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Axisymmetric nuclei of strain and their applications for multilayered solids

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Abstract

In this paper, the effect of several axisymmetric elastic singularities (i.e., point forces, double forces, sum of two double forces and centers of dilatation) on the elastic response of a multilayered solid is investigated. The boundary conditions in an infinite solid at the plane passing through the singularity are derived first using Papkovich–Neuber harmonic functions. Then, a Green's function solution for multilayered solids is obtained by solving a set of simultaneous linear algebraic equations using both the boundary conditions for the singularity and the layer interfaces. Finally, the elastic solutions in a single layer on an infinite substrate due to point defects and infinitesimal prismatic dislocation loops are presented to illustrate the application of these Green's function solutions.
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Keywords: Elastic singularity; Point defects; Dislocation loops; Multilayered materials; Thin films

1. Introduction

The elastic solutions of elastic singularities (nuclei of strain) such as double forces, centers of dilatation, and centers of rotation, etc., in a solid of infinite extent may be obtained from Lord Kelvin's point force solution (1848) by methods of synthesis and superposition. The number of elastic singularities that may be obtained by this method is virtually unlimited and leads to solutions for a number of problems of practical importance. The solution for a point force acting in the interior of a semi-infinite solid was first solved by Mindlin (1936). By using Galerkin vector stress functions, solutions for the complete set of 40 nuclei of strain that have physical significance have been presented by Mindlin and Cheng (1950a) for a half-space, and by Yu and Sanday (1991a) for two joined half-spaces (bimaterials). By using a matrix representation of stresses and displacements, and solving the matrix equations, Vijayakumar and Cormack (1987a,b) obtained the stresses and displacements for different nuclei of strain in bimaterials. By superposition of the Mindlin's solution, infinite series solutions are found for a point force (Ling, 1992) and a center of dilatation (Yu and Sanday, 1992) in a plate.

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A general solution for two- and three-layer elastic half-spaces (in integral form) was first given by Burmister (1945a,b,c). Chen (1971) presented a general formulation using Fourier integrals for a three-layer system. By using a Fourier transform, Benitez and Rosakis (1987) obtained the point force solutions for a single and multilayered media. Yu and Sanday (1993) presented a solution for the center of dilatation in three-layer materials using the mirror image method. Since then, extensive study has been performed on the boundary value problems in multilayered elastic materials (e.g., Yue, 1996; Pan, 1997; Ai et al., 2002, additional references can be found in these papers). In previous investigations, a number of methods including propagator matrix, forward transfer matrix, backward transfer matrix, stiffness matrix, flexibility matrix, thin layer method, finite layer method, and Sneddon and Muki's method have been proposed. However, most of these investigations are on problems for point forces or surface loading.

In this paper, a general method for solving problems involving axisymmetric elastic singularities in multilayered materials is presented. The solution involves using boundary conditions (derived from the prior work cited above) to solve these problems. In Section 2, the boundary conditions at the plane passing through the singularities in an infinite solid will be derived from known solutions. The applicable singularities discussed here include point force, double forces, sum of two double forces and center of dilatation. In Section 3, additional boundary conditions due to layer interfaces will be used to obtain the elastic response in a multilayered system due to these singularities. Applications of this method will be demonstrated in the following two sections for a single layer, perfectly bonded to a homogeneous, semi-infinite substrate. The problems of practical importance reported in Sections 4 and 5 involve point defects and infinitesimal prismatic dislocation loops, respectively. The point defect is approximated by a center of dilatation. The elastic response due to a prismatic dislocation loop is the linear superposition of the double forces in the z-direction and the center of dilatation.

2. The fundamental equations

To establish the method, we will consider an infinite homogeneous elastic solid (Fig. 1a) with shear modulus μ and Poisson's ratio v with an elastic singularity at the object point $(0, c)$. Since the problems are axisymmetric, a cylindrical coordinate (r, θ, z) will be used. This allows the simplification of most expressions since the solutions are independent of θ (i.e., $(r, z) = (r, \theta, z)$). The Papkovich–Neuber functions $\varphi(r, z)$ and $\psi(r, z)$ are used to formulate the problems. The displacements and stresses are

$$\begin{aligned}
&2\mu u_r = -z\psi_{,r} - \varphi_{,r}, \quad 2\mu u_z = \kappa\psi - z\psi_{,z} - \varphi_{,z},\\
&\sigma_r = 2v\varphi_{,z} - z\varphi_{,rr} - \psi_{,rr}, \quad \sigma_\theta = 2v\varphi_{,z} - \frac{z}{r}\varphi_{,r} - \frac{1}{r}\psi_{,r},\\
&\sigma_z = 2(1-v)\psi_{,z} - z\psi_{,zz} - \varphi_{,zz}, \quad \sigma_{rz} = (1-2v)\psi_{,r} - z\psi_{,rz} - \varphi_{,rz},
\end{aligned} \tag{2.1}$$

where $\kappa = 3 - 4v$. The suffixes following a comma denote differentiation with respect to the indicated cylindrical coordinates, e.g., $\varphi_{,rz} = \partial^2\varphi/\partial r\partial z$. The harmonic functions $\varphi(r, z)$ and $\psi(r, z)$ are expressed in terms of a set of unknown functions $A_i(\xi)$, $(i = 1, 2, 3, 4)$ as given in the following Hankel integrals

$$\psi_- = \int_0^\infty A_1 e^{\xi z} J_0(r\xi)d\xi, \quad \varphi_- = \int_0^\infty A_2 e^{\xi z}\xi^{-1} J_0(r\xi)d\xi, \tag{2.2}$$

for $z \leqslant c$, and

$$\psi_+ = \int_0^\infty A_3 e^{-\xi z} J_0(r\xi)d\xi, \quad \varphi_+ = \int_0^\infty A_4 e^{-\xi z}\xi^{-1} J_0(r\xi)d\xi, \tag{2.3}$$

for $z \geqslant c$, where $J_n(r\xi)$ is the Bessel function of the first kind of order n. Substituting (2.2) and (2.3) into (2.1), one obtains

$$\begin{aligned}
&u_{r-} = \int_0^\infty \hat{u}_{r-}(z,\xi) J_1(r\xi)d\xi, \quad u_{z-} = \int_0^\infty \hat{u}_{z-}(z,\xi) J_0(r\xi)d\xi,\\
&\sigma_{z-} = \int_0^\infty \hat{\sigma}_{z-}(z,\xi) J_0(r\xi)\xi d\xi, \quad \sigma_{rz-} = \int_0^\infty \hat{\sigma}_{rz-}(z,\xi) J_1(r\xi)\xi d\xi,
\end{aligned} \tag{2.4}$$

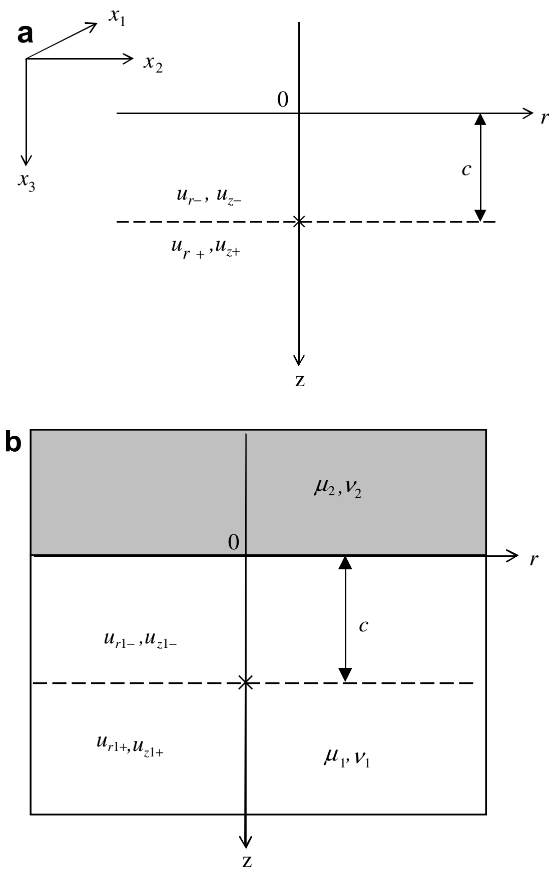

Fig. 1. (a) An infinite solid with an elastic singularity at the point $(0, c)$. (b) Two perfectly bonded semi-infinite solids with an elastic singularity at the point $(0, c)$.

for $z \leqslant c$, and

$$\begin{aligned} u_{r+} &= \int_0^\infty \hat{u}_{r+}(z,\xi) J_1(r\xi) \mathrm{d}\xi, \quad u_{z+} = \int_0^\infty \hat{u}_{z+}(z,\xi) J_0(r\xi) \mathrm{d}\xi, \\ \sigma_{z+} &= \int_0^\infty \hat{\sigma}_{z+}(z,\xi) J_0(r\xi)\xi \mathrm{d}\xi, \quad \sigma_{rz+} = \int_0^\infty \hat{\sigma}_{rz+}(z,\xi) J_1(r\xi)\xi \mathrm{d}\xi, \end{aligned} \tag{2.5}$$

for $z \geqslant c$, where

$$\begin{aligned} \hat{u}_{r-} &= \frac{1}{2\mu}(z\xi A_1 + A_2)\mathrm{e}^{z\xi}, \\ \hat{\sigma}_{z-} &= \{[2(1-v) - z\xi]A_1 - A_2\}\mathrm{e}^{z\xi}, \quad \hat{\sigma}_{rz-} = \{[z\xi - (1-2v)]A_1 + A_3\}\mathrm{e}^{z\xi}, \\ \hat{u}_{r+} &= \frac{1}{2\mu}(z\xi A_3 + A_4)\mathrm{e}^{-z\xi}, \quad \hat{u}_{z+} = \frac{1}{2\mu}[(\kappa + z\xi)A_3 + A_4]\mathrm{e}^{-z\xi}, \\ \hat{\sigma}_{z+} &= -\{[2(1-v) + z\xi]A_3 + A_4\}\mathrm{e}^{-z\xi}, \quad \hat{\sigma}_{rz+} = -\{[(1-2v) + z\xi]A_3 + A_4\}\mathrm{e}^{-z\xi}. \end{aligned} \tag{2.6}$$

To derive the boundary conditions at the plane $z = c$ for different singularities, the following existing solutions will be used. The Galerkin vectors (g_1, g_2, g_3) expressed in Cartesian coordinates (x_1, x_2, x_3) for different singularities are (Mindlin and Cheng, 1950a; Yu and Sanday, 1991a)

$$g_1 = g_2 = 0, \quad g_3 = R, \tag{2.7}$$

for the point force in the z-direction,

$$g_1 = g_2 = 0, \quad g_3 = R_{,z}, \tag{2.8}$$

for the double forces in the z-direction,

$$g_1 = R_{,1}, \quad g_2 = R_{,2}, \quad g_3 = 0, \tag{2.9}$$

for sum of the double forces in the x_1-direction and x_2-direction where $R_{,i} = \partial R/\partial x_i$, $i = 1, 2$, and

$$g_1 = g_2 = 0, \quad g_3 = \log[R - (z - c)], \tag{2.10}$$

for a center of dilatation where

$$R = \sqrt{r^2 + (z - c)^2}.$$

The relationship between the displacement and Galerkin vectors (Mindlin, 1936) is

$$u_i = \frac{1}{2\mu}[2(1 - v)g_{i,jj} - g_{k,ki}], \tag{2.11}$$

where the repeated suffix indicates summation over the values 1, 2, 3. The stress components can be obtained from (2.11) using Hooke's law. Coefficients A_i in (2.6) are obtained by substituting (2.7)–(2.10) into (2.11), and using the integral form (Gradshteyn and Ryzhik, 1980)

$$\begin{aligned} \frac{1}{R} &= \int_0^\infty e^{(z-c)\xi} J_0(\xi r) d\xi, \quad z \leqslant c, \\ &= \int_0^\infty e^{-(z-c)\xi} J_0(\xi r) d\xi, \quad z \geqslant c. \end{aligned} \tag{2.12}$$

Substituting the coefficients A_i just obtained into (2.4)–(2.6), one has the following boundary conditions at the plane $z = c$.

Case A: point force in the z-direction

$$\begin{aligned} &\hat{u}_{r-}(c, \xi) = \hat{u}_{r+}(c, \xi), \quad \hat{u}_{z-}(c, \xi) = \hat{u}_{z+}(c, \xi), \\ &\hat{\sigma}_{z-}(c, \xi) - \hat{\sigma}_{z+}(c, \xi) = 4(1 - v), \quad \hat{\sigma}_{rz-}(c, \xi) = \hat{\sigma}_{rz+}(c, \xi), \end{aligned} \tag{2.13}$$

Case B: double forces in the z-direction

$$\begin{aligned} &\hat{u}_{r-}(c, \xi) = \hat{u}_{r+}(c, \xi), \quad \hat{u}_{z-}(c, \xi) - \hat{u}_{z+}(c, \xi) = \frac{2(1 - 2v)}{\mu}\xi, \\ &\hat{\sigma}_{z-}(c, \xi) = \hat{\sigma}_{z+}(c, \xi), \quad \hat{\sigma}_{rz-}(c, \xi) - \hat{\sigma}_{rz+}(c, \xi) = 4v\xi, \end{aligned} \tag{2.14}$$

Case C: double forces in the r-direction (sum of the double forces in the x_1- and x_2-directions)

$$\begin{aligned} &\hat{u}_{r-}(c, \xi) = \hat{u}_{r+}(c, \xi), \quad \hat{u}_{z-}(c, \xi) = \hat{u}_{z+}(c, \xi), \\ &\hat{\sigma}_{z-}(c, \xi) = \hat{\sigma}_{z+}(c, \xi), \quad \hat{\sigma}_{rz-}(c, \xi) - \hat{\sigma}_{rz+}(c, \xi) = -4(1 - v)\xi, \end{aligned} \tag{2.15}$$

Case D: center of dilatation

$$\begin{aligned} &\hat{u}_{r-}(c, \xi) = \hat{u}_{r+}(c, \xi), \quad \hat{u}_{z-}(c, \xi) - \hat{u}_{z+}(c, \xi) = \frac{1}{\mu}\xi, \\ &\hat{\sigma}_{z-}(c, \xi) = \hat{\sigma}_{z+}(c, \xi), \quad \hat{\sigma}_{rz-}(c, \xi) - \hat{\sigma}_{rz+}(c, \xi) = -2\xi. \end{aligned} \tag{2.16}$$

The relationships between the displacements due to the center of dilatation, $u_{ij}(g_{cd})$, and the sum of three double forces in the x_1-, x_2-, and x_3-directions, u_{ij} (g_{mm}), are

$$u_{ij}(g_{mm}) = 2(1 - 2v)u_{ij}(g_{cd}). \tag{2.17}$$

Some of these boundary conditions could be obtained using a different approach. For Case A, the displacement components u_r and u_z, and the shear stress σ_{rz} are continuous at the plane $z = c$. The normal stress σ_z is discontinuous at the point where the point force is applied, and the surface integral of the normal traction $(\sigma_{z-} - \sigma_{z+})$ should be equal to the point forcef_z, i.e.,

$$2\pi \int_0^\infty (\sigma_{z-} - \sigma_{z+}) r\, \mathrm{d}r = f_z. \tag{2.18}$$

Eq. (2.18) can be expressed as (Pekeris, 1955)

$$\sigma_{z-}(r,c) - \sigma_{z+}(r,c) = \lim_{\varepsilon \to 0} f(\varepsilon, r), \tag{2.19}$$

where

$$f(\varepsilon, r) = 4(1-v) \int_0^\infty \mathrm{e}^{-\varepsilon\xi} J_0(r\xi)\xi \mathrm{d}\xi, \tag{2.20}$$

and ε is the infinitesimal distance in the z-direction between points (r, z_+) and (r, z_-). Without loss of generality and to simplify the expression, the magnitude of f_z is assumed to be $8\pi(1-v)$. Substituting (2.20) into (2.19), one obtains (2.13). The same result has been obtained by Sneddon (1951) with a different approach. For Case C, the displacement components u_r and u_z, and the normal stress σ_z are continuous at the plane $z = c$. The shear stress σ_{rz} is discontinuous, and the discontinuity can be expressed as

$$\sigma_{rz-}(r,c) - \sigma_{rz+}(r,c) = \frac{\partial}{\partial r} f(0, r), \tag{2.21}$$

which gives the boundary conditions (2.15). By using the relationship (2.17), the sum of (2.14) and (2.15) gives the boundary conditions (2.16) for Case D.

It can be easily shown that the elastic response due to elastic singularities in bimaterials can be derived using the boundary conditions (2.13)–(2.16). As shown in Fig. 1b, the two half-spaces are perfectly bonded together. The shear modulus and Poisson's ratios for half-space N are μ_N and v_N $(N = 1, 2)$, respectively. The singularity is at point $(0, c)$ in half-space 1. The boundary conditions on plane $z = c$ for four axisymmetric nuclei of strain are given in (2.13)–(2.16) where u_{r-}, u_{z-}, σ_{z-}, σ_{rz-}, u_{r+}, u_{z+}, σ_{z+} and σ_{rz+} are replaced by u_{r1-}, u_{z1-}, σ_{z1-}, σ_{rz1-}, u_{r1+}, u_{z1+}, σ_{z1+} and σ_{rz1+}, respectively. The boundary conditions at the interface $z = 0$ are

$$\begin{aligned} &u_{r1-}(r,0) = u_{r2}(r,0), \quad u_{z1-}(r,0) = u_{z2}(r,0), \\ &\sigma_{z1-}(r,0) = \sigma_{z2}(r,0), \quad \sigma_{rz1-}(r,0) = \sigma_{rz2}(r,0). \end{aligned} \tag{2.22}$$

The harmonic functions in (2.1) are

$$\psi_{1-} = \int_0^\infty (A_{11}\mathrm{e}^{\xi z} + A_{12}\mathrm{e}^{-\xi z}) J_0(r\xi)\mathrm{d}\xi, \quad \varphi_{1-} = \int_0^\infty (A_{13}\mathrm{e}^{\xi z} + A_{14}\mathrm{e}^{-\xi z})\xi^{-1} J_0(r\xi)\mathrm{d}\xi, \tag{2.23}$$

for $0 \leqslant z \leqslant c$ in half-space 1,

$$\psi_{1+} = \int_0^\infty A_{15}\mathrm{e}^{-\xi z} J_0(r\xi)\mathrm{d}\xi, \quad \varphi_{1+} = \int_0^\infty A_{16}\mathrm{e}^{-\xi z}\xi^{-1} J_0(r\xi)\mathrm{d}\xi, \tag{2.24}$$

for $z \geqslant c$ in half-space 1, and

$$\psi_2 = \int_0^\infty A_{21}\mathrm{e}^{\xi z} J_0(r\xi)\mathrm{d}\xi, \quad \varphi_2 = \int_0^\infty A_{22}\mathrm{e}^{\xi z}\xi^{-1} J_0(r\xi)\mathrm{d}\xi, \tag{2.25}$$

for $z \leqslant 0$ in half-space 2. The functions $A_{ji}(\xi)$ are obtained by solving the simultaneous equations obtained by substituting (2.23)–(2.25) into (2.1), (2.13)–(2.16) and (2.22). The elastic solutions obtained by substituting these $A_{ji}(\xi)$ into (2.23)–(2.25) are the same as those obtained using the Galerkin vectors method (Yu and Sanday, 1991a). Using the center of dilatation as an example, the coefficients $A_{ji}(\xi)$ obtained by solving the simultaneous equations are

$$\begin{aligned}
&A_{11} = A_{21} = 0, \quad A_{12} = A_{15} = 2(\mu_1 - \mu_2)\beta_0 \xi \mathrm{e}^{-c\xi}, \\
&A_{13} = -\xi \mathrm{e}^{-c\xi}, \quad A_{14} = -\kappa_1(\mu_1 - \mu_2)\beta_0 \xi \mathrm{e}^{-c\xi}, \\
&A_{16} = -\xi \mathrm{e}^{c\xi} - \kappa_1(\mu_1 - \mu_2)\beta_0 \xi \mathrm{e}^{-c\xi}, \quad A_{22} = -4(1 - v_1)\mu_2 \beta_0 \xi \mathrm{e}^{-c\xi},
\end{aligned} \tag{2.26}$$

where

$$\beta_0 = \frac{1}{\mu_1 + \kappa_1 \mu_2}, \quad \kappa_1 = 3 - 4v_1. \tag{2.27}$$

Substituting (2.26) into (2.23)–(2.25) and (2.2), one has

$$\begin{aligned}
u_{r1+} &= -\frac{1}{2\mu_1} \int_0^\infty [\mathrm{e}^{c\xi} + (\mu_1 - \mu_2)\beta_0(\kappa_1 - 2z\xi)\mathrm{e}^{-c\xi}]\mathrm{e}^{-z\xi} J_1(\xi r)\xi \mathrm{d}\xi, \\
u_{z1+} &= -\frac{1}{2\mu_1} \int_0^\infty [\mathrm{e}^{c\xi} - (\mu_1 - \mu_2)\beta_0(\kappa_1 + 2z\xi)\mathrm{e}^{-c\xi}]\mathrm{e}^{-z\xi} J_0(\xi r)\xi \mathrm{d}\xi,
\end{aligned} \tag{2.28}$$

for $z \geqslant c$,

$$\begin{aligned}
u_{r1-} &= -\frac{1}{2\mu_1} \int_0^\infty [\mathrm{e}^{z\xi} + (\mu_1 - \mu_2)\beta_0(\kappa_1 - 2z\xi)\mathrm{e}^{-z\xi}]\mathrm{e}^{-c\xi} J_1(\xi r)\xi \mathrm{d}\xi, \\
u_{z1-} &= \frac{1}{2\mu_1} \int_0^\infty [\mathrm{e}^{z\xi} + (\mu_1 - \mu_2)\beta_0(\kappa_1 + 2z\xi)\mathrm{e}^{-z\xi}]\mathrm{e}^{-c\xi} J_0(\xi r)\xi \mathrm{d}\xi,
\end{aligned} \tag{2.29}$$

for $0 \leqslant z \leqslant c$, and

$$\begin{aligned}
u_{r2} &= -2(1 - v_1)\beta_0 \int_0^\infty \mathrm{e}^{(z-c)\xi} J_1(\xi r)\xi \mathrm{d}\xi, \\
u_{z2} &= 2(1 - v_1)\beta_0 \int_0^\infty \mathrm{e}^{(z-c)\xi} J_0(\xi r)\xi \mathrm{d}\xi,
\end{aligned} \tag{2.30}$$

for $z \leqslant 0$.

The Galerkin vectors for a center of dilatation in a bimaterial with perfect bonding are (Yu and Sanday, 1991a)

$$g_1^I = g_2^I = 0, \quad g_3^I = -\phi^I + (\mu_1 - \mu_2)\beta_0[(1 - 4v)\phi^{II} + 2z\varphi^{II}], \tag{2.31}$$

and

$$g_1^{II} = g_2^{II} = 0, \quad g_3^{II} = -4(1 - v)\mu_2 \beta_0 \phi^I, \tag{2.32}$$

for half-space 1 and 2, respectively, where

$$\begin{aligned}
&\phi^I = \log[R^I - (z - c)], \quad \phi^{II} = \log[R^{II} + (z + c)], \quad \varphi^{II} = \frac{1}{R^{II}}, \\
&R^I = \sqrt{r^2 + (z - c)^2}, \quad R^{II} = \sqrt{r^2 + (z + c)^2}.
\end{aligned} \tag{2.33}$$

The relationship between the displacement and Galerkin vectors is

$$u_i^L = \frac{1}{2\mu}[2(1 - v)g_{i,jj}^L - g_{k,ki}^L], \quad L = I, II. \tag{2.34}$$

By substituting (2.31)–(2.33) into (2.34) and using the relationship (2.12), one has the same results as given in (2.28)–(2.30).

3. Multilayered solids

The method presented in the previous section will now be extended to multilayered solids. The problems treated here are those of an elastic multilayered solid consisting of N homogeneous and isotropic layers sandwiched between two semi-infinite solids as shown in Fig. 2. The shear modulus, Poisson's ratio and thickness of the jth layer are μ_j, ν_j and h_j, respectively. The elastic singularity is at the point $(0, c)$ in the nth layer where the origin of the coordinate system is set on the top surface of this layer. Using a similar approach to the infinite solids, the Papkovich–Neuber functions $\varphi_j(r,z)$ and $\psi_j(r,z)$ are used. The displacement and stress components are

$$\begin{aligned}
&2\mu_j u_{rj} = -z\psi_{j,r} - \varphi_{j,r}, \quad 2\mu_j u_{zj} = k_j\psi_j - z\psi_{j,z} - \varphi_{j,z},\\
&\sigma_{rj} = 2\nu_j\varphi_{j,z} - z\varphi_{j,rr} - \psi_{j,rr}, \quad \sigma_{\theta j} = 2\nu\varphi_{j,z} - z\frac{1}{r}\varphi_{j,r} - \frac{1}{r}\psi_{j,r},\\
&\sigma_{zj} = 2(1-\nu_j)\psi_{j,z} - z\psi_{j,zz} - \varphi_{j,zz}, \quad \sigma_{rzj} = (1-2\nu_j)\psi_{j,r} - z\psi_{j,rz} - \varphi_{j,rz},
\end{aligned} \tag{3.1}$$

where

$$j = 1, 2, \ldots, N, \text{ and } \kappa_j = 3 - 4\nu_j. \tag{3.2}$$

The harmonic functions $\varphi_j(r,z)$, $\psi_j(r,z)$ are expressed in terms of the unknown functions $A_{ji}(\xi)$ as

$$\begin{aligned}
\psi_j &= \int_0^\infty (A_{j1}e^{\xi z} + A_{j2}e^{-\xi z})J_0(r\xi)d\xi,\\
\varphi_j &= \int_0^\infty (A_{j3}e^{\xi z} + A_{j4}e^{-\xi z})\xi^{-1}J_0(r\xi)d\xi,
\end{aligned} \tag{3.3}$$

for $j = 1, 2, \ldots, n-1, n+1, \ldots, N$. To satisfy the boundary condition that the displacements and their derivatives (in the two semi-infinite solids 1 and N ($h_1 = h_N = \infty$)) vanish for $z \to \pm\infty$, one has

$$A_{12} = A_{14} = A_{N1} = A_{N3} = 0. \tag{3.4}$$

The functions $\varphi_j(r,z)$ and $\psi_j(r,z)$ for $j = n$ are

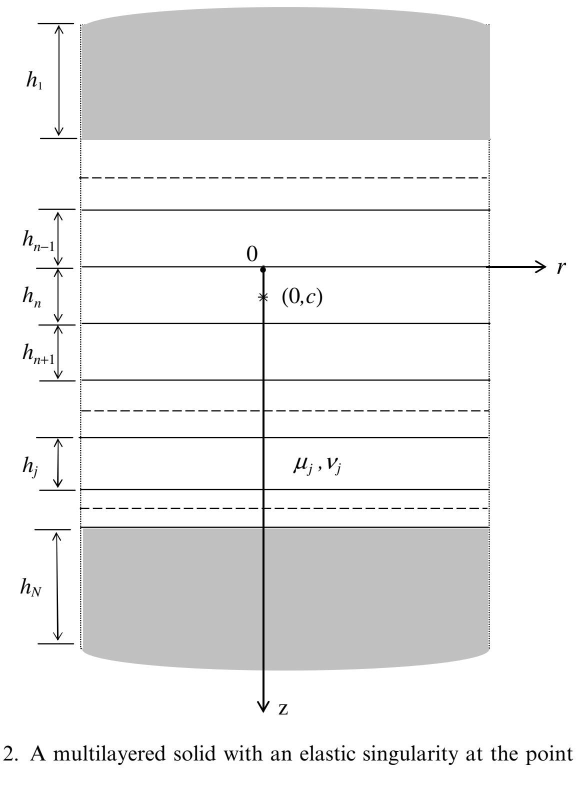

Fig. 2. A multilayered solid with an elastic singularity at the point $(0, c)$.

$$\begin{aligned}\psi_{n-} &= \int_0^\infty (A_{n1-}\mathrm{e}^{\xi z} + A_{n2-}\mathrm{e}^{-\xi z})J_0(r\xi)\mathrm{d}\xi,\\ \varphi_{n-} &= \int_0^\infty (A_{n3-}\mathrm{e}^{\xi z} + A_{n4-}\mathrm{e}^{-\xi z})\xi^{-1}J_0(r\xi)\mathrm{d}\xi,\end{aligned} \tag{3.5}$$

for $0 \leqslant z \leqslant c$, and

$$\begin{aligned}\psi_{n+} &= \int_0^\infty (A_{n1+}\mathrm{e}^{\xi z} + A_{n2+}\mathrm{e}^{-\xi z})J_0(r\xi)\mathrm{d}\xi,\\ \varphi_{n+} &= \int_0^\infty (A_{n3+}\mathrm{e}^{\xi z} + A_{n4+}\mathrm{e}^{-\xi z})\xi^{-1}J_0(r\xi)\mathrm{d}\xi,\end{aligned} \tag{3.6}$$

for $c \leqslant z \leqslant h_n$. Substituting (3.3) into (3.1), one has for $j = 1, 2, \ldots, n-1, n+1, \ldots, N$,

$$\begin{aligned}u_{rj} &= \int_0^\infty \hat{u}_{rj}(z,\xi)J_1(r\xi)\mathrm{d}\xi, \quad u_{zj} = \int_0^\infty \hat{u}_{zj}(z,\xi)J_0(r\xi)\mathrm{d}\xi,\\ \sigma_{zj} &= \int_0^\infty \hat{\sigma}_{zj}(z,\xi)J_0(r\xi)\xi\mathrm{d}\xi, \quad \sigma_{rj} = \int_0^\infty \hat{\sigma}_{rj}(z,\xi)J_1(r\xi)\xi\mathrm{d}\xi,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}\hat{u}_{rj} &= \frac{1}{2\mu_j}[(z\xi A_{j1} + A_{j3})\mathrm{e}^{\xi z} + (z\xi A_{j2} + A_{j4})\mathrm{e}^{-\xi z}],\\ \hat{u}_{zj} &= \frac{1}{2\mu_j}\{[(\kappa_j - z\xi)A_{j1} - A_{j3}]\mathrm{e}^{\xi z} + [(\kappa_j + z\xi)A_{j2} + A_{j4}]\mathrm{e}^{-\xi z}\},\\ \hat{\sigma}_{zj} &= \{[2(1-v_j) - z\xi]A_{j1} - A_{j3}\}\mathrm{e}^{\xi z} - \{[2(1-v_j) + z\xi]A_{j2} + A_{j4}\}\mathrm{e}^{-\xi z},\\ \hat{\sigma}_{rzj} &= \{[z\xi - (1-2v_j)]A_{j1} + A_{j3}\}\mathrm{e}^{\xi z} - \{[z\xi + (1-2v_j)]A_{j2} + A_{j4}\}\mathrm{e}^{-\xi z}.\end{aligned} \tag{3.8}$$

For $j = n$, from (3.1), (3.5) and (3.6), one has

$$\begin{aligned}u_{rn\pm} &= \int_0^\infty \hat{u}_{rn\pm}(z,\xi)J_1(r\xi)\mathrm{d}\xi, \quad u_{zn\pm} = \int_0^\infty \hat{u}_{zn\pm}(z,\xi)J_0(r\xi)\mathrm{d}\xi,\\ \sigma_{zn\pm} &= \int_0^\infty \hat{\sigma}_{zn\pm}(z,\xi)J_0(r\xi)\xi\mathrm{d}\xi, \quad \sigma_{rn\pm} = \int_0^\infty \hat{\sigma}_{rn\pm}(z,\xi)J_1(r\xi)\xi\mathrm{d}\xi,\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}\hat{u}_{rn\pm} &= \frac{1}{2\mu_n}[(z\xi A_{n1\pm} + A_{n3\pm})\mathrm{e}^{\xi z} + (z\xi A_{n2\pm} + A_{n4\pm})\mathrm{e}^{-\xi z}],\\ \hat{u}_{zn\pm} &= \frac{1}{2\mu_n}\{[(\kappa_n - z\xi)A_{n1\pm} - A_{n3\pm}]\mathrm{e}^{\xi z} + [(\kappa_n + z\xi)A_{n2\pm} + A_{n4\pm}]\mathrm{e}^{-\xi z}\},\\ \hat{\sigma}_{zn\pm} &= \{[2(1-v_n) - z\xi]A_{n1\pm} - A_{n3\pm}\}\mathrm{e}^{\xi z} - \{[2(1-v_n) + z\xi]A_{n2\pm} + A_{n4\pm}\}\mathrm{e}^{-\xi z},\\ \hat{\sigma}_{rzn\pm} &= \{[z\xi - (1-2v_n)]A_{n1\pm} + A_{n3\pm}\}\mathrm{e}^{\xi z} - \{[z\xi + (1-2v_n)]A_{n2\pm} + A_{n4\pm}\}\mathrm{e}^{-\xi z}.\end{aligned} \tag{3.10}$$

In (3.9) and (3.10), the subscript "+" is for $c \leqslant z \leqslant h_n$ and the subscript "−" is for $0 \leqslant z \leqslant c$.

For layers perfectly bonded together, the boundary conditions at the interfaces $z = z_j$, $0 \leqslant r \leqslant \infty$ between layer j and layer $j+1$ are

$$\begin{aligned}&u_{rj}(r, z_j) = u_{rj+1}(r, z_j), \quad u_{zj}(r, z_j) = u_{zj+1}(r, z_j),\\ &\sigma_{zj}(r, z_j) = \sigma_{zj+1}(r, z_j), \quad \sigma_{rzj}(r, z_j) = \sigma_{rzj+1}(r, z_j),\end{aligned} \tag{3.11}$$

where

$$z_j = \sum_{k=1}^{j} h_k. \tag{3.12}$$

If, the layers are not perfectly bonded together, the boundary conditions can be described differently as given for the imperfect interface models; such as the free sliding model (e.g., Ghahremani, 1980; Mura and Furuhashi, 1984), the linear spring model (e.g., Aboudi, 1987; Hashin, 1991; Lipton and Vernescu, 1996; Shen et al., 2000) and the dislocation-like model (e.g., Yu, 1998; Yu et al., 2002; Pan, 2003; Duan et al., 2005).

The boundary conditions in the nth layer where the elastic singularity is located are given in (2.13)–(2.16) by replacing u_{r-}, u_{z-}, σ_{z-}, σ_{rz-}, u_{r+}, u_{z+}, σ_{z+} and σ_{rz+} with u_{rn-}, u_{zn-}, σ_{zn-}, σ_{rzn-} , u_{rn+}, u_{zn+}, σ_{zn+} and σ_{rzn+}, respectively. By substituting (3.7) into boundary conditions (3.11), and substituting (3.9) into any one of the boundary conditions (2.13)–(2.16), a set of $4(N+1)$ linear simultaneous algebraic equations of unknown $A_{ji}(\xi)$ are obtained, which can be solved symbolically by commercially available software such as *MATHEMATICA*. The elastic deformation due to these nuclei of strain can then be computed by substituting $A_{ji}(\xi)$ into (3.7)–(3.10). The formulation of the problem is now complete for each of the four axisymmetric elastic singularities in a multilayered solid. Examples of practically important applications of these singularities are presented in the following sections.

4. Point defects

The interaction between a point defect and the interface in a solid has been studied extensively because it plays an important role in material behavior related to diffusion, oxidation and corrosion. In continuum mechanics, the point defect is approximated by a center of dilatation. In the case where a planar, uniform elastic layer is bonded to an elastic semi-infinite solid (with different elastic properties), the problem has been studied by Dundurs and Stippes (1966), Bacon (1972) and Yu et al. (1996) using the method of Hankel transforms (Sneddon, 1951). The solution for a center of dilation also plays an important role in thermal stress analysis. A general method for the analysis of thermal stresses developed by Goodier (1937) for an infinite solid is based on the integration of properly weighted centers of dilatation over the volume occupied by the body (Boley and Weiner, 1960; Dundurs and Guell, 1965). By using Galerkin vectors, Mindlin and Cheng (1950b) extended the theory to a semi-infinite solid and Yu and coworkers (Yu et al., 1992; Yu and Sanday, 1992, 1993) extended the method to a bimaterial, a plate, and a layer sandwiched between two half-spaces.

The elastic deformation caused by a point defect in a planar, elastic layer perfectly bonded to a semi-infinite solid will be given in this section. It should be noted that this method could be easily extended to a multilayer system as described in Section 3. As shown in Fig. 3, solid 1 is a free space where $\mu_1 = v_1 = 0$, solid 2 is the layer with thickness $h_2 = h$, and solid 3 is the substrate where $h_3 = \infty$. The point defect is at the point $(0, c)$ in the layer. Letting $N = 3$, $n = 2$ and $j = 2$, 3 in (3.1)–(3.12), and substituting $\hat{u}_{r-}, \hat{u}_{z-}, \ldots, \hat{\sigma}_{z+}, \hat{\sigma}_{rz+}$ in (2.17) by $\hat{u}_{r2-}, \hat{u}_{z2-}, \ldots, \hat{\sigma}_{z2+}, \hat{\sigma}_{rz2+}$, one has the simultaneous algebraic equations, given in Appendix A for the unknown coefficients A_{2i+}, A_{2i-}, A_{32} and A_{34} . The solution of the simultaneous Eqs. (A.1)–(A.10) is

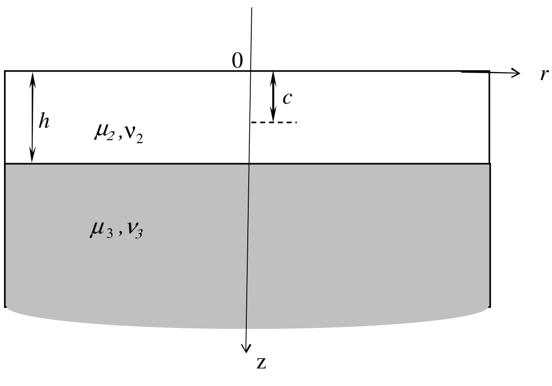

Fig. 3. An elastic layer of thickness h attached to a semi-infinite substrate with a point defect or an infinitesimal prismatic dislocation loop at the point $(0, c)$.

$$
\begin{aligned}
A_{21-} &= 2(\beta-1)[(\kappa_2-\kappa_3\beta)e^{2c\xi}+(1+\kappa_3\beta)(e^{2c\xi}-2h\xi)e^{2h\xi}]\xi e^{-c\xi}/\Delta_0,\\
A_{22-} &= -2(1+\kappa_3\beta)[(1-\beta)(1+2h\xi e^{2c\xi})+(\kappa_2+\beta)e^{2h\xi}]\xi e^{(2h-c)\xi}/\Delta_0,\\
A_{23-} &= \{(\beta-1)[\kappa_2(\kappa_2-\kappa_3\beta)+(1+\kappa_3\beta)(\kappa_2-2h\xi)e^{2h\xi}]e^{2c\xi}\\
&\quad+(1+\kappa_3\beta)[(1-\beta)(1+2\kappa_2h\xi)+(\kappa_2+\beta)e^{2h\xi}]e^{2h\xi}\}\xi e^{-c\xi}/\Delta_0,\\
A_{24-} &= \{(\beta-1)[(\kappa_2-\kappa_3\beta)+(1-2\kappa_2h\xi)(1+\kappa_3\beta)e^{2h\xi}]e^{2c\xi}\\
&\quad+(1+\kappa_3\beta)[(1-\beta)(\kappa_2+2h\xi)+\kappa_2(\kappa_2+\beta)e^{2h\xi}]e^{2h\xi}\}\xi e^{-c\xi}/\Delta_0,\\
A_{21+} &= 2(\beta-1)[(\kappa_2-\kappa_3\beta)e^{2c\xi}+(1+\kappa_3\beta)(e^{2c\xi}-2h\xi)e^{2h\xi}]\xi e^{-c\xi}/\Delta_0,\\
A_{22+} &= 2(1+\kappa_3\beta)[(\beta-1)(1+2h\xi e^{2c\xi})-(\kappa_2+\beta)e^{2h\xi}]\xi e^{(2h-c)\xi}/\Delta_0,\\
A_{23+} &= \{[2(\kappa_2-2h\xi)h\xi-\kappa_2^2+2b_3\beta(\kappa_2+2\kappa_2h\xi-4h^2\xi^2)+\kappa_3\beta^2(1-2\kappa_2h\xi+4h^2\xi^2)]e^{2h\xi}\\
&\quad+(\beta-1)[(1+\kappa_3\beta)(\kappa_2-2h\xi)e^{2(c+h)\xi}+(\kappa_2-\kappa_3\beta)(1+\kappa_2e^{2c\xi})]\}\xi e^{-c\xi}/\Delta_0,\\
A_{24+} &= \{[2(\kappa_2+2h\xi)h\xi+\kappa_2^2-2b_3\beta(\kappa_2-2\kappa_2h\xi-4h^2\xi^2)-\kappa_3\beta^2(1+2\kappa_2h\xi+4h^2\xi^2)]e^{2c\xi}\\
&\quad+(1+\kappa_3\beta)[\kappa_2(\kappa_2+\beta)e^{2h\xi}+(\kappa_2+\beta)e^{2(c+h)\xi}-(\beta-1)(\kappa_2+2h\xi)]\}\xi e^{(2h-c)\xi}/\Delta_0,\\
A_{32} &= 8a_2[(\beta-1)(1+2h\xi e^{2c\xi})-(\kappa_2+\beta)e^{2h\xi}]\xi e^{-(c-2h)\xi}/\Delta_0,\\
A_{34} &= 4a_2\{(1+\kappa_3\beta)e^{2(c+h)\xi}-[\kappa_3\beta-\kappa_2+2(\beta-1)(\kappa_3+2h\xi)h\xi]e^{2c\xi}\\
&\quad+[\kappa_3(\kappa_2+\beta)+4(b_2-b_3\beta)h\xi]e^{2h\xi}-(\beta-1)(\kappa_3+2h\xi)\}\xi e^{-(c-2h)\xi}/\Delta_0,
\end{aligned}
\tag{4.1}
$$

where

$$
\Delta_0=(\beta-1)(\kappa_2-\kappa_3\beta)-2[\gamma_1-2b_2b_3\beta-\kappa_3\beta^2+2(1-\beta)(1+\kappa_3\beta)h^2\xi^2]e^{2h\xi}-(\beta+\kappa_2)(1+\kappa_3\beta)e^{4h\xi}, \tag{4.2}
$$

$$
\beta=\frac{\mu_2}{\mu_3},\quad \gamma_1=5-12\nu_2+8\nu_2^2. \tag{4.3}
$$

The solution for the problem is obtained by substituting (4.1) into (3.7)–(3.10). For a thin film, one would have $\mu_3=\nu_3=0$, $A_{32}=A_{34}=0$, and the boundary conditions become (A.1)–(A.6), (A.9) and (A.10) given in Appendix A.

5. Infinitesimal prismatic dislocation loops

Over the past few decades, studies of dislocations and their effect on materials properties have been greatly expanded. For example, dislocation loops have been shown to have a great effect on dopant diffusion in preamorphized silicon and on electronics device performance (Skarlatos et al., 2006). It also affects void growth in the ductile fracture of materials at room temperature (Ahn et al., 2006). The elastic solution for an infinitesimal dislocation loop lying in its glide plane was first obtained by Nabarro (1951). Bacon and Groves (1969) provided the solution for an infinitesimal dislocation loop in an isotropic, semi-infinite elastic medium. For elastic interactions between dislocations and surface layers, the first analytical work was accomplished by Head (1953a) for a screw dislocation near a layer coated on a half-space using the analog between electrostatic image treatments and those of elasticity. The elastic field for a screw dislocation traversing a plate has been given by Eshelby (1979). Chu (1982) provided a solution for a screw dislocation in a two-phase thin film of equal thickness in each phase. And recently, Wang (1999) solved the problem of screw dislocations in a two-phase, thin film of equal thickness in each phase. The elastic stress field due to an edge dislocation in a bimaterial was also first investigated by Head (1953b) and re-examined by Dundurs and Sendeckyj (1965), and Mura (1968). The elastic fields for an edge dislocation in the substrate and in the film of a film/substrate system have been given by Weeks et al. (1968) and Lee and Dundurs (1973), respectively. By linear superposition of the solutions for the elastic singularities, Yu and Sanday (1991b, 1994) provided the solutions for a dislocation line, and a dislocation loop of arbitrary shape, orientation and Burgers vector. By extending their method, double forces in the z-direction and center of dilatation will be used to obtain the elastic solution for a prismatic loop traversing the layer with Burger vector in the z-direction. The elastic field of an edge dislocation can be obtained by integrating the result of the prismatic loop over the surface of the cut formally used to generate the dislocation (Nabarro, 1951).

Consider an infinitesimal prismatic loop at the point $(0, c)$ in layer n with a Burgers vector b_z, and the surface of the cut formally used to generate the dislocation loop is $\mathrm{d}S_z$. The elastic displacement due to this infinitesimal loop is the linear superposition of the field due to the double forces in the z-direction $u_{ij}\,(g_{zz})$ and due to the center of dilatation $u_{ij}(g_c)$ (Yu and Sanday, 1991b)

$$\begin{aligned} u_{rj} &= [u_{rj}(g_{zz}) - 2v_j u_{rj}(g_{\mathrm{cd}})]Q\,\mathrm{d}S_z, \\ u_{zj} &= [u_{zj}(g_{zz}) - 2v_j u_{zj}(g_{\mathrm{cd}})]Q\,\mathrm{d}S_z, \end{aligned} \tag{5.1}$$

where $j = 1, 2, \ldots, N$, and

$$Q = \frac{\mu_n b_z}{4\pi(1 - v_n)}. \tag{5.2}$$

We will again illustrate this method by using the dislocation loop in a layer, which is perfectly bonded to a semi-infinite substrate (Fig. 3). In addition to the elastic displacements $u_{rj}(g_{\mathrm{cd}})$ for a center of dilation (4.1), the solution $u_{rj}(g_{zz})$ for the double forces in the z-direction are also needed. The simultaneous equations needed for the double forces in the z-direction are obtained by substituting (A.4) and (A.6) (in Appendix A) into (2.15), thus

$$\begin{aligned} &[(\kappa_2 - c\xi)A_{21-} - A_{23-}]\mathrm{e}^{\xi c} + [(\kappa_2 + c\xi)A_{22-} + A_{24-}]\mathrm{e}^{-\xi c} \\ &\quad - [(\kappa_2 - c\xi)A_{21+} - A_{23+}]\mathrm{e}^{\xi c} - [(\kappa_2 + c\xi)A_{22+} + A_{24+}]\mathrm{e}^{-\xi c} = 4(1 - 2v_2)\xi, \end{aligned} \tag{5.3}$$

for $\hat{u}_{2z-}(r, c) - \hat{u}_{2z+}(r, c) = 2(1 - 2v_2)\xi/\mu_2$, and

$$\begin{aligned} &[(c\xi - b_2)A_{21-} + A_{23-}]\mathrm{e}^{c\xi} - [(c\xi + b_2)A_{22-} + A_{24-}]\mathrm{e}^{-c\xi} \\ &\quad - [(c\xi - b_2)A_{21+} + A_{23+}]\mathrm{e}^{c\xi} + [(c\xi + b_2)A_{22+} + A_{224+}]\mathrm{e}^{-\xi c} = 4v_2\xi, \end{aligned} \tag{5.4}$$

for $\hat{\sigma}_{rz2-}(r, c) - \hat{\sigma}_{rz2+}(r, c) = 4v_2\xi$. The coefficients for the double forces in the z-direction obtained by solving the simultaneous Eqs. (5.3), (5.4), (A.1)–(A.3), (A.5) and (A.7)–(A.10) are

$$\begin{aligned} A_{21-} = \{&(\beta - 1)(\kappa_2 - \kappa_3\beta)(1 - 4v_2 - 2c\xi)\mathrm{e}^{2c\xi} + (\beta - 1)(1 + \kappa_3\beta)[1 - 4v_2 + 2(h - c)\xi]\mathrm{e}^{2(c+h)\xi} \\ &- [(\kappa_2 + \beta)(\kappa_2 - \kappa_3\beta) - 2(\beta - 1)(1 + \kappa_3\beta)(1 - 4v_2 - 2(h - c)\xi)h\xi]\mathrm{e}^{2h\xi} \\ &- (\kappa_2 + \beta)(1 + \kappa_3\beta)\mathrm{e}^{4h\xi}\}\xi\mathrm{e}^{-c\xi}/\Delta_0, \\ A_{22-} = \{&(\beta - 1)(\kappa_2 - \kappa_3\beta)\mathrm{e}^{2c\xi} + [2(\beta - 1)(1 + \kappa_3\beta)(1 - 4v_2 + 2(h - c)\xi)h\xi \\ &- (\kappa_2 + \beta)(\kappa_2 - \kappa_3\beta)]\mathrm{e}^{2(c+h)\xi} + (\beta - 1)(1 + \kappa_3\beta)(1 - 4v_2 - 2(h - c)\xi)\mathrm{e}^{2h\xi} \\ &- (\kappa_2 + \beta)(1 + \kappa_3\beta)(1 - 4v_2 + 2c\xi)\mathrm{e}^{4h\xi}\}\xi\mathrm{e}^{-c\xi}/\Delta_0, \\ A_{23-} = \{&(\beta - 1)(\kappa_2 - \kappa_3\beta)(\gamma_3 + \kappa_2 c\xi)\mathrm{e}^{2c\xi} + [\kappa_2 - 2\gamma_4 b_3\beta + \alpha_3\kappa_3\beta^2 \\ &+ (1 - \beta)(1 + \kappa_3\beta)(c + 2\gamma_3 h - 2\kappa_2(h - c)h\xi)\xi]\mathrm{e}^{2(c+h)\xi}\}\xi\mathrm{e}^{-c\xi}/\Delta_0, \\ A_{24-} = \{&(1 - \beta)(\kappa_2 - \kappa_3\beta)(1 + c\xi)\mathrm{e}^{2c\xi} + [\gamma_5 - 2\gamma_2 b_3\beta - \kappa_3\beta^2 \\ &+ (1 - \beta)(1 + \kappa_3\beta)((1 - 2\kappa_3 h\xi)c\xi + 2(\gamma_3 + \kappa_2 h\xi)h\xi)]\mathrm{e}^{2(c+h)\xi} \\ &- [\kappa_2 - 2\gamma_4 b_3\beta + \gamma_3\kappa_3\beta^2 + (1 - \beta)(1 + \kappa_3\beta)(2h + 2h^2\xi - \kappa_2 c - 2ch\xi)\xi]\mathrm{e}^{2h\xi} \\ &+ (\kappa_2 + \beta)(1 + \kappa_3\beta)(\gamma_3 + \kappa_2 c\xi)\mathrm{e}^{4h\xi}\}\xi\mathrm{e}^{-c\xi}/\Delta_0, \\ A_{21+} = (\beta - 1)\{&(\kappa_2 - \kappa_3\beta)[(1 - 4v_2 - 2c\xi)\mathrm{e}^{2c\xi} - 1] + (1 + \kappa_3\beta)[1 - 4v_2 + 2(h - c)\xi]\mathrm{e}^{2(c+h)\xi} \\ &- (1 + \kappa_3\beta)[1 + 2(1 - 4v_2 + 2c\xi)h\xi]\mathrm{e}^{2h\xi}\}\xi\mathrm{e}^{-c\xi}/\Delta_0, \end{aligned}$$

$$
\begin{aligned}
A_{22+} =& (1+\kappa_3\beta)\{(\beta-1)[1-4v_2-2(h-c)\xi]-(\beta-1)[1-2(1-4v_2-2c\xi)h\xi]\mathrm{e}^{2c\xi}\\
&+(\kappa_2+\beta)\mathrm{e}^{2(c+h)\xi}-(\kappa_2+\beta)(1-4v_2+2c\xi)\mathrm{e}^{2h\xi}\}\xi\mathrm{e}^{(2h-c)\xi}/\Delta_0,\\
A_{23+} =& \{(1-\beta)(\kappa_2-\kappa_3\beta)[1-c\xi-(\gamma_3-\kappa_2c\xi)\mathrm{e}^{2c\xi}]+[\kappa_2-2\gamma_4b_3\beta+\gamma_3\kappa_3\beta^2\\
&+(1-\beta)(1+\kappa_3\beta)(\kappa_2c-2h+2(h-c)h\xi)\xi]\mathrm{e}^{2(c+h)\xi}+[\gamma_6-\kappa_3\beta+2\gamma_4b_3\beta\\
&-2(1-\beta)(1+\kappa_3\beta)(2ch\xi-\kappa_2c+(1-4v_2)h)h\xi^2-(\kappa_2^2c-2(1-\beta)(1+\kappa_3\beta)\gamma_3h\\
&-\beta(\kappa_3\beta+2b_3\kappa_2)c)\xi]\mathrm{e}^{2h\xi}\}\xi\mathrm{e}^{-c\xi}/\Delta_0,\\
A_{24+} =& -\{[\kappa_2+\gamma_3\kappa_3\beta^2-2\gamma_4b_3\beta-(1-\beta)(1+\kappa_3\beta)(\kappa_2-2(1+(h-c)\xi)h\xi)]\mathrm{e}^{c\xi}\\
&+[\gamma_6-\kappa_3\beta^2+2\gamma_4b_3\beta+2(1-\beta)(1+\kappa_3\beta)(\kappa_2c-(1-4v_2-2c\xi)h)h\xi^2\\
&+((\kappa_2+\beta)(\kappa_2-\kappa_3\beta)c-2\gamma_3(1-\beta)(1+\kappa_3\beta)h)\xi]\mathrm{e}^{3c\xi}\\
&+(1+c\xi)[\kappa_2+2(5-6v_3-2v_2\kappa_3)\beta+\kappa_3\beta^2]\mathrm{e}^{(3c+2h)\xi}\\
&-(\kappa_2+\beta)(1+\kappa_3\beta)(\gamma_3+\kappa_2c\xi)\mathrm{e}^{(c+2h)\xi}\}\xi\mathrm{e}^{2(h-c)\xi}/\Delta_0,\\
A_{32} =& 4a_2\{(\beta-1)[1-4v_2-2(h-c)\xi]-(\kappa_2+\beta)(1-4v_2+2c\xi)\mathrm{e}^{2h\xi}\\
&-(\beta-1)[1-2(1-4v_2)h\xi+4ch\xi^2]\mathrm{e}^{2c\xi}+(\kappa_2+\beta)\mathrm{e}^{2(c+h)\xi}\}\xi\mathrm{e}^{(2h-c)\xi}/\Delta_0,\\
A_{34} =& 4a_2\{2(v_2\kappa_3\beta-2v_2b_3-v_3)-(1-\beta)[2(b_3+2v_2)h-\kappa_3c]\xi+2(1-\beta)(c-h)h\xi^2\\
&-[4a_2-2v_3\kappa_2+2v_2\kappa_2\beta+(1+\kappa_3\beta)c\xi+2(b_2-b_3\beta)h\xi]\mathrm{e}^{2(c+h)\xi}\\
&+[4(\beta-1)ch^2\xi^3-2(1-\beta)(\kappa_3c-(1-4v_2)h)h\xi^2+(\kappa_3\beta-\kappa_2)c\xi+2(\beta-1)(2v_2\kappa_3-b_3)h\xi\\
&+2(v_2\kappa_3\beta-4v_2a_2+v_3)]\mathrm{e}^{2c\xi}+[2(2-3v_3-(4+\beta)v_2\kappa_3+4v_2^2\kappa_3)+\kappa_3(\kappa_2+\beta)c\xi\\
&+(b_2-b_3\beta)(2(1-4v_2)+4c\xi)h\xi]\mathrm{e}^{2h\xi}\}\xi\mathrm{e}^{(2h-c)\xi}/\Delta_0,
\end{aligned}
\tag{5.5}
$$

where

$$
\begin{aligned}
\gamma_2 &= 5-14v_2+8v_2^2,\quad \gamma_3=1-8v_2+8v_2^2,\quad \gamma_4=3-10v_2+8v_2^2,\\
\gamma_5 &= 13-52v_2+72v_2^2-32v_2^3,\quad \gamma_6=-3+28v_2-56v_2^2+32v_2^3,
\end{aligned}
$$

Δ_0 and β are given in (4.2) and (4.3), respectively. For a thin film, one would have $\mu_3 = v_3 = 0$, $A_{32} = A_{34} = 0$, and the boundary conditions become (5.3), (5.4), (A.1)–(A.3), (A.5), (A.9) and (A.10).

From (4.1), the coefficients needed in (3.3)–(3.5) for the complete solution are

$$
\begin{aligned}
A_{2i\pm} &= [A_{2i\pm}(g_{zz})-2v_2A_{2i\pm}(g_{\mathrm{cd}})]Q\,\mathrm{d}S_z,\quad (i=1,2,3,4),\\
A_{3i\pm} &= [A_{3i}(g_{zz})-2v_2A_{3i}(g_{\mathrm{cd}})]Q\,\mathrm{d}S_z,\quad (i=2,4),
\end{aligned}
\tag{5.6}
$$

where $A_{2i\pm}(g_{zz})$ and $A_{3i}(g_{zz})$ are for the double forces in the z-direction as given in (5.5), and $A_{2i\pm}$ (g_{cd}) and $A_{3i}(g_{\mathrm{cd}})$ are for the center of dilatation as given in (4.1). The elastic field in the solids due to a straight edge dislocation, or a prismatic dislocation loop with finite area can be obtained by integrating the results given in (5.1) for the infinitesimal dislocation loop over the surface of the cut that generates the dislocation. For an edge dislocation at $x_1 = x_2 = 0$, $x_3 = c$ with Burgers vector b_z and dislocation line vector parallel to the x_2-direction, the surface of the cut is the half plane $0 \leqslant x_1 \leqslant \infty$, $-\infty \leqslant x_2 \leqslant \infty$. The elastic displacement due to this edge dislocation can be found by integrating (5.1) over $\mathrm{d}S_z = \mathrm{d}x_1'\mathrm{d}x_2'$ from $x_1' = 0$ to ∞ and from $x_2' = -\infty$ to ∞, where r in the solution for the infinitesimal dislocation loop is replaced by $\sqrt{(x_1-x_1')^2+(x_2-x_2')^2}$. Similarly, for a prismatic loop with area S and Burgers vector b_z, the solution is obtained by integrating (5.1) over the surface area S.

6. Summary

A method is proposed for solving some axisymmetric problems in multilayered solids. The boundary conditions for the point force in the z-direction, the double forces in the z-direction, the double forces in the r-direction, and the centers of dilatation are derived. These boundary conditions are then used to derive the elastic response in multilayered solids. The unknown coefficients for the solution are expressed in terms of a set of simultaneous linear algebraic equations. As examples of this method, the elastic responses of point defects and dislocation loops are presented for a layer perfectly bonded to a semi-infinite substrate. For more than one layer, bonded to the semi-infinite solid, the expression for the solution is too long to be presented here. However, by using symbolic mathematics software such as *MATHEMATICA*, solving the simultaneous linear algebraic equations becomes straight forward.

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Appendix A. Boundary conditions for centers of dilatation

By substituting (3.8) and (3.10) into (2.16) and (3.11), one has

$$2a_2(A_{21-} - A_{22-}) - A_{23-} - A_{24-} = 0, \tag{A.1}$$

for $\hat{\sigma}_{z2-}(r,0) = 0$,

$$b_2(A_{21-} + A_{2-}) - A_{23-} + A_{24-} = 0, \tag{A.2}$$

for $\hat{\sigma}_{rz2-}(r,0) = 0$,

$$\begin{aligned}&(c\xi A_{21-} + A_{23-})e^{\xi c} + (c\xi A_{22-} + A_{24-})e^{-\xi c}\\ &\quad = (c\xi A_{21+} + A_{23+})e^{\xi c} + (c\xi A_{22+} + A_{24+})e^{-\xi c},\end{aligned} \tag{A.3}$$

for $\hat{u}_{r2-}(r,c) = \hat{u}_{r2+}(r,c)$,

$$\begin{aligned}&[(\kappa_2 - c\xi)A_{21-} - A_{23-}]e^{\xi c} + [(\kappa_2 + c\xi)A_{22-} + A_{24-}]e^{-\xi c}\\ &\quad - [(\kappa_2 - c\xi)A_{21+} - A_{23+}]e^{\xi c} - [(\kappa_2 + c\xi)A_{22+} + A_{24+}]e^{-\xi c} = 2\xi,\end{aligned} \tag{A.4}$$

for $\hat{u}_{2z-}(r,c) - \hat{u}_{2z+}(r,c) = \xi/\mu_2$,

$$\begin{aligned}&[(2a_2 - c\xi)A_{21-} - A_{23-}]e^{c\xi} - [(2a_2 + c\xi)A_{22-} + A_{24-}]e^{-c\xi}\\ &\quad = [(2a_2 - c\xi)A_{21+} - A_{23+}]e^{c\xi} - [(2a_2 + c\xi)A_{22+} + A_{24+}]e^{-c\xi},\end{aligned} \tag{A.5}$$

for $\hat{\sigma}_{2z-}(r,c) = \hat{\sigma}_{2z+}(r,c)$,

$$\begin{aligned}&[(c\xi - b_2)A_{21-} + A_{23-}]e^{c\xi} - [(c\xi + b_2)A_{22-} + A_{24-}]e^{-c\xi}\\ &\quad - [(c\xi - b_2)A_{21+} + A_{23+}]e^{c\xi} + [(c\xi + b_2)A_{22+} + A_{224+}]e^{-\xi c} = -2\xi,\end{aligned} \tag{A.6}$$

for $\hat{\sigma}_{rz2-}(r,c) - \hat{\sigma}_{rz2+}(r,c) = -2\xi$,

$$(h\xi A_{21+} + A_{23+})e^{\xi h} + (h\xi A_{22+} + A_{24+})e^{-\xi h} = \frac{\mu_2}{\mu_3}(h\xi A_{32} + A_{34})e^{-h\xi}, \tag{A.7}$$

for $\hat{u}_{r2+}(r,h) = \hat{u}_{r3}(r,h)$,

$$\begin{aligned}&[(\kappa_2 - h\xi)A_{21+} - A_{23+}]e^{\xi h} + [(\kappa_2 + h\xi)A_{22+} + A_{24+}]e^{-\xi h}\\ &\quad = \frac{\mu_2}{\mu_3}[(\kappa_3 + h\xi)A_{32} + A_{34}]e^{-h\xi},\end{aligned} \tag{A.8}$$

for $\hat{u}_{z2+}(r,h) = \hat{u}_{z3}(r,h)$,

$$\begin{aligned}&[(2a_2 - h\xi)A_{21+} - A_{23+}]e^{\xi h} - [(2a_2 + h\xi)A_{22+} + A_{24+}]e^{-\xi h}\\ &\quad = -[(2a_3 + h\xi)A_{32} + A_{34}]e^{-\xi h},\end{aligned} \tag{A.9}$$

for $\hat{\sigma}_{z2+}(r,h) = \hat{\sigma}_{z3}(r,h)$,

$$[(h\xi - b_2)A_{21+} + A_{23+}]e^{\xi h} - [(h\xi + b_2)A_{22+} + A_{24+}]e^{-\xi h} = -[(b_3 + h\xi)A_{32} + A_{34}]e^{-\xi h}, \tag{A.10}$$

for $\hat{\sigma}_{rz2+}(r,h) = \hat{\sigma}_{rz3}(r,h)$, where

$$\kappa_i = 3 - 4\nu_i, \quad a_i = 1 - \nu_i, \quad b_i = 1 - 2\nu_i. \tag{A.11}$$

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