Alexandria Engineering Journal (2016) 55, 87–91

HOSTED BY
ELSEVIER

Alexandria University
Alexandria Engineering Journal
www.elsevier.com/locate/aej
www.sciencedirect.com

ORIGINAL ARTICLE

Numerical simulation of a fractional model of temperature distribution and heat flux in the semi infinite solid

Anupama Choudhary [a,*], Devendra Kumar [b], Jagdev Singh [a]

[a] *Department of Mathematics, Jagan Nath University, Jaipur 303901, Rajasthan, India*
[b] *Department of Mathematics, JECRC University, Jaipur 303905, Rajasthan, India*

Received 4 December 2014; revised 3 January 2016; accepted 16 January 2016
Available online 12 February 2016

KEYWORDS
Fractional differential equation;
Caputo derivative;
Wright function;
Mittag-Leffler function;
Integral transforms;
Heat flux

Abstract In this paper, a fractional model for the computation of temperature and heat flux distribution in a semi-infinite solid is discussed which is subjected to spatially decomposing, time-dependent laser source. The apt dimensionless parameters are identified and the reduced temperature and heat flux as a function of these parameters are presented in a numerical form. Some special cases of practical interest are also discussed. The solution is derived by the application of the Laplace transform, the Fourier sine transform and their derivatives. Also, we developed an alternative solution of it by using the Sumudu transform, the Fourier transform and their derivatives. These results are received in compact and graceful forms in terms of the generalized Mittag-Leffler function, which are suitable for numerical computation.

1. Introduction

In the modeling of many physical and chemical processes and engineering systems fractional differentiation has been widely used. The instances are electrochemistry and electromagnetic waves, diffusion waves, fractal electrical networks, electrical machines, nanotechnology, viscoelastic supplies and systems, quantum evolution of complex systems [1], and heat conduction [2]. Automatic control is also a field in which many applications of fractional differentiations have been anticipated. Recently, it is demonstrated that the real state of a fractional order system is not exactly observable [3]. However, the authors have also have demonstrated that the pseudo state vector of the pseudo state space description can be estimated using a Luenberger like observer. As fractional order derivatives and integrals explain the memory and genetic properties of different substances, the above mentioned new models are more sufficient than the earlier used integer order models [4]. This is the biggest advantage of the fractional order models in comparison with integer order models in which such effects are neglected. A semi-infinite solid is an idealized body that has a single plane surface and extends to infinity in all directions. This idealized body is used to specify that the temperature

* Corresponding author.
E-mail addresses: choudhary.anupama9@gmail.com (A. Choudhary), devendra.maths@gmail.com (D. Kumar), jagdevsinghrathore@gmail.com (J. Singh).
Peer review under responsibility of Faculty of Engineering, Alexandria University.

http://dx.doi.org/10.1016/j.aej.2016.01.007
1110-0168

change in the part of the body in which we are interested is due to the thermal situation on a single surface. The earth, for instance, can be considered as a semi-infinite medium in determining the variation of temperature close to its surface. A thick wall can also be modeled as a semi-infinite medium if we are interested in the variation of temperature in the region near one of the surfaces, and the other surface is extreme to have any impact on the region of interest during the time of surveillance. In view of great importance of fractional differential equations many authors have paid attention for handling linear and nonlinear fractional differential equations [5–7]. In recent years many authors have employed various analytical schemes to investigate nonlinear problems arising in scientific and technological fields such as nonlinear oscillation of a centrifugal governor system [8], dynamic analysis of generalized conservative nonlinear oscillators [9], nonlinear vibrating systems [10], and frequency analysis of strongly nonlinear generalized duffing oscillators [11].

2. Preliminary results

Consider semi-infinite solid initially at temperature T_0. The left face of the solid is suddenly raised to temperature T_s at time zero and defining $\theta = \frac{T-T_0}{T_s-T_0}$. If we suppose, constant thermal conductivity, no internal heat generation and insignificant temperature variation in the y and z directions. The relevant differential equation is given by classical non-homogenous heat equation defined by [12]:

$$\frac{\partial\theta}{\partial t} = C\frac{\partial^2\theta}{\partial x^2}, \tag{1}$$

where K is the thermal diffusivity. Subject to boundary conditions are

$$\left.\begin{array}{l} t = 0;\ \theta = 0 \\ x = 0;\ \theta = 1 \\ x \to \infty;\ \theta \to 0 \end{array}\right\} \tag{2}$$

The following well-known facts are considered to study the temperature distribution and heat flux in the semi infinite solid.

The Laplace transform is defined by [13]

$$L\{f(x)\} = \int_0^\infty e^{-st}f(t)dt; \quad Re(s) > 0. \tag{3}$$

The Fourier sine transform is defined by [14]

$$\overline{F}(n,\ t) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x,\ t)\sin nx\, dx. \tag{4}$$

The error function of x is defined by [15]

$$erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-z^2)dz \tag{5}$$

and the complimentary error function of x is defined as

$$erf_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty \exp(-z^2)dz. \tag{6}$$

A generalization of the Mittag-Leffler function [16,17]

$$E_\alpha(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+1)}, (\alpha \in C, \mathbb{R}(\alpha) > 0) \tag{7}$$

was introduced by [18] in the general form

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(n\alpha+\beta)}, (\alpha,\beta \in C, \mathbb{R}(\alpha) > 0) \tag{8}$$

also derived by [19] in the following integral

$$\int_0^\infty e^{-st}t^{\beta-1}\frac{d^k}{dz^k}E_{\alpha,\beta}(xt^\alpha)dt = \frac{k!s^{\alpha-\beta}}{(s^\alpha - x)^{k+1}}. \tag{9}$$

The fractional derivative of order $\alpha > 0$ is presented by Caputo [20] in the form

$$\begin{aligned} {}_0^cD_x^\alpha f(x) &= \frac{1}{\Gamma(m-\alpha)}\int_0^x \frac{f^{(m)}(\tau)}{(x-\tau)^{\alpha-m+1}}d\tau,\ m-1<\alpha<m \\ &= \frac{d^mf(t)}{dt^m}, if\ \alpha = m; m \in N \end{aligned} \tag{10}$$

where $\frac{d^mf(t)}{dt^m}$ is the mth derivative of order m of the function $f(t)$ with respect to t. The Laplace transform of this derivative is given by [4]

$$L\left[{}_0^cD_x^\alpha f(x); s\right] = s^\alpha\bar{f}(s) - \sum_{k=0}^{m-1}s^{\alpha-k-1}f^{(k)}(0+),\ m-1<\alpha\leq m. \tag{11}$$

A generalization of the Caputo fractional derivative operator Eq. (10) is given by [21], by introducing a right-sided fractional derivative operator of two parameters of order $0 < \alpha < 1$ and $0 \leqslant \beta \leqslant 1$ as

$$_0D_{a+}^{\alpha,\beta}f(x) = I_{a+}^{\beta(1-\alpha)}\frac{d}{dx}\left(I_{a+}^{(1-\beta)(1-\alpha)}f(x)\right). \tag{12}$$

If we put $\beta = 1$, Eq. (12) reduces the Caputo fractional derivative operator assigned from Eq. (10).

Laplace transform formula for this operator [21] is given by

$$L\left[{}_0D_x^{\alpha,\beta} f(x); s\right] = s^\alpha\tilde{f}(s) - s^{\beta(\alpha-1)}I_{0+}^{(1-\beta)(1-\alpha)}f(0+);\ 0<\alpha\leq 1. \tag{13}$$

Sumudu transform formula for this operator [21,22], holds the formula

$$S\left[{}_0D_x^{\alpha,\beta} f(x); s\right] = u^{-\alpha}\tilde{f}(u) - u^{-\beta(\alpha-1)+1}I_{0+}^{(1-\beta)(1-\alpha)}f(0+);\ 0<\alpha\leq 1, \tag{14}$$

where the initial value term

$$I_{0+}^{(1-\beta)(1-\alpha)}f(0+), \tag{15}$$

involves the Riemann–Liouville fractional integral operator of order $(1-\beta)(1-\alpha)$ evaluated in the limit as $x \to 0+$. For more details and properties of this operator see in [23].

The simplest Wright function is defined by [24]

$$\mathcal{W}(\alpha,\beta;z) = \sum_{k=0}^{\infty}\frac{1}{\Gamma(\alpha k+\beta)}\frac{z^k}{k!},\ \text{where } \alpha,\beta,z \in C. \tag{16}$$

Generalized k-Wright function is an exciting generalization of Wright function Eq. (16). Some exciting properties of the generalized k-Wright function are obtained by [25].

Following integral [26] is required for simplification

$$\int_0^\infty n\ sinnx\ E_{\alpha,\alpha+1}\left(-n^2Kt^\alpha\right)dn = \frac{\pi}{2Kt^\alpha}\mathcal{W}\left(-\frac{\alpha}{2},1;\frac{-x}{\sqrt{Kt^\alpha}}\right). \tag{17}$$

The correlation between the Wright function and the complementary Error function is given as

$$\mathcal{W}\left(-\frac{1}{2},1;z\right)=erf_c\left(\frac{z}{2}\right). \tag{18}$$

We will also use the following consequence obtained by Chaurasia and Singh [27] as

$$S^{-1}\left[u^{\gamma-1}\left(1-\omega u^{\beta}\right)^{-\delta}\right]=t^{\gamma-1}E^{\delta}_{\beta,\gamma}\left(\omega t^{\beta}\right). \tag{19}$$

3. Mathematical modeling

Now, we consider a new model in the form of fractional partial differential equation

$$D_t^{\alpha,\beta}\theta(x,t)=C\frac{\partial^2\theta}{\partial x^2}, \tag{20}$$

where $0<\alpha\leqslant 1; 0\langle\beta\leqslant 1; t\rangle 0, x\in\mathbb{R}$ and $\theta=\frac{T-T_0}{T_s-T_0}$. (21)

The significant boundary conditions are as follows

$$\left.\begin{array}{l}\theta(x,0)=0\\ \theta(0,t)=1\\ \lim\limits_{x\to\infty}\theta(x,t)=0\end{array}\right\} \tag{22}$$

If we put $\beta=1$, then Eq. (20) reduces into the Caputo fractional derivative operator, and then we arrive at recently obtained result by [28].

If we consider $\alpha=1$ and $\beta=1$ then Eq. (20) reduces in standard heat Eq. (1).

4. Solution of problem

Applying the Fourier Sine transform on Eq. (20), yields

$$D_t^{\alpha,\beta}\theta_s(n,t)=C\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\partial^2\theta}{\partial x^2}\sin\ nx\ dx. \tag{23}$$

Integrating by parts gives

$$D_t^{\alpha,\beta}\theta_s(n,t)=C\sqrt{\frac{2}{\pi}}\left[\sin nx\frac{\partial\theta}{\partial x}\right]_0^{\infty}-nC\sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{\partial\theta}{\partial x}\cos nxdx$$

or $$D_t^{\alpha,\beta}\theta_s(n,t)=0-nC\sqrt{\frac{2}{\pi}}[\theta\cos nx]_0^{\infty}-n^2C\sqrt{\frac{2}{\pi}}\times\int_0^{\infty}\theta\sin nxdx$$

or $$D_t^{\alpha,\beta}\theta_s(n,t)=-nC\sqrt{\frac{2}{\pi}}(0-1)-n^2C\theta_s(n,t) \tag{24}$$

Now, using Eq. (13), taking Laplace transform of Eq. (24), gives

$$s^{\alpha}\tilde{\theta}_s(n,s)-s^{\beta(\alpha-1)}I_0^{(1-\beta)(1-\alpha)}\tilde{\theta}_s(0+)=nC\sqrt{\frac{2}{\pi}}\cdot\frac{1}{s}-n^2C\tilde{\theta}_s(n,s) \tag{25}$$

This reduces to

$$\tilde{\theta}_s(n,s)=nC\sqrt{\frac{2}{\pi}}\left[\frac{1}{s(s^{\alpha}+n^2C)}\right]. \tag{26}$$

The inverse Laplace transform of Eq. (26) is given by [29]

$$\theta_s(n,t)=nC\sqrt{\frac{2}{\pi}}t^{\alpha}E_{\alpha,\alpha+1}\left(-n^2Ct^{\alpha}\right). \tag{27}$$

Now, taking inverse Fourier sine transform of Eq. (27), we get

$$\begin{aligned}\theta(x,t)&=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\theta_s(n,t)\sin nxdn,\\ &=\sqrt{\frac{2}{\pi}}\int_0^{\infty}nC\sqrt{\frac{2}{\pi}}t^{\alpha}E_{\alpha,\alpha+1}\left(-n^2Ct^{\alpha}\right)\sin nxdn,\\ &=\frac{2}{\pi}Ct^{\alpha}\int_0^{\infty}n\sin nxE_{\alpha,\alpha+1}\left(-n^2Ct^{\alpha}\right)dn.\end{aligned} \tag{28}$$

Using Eq. (17), then Eq. (28) can be reduced in the form of Wright function as

$$\theta(x,t)=\mathcal{W}\left(\frac{-\alpha}{2},1,\frac{-x}{\sqrt{Ct^{\alpha}}}\right). \tag{29}$$

If we assume $\alpha=1$, then Eq. (29) reduces to

$$\theta(x,t)=erf_c\left(\frac{x}{2\sqrt{Ct^{\alpha}}}\right). \tag{30}$$

5. Alternative method for solving fractional partial differential equation

In this section, we solve the fractional partial differential Eq. (20) by an alternative method by using Sumudu transform.

Now, using Eq. (14), taking Sumudu transform of Eq. (24), we get

$$u^{-\alpha}\bar{\theta}_s(n,u)-u^{-\beta(\alpha-1)+1}I_0^{(1-\beta)(1-\alpha)}\theta_s(0+)=nC\sqrt{\frac{2}{\pi}}-n^2C\bar{\theta}_s(n,u). \tag{31}$$

This reduces to

$$\bar{\theta}_s(n,u)=nC\sqrt{\frac{2}{\pi}}\left[\frac{1}{(u^{-\alpha}+n^2C)}\right]. \tag{32}$$

Taking inverse Sumudu transform both sides in Eq. (32) and using Eq. (19), we get

$$\theta_s(n,t)=nC\sqrt{\frac{2}{\pi}}t^{\alpha}E_{\alpha,\alpha+1}\left(-n^2Ct^{\alpha}\right). \tag{33}$$

Now, taking inverse Fourier Sine transform both sides in Eq. (33), yields

$$\begin{aligned}\theta(x,t)&=\sqrt{\frac{2}{\pi}}\int_0^{\infty}\theta_s(n,t)\sin nxdn,\\ &=\sqrt{\frac{2}{\pi}}\int_0^{\infty}nC\sqrt{\frac{2}{\pi}}t^{\alpha}E_{\alpha,\alpha+1}\left(-n^2Ct^{\alpha}\right)\sin nxdn,\\ &=\frac{2}{\pi}Ct^{\alpha}\int_0^{\infty}n\sin nxE_{\alpha,\alpha+1}\left(-n^2Ct^{\alpha}\right)dn.\end{aligned} \tag{34}$$

Using Eq. (17), then Eq. (34) can be reduced in the form of Wright function as

$$\theta(x,t)=\mathcal{W}\left(\frac{-\alpha}{2},1,\frac{-x}{\sqrt{Ct^{\alpha}}}\right). \tag{35}$$

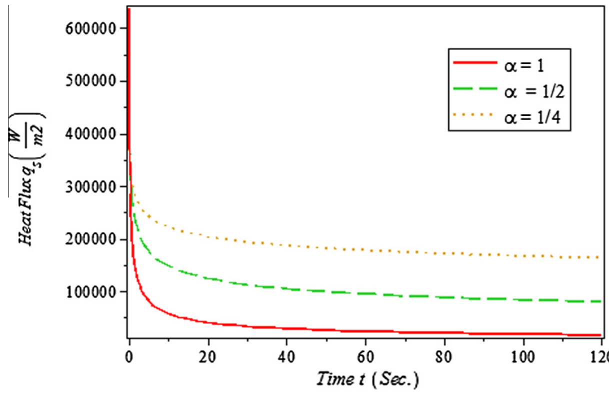

Figure 1 Plot of Heat flux v/s Time at $k = 1.4\ \mathrm{W/mK}$; $C = 0.75 \times 10^{-6}\ \mathrm{m^2/s}$ for different values of α (Eq. (36)).

6. The surface heat flux

The heat flux at the surface is identified as

$$q_s = -k\left(\frac{\partial T}{\partial x}\right)_{x=0}$$
$$= -k\frac{\partial}{\partial x}\left\{T_0 + (T_s - T_0)\mathcal{W}\left(\frac{-\alpha}{2}, 1, \frac{-x}{\sqrt{Ct^\alpha}}\right)\right\}_{x=0}$$

where $\theta = \frac{T-T_0}{T_s-T_0}$,

$$q_s = -k\left\{0 + (T_s - T_0)\frac{\partial}{\partial x}\left(\sum_{r=0}^{\infty}\frac{\left(\frac{-x}{\sqrt{Ct^\alpha}}\right)^r}{\Gamma(-\frac{\alpha r}{2}+1)r!}\right)\right\}_{x=0}$$

$$\text{or}\quad q_s = -k(T_s - T_0)\frac{\partial}{\partial x}\left(1 - \frac{\left(\frac{x}{\sqrt{Ct^\alpha}}\right)}{\Gamma\left(-\frac{\alpha}{2}+1\right)} + \frac{\left(\frac{-x}{\sqrt{Ct^\alpha}}\right)^2}{\Gamma(-\alpha+1)2!} - \dots\right)_{x=0}$$

$$\text{or}\quad q_s = -k(T_s - T_0)\left(\frac{\frac{-1}{\sqrt{Ct^\alpha}}}{\Gamma\left(-\frac{\alpha}{2}+1\right)}\right),$$

$$\text{i.e.}\quad q_s = \frac{k(T_s - T_0)}{\sqrt{Ct^\alpha}\cdot\Gamma\left(-\frac{\alpha}{2}+1\right)}. \tag{36}$$

7. Numerical evaluation

A 15 cm thick concrete firewall has a black silicone paint surface. The wall is approximated as a black body at 1000 K. It will take 2 min for the surface to reach 500 K, if the initial temperature of the wall is 300 K. Find the surface heat flux.

Solution: We have $T_s = 500$ K; $T_0 = 300$ K.

The required concrete properties are

$$k = 1.4\ \mathrm{W/mK};\ C = 0.75 \times 10^{-6}\ \mathrm{m^2/s};\ t = 2\ \mathrm{min} = 120\ \mathrm{s}.$$

In particular for $\alpha = \frac{1}{4} = 0.25$, the heat flux is obtained by using Eq. (36) as follow

$$q_s = \frac{1.4(500 - 300)}{\sqrt{0.75 \times 10^{-6}(120)^{1/4}}\cdot\Gamma\left(\frac{7}{8}\right)}$$
$$= \frac{1.4 \times 200}{0.001575536 \times 1.08965235742}$$
$$= \frac{280}{0.001717}$$
$$= 163075.131\ \mathrm{W/m^2}.$$

The numerical results for the heat flux (36) for different values of t and α at $T_s = 500$ K, $T_0 = 300$ K, $k = 1.4$ W/mK and $C = 0.75 \times 10^{-6}\ \mathrm{m^2/s}$ are shown in Fig. 1. It can be observed from Fig. 1 that heat flux (q_s) decreases with increase in time t and increases with the increase in α but afterward its nature is opposite.

8. Conclusions

In this paper, we have presented a solution of a fractional partial differential equation. The solution has been developed in terms of the generalized Mittag-Leffler and Wright function form with the help of Fourier transform, Laplace transform and its inverse after deriving the related formulae for fractional integrals, and derivatives. The modifications to generalization of the Caputo fractional derivative operator proposals carried out here by developing and discussing an alternate mathematical model with the help of Fourier transform, the Sumudu transform and their derivatives to psychoanalyze the behavior of temperature distribution and heat flux in semi infinite solid represent just an example of what needs to be done to increase our general understanding and use of these concepts. The existence and uniqueness of solution have been discussed in both strong and weak senses. Furthermore, a numerical method based on the boundary conditions has been devised in order to obtain constant and exact numerical solutions.

References

[1] J. Sabatier, O.P. Agrawal, J.A. Tenreiro Machado (Eds.), Advances in Fractional Calculus, Theoretical Developments and Applications in Physics and Engineering, Springer, Dordrecht, The Netherlands, 2007, p. xiv + 552.

[2] J. Sabatier, P. Melchior, A. Oustaloup, A testing bench for fractional order systems education, J. Euro. Syst. Autom. 42 (6–8) (2008) 839–861.

[3] J. Sabatier, M. Merveillaut, L. Fenetau, A. Oustaloup, On observability of fractional order systems. In: Proceedings of the ASME International Design Engineering Technical Conferences & Computers and Information in Engineering Conference (IDETC/CIE '09), 2009, San Diego, Calif, USA.

[4] I. Podulbny, Fractional Differential Equations, Academic Press, New York, 1999.

[5] S. Kumar, A numerical study for solution of time fractional nonlinear shallow-water equation in oceans, Z. Naturforsch. 68a (2013) 1–7.

[6] S. Kumar, H. Kocak, A. Yildirim, A fractional model of gas dynamics by using Laplace transform, Z. Naturforsch. 67a (2012) 389–396.

[7] K. Vishal, S. Kumar, S. Das, Application of homotopy analysis method for fractional Swift Hohenberg equation – revisited, Appl. Math. Model. 36 (8) (2012) 3630–3637.

[8] D. Younesian, H. Askari, Z. Saadatnia, M.K. Yazdi, Periodic solutions for nonlinear oscillation of a centrifugal governor

system using the He's frequency–amplitude formulation and He's energy balance method, Nonlinear Sci. Lett. A 2 (2011) 143–148.
[9] Y. Khan, M.K. Yazdi, H. Askari, Z. Saadatnia, Dynamic analysis of generalized conservative nonlinear oscillators via frequency amplitude formulation, Arab. J. Sci. Eng. 38 (1) (2013) 175–179.
[10] H. Askari, Z. Saadatnia, A. Yildirim, M.K. Yazdi, Y. Khan, Application of higher order hamiltonian approach to nonlinear vibrating systems, J. Theor. Appl. Mech. 51 (20) (2013) 287–296.
[11] D. Younesian, H. Askari, Z. Saadatnia, M.K. Yazdi, Frequency analysis of strongly nonlinear generalized Duffing oscillators using He's frequency–amplitude formulation and He's energy balance method, Comput. Math. Appl. 59 (9) (2010) 3222–3228.
[12] A.F. Mills, V. Ganesan, Heat Transfer, Dorling Kindersley (India) Pvt. Ltd., 2009.
[13] I.N. Sneddon, Fourier Transform, MacGraw-Hill, New York, 1951.
[14] L. Debnath, Integral Transforms and Their Applications, CRC Press, New York–London–Tokyo, 1995.
[15] E.D. Rainville, Special Functions, The Macmillan Company, New York, 1960.
[16] G.M. Mittage-Leffler, Sur la nouvelle function $E\alpha(x)$, C.R. Acad. Sci., Paris, (Ser.II) 137 (1903) 554–558.
[17] G.M. Mittag-Leffler, Sur la representatin analytique d'une branche uniforme d'une function monogene, Acta Math. 29 (1905) 101–181.
[18] A. Wiman, Uber den fundamentalsatz in der Theorie der Functionen $E\alpha(x)$, Acta Math. 29 (1905) 191–201.
[19] A.K. Shukla, J.C. Prajapati, on a generalization of Mittag-Leffler function and its properties, J. Math. Anal. Appl. 336 (2007) 797–811.
[20] M. Caputo, Linear models of dissipation whose Q is almost frequency independent – II, Geophys. J. Roy. Astron. Soc. 13 (1967) 529–539, Reprinted in: Fract. Calc. Appl. Anal. 11(1) (2008) 3–14.
[21] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore–New Jersey–Hong Kong, 2000, pp. 87–130.
[22] F.B.M. Belgacem, A.A. Karaballi, S.L. Kalla, Analytical investigations of the Sumudu transform and applications to integral production equations, Math. Probl. Eng. 3 (2003) 103–118.
[23] Z. Tomovski, R. Hilfer, H.M. Srivastava, Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions, Integr. Transf. Spec. Funct. 21 (11) (2010) 797–814.
[24] H.M. Srivastava, R.K. Parmar, P. Chopra, A class of extended fractional derivative operators and associated generating relations involving hypergeometric functions, Axioms 1 (3) (2012) 238–258.
[25] K.S. Gehlot, J.C. Prajapati, Fractional calculus of generalized k-Wright function, J. Fract. Calculus Appl. 4 (2) (2013) 283–289.
[26] El-Shahed Moustafa, Salem Ahmed, Decay of vortex velocity and diffusion of the temperature for fractional viscoelastic fluid through porous medium, Int. J. Comm. Heat Mass Transfer 33 (2006) 240–248.
[27] V.B.L. Chaurasia, J. Singh, Application of Sumudu transform in fractional kinetic equations, Gen. Math. Notes 2 (1) (2011) 86–95.
[28] J.C. Prajapati, K.B. Kachhia, fractional modeling of temperature distribution and heat flux in the semi infinite solid, J. Fract. Calculus Appl. 5 (2) (2014) 38–43.
[29] J.C. Prajapati, A.D. Patel, K.N. Pathak, A.K. Shukla, Fractional calculus approach in the study of instability phenomenon in fluid dynamics, Palestine J. Math. 1 (2) (2012) 95–103.