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Homoclinic solutions for a class of subquadratic second-order Hamiltonian systems

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ABSTRACT

In this paper, we study the existence of infinitely many homoclinic solutions for a class of subquadratic second-order Hamiltonian systems. By using the variant fountain theorem, we obtain a new criterion for guaranteeing that second-order Hamiltonian systems has infinitely many homoclinic solutions. Recent results from the literature are generalized and significantly improved. An example is also given in this paper to illustrate our main results.

1. Introduction

Consider the following second-order Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + W_u\big(t, u(t)\big) = 0, \quad t \in \mathbb{R}, \tag{HS}$$

where $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix-valued function. As usual we say that a solution u of (HS) is homoclinic (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^{N\times N})$, $u \neq 0$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$.

Inspired by the excellent monographs [1,2], by now, the existence and multiplicity of periodic and homoclinic solutions for second-order Hamiltonian systems have been extensively investigated in many papers (see [3–19] and the references therein) via variational methods. Also second-order Hamiltonian systems with impulses via variational methods have been recently considered in [20–22].

More precisely, many authors studied the existence and multiplicity of homoclinic solutions for (HS), such as [7–19]. Some of them treated the case where $L(t)$ and $W(t,u)$ are either independent of t or periodic in t, see for instance [7–9], and a more general case is considered in the recent paper [9]. In this case, the existence of homoclinic solutions can be obtained by going to the limit of periodic solutions of approximating problems.

If $L(t)$ is neither a constant nor periodic in t, the problem of existence of homoclinic solutions for (HS) is quite different from the one just described, due to the lack of compactness of the Sobolev embedding. After the work of Rabinowitz and Tanaka [10], many results [11–19] were obtained for the case where $L(t)$ is neither a constant nor periodic in t. More precisely, recently, Zhang and Yuan [17] studied existence of homoclinic solutions for (HS) and obtained the existence of a nontrivial homoclinic solution for (HS) by using a standard minimizing argument.

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Theorem 1.1. *(See [17, Theorem 1.1].) Assume that L and W satisfy the following conditions*:

(H_1) *$L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ and $L(t)$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there is a continuous function $\alpha : \mathbb{R} \to \mathbb{R}$ such that $\alpha(t) > 0$ for all $t \in \mathbb{R}$ and $(L(t)u, u) \geqslant \alpha(t)|u|^2$ and $\alpha(t) \to +\infty$ as $|t| \to +\infty$;*

(H_2) *$W(t, u) = a(t)|u|^\gamma$ where $a : \mathbb{R} \to \mathbb{R}^+$ is a positive continuous function such that $a \in L^2(\mathbb{R}, \mathbb{R}) \cap L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R})$ and $1 < \gamma < 2$ is a constant.*

Then (HS) *possesses a nontrivial homoclinic solution.*

In [17], authors pointed out that under the assumptions of Theorem 1.1, they were not sure whether (HS) has infinitely many homoclinic solutions, though $W(t, u)$ is even with respect to u. Motivated by the above fact, in this paper our aim is to study the existence of infinitely many homoclinic solutions for (HS) under some conditions weaker than those in the previous theorem. Our tool is the variant fountain theorem established in [23].

Now, we state our main result.

Theorem 1.2. *Let the above condition (H_1) holds. Moreover, assume that the following condition holds*:

$(H_2)'$ *$W(t, u) = a(t)|u|^\gamma$ where $a : \mathbb{R} \to \mathbb{R}^+$ is a continuous function such that $a(t) \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R})$ and $1 < \gamma < 2$ is a constant.*

Then (HS) *possesses infinitely many homoclinic solutions.*

Remark 1.1. Obviously, the condition $(H_2)'$ is weaker than the condition (H_2), and it also implies that $\inf_{t\in\mathbb{R}} a(t) = 0$. Indeed, if we choose $a(t) = (1 + |t|)^{-\frac{1}{2}}$, then $a \in L^{\frac{2}{2-\gamma}}(\mathbb{R}, \mathbb{R})$ and $\inf_{t\in\mathbb{R}} a(t) = 0$, but $a \notin L^2(\mathbb{R}, \mathbb{R})$. In addition, we obtain infinitely many homoclinic solutions for (HS), not a nontrivial homoclinic solution. So we generalize and significantly improve Theorem 1.1 in [17].

Remark 1.2. Compared with the case where $W(t, u)$ is superquadratic as $|u| \to +\infty$, the case where $W(t, u)$ is subquadratic as $|u| \to +\infty$ has been considered only by a few authors. For example, in [12,19] the authors studied this case. They all obtained (HS) has infinitely many homoclinic solutions. But due to $\inf_{t\in\mathbb{R}} a(t) = 0$, there is no constant $b > 0$ such that $W(t, u) = a(t)|u|^\gamma \geqslant b|u|^\gamma$ for any $(t, u) \in \mathbb{R} \times \mathbb{R}^N$. $W(t, u) = a(t)|u|^\gamma$ does not satisfy the conditions (W_3) and (W_5) in [12]. Also W does not satisfy the conditions (W_3) and (W_4) in [19]. Therefore we also extend Theorem 1.2 in [12] and Theorem 1.1 in [19].

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of Theorem 1.2 and an example is also given to illustrate our main results.

2. Preliminaries

In this section, the following theorem will be needed in our argument. Let E be a Banach space with the norm $\|\cdot\|$ and $E = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$ with $\dim X_j < \infty$ for any $j \in \mathbb{N}$. Set $Y_k = \bigoplus_{j=0}^k X_j$, $Z_k = \overline{\bigoplus_{j=k}^\infty X_j}$. Consider the following C^1-functional $\varphi_\lambda : E \to \mathbb{R}$ defined by

$$\varphi_\lambda(u) = A(u) - \lambda B(u), \quad \lambda \in [1, 2].$$

Theorem 2.1. *(See [23, Theorem 2.2].) Suppose that the functional $\varphi_\lambda(u)$ defined above satisfies*:

(C_1) *φ_λ maps bounded sets to bounded sets uniformly for $\lambda \in [1, 2]$. Furthermore, $\varphi_\lambda(-u) = \varphi_\lambda(u)$ for all $(\lambda, u) \in [1, 2] \times E$.*

(C_2) *$B(u) \geqslant 0$; $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E.*

(C_3) *There exist $\rho_k > r_k > 0$ such that $a_k(\lambda) := \inf_{u\in Z_k, \|u\|=\rho_k} \varphi_\lambda(u) \geqslant 0 > b_k(\lambda) := \max_{u\in Y_k, \|u\|=r_k} \varphi_\lambda(u)$ for $\lambda \in [1, 2]$, $d_k(\lambda) := \inf_{u\in Z_k, \|u\|\leqslant\rho_k} \varphi_\lambda(u) \to 0$ as $k \to \infty$ uniformly for $\lambda \in [1, 2]$.*

Then there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that $\varphi'_{\lambda_n}|_{Y_n}(u(\lambda_n)) = 0$, $\varphi_{\lambda_n}(u(\lambda_n)) \to c_k \in [d_k(2), b_k(1)]$ as $n \to \infty$. In particular, if $\{u(\lambda_n)\}$ has a convergent subsequence for every k, then φ_1 has infinitely many nontrivial critical points $\{u_k\} \subset E \setminus \{0\}$ satisfying $\varphi_1(u_k) \to 0^-$ as $k \to \infty$.

Like in [17], let

$$E = \left\{ u \in H^1(\mathbb{R}, \mathbb{R}^N) \colon \int_{\mathbb{R}} \left[|\dot{u}(t)|^2 + (L(t)u(t), u(t)) \right] dt < +\infty \right\}.$$

Then the space E is a Hilbert space with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}} \big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), v(t)\big)\, dt,$$

where $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^N$. Denote by E^* its dual space with the associated operator norm $\|\cdot\|_{E^*}$. Note that E is continuously embedded into $L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in [2, +\infty]$. Therefore, there exists a constant $C > 0$ such that

$$\|u\|_p \leqslant C\|u\|, \quad \forall u \in E. \tag{2.1}$$

Lemma 2.1. *(See [11, Lemma 1].) Suppose that $L(t)$ satisfies (H_1). Then the embedding of E in $L^2(\mathbb{R}, \mathbb{R}^N)$ is compact.*

Lemma 2.2. *Suppose that (H_1) and $(H_2)'$ are satisfied. If $u_k \rightharpoonup u$ (weakly) in E, then $W_u(t, u_k) \to W_u(t, u)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$.*

Proof. Assume that $u_k \rightharpoonup u$ in E. Then $u_k \to u$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ by Lemma 2.1, passing to a subsequence if necessary, we have

$$\|u_k - u\|_2 \to 0, \quad \text{as } k \to \infty.$$

So it can be assumed that

$$\sum_{k=1}^{\infty} \|u_k - u\|_2 < +\infty,$$

which implies that $u_k(t) \to u(t)$ for almost every $t \in \mathbb{R}$ and

$$\sum_{k=1}^{\infty} \big|u_k(t) - u(t)\big| = h(t) \in L^2(\mathbb{R}, \mathbb{R}). \tag{2.2}$$

Furthermore, by the fact that $W_u(t, u) = \gamma a(t)|u|^{\gamma-2}u$, one has

$$\big|W_u(t, u_k) - W_u(t, u)\big| \leqslant \gamma a(t)\big(|u_k|^{\gamma-1} + |u|^{\gamma-1}\big),$$

which yields that

$$\begin{aligned}\big|W_u(t, u_k) - W_u(t, u)\big|^2 &\leqslant 2\gamma^2 a(t)^2\big(|u_k|^{2\gamma-2} + |u|^{2\gamma-2}\big) \\ &\leqslant 2\gamma^2 a(t)^2\big(|u_k - u|^{2\gamma-2} + 2|u|^{2\gamma-2}\big).\end{aligned} \tag{2.3}$$

Therefore, by (2.2) and (2.3), we have

$$\big|W_u(t, u_k) - W_u(t, u)\big|^2 \leqslant 2\gamma^2 a(t)^2\big(\big|h(t)\big|^{2\gamma-2} + 2|u|^{2\gamma-2}\big),$$

which yields that, combining (2.1) and the Hölder inequality,

$$\begin{aligned}\int_{\mathbb{R}} \big|W_u(t, u_k) - W_u(t, u)\big|^2 dt &\leqslant 2\gamma^2 \int_{\mathbb{R}} a(t)^2\big(\big|h(t)\big|^{2\gamma-2} + 2|u|^{2\gamma-2}\big) \\ &\leqslant 2\gamma^2 \|a\|^2_{\frac{2}{2-\gamma}} \big(\|h\|_2^{2\gamma-2} + 2\|u\|_2^{2\gamma-2}\big) \\ &\leqslant 2\gamma^2 \|a\|^2_{\frac{2}{2-\gamma}} \big(\|h\|_2^{2\gamma-2} + 2C^{2\gamma-2}\|u\|^{2\gamma-2}\big).\end{aligned}$$

By using the Lebesgue dominated convergence theorem, the lemma is proved. □

Define the functional φ on E by

$$\begin{aligned}\varphi(u) &= \frac{1}{2}\int_{\mathbb{R}} \big[\big|\dot{u}(t)\big|^2 + \big(L(t)u(t), u(t)\big)\big]\, dt - \int_{\mathbb{R}} W\big(t, u(t)\big)\, dt \\ &= \frac{1}{2}\|u\|^2 - \int_{\mathbb{R}} W\big(t, u(t)\big)\, dt \\ &= \frac{1}{2}\|u\|^2 - \Phi(u),\end{aligned} \tag{2.4}$$

where $\Phi(u) := \int_{\mathbb{R}} W(t, u(t))\, dt$.

Lemma 2.3. *Under the conditions* (H_1) *and* $(H_2)'$, *we have*

$$\varphi'(u)v = \int_{\mathbb{R}} \big[\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), v(t)\big)\big]\,dt - \int_{\mathbb{R}} \big(W_u\big(t, u(t)\big), v(t)\big)\,dt$$
$$= \int_{\mathbb{R}} \big[\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), v(t)\big)\big]\,dt - \Phi'(u)v \tag{2.5}$$

for any $u, v \in E$, *which yields that*

$$\varphi'(u)u = \|u\|^2 - \int_{\mathbb{R}} \big(W_u\big(t, u(t)\big), u(t)\big)\,dt. \tag{2.6}$$

Moreover, $\varphi \in C^1(E, \mathbb{R})$, $\Phi' : E \to E^*$ *is compact, and any critical point of* φ *on* E *is a classical solution of* (HS) *satisfying* $u(t) \to 0$ *and* $\dot{u}(t) \to 0$ *as* $|t| \to +\infty$.

Proof. We firstly show that $\varphi : E \to \mathbb{R}$. By (2.1) and the Hölder inequality, we have

$$0 \leqslant \int_{\mathbb{R}} W\big(t, u(t)\big)\,dt = \int_{\mathbb{R}} a(t)\big|q(t)\big|^{\gamma}\,dt \leqslant \|a\|_{\frac{2}{2-\gamma}} \|u\|_2^{\gamma} \leqslant C^{\gamma} \|a\|_{\frac{2}{2-\gamma}} \|u\|^{\gamma}. \tag{2.7}$$

Combining (2.4) and (2.7), we show that $\varphi : E \to \mathbb{R}$.

Next we prove that $\varphi \in C^1(E, \mathbb{R})$. By (2.4) we have

$$\varphi(u) = \frac{1}{2}\|u\|^2 - \Phi(u),$$

where

$$\Phi(u) = \int_{\mathbb{R}} W\big(t, u(t)\big)\,dt.$$

It is sufficient to show that $\Phi \in C^1(E, \mathbb{R})$. In the process we will see that

$$\Phi'(u)v = \int_{\mathbb{R}} \big(W_u\big(t, u(t)\big), v(t)\big)\,dt, \tag{2.8}$$

which is defined for all $u, v \in E$. For any given $u \in E$, let us define $J(u) : E \to \mathbb{R}$ as follows:

$$J(u)v = \int_{\mathbb{R}} \big(W_u\big(t, u(t)\big), v(t)\big)\,dt, \quad v \in E.$$

It is clear that $J(u)$ is linear. Now we show that $J(u)$ is bounded. Indeed, for any given $u \in E$, by (2.1) and the Hölder inequality, we obtain

$$\big|J(u)v\big| = \left|\int_{\mathbb{R}} \big(W_u\big(t, u(t)\big), v(t)\big)\,dt\right| \leqslant \int_{\mathbb{R}} a(t)\big|u(t)\big|^{\gamma-1}\big|v(t)\big|\,dt$$
$$\leqslant \left(\int_{\mathbb{R}} a(t)\big|u(t)\big|^{\gamma-1}\,dt\right)^{\frac{1}{2}} \|v\|_2$$
$$\leqslant \|a\|_{\frac{2}{2-\gamma}} \|u\|_2^{\gamma-1} \|v\|_2$$
$$\leqslant C^{\gamma} \|a\|_{\frac{2}{2-\gamma}} \|u\|^{\gamma-1} \|v\|. \tag{2.9}$$

Moreover, for any $u, v \in E$, by the Mean Value Theorem, we have

$$\int_{\mathbb{R}} W\big(t, u(t) + v(t)\big)\,dt - \int_{\mathbb{R}} W\big(t, u(t)\big)\,dt = \int_{\mathbb{R}} \big(W_u\big(t, u(t) + \theta(t)v(t)\big), v(t)\big)\,dt,$$

where $\theta(t) \in (0, 1)$. Therefore, by Lemma 2.2 and the Hölder inequality, one has

$$\int_{\mathbb{R}} \big(W_u\big(t, u(t) + \theta(t)v(t)\big), v(t)\big)\,dt - \int_{\mathbb{R}} \big(W_u\big(t, u(t)\big), v(t)\big)\,dt$$
$$= \int_{\mathbb{R}} \big(W_u\big(t, u(t) + \theta(t)v(t)\big) - W_u\big(t, u(t)\big), v(t)\big)\,dt \to 0 \tag{2.10}$$

as $v \to 0$ in E. Combining (2.9) and (2.10), we see that (2.8) holds. Similar to the proof of Lemma 3.1 in [17], we obtain Φ' is continuous. Therefore, $\varphi \in C^1(E, \mathbb{R})$ and

$$\varphi'(u)v = \int_{\mathbb{R}} \big[\big(\dot{u}(t), \dot{v}(t)\big) + \big(L(t)u(t), v(t)\big)\big]\,dt - \int_{\mathbb{R}} \big(W_u\big(t, u(t)\big), v(t)\big)\,dt$$

for any $u, v \in E$, which yields that

$$\varphi'(u)u = \|u\|^2 - \int_{\mathbb{R}} \big(W_u\big(t, u(t)\big), u(t)\big)\,dt.$$

Now we verify that $\Phi' : E \to E^*$ is compact. Letting $u_k \rightharpoonup u$ (weakly) in E, by Lemma 2.2, we have $W_u(t, u_k) \to W_u(t, u)$ in $L^2(\mathbb{R}, \mathbb{R}^N)$, i.e.,

$$\left(\int_{\mathbb{R}} \big|W_u(t, u_k) - W_u(t, u)\big|^2\,dt\right)^{\frac{1}{2}} \to 0, \quad \text{as } k \to \infty. \tag{2.11}$$

Then by (2.1), (2.11) and the Hölder inequality, we have

$$\begin{aligned}
\big\|\Phi'(u_k) - \Phi'(u)\big\|_{E^*} &= \sup_{\|v\|=1} \big\|\big(\Phi'(u_k) - \Phi'(u)\big)v\big\| \\
&= \sup_{\|v\|=1} \left|\int_{\mathbb{R}} \big\langle W_u(t, u_k) - W_u(t, u), v\big\rangle\,dt\right| \\
&\leqslant \sup_{\|v\|=1} \left[\left(\int_{\mathbb{R}} \big|W_u(t, u_k) - W_u(t, u)\big|^2\,dt\right)^{\frac{1}{2}} \|v\|_2\right] \\
&\leqslant C\left(\int_{\mathbb{R}} \big|W_u(t, u_k) - W_u(t, u)\big|^2\,dt\right)^{\frac{1}{2}} \to 0, \quad \text{as } k \to \infty.
\end{aligned}$$

Consequently, Φ' is weakly continuous. Therefore, Φ' is compact by the weakly continuity of Φ' since E is a Hilbert space.

Finally, as the discussion in Lemma 3.1 of [17], we obtain that the critical points of φ are classical solutions of (HS) satisfying $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $|t| \to \infty$. The proof is complete. □

3. Main results

In order to apply Theorem 2.1, we define the functionals A, B and φ_λ on our working space E by

$$A(u) = \frac{1}{2}\|u\|^2, \qquad B(u) = \int_{\mathbb{R}} W(t, u)\,dt,$$

and

$$\begin{aligned}
\varphi_\lambda(u) &= A(u) - \lambda B(u) \\
&= \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}} W(t, u)\,dt
\end{aligned}$$

for all $u \in E$ and $\lambda \in [1, 2]$. From Lemma 2.3, we know that $\varphi_\lambda(u) \in C^1(E, \mathbb{R})$ for all $\lambda \in [1, 2]$. We choose a completely orthonormal basis $\{e_j\}$ of E and define $X_j := \mathbb{R}e_j$. Then Z_k, Y_k can be defined as that in the beginning of Section 2. Note that $\varphi_1 = \varphi$, where φ is the functional defined in (2.4).

Lemma 3.1. *Let (H_1) and $(H_2)'$ be satisfied, then $B(u) \geqslant 0$. Moreover, $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E.*

Proof. Obviously, $B(u) \geqslant 0$ follows by the definition of the functional B and $(H_2)'$. Now we claim that $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E. Due to $\inf_{t\in\mathbb{R}} a(t) = 0$, there is no constant $b > 0$ such that $W(t,u) = a(t)|u|^\gamma \geqslant b|u|^\gamma$ for any $(t,u) \in \mathbb{R} \times \mathbb{R}^N$. Therefore, we have to find another way to overcome this difficulty. For any finite dimensional subspace $F \subset E$, there exists $\epsilon_1 > 0$ such that

$$\operatorname{meas}\{t \in \mathbb{R}\colon a(t)|u(t)|^\gamma \geqslant \epsilon_1\|u\|^\gamma\} \geqslant \epsilon_1, \quad \forall u \in F \setminus \{0\}. \tag{3.1}$$

Otherwise, for any positive integer n, there exists $u_n \in F \setminus \{0\}$ such that

$$\operatorname{meas}\left\{t \in \mathbb{R}\colon a(t)|u_n(t)|^\gamma \geqslant \frac{1}{n}\|u_n\|^\gamma\right\} < \frac{1}{n}.$$

Set $v_n(t) := \frac{u_n(t)}{\|u_n\|} \in F \setminus \{0\}$, then $\|v_n\| = 1$ for all $n \in \mathbb{N}$ and

$$\operatorname{meas}\left\{t \in \mathbb{R}\colon a(t)|v_n(t)|^\gamma \geqslant \frac{1}{n}\right\} < \frac{1}{n}. \tag{3.2}$$

Since $\dim F < \infty$, it follows from the compactness of the unit sphere of F that there exists a subsequence, say $\{v_n\}$, such that v_n converges to some v_0 in F. Hence, we have $\|v_0\| = 1$. By the equivalence of the norms on the finite dimensional space F, we have $v_n \to v_0$ in $L^2(\mathbb{R}, \mathbb{R}^N)$, i.e.,

$$\int_{\mathbb{R}} |v_n - v_0|^2\,dt \to 0, \quad \text{as } n \to \infty. \tag{3.3}$$

By (3.3) and the Hölder inequality, we have

$$\begin{aligned}\int_{\mathbb{R}} a(t)|v_n - v_0|^\gamma\,dt &\leqslant \left(\int_{\mathbb{R}} a(t)^{\frac{2}{2-\gamma}}\,dt\right)^{\frac{2-\gamma}{2}} \left(\int_{\mathbb{R}} |v_n - v_0|^2\,dt\right)^{\frac{\gamma}{2}} \\ &= \|a\|_{\frac{2}{2-\gamma}} \left(\int_{\mathbb{R}} |v_n - v_0|^2\,dt\right)^{\frac{\gamma}{2}} \to 0, \quad \text{as } n \to \infty.\end{aligned} \tag{3.4}$$

Thus there exist $\xi_1, \xi_2 > 0$ such that

$$\operatorname{meas}\{t \in \mathbb{R}\colon a(t)|v_0(t)|^\gamma \geqslant \xi_1\} \geqslant \xi_2. \tag{3.5}$$

In fact, if not, for all positive integers n, we have

$$\operatorname{meas}\left\{t \in \mathbb{R}\colon a(t)|v_0(t)|^\gamma \geqslant \frac{1}{n}\right\} = 0.$$

It implies that

$$0 \leqslant \int_{\mathbb{R}} a(t)|v_0(t)|^{\gamma+2}\,dt < \frac{1}{n}\|v_0\|_2^2 \leqslant \frac{C^2}{n}\|v_0\|^2 = \frac{C^2}{n} \to 0,$$

as $n \to \infty$ by (2.1). Hence $v_0 = 0$ which contradicts that $\|v_0\| = 1$. Therefore, (3.5) holds.

Now let

$$\Omega_0 = \{t \in \mathbb{R}\colon a(t)|v_0(t)|^\gamma \geqslant \xi_1\}, \qquad \Omega_n = \left\{t \in \mathbb{R}\colon a(t)|v_n(t)|^\gamma < \frac{1}{n}\right\}$$

and $\Omega_n^c = \mathbb{R} \setminus \Omega_n = \{t \in \mathbb{R}\colon a(t)|v_n(t)|^\gamma \geqslant \frac{1}{n}\}$. By (3.2) and (3.5), we have

$$\begin{aligned}\operatorname{meas}(\Omega_n \cap \Omega_0) &= \operatorname{meas}(\Omega_0 \setminus (\Omega_n^c \cap \Omega_0)) \\ &\geqslant \operatorname{meas}(\Omega_0) - \operatorname{meas}(\Omega_n^c \cap \Omega_0) \\ &\geqslant \xi_2 - \frac{1}{n}\end{aligned}$$

for all positive integers n. Let n be large enough such that $\xi_2 - \frac{1}{n} \geqslant \frac{1}{2}\xi_2$ and $\frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n} \geqslant \frac{1}{2^\gamma}\xi_1$. Then we have

$$\begin{aligned}\int_{\mathbb{R}} a(t)|v_n - v_0|^\gamma \, dt &\geqslant \int_{\Omega_n\cap\Omega_0} a(t)|v_n - v_0|^\gamma \, dt \\ &\geqslant \frac{1}{2^{\gamma-1}} \int_{\Omega_n\cap\Omega_0} a(t)|v_0|^\gamma \, dt - \int_{\Omega_n\cap\Omega_0} a(t)|v_n|^\gamma \, dt \\ &\geqslant \left(\frac{1}{2^{\gamma-1}}\xi_1 - \frac{1}{n}\right)\text{meas}(\Omega_n \cap \Omega_0) \\ &\geqslant \frac{\xi_1}{2^\gamma} \cdot \frac{\xi_2}{2} = \frac{\xi_1\xi_2}{2^{\gamma+1}} > 0\end{aligned}$$

for all large n, which is a contradiction to (3.4). Therefore, (3.1) holds. For the ϵ_1 given in (3.1), let

$$\Omega_u = \left\{t \in \mathbb{R}\colon a(t)\left|u(t)\right|^\gamma \geqslant \epsilon_1 \|u\|^\gamma\right\}, \quad \forall u \in F \setminus \{0\}. \tag{3.6}$$

Then by (3.1),

$$\text{meas}(\Omega_u) \geqslant \epsilon_1, \quad \forall u \in F \setminus \{0\}. \tag{3.7}$$

Combining $(H_2)'$ and (3.7), for any $u \in F \setminus \{0\}$, we have

$$\begin{aligned}B(u) = \int_{\mathbb{R}} W(t,u)\, dt &= \int_{\mathbb{R}} a(t)\left|u(t)\right|^\gamma dt \\ &\geqslant \int_{\Omega_u} a(t)\left|u(t)\right|^\gamma dt \\ &\geqslant \epsilon_1 \|u\|^\gamma \text{meas}(\Omega_u) \geqslant \epsilon_1^2 \|u\|^\gamma.\end{aligned}$$

This implies $B(u) \to \infty$ as $\|u\| \to \infty$ on any finite dimensional subspace of E. The proof is complete. □

Lemma 3.2. *Under the assumptions of Theorem 1.2, there exists a sequence $\rho_k \to 0^+$ as $k \to \infty$ such that*

$$a_k(\lambda) := \inf_{u\in Z_k,\, \|u\|=\rho_k} \varphi_\lambda(u) \geqslant 0,$$

and

$$d_k(\lambda) := \inf_{u\in Z_k,\, \|u\|\leqslant\rho_k} \varphi_\lambda(u) \to 0, \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2],$$

where $Z_k = \overline{\bigoplus_{j=k}^\infty X_j} = \overline{\text{span}\{e_k, \ldots\}}$ for all $k \in \mathbb{N}$.

Proof. Set $\beta_k := \sup_{u\in Z_k,\, \|u\|=1} \|u\|_2$. Then $\beta_k \to 0$ as $k \to \infty$. Indeed, it is clear that $0 < \beta_{k+1} \leqslant \beta_k$, so that $\beta_k \to \beta \geqslant 0$, as $k \to \infty$. For every $k \geqslant 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_2 > \beta_k/2$. By definition of Z_k, $u_k \rightharpoonup 0$ in E. Then it implies that $u_k \to 0$ in $L^2(\mathbb{R}, \mathbb{R}^N)$ by Lemma 2.1. Thus we have proved that $\beta = 0$. By $(H_2)'$ and the Hölder inequality, we have

$$\begin{aligned}\varphi_\lambda(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}} W(t,u)\, dt \\ &\geqslant \frac{1}{2}\|u\|^2 - 2\int_{\mathbb{R}} W(t,u)\, dt \\ &\geqslant \frac{1}{2}\|u\|^2 - 2\|a\|_{\frac{2}{2-\gamma}} \|u\|_2^\gamma \\ &\geqslant \frac{1}{2}\|u\|^2 - 2\beta_k^\gamma \|a\|_{\frac{2}{2-\gamma}} \|u\|^\gamma.\end{aligned} \tag{3.8}$$

Let $\rho_k = (8\beta_k^\gamma \|a\|_{\frac{2}{2-\gamma}})^{1/(2-\gamma)}$. Obviously, $\rho_k \to 0$ as $k \to \infty$. Combining this with (3.8), straightforward computation shows that

$$a_k(\lambda) := \inf_{u\in Z_k,\, \|u\|=\rho_k} \varphi_\lambda(u) \geqslant \frac{1}{4}\rho_k^2 > 0.$$

Furthermore, by (3.8), for any $u \in Z_k$ with $\|u\| \leqslant \rho_k$, we have

$$\varphi_\lambda(u) \geqslant -2\beta_k^\gamma \|a\|_{\frac{2}{2-\gamma}} \|u\|^\gamma.$$

Therefore,

$$0 \geqslant \inf_{u\in Z_k,\, \|u\|\leqslant\rho_k} \varphi_\lambda(u) \geqslant -2\beta_k^\gamma \|a\|_{\frac{2}{2-\gamma}} \|u\|^\gamma.$$

Since $\beta_k, \rho_k \to 0$ as $k \to \infty$, we have

$$d_k(\lambda) := \inf_{u\in Z_k,\, \|u\|\leqslant\rho_k} \varphi_\lambda(u) \to 0, \quad \text{as } k \to \infty \text{ uniformly for } \lambda \in [1,2].$$

The proof is complete. □

Lemma 3.3. *Under the assumptions of Theorem* 1.2*, for the sequence* $\{\rho_k\}_{k\in\mathbb{N}}$ *obtained in Lemma* 3.2*, there exist* $0 < r_k < \rho_k$ *for all* $k \in \mathbb{N}$ *such that*

$$b_k(\lambda) := \max_{u\in Y_k,\, \|u\|=r_k} \varphi_\lambda(u) < 0 \quad \text{for all } \lambda \in [1,2],$$

where $Y_k = \overline{\bigoplus_{j=1}^k X_j} = \overline{\operatorname{span}\{e_1,\dots,e_k\}}$ *for all* $k \in \mathbb{N}$.

Proof. For any $u \in Y_k$ (a finite dimensional subspace of E) and $\lambda \in [1,2]$, by $(H_2)'$, (3.6) and (3.7), we have

$$\begin{aligned}
\varphi_\lambda(u) &= \frac{1}{2}\|u\|^2 - \lambda \int_{\mathbb{R}} W(t,u)\,dt \\
&\leqslant \frac{1}{2}\|u\|^2 - \int_{\Omega_u} a(t)\big|u(t)\big|^\gamma\,dt \\
&\leqslant \frac{1}{2}\|u\|^2 - \epsilon_1\|u\|^\gamma \operatorname{meas}(\Omega_u) \\
&\leqslant \frac{1}{2}\|u\|^2 - \epsilon_1^2\|u\|^\gamma.
\end{aligned} \tag{3.9}$$

Choose $0 < r_k < \min\{\rho_k, \epsilon_1^{\frac{2}{2-\gamma}}\}$. By (3.9), direct computation shows that

$$b_k(\lambda) := \max_{u\in Y_k,\, \|u\|=r_k} \varphi_\lambda(u) \leqslant -\frac{r_k^2}{2} < 0, \quad \forall k \in \mathbb{N}.$$

The proof is complete. □

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Evidently, the condition (C_1) in Theorem 2.1 holds. By Lemmas 3.1, 3.2 and 3.3, conditions (C_2) and (C_3) in Theorem 2.1 are also satisfied. Therefore, by Theorem 2.1 there exist $\lambda_n \to 1$, $u(\lambda_n) \in Y_n$ such that

$$\varphi'_{\lambda_n}\big|_{Y_n}\big(u(\lambda_n)\big) = 0, \quad \varphi_{\lambda_n}\big(u(\lambda_n)\big) \to c_k \in \big[d_k(2), b_k(1)\big], \quad \text{as } n \to \infty. \tag{3.10}$$

For the sake of notational simplicity, in what follows we always set $u_n = u_{\lambda_n}$ for all $n \in \mathbb{N}$.

Now we show that $\{u_n\}$ is bounded in E. Indeed, by $(H_2)'$, (2.1), (3.10) and the Hölder inequality, we have

$$\begin{aligned}
\|u_n\|^2 &= 2\varphi_{\lambda_n}(u_n) + 2\lambda_n \int_{\mathbb{R}} a(t)\big|u_n(t)\big|^\gamma\,dt \\
&\leqslant M + 4\|a\|_{\frac{2}{2-\gamma}} \|u_n\|_2^\gamma \\
&\leqslant M + 4C^\gamma \|a\|_{\frac{2}{2-\gamma}} \|u_n\|^\gamma, \quad \forall n \in \mathbb{N},
\end{aligned} \tag{3.11}$$

for some $M > 0$. Since $1 < \gamma < 2$, (3.11) yields $\{u_n\}$ is bounded in E.

Finally, we show that $\{u_n\}$ possesses a strong convergent subsequence in E. In fact, in view of the boundedness of $\{u_n\}$, without loss of generality, we may assume

$$u_n \rightharpoonup u_0, \quad \text{as } n \to \infty, \tag{3.12}$$

for some $u_0 \in E$. By virtue of the Riesz Representation Theorem, $\varphi'_{\lambda_n}|_{Y_n} : Y_n \to Y_n^*$ and $\Phi' : E \to E^*$ can be viewed as $\varphi'_{\lambda_n}|_{Y_n} : Y_n \to Y_n$ and $\Phi' : E \to E$ respectively, where Y_n^* is the dual space of Y_n. Note that

$$0 = \varphi'_{\lambda_n}\big|_{Y_n}(u_n) = u_n - \lambda_n P_n \Phi'(u_n), \quad \forall n \in \mathbb{N},$$

where $P_n : E \to Y_n$ is the orthogonal projection for all $n \in \mathbb{N}$. That is,

$$u_n = \lambda_n P_n \Phi'(u_n), \quad \forall n \in \mathbb{N}. \tag{3.13}$$

By Lemma 2.3, we obtain $\Phi' : E \to E$ is also compact. Combining this with the boundedness of $\{u_n\}$ and (3.12), one has the right-hand side of (3.13) converges strongly in E and hence $u_n \to u_0$ in E.

Now from the last assertion of Theorem 2.1, we know that $\varphi = \varphi_1$ has infinitely many nontrivial critical points. Therefore, (HS) possesses infinitely many nontrivial homoclinic solutions by Lemma 2.3. The proof of Theorem 1.2 is complete. □

Example 3.1. Consider the following Hamiltonian system with $N = 3$:

$$\ddot{u}(t) - L(t)u(t) + W_u\big(t, u(t)\big) = 0, \quad \forall t \in \mathbb{R}, \tag{3.14}$$

where

$$L(t) = \begin{pmatrix} 1+t^2 & 0 & 0 \\ 0 & 1+t^2 & 0 \\ 0 & 0 & 1+t^2 \end{pmatrix}, \qquad W(t,u) = \left(\frac{1}{1+|t|}\right)^{\frac{1}{2}} |u|^{\frac{3}{2}}.$$

Let $\alpha(t) = t^2$, $\gamma = \frac{3}{2}$ and

$$a(t) = \left(\frac{1}{1+|t|}\right)^{\frac{1}{2}}.$$

Clearly, (H_1) and $(H_2)'$ in Theorem 1.2 hold. Therefore, by applying Theorem 1.2, we obtain that Hamiltonian system (3.14) possesses infinitely many homoclinic solutions. However, it is easy to see that (H_2) in Theorem 1.1 is not satisfied. So we cannot obtain the existence of homoclinic solutions for Hamiltonian system (3.14) by Theorem 1.1.

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