f(x, y; 0), 0) \notin T_{(x,y)}D^L$ *and* $(f(x, y; 0), 0) \notin T_{(x,y)}D^R$.

(c) (A1′)–(A3′) *hold for* M_0^L *and* $N_0^L = M_0^L \cap W^s(S_0)$.

(d) M_0^R *intersects* $W^{cu}(S_+)$ *transversally, the set* $\alpha(N_0^R)$ *is a* $(\dim N_0^R - 1)$*-dimensional submanifold of* S_+ *where* $N_0^R = M_0^R \cap W^{cu}(S_0)$;

(e) *On* S_0, *the sets* $\alpha(N_0^R) \cdot \mathbb{R}_-$ *and* $\omega(N_0^L) \cdot \mathbb{R}_+$ *intersect transversally. Thus, there exist* $(0, y^L) \in \omega(N_0^L)$, $(0, y^R) \in \alpha(N_0^R)$, *and* τ_0 *such that* $y^L \cdot \tau_0 = y^R$.

If $(0, y^R) \succ P_0(0, y^L)$, *then, for* $\varepsilon > 0$ *small, there exists a* τ_ε *which is* $O(\varepsilon)$ *close to* τ_0 *such that the connecting problem* (39) *and* (40) *has a locally unique solution.*

Proof. As in the proof of Theorem 5.1, we have a locally unique singular solution $\Gamma := \Gamma_L \cup \Gamma_0 \cup \Gamma_R$ to the boundary value problem, where

$$\Gamma_L := \{\gamma_L(t) : \gamma_L(0) \in D^L,\ \ \gamma_L(t) \to (0, y^L) \text{ as } t \to +\infty\};$$

$$\Gamma_0 := \{\gamma_0(\tau) : \gamma_0(0) = (0, y^L),\ \ \gamma_0(\tau_0) = (0, y^R)\};$$

$$\Gamma_R := \{\gamma_R(t) : \gamma_R(t) \to (0, y^R) \text{ as } t \to -\infty,\ \ \gamma_R(0) \in D^R\}.$$

Since γ_L and γ_R are fast "jump" solutions, we use the fast time t to parameterize the solutions. The singular solution γ_0 on the slow manifold is then naturally parameterized by the slow time τ.

Let $q^L = (u^L, v^L, w^L, y^L) = \Gamma_L \cap \partial\mathscr{B}$ and $q^R = (u^R, v^R, w^R, y^R) = \Gamma_R \cap \partial\mathscr{B}$. Then $u^L = w^L = v^R = 0$. If $(0, y^R) \succ P_0(0, y^L)$, then there exists $q_\varepsilon^L \in M_\varepsilon^L \cap \partial\mathscr{B}$ which is $O(\varepsilon)$ close to q^L such that $\phi^\tau(q_\varepsilon^L) = q_\varepsilon^R$ is $O(\varepsilon)$ close to q^R and τ is $O(\varepsilon)$ close to τ_0. The conditions of the Third Exchange Lemma (Theorem 4.10) are satisfied for M_ε^L entering $\mathscr{B}$ at q_ε^L and exiting $\mathscr{B}$ at q_ε^R. We then conclude that, at q_ε^R, M_ε^L is C^1 $O(\varepsilon)$ close to $W^{cu}(\omega(N_0^L) \cdot (\tau_0 - \delta, \tau_0 + \delta))$ for some $\delta > 0$. Condition (d) and Proposition 1 in [30] then imply that M_ε^R intersects M_ε^L transversally which gives rise to a solution connecting D^L to D^R. Note that the time spent by γ_L and γ_R are $O(\varepsilon)$ close to zero in the time scale of τ. Thus, there exists a locally unique solution to the connecting problem with τ_ε $O(\varepsilon)$ close to τ_0. ∎

The related non-existence result is as follows.

Corollary 5.6. *Assume* (a)–(e) *in Theorem* 5.5. *Let* Γ *be the singular orbit constructed in the proof of Theorem* 5.5. *Suppose* $\gamma_R(0) = (u^R(0), 0, w^R(0), y^R) \notin W^u(S_0)$; *that is,* $w^R(0) \neq 0$. *If* $(0, y^R) \prec P_0(0, y^L)$ *but close, then there is no solution for the connecting problem* (39) *and* (40) *in a neighborhood of* Γ; *that is, the singular orbit* Γ *is not shadowed by any true solution.*

Proof. Since $\gamma_L(0) = (0, v^L(0), 0, y^L)$, for any q_ε^0 close to $\gamma_L(0)$, the solution $\phi_\varepsilon^\tau(q_\varepsilon^0)$ with the initial condition q_ε^0 leaves $\mathscr{B}$ either along $W^u(S_0)$ or near $\{y = P_0(y^L)\}$. Note that $\Gamma_R \cap W^u(S_0) = \varnothing$. Therefore, if $(0, y^R) \prec P_0(0, y^L)$, then Corollary 2.3 and Lemma 4.8 imply that M_ε^L will not intersect Γ_R. This completes the proof. ▮

APPENDIX

In this appendix we construct a locally invariant foliation of the center-unstable $W_\varepsilon^{cu}(S_0)$ (resp., center-stable manifold $W_\varepsilon^{cs}(S_0)$) over the center manifold $W_\varepsilon^c(S_0)$ for the singular perturbation problem 2. This result is used in the proof of Lemma 3.1.

Lemma A.1. *Assume that the hypotheses* (H1)–(H3) *are satisfied by system* (2). *Then there exists a locally invariant foliation of the center-unstable manifold* $W_\varepsilon^{cu}(S_0)$ (*resp. center-stable manifold* $W_\varepsilon^{cs}(S_0)$) *over the center manifold* $W_\varepsilon^c(S_0)$.

Proof. The idea is to embed the local dynamic into a system of the same dimension with a normally hyperbolic invariant manifold M such that locally $W_\varepsilon^c(S_0)$ is an open subset of M. The standard normally hyperbolic invariant manifold theory [7, 11] implies the result.

For simplicity, we provide the procedure for the case that $n = 1$ and $l = 0$. In Step 2 of the proof of Lemma 3.1 we show that there exists a local coordinate system such that system (2) has the form

$$\begin{aligned} u' &= U(u, w, y; \varepsilon)\, u, \\ w' &= \lambda(u, w, y; \varepsilon)\, w, \\ y' &= \varepsilon(h(y; \varepsilon) + a(u, w, y; \varepsilon)\, w), \end{aligned} \tag{41}$$

with $\Re(\beta_j)>\beta_0>\lambda$, where β_j are eigenvalues of $U(0, 0, y; \varepsilon)$ for $j=1, ..., k$. Assume that system (41) holds true in

$$N := \{(u, w, y) : |u| < \Delta,\ |w| < 2\Delta,\ |y| < 2\}$$

for some small Δ, and the neighborhood of interest is

$$\mathscr{B} := \{(u, w, y) : |u| < \Delta,\ |w| < \Delta,\ |y| < 1\}.$$

Choose a smooth cut-off function $\chi: (-2\Delta, 2\Delta) \times (-2, 2) \to [0, 1]$ so that

$$\chi(w, y) = \begin{cases} 1, & \text{if } \ (w, y) \in (-\Delta, \Delta) \times (-1, 1) \\ 0, & \text{if } \ |w| > \dfrac{3\Delta}{2} \text{ or } |y| > \dfrac{3}{2}, \end{cases}$$

and define a new system as

$$\begin{aligned} u' &= \chi(w, y)\, U(u, w, y; \varepsilon)\, u + (1-\chi(w, y))\, \beta_0 u, \\ w' &= \chi(w, y)\, \lambda(u, w, y; \varepsilon)\, w, \\ y' &= \varepsilon \chi(w, y)(h(y; \varepsilon) + a(u, w, y; \varepsilon)\, w). \end{aligned} \tag{42}$$

Let us denote the vector field of system (42) by $F(u, w, y; \varepsilon)$. Note that the plane $\{u=0\}$ is invariant under system (42) and

$$F(u, \pm 2\Delta, y; \varepsilon) = F(u, w, \pm 2; \varepsilon) = (\beta_0 u, 0, 0).$$

We may then identify the point $(u, 2\Delta, y)$ with $(u, -2\Delta, y)$ and the point $(u, w, 2)$ with $(u, w, -2)$ to obtain a system on $\mathbb{R} \times \mathbb{T}^2$ where $\mathbb{T}^2$ denotes the two dimensional torus. After this identification, $\{u=0\}$ becomes an invariant torus. We now show that, for ε small, the invariant torus is normally hyperbolic.

At $\varepsilon=0$, the system (42) reduces to

$$\begin{aligned} u' &= \chi(w, y)\, U(u, w, y; 0)\, u + (1-\chi(w, y))\, \beta_0 u, \\ w' &= \chi(w, y)\, \lambda(u, w, y; 0)\, w, \\ y' &= 0. \end{aligned} \tag{43}$$

From the last equation, we see that the invariant torus is foliated by the family of invariant circles $S_y := \{(0, w, y)\}$ parameterized by y. On each invariant circle there is at least one equilibrium and all non-equilibrium solutions are heteroclinic orbits. Thus, to check the normal hyperbolicity of the invariant torus at $\varepsilon=0$, it suffices to do so at equilibria.

The linearization of F at a point $(0, w, y)$ is

$$\begin{pmatrix} \chi U+(1-\chi)\beta_0 & 0 & 0 \\ \chi\lambda_u w & (\chi_w w+\chi)\lambda+\chi\lambda_w w & \chi_y \lambda w+\chi\lambda_y w \\ 0 & 0 & 0 \end{pmatrix}.$$

If Δ is small enough, then the real parts of the eigenvalues of $\chi U+(1-\chi)\beta_0$ are greater than β_0. Therefore, it suffices to show that $e:=(\chi_w w+\chi)\lambda+\chi\lambda_w w<\beta_0$ at equilibria. The equilibria are determined by $\chi(w, y)\lambda(0, w, y; 0) w=0$. If $\chi(w, y)=0$, then $\chi_w(w, y)=0$ and hence $e=0<\beta_0$. If $\lambda=0$, then $e=\chi\lambda_w w<2|\lambda_w|\Delta<\beta_0$ if Δ is small enough. Finally, if $w=0$, we have $e=\chi\lambda<\beta_0$. Therefore, the invariant torus is normally hyperbolic at $\varepsilon=0$. The standard theory implies that the invariant torus for ε small is also normally hyperbolic and thus there exists an invariant unstable foliation over it. Systems (41) and (42) agree in $\mathscr{B}$, and hence system (41) has a locally invariant foliation over the center manifold $\{u=0, |w|<\Delta, |y|<1\}$. ∎

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