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Determinantal Method and the Littlewood–Richardson Rule

TAKESHI TOKUYAMA

IBM Research, Tokyo Research Laboratory, 5–19, Sanban-Cho, Tokyo, 102 Japan

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1. INTRODUCTION

Determinantal method is a powerful tool to study invariants and representations of classical groups. Let $R_1 = \mathbb{Z}[x_{i,j}]_{i,j=1,2,...,n}$ be the integral coordinate ring of the module of all $n \times n$ integral matrices. In [7], Doubilet, Rota, and Stein gave a beautiful theory on combinatorial approach to invariant theory using a nice basis of R_1 called standard basis whose elements are simply parametrized by means of "standard tableaux." This standard basis also occurs in the theory of flag manifolds (cf. [12], [17]). As application of this theory, Procesi and DeConcini gave many results on the characteristic free approach to the invariants and representation spaces of classical groups [4], [5], [6].

In particular, a representation space of each finite dimensional irreducible representation of $GL(n, \mathbb{C})$ is realized in $R_1 \otimes \mathbb{C}$ canonically. This representation space plays a key roll in application to physics, since each weight vector in this representation space is actually an element of standard basis. Now let us consider the tensor product representation of two arbitrarily irreducible representations ρ_λ and ρ_μ of $GL(n, \mathbb{C})$. Tensor product representations have deep relation with the interaction of particles (cf. [2]). The branching rule of the tensor product representation $\rho_\lambda \otimes \rho_\mu$ into its irreducible constituents is described using combinatorial methods on Young diagrams. This rule, the Littlewood–Richardson rule, was first found by Littlewood [14] and proved rather recently (cf. [15]). However, in application to physics, not only the branching rule but also more detailed information of the representation spaces of tensor product representations is required. In case of $GL(2, \mathbb{C})$, the decomposition of $\rho_\lambda \otimes \rho_\mu$ is multiplicity free. So in this case, each weight vector of each irreducible constituent of $\rho_\lambda \otimes \rho_\mu$ is canonically described in terms of the standard bases of ρ_λ and ρ_μ, whose coefficients are called Wigner coef-

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ficients (cf. [2]). Wigner coefficients are generalized and known for many other multiplicity-free branchings. However, in case of the branchings with nontrivial multiplicities, even the definition of the Wigner coefficients itself is obscure. Unfortunately, if n is larger than 2, the tensor product representations of $GL(n, \mathbb{C})$ usually branch with nontrivial multiplicities. In this paper, we shall give an expression of the highest weight vectors of irreducible constituents occurring in this branching. If we apply the results of Li, Moody, Nicolscu, and Patera [13], we can get all other weight vectors of each irreducible constituent canonically from our result.

We shall realize a representation space of $\rho_\lambda \otimes \rho_\mu$ in $R_1 \otimes R_1 \approxeq R = \mathbb{Z}[x_{\varepsilon,j,k}]_{\varepsilon=1,2;j,k=1,2,\ldots,n}$. In Section 3, we shall give an action of the reductive Lie algebra $\mathbb{Z}^2 + \mathfrak{gl}(n, \mathbb{Z}) + \mathfrak{gl}(n, \mathbb{Z})$ and its Borel subalgebra $\mathfrak{b}$ on R. Using the determinantal method and the Young diagramatical method, we construct a relative invariant of $\mathfrak{b}$ in R as a mean value of the orbit of an element of R with respect to the horizontal transformation group of a skew diagram. Our main theorem is Theorem 3.13, which states that any relative invariants can be described in terms of the relative invariants constructed as above. In Section 5, we apply our results in Section 3 to the branching of tensor product representations. As a corollary, we have a new proof of the Littlewood–Richardson rule (cf. [10], [11], [15]).

Since our argument is done on $\mathbb{Z}$, the integral numbers, our result is a characteristic free approach to the Littlewood–Richardson rule and we expect some applications of our results to the representation theory of Chevalley groups of type A. More systematic characteristic free approaches to the representations of general linear grous were given by Akin and Buchsbaum using Shur modules and Weyl modules, and particularly they gave a characteristic free approach to the Littlewood–Richardson rule in [1] from a different point of view from ours. Also, Boffi studied relative invariants of a Borel subalgebra of general linear Lie algebras to give a universal form of the Littlewood–Richardson rule in [3] in a different way.

2. Combinatorial Notations

2.1. *Diagrams*

A partition is a nonincreasing finite sequence of nonnegative integers. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a partition. The summation $|\lambda| = \sum_{i=1}^{r} \lambda_i$ is called the size of λ. We usually regard a partition as a sequence of positive integers identifying $(\lambda_1, \lambda_2, \ldots, \lambda_r, 0)$ and $(\lambda_1, \lambda_2, \ldots, \lambda_r)$.

Adopting the notation of Macdonald [15], we correspond a diagram $Y(\lambda)$ to λ as seen in the following example. (A diagram is a subset of $\mathbb{Z} \times \mathbb{Z}$ defined on p. 1 in [15].)

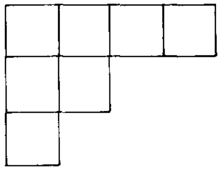

FIGURE 1

EXAMPLE. The diagram of the partition $\lambda=(4, 2, 1)$ is $Y(\lambda)=$ Fig. 1.

We usually denote $Y(\lambda)$ simply by λ identifying a Young diagram to its corresponding partition. ϕ denotes the vacant partition (or its diagram). For a diagram λ, ${}^t\lambda$ denotes the transposed diagram of λ defined by $({}^t\lambda)_i = \# \{j \mid \lambda_j = i\}$.

For diagrams λ and μ, we write $\lambda \supset \mu$ if $\lambda_i \geqslant \mu_i$ for any i. When $\lambda \supset \mu$, the set theoretic difference $\lambda - \mu$ of the diagrams is called a skewdiagram. For example, if $\lambda = (5, 4, 2, 1)$ and $\mu = (3, 2, 2)$ then $\lambda - \mu$ is the shaded region in the following figure.

We call μ the deficiency of $\lambda - \mu$. A diagram can be regarded as a skew-diagram with deficiency ϕ.

2.2. *Tableaux*

DEFINITION 2.2. Let $\lambda - \mu$ be a skewdiagram. A skewtableau T of shape $\lambda - \mu$ is a map from $\lambda - \mu$ into the set of positive integers. Graphically, T may be described by numbering each square of the skewdiagram $\lambda - \mu$. $T(i, j)$ is the integer written into (i, j)-place of T in the above descrition, which we often write $t_{i,j}$ for abbreviation. A skewtableau whose shape is a diagram is called a tableau. If the range of T is involved in $\{1, 2, ..., n\}$, T is called n-skewtableau.

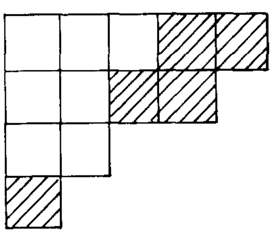

FIGURE 2

EXAMPLE.

$$T = \begin{matrix} 3 & 2 & 5 \\ 1 & & \end{matrix}$$

is a tableau of shape (3, 1) such that $t_{1,1}=3$, $t_{1,2}=2$, $t_{1,3}=5$ and $t_{2,1}=1$.

A skewtableau is called normal skewtableau if it is a row strictly increasing array. A skewtableau (resp. tableau) is called standard skewtableau (resp. standard tableau) if it is a row strictly and column weakly increasing array.

DEFINITION 2.3. The content of a skewtableau T is the sequence $(m(T)_i)_{i\in\mathbb{Z}}$ where $m(T)_i$ is the number of occurrence of the positive integer i written in T as symbols.

DEFINITION 2.4. Let λ be a diagram. The canonical tableau of shape λ is the (standard) tableau C_λ defined by $(C_\lambda)(i,j)=j$ for any square (i,j) involved in λ.

EXAMPLE.

$$C_{(4,2,2)} = \begin{matrix} 1 & 2 & 3 & 4 \\ 1 & 2 & & \\ 1 & 2 & & \end{matrix}$$

DEFINITION 2.5 (Bitableau). A bitableau of shape λ is a pair of tableaux of shape λ. A bitableau (S, T) is called a standard bitableau if both S and T are standard tableaux.

EXAMPLE.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \quad & 1 & 3 & 4 & 5 \\ 1 & 2 & 4 & & \quad & 2 & 3 & 4 & \\ 1 & & & & \quad & 6 & & & \end{pmatrix}$$

is a standard bitableau of shape (4, 3, 1).

Note. We adopt the notations in [7] here. Our notations for tableaux are different from those in [15].

3

We consider the Lie algebra

$$\mathfrak{g}_0 := \mathfrak{gl}(n_1, \mathbb{Z}) + \mathfrak{gl}(n_2, \mathbb{Z}) + \mathfrak{gl}(n_3, \mathbb{Z}).$$

Let $\mathfrak{h}$ be the diagonal Cartan subalgebra of $\mathfrak{gl}(n_1, \mathbb{Z})$ and let $\mathfrak{b}_2$ (resp. $\mathfrak{b}_3$) be the upper triangular Borel subalgebra of $\mathfrak{gl}(n_2, \mathbb{Z})$ (resp. $\mathfrak{gl}(n_3, \mathbb{Z})$). Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{gl}(n_2, \mathbb{Z}) + \mathfrak{gl}(n_3, \mathbb{Z})$ is a reductive subalgebra of $\mathfrak{g}_0$ and $\mathfrak{b} = \mathfrak{b} + \mathfrak{b}_2 + \mathfrak{b}_3$ is a Borel subalgebra of $\mathfrak{g}$. Since $\mathfrak{gl}(n_i, \mathbb{Z})$ acts on $\mathbb{Z}^{n_i}$ naturally, we have a Lie algebra action of $\mathfrak{g}_0$ on the tensor space $\mathbb{Z}^{n_1 n_2 n_3} = \mathbb{Z}^{n_1} \otimes \mathbb{Z}^{n_2} \otimes \mathbb{Z}^{n_3}$ in the usual way. This action induces a representation ρ of the Lie algebra $\mathfrak{g}_0$ on the integral coordinate ring $R_{n_1, n_2, n_3} := \mathbb{Z}[x_{i,j,k}]_{i=1,2,\ldots,n_1; j=1,2,\ldots,n_2; k=1,2,\ldots,n_3}$ of $\mathbb{Z}^{n_1 n_2 n_3}$.

From now on, we assume that $n_1 = 2$ and $n_2 = n_3 = n$. We denote $R := R_{2,n,n}$. The aim of this paper is to find all relative invariants of the Lie algebra $\mathfrak{b}$ with respect to the restricted representation $\rho|_{\mathfrak{b}}$ on R.

Also from now on, we shall mainly use n-skewtableaux. So we simply write skewtableau, tableau, or bitableau instead of writing n-skewtableau, n-tableau, or n-bitableau. We shall notice especially if we shall use tableaux with some symbols larger than n.

Notation 3.1. Let i_s and j_t be positive integers not larger than n for s and $t = 1, 2, 3, \ldots, p$. Then we use the notation $(i_1, i_2, \ldots, i_p \mid j_1, j_2, \ldots, j_p)_\varepsilon$ to represent for $\det(x_{\varepsilon, i_s, j_t})_{s,t=1,2,\ldots,p}$ where ε is 1 or 2.

The following lemma is well known (cf. [7], [8]) and our key lemma.

LEMMA S (Straightening law for minors; (Lemma 1, Sect. 8 in [7])). *Given two minors $P = (i_1, i_2, \ldots, i_s \mid j_1, j_2, \ldots, j_s)_\varepsilon$ and $Q = (k_1, k_2, \ldots, k_r \mid l_1, l_2, \ldots, l_r)_\varepsilon$ such that $I = i_1, i_2, \ldots, i_s$, $J = j_1, j_2, \ldots, j_s$, $K = k_1, k_2, \ldots, k_r$, and $L = l_1, l_2, \ldots, l_r$ are increasing sequences. Then for any index b satisfying $i_b < k_b$, the following formula $(*)$ holds. Suppose that the symmetric group $\mathfrak{S}_{r+1}$ of degree $r+1$ acts on the sequence $(m_1, m_2, \ldots, m_{r+1}) = (i_1, i_2, \ldots, i_b, \; k_b, k_{b+1}, \ldots, k_r)$ as the transformation of indices.*

$$\sum_{\alpha \in \mathfrak{S}_{r+1}} \operatorname{Sgn}(\alpha)(m_{\alpha(1)}, m_{\alpha(2)}, \ldots, m_{\alpha(b)}, i_{b+1}, i_{b+2}, \ldots, i_s \mid j_1, j_2, \ldots, j_s)_\varepsilon$$
$$\times (k_1, k_2, \ldots, k_{b-1}, m_{\alpha(b+1)}, m_{\alpha(b+2)}, \ldots, m_{\alpha(r+1)} \mid l_1, \ldots, l_r)_\varepsilon = A. \quad (*)$$

A, the right side of $(*)$, is an integral linear combination of some suitable products $P'Q'$ such that the size of the minor Q' is bigger than r and the sum of the sizes of the minors Q' and P' equals $r + s$.

Let (T, S) be a (n-)bitableau of shape λ. We correspond the following two elements of R to it.

$$(T|S)_\varepsilon = \sum_{i=1,2,\ldots,h} (t_{i,1}, t_{i,2}, \ldots, t_{i,\lambda_i} | s_{i,1}, s_{i,2}, \ldots, s_{i,\lambda_i})_\varepsilon,$$

where $h = ({}^t\lambda)_1$ is the depth of λ, $\varepsilon = 1$ or 2, and $t_{i,j}$ (resp. $s_{i,j}$) are the symbols at (i, j)'s in T (resp. S).

The following Theorem is fundamental.

THEOREM 3.2 (Standard basis (Doubilet–Rota–Stein [7])).

(1) *$L_\varepsilon := \{(T|S)_\varepsilon \mid (T, S)$ are standard bitableaux$\}$ is a $\mathbb{Z}$-linear basis of $R_\varepsilon := \mathbb{Z}[x_{\varepsilon,i,j}]_{i,j=1,2,\ldots,n}$.*

(2) *$L := \{(T|S)_1 \times (T'|S')_2 \mid (T, S)$ and (T', S') are standard bitableaux$\}$ is a $\mathbb{Z}$-linear basis of R.*

We shall call standard polynomials for the elements of L_ε and L.

3.3. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$ be a diagram of size m. The Young subgroup $\mathfrak{S}_\lambda$ of the symmetric group $\mathfrak{S}_m$ of degree m acts on the set of all tableaux of shape λ as the horizontal transformation group as follows.

$\mathfrak{S}_\lambda := \mathfrak{S}_{\lambda_1} \times \mathfrak{S}_{\lambda_2} \times \cdots \times \mathfrak{S}_{\lambda_r}$ (direct product). Let $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_r)$ be an element of $\mathfrak{S}_\lambda$ where $\sigma_i \in \mathfrak{S}_{\lambda_i}$. Then for a tableau T of shape λ, $\sigma(T)$ is the tableau given as

$$(\sigma(T))(i, j) = T(i, \sigma_i(j)) \qquad \text{for} \quad (i, j) \in \lambda.$$

For a skewdiagram $\lambda - \mu$, we define the horizontal transformation group $\mathfrak{S}_{\lambda-\mu}$ of $\lambda - \mu$ similarly. Actually, $\mathfrak{S}_{\lambda-\mu}$ is the subgroup of $\mathfrak{S}_\lambda$ consisting of all horizontal transformations which stabilize the first μ_i symbols of any ith row of any tableau of shape λ.

DEFINITION 3.4. Let $\mu - \gamma$ be a skewdiagram and let T be a tableau of shape λ such that μ is a subdiagram of λ. Then the subskewtableau $T|_{\mu-\gamma}$ is the skewtableau of shape $\mu - \gamma$ defined by

$$T|_{\mu-\gamma}(i, j) = T(i, j) \qquad \text{for any} \quad (i, j) \in \mu - \gamma.$$

Especially, when σ is a horizontal transformation of λ, we shall denote $C(\sigma; \mu)$ for $\sigma(C_\lambda)|_\mu$ (here $\gamma = \phi$).

EXAMPLE. Let $\lambda = (3, 2)$, $\mu = (2, 1)$, and let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \times \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \in \mathfrak{S}_3 \times \mathfrak{S}_2 = \mathfrak{S}_\lambda.$$

Then

$$\sigma(C_\lambda)=\begin{matrix}3 & 1 & 2\\ 2 & 1 & \end{matrix} \quad \text{and} \quad C(\sigma;\mu)=\begin{matrix}3 & 1\\ 2 & \end{matrix}\;.$$

DEFINITION 3.5. Let $\lambda-\mu$ be a skewdiagram and let T be a standard skewtableau of shape $\lambda-\mu$ and content τ. We assume that τ can be regarded as a diagram, that is, $\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_n$. For a tableau S of shape λ, we define a tableau T^{*S} of shape ${}^t\tau$ as follows. We shall keep our eyes upon a symbol k. Since T is standard, k can occur at most once in each row of T. So $T^{-1}(k)$, the inverse image of k with respect to the map T, is a vertical strip of size τ_k. We have a skewtableau $S|_{T^{-1}(k)}$ of shape $T^{-1}(k)$. Then the kth column of T^{*S} is defined as the sequence got by reading the symbols in $S|_{T^{-1}(k)}$ from the bottom to the top.

Especially if $\sigma \in \mathfrak{S}_\lambda$, we denote $T^{*\sigma}$ for $T^{*\sigma(C_\lambda)}$ for abbreviation.

EXAMPLE. Let

$$T=\begin{matrix}\cdot & \cdot & \cdot & 1,\\ \cdot & 1 & 2 & \\ 1 & & & \\ 1 & & & \end{matrix}$$

which is a standard skewtableau of shape $(4, 3, 1, 1)-(3, 1)$ and content $\tau=(4, 1)$. The transposed diagram of τ is ${}^t\tau=(2, 1, 1, 1)$.

$$\sigma=\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 1 & 3 & 4\end{pmatrix}\times\begin{pmatrix}1 & 2 & 3\\ 2 & 1 & 3\end{pmatrix}\times\begin{pmatrix}1\\ 1\end{pmatrix}\times\begin{pmatrix}1\\ 1\end{pmatrix}$$

is a horizontal transformation of $\lambda=(4, 3, 1, 1)$. Then

$$T^{*\sigma}=\begin{matrix}1 & 3\\ 1 & \\ 1 & \\ 4 & \end{matrix}$$

DEFINITION 3.6. A pair (T_1, T_2) of standard skewtableaux is called a good pair of type $(\lambda, \lambda'; \mu; \tau)$ if it satisfies the following conditions.

(1) T_1 and T_2 have shapes $\lambda-\mu$ and $\lambda'-\mu$ respectively.

(2) Both T_1 and T_2 have content τ.

(3) τ is a diagram (i.e., $\tau_1 \geqslant \tau_2 \geqslant \cdots \geqslant \tau_n$).

DEFINITION 3.7. Let (T_1, T_2) be a good pair of type $(\lambda, \lambda'; \mu; \tau)$. Then we define an element $f(T_1, T_2)$ of R by

$$f(T_1, T_2) := \sum_{\sigma \in \mathfrak{S}_\lambda} \sum_{\alpha \in \mathfrak{S}_{\lambda'}} \operatorname{sgn}(\sigma) \operatorname{sgn}(\alpha)(C(\sigma; \mu) \mid C(\alpha; \mu))_1 \times (T_1^{*\sigma} \mid T_2^{*\alpha})_2.$$

An element f of R is called a relative invariant of the Lie algebra $\mathfrak{b}$ if there exists a linear character χ of $\mathfrak{b}$ such that $b \cdot f = \rho(b) f = \chi(b) f$ for any element b of $\mathfrak{b}$. Since the characters of a Borel subalgebra of $\mathfrak{gl}(n, \mathbb{Z})$ are parametrized by the set of partitions of length n, a character of $\mathfrak{b} = \mathfrak{h} + \mathfrak{b}_2 + \mathfrak{b}_3$ is parametrized by $(p, q; \lambda; \lambda')$ where p and q are integers and λ and λ' are partitions of length n (recall that $\mathfrak{h} \approx \mathbb{Z} + \mathbb{Z}$). Here, the length of a partition means the length of it regarded as a sequence of nonnegative integers. (So the length of it as a sequence of positive integers may be shorter.)

$(p, q; \lambda, \lambda')$ corresponding to χ is called the weight of f. It is easily seen that p and q are nonnegative integers if $(p, q; \lambda, \lambda')$ corresponds to a relative invariant in R.

THEOREM 3.8. *Let (T_1, T_2) be a good pair of type $(\lambda, \lambda'; \mu; \tau)$. Then $f(T_1, T_2)$ is a relative invariant of $\mathfrak{b}$ of weight $(|\mu|, |\tau|; \lambda, \lambda')$.*

Proof. Let $\mathfrak{b} = \mathfrak{t} + \mathfrak{n}$ be the Levi decomposition of $\mathfrak{b}$. First we show that the nilpotent radical $\mathfrak{n}$ kills $f(T_1, T_2)$. Since $\mathfrak{n}$ is the direct sum of the nilpotent radicals $\mathfrak{n}_2$ if $\mathfrak{b}_2$ and $\mathfrak{n}_3$ of $\mathfrak{b}_3$, it suffices to show that both $\mathfrak{n}_2$ and $\mathfrak{n}_3$ kill $f(T_1, T_2)$.

Let $A_i = E_{i,i+1}$ be the $n \times n$ fundamental matrix such as its $(i, i+1)$ component is 1 and the other components are zero. Then $\{A_i \mid i = 1, 2, ..., n-1\}$ generates $\mathfrak{n}_2$, the Lie algebra consisting of upper triangular matrices of size n. We shall show that $A_i \cdot f(T_1, T_2) = 0$ for any $i = 1, 2, ..., n-1$. Suppose $\lambda_k > i$ and let $C_\lambda^{i,k}$ be the tableau got by replacing a symbol $i+1$ in the kth row of C_λ with an i. We define a polynomial $f^{i,k}(T_1, T_2)$ by

$$f^{i,k}(R_1, T_2) = \sum_{\alpha \in \mathfrak{S}_\lambda} \sum_{\beta \in \mathfrak{S}_{\lambda'}} \operatorname{sgn}(\alpha) \operatorname{sgn}(\beta)(\alpha(C_\lambda^{i,k})|_\mu | C(\beta, \mu))_1 \, (T_1^{*\alpha(C_\lambda^{i,k})} \mid T_2^{*\beta})_2.$$

Since there are two i's in the kth row of $C_\lambda^{i,k}$, there exists a horizontal transformation of λ which stabilizes $C_\lambda^{i,k}$. So $f^{i,k}(T_1, T_2) = 0$. On the other hand, because of the definition of $f(T_1, T_2)$, $A^i \cdot f(T_1, T_2) = \sum_{k=1,2,...,m} f^{i,k}(T_1, T_2)$ where m is the length of the $i+1$th column of λ. So n_2 annihilates $f(T_1, T_2)$. Similarly n_3 annihilates $f(T_1, T_2)$. The rest to be shown is the determination of the weight, which is easily got from the action of $\mathfrak{t}$ on R.

DEFINITION 3.9 (Word of a tableau). Let T be a skewtableau. The word of T is the sequence $w(T)$ of positive integers derived from T by reading the symbols in T from bottom to top in successive columns, starting with the first column.

A word $w = a_1, a_2, ..., a_r$ is said to be a lattice permutation (or Yamanouchi word) if for any positive integer p and i such that $p < r$, the number of occurrence of the symbol i in the subsequence $a_1, a_2, ..., a_p$ is not less than that of the symbol $i+1$ in the same subsequence.

We shall say that a skewtableau T has *L–R property* if $w(T)$ is a lattice permutation.

For a word $w = a_1, a_2, ..., a_r$, the dual word $w^+ = b_1, b_2, ..., b_r$ is defined such that b_i is the number of occurrence of the symbol a_i in the subsequence $a_1, a_2, ..., a_i$ of w. A word is a lattice permutation if and only if $(w^+)^+ = w$.

DEFINITION 3.10. The dual tableau T^+ of a skewtableau T is the tableau of same shape as T such that $w(T^+) = w(T)^+$.

EXAMPLE. If

$$T = \begin{matrix} & & 1 & 2 & 3 & 4, \\ & 1 & 2 & 3 & & \\ & 1 & & & & \end{matrix}$$

then

$$T^+ = \begin{matrix} & & 3 & 2 & 2 & 1. \\ & 2 & 1 & 1 & & \\ & 1 & & & & \end{matrix}$$

Remark. T^+ may not be an n-tableau.

Let T be a standard skewtableau of shape $\lambda - \mu$. Then we define $N(T) = \#(\mathrm{stab}_{\mathfrak{G}_\lambda} T^+) = \#(\mathrm{stab}_{\mathfrak{G}_{\lambda/\mu}} T^+)\ \#(\mathfrak{G}_\mu)$, where $\mathrm{stab}_G T^+$ means the stabilizer of T^+ in G and $\#(G)$ means the order of G.

LEMMA 3.11. *Let (T_1, T_2) be a good pair. Then $f(T_1, T_2)$ is divisible by $N(T_1) \cdot N(T_2)$ in R.*

Proof. Suppose that (T_1, T_2) is of type $(\lambda, \lambda'; \mu, \tau)$. Let σ be an element of $\mathrm{stab}_{\mathfrak{G}_\lambda} T^+$ and α and β be elements of $\mathfrak{G}_\lambda$. Since $\sigma = \sigma_1 \times \sigma_2$ where σ_1 and σ_2 belong to $\mathfrak{G}_\mu$ and $\mathfrak{G}_{\lambda/\mu}$ respectively, $(C(\sigma \cdot \alpha; \mu) \mid C(\beta; \mu))_1 = \mathrm{sgn}(\sigma_1)(C(\alpha; \mu) \mid C(\beta; \mu))_1$. Further, we claim that $(T_1^{*\sigma \cdot \alpha} \mid T_2^{*\beta})_2 = \mathrm{sgn}(\sigma_2)$ $(T_1^{*\alpha} \mid T_2^{*\beta})_2$. To prove this claim we recall the definition of $T_1^{*\alpha}$. The kth column of $T_1^{*\alpha}$ consists of the sequence written in $\alpha(C_\lambda)|_{T_1^{-1}(k)}$. So we find that the kth row of $T_1^{*\alpha}$ consists of the sequence written in $\alpha(C_\lambda)|_{(T_1^+)^{-1}(k)}$ read from left to right. Since $\sigma \cdot \alpha(C_\lambda)|_{(T_1^+)^{-1}(k)}$ and $\alpha(C_\lambda)|_{(\sigma T_1^+)^{-1}(k)}$ are the sequences which have the same members and the difference of the orderings of them corresponds to the permutation σ_2, the claim follows. Similar results holds for the stabilizer of T_2^+.

So from the definition of $f(T_1, T_2)$, $f(T_1, T_2)$ is divisible by $N(T_1)\, N(T_2)$.

DEFINITION 3.12. $F(T_1, T_2) := f(T_1, T_2)/N(T_1)\, N(T_2)$ is called the *L–R polynomial* corresponding to the good pair (T_1, T_2).

THEOREM 3.13 (Main theorem). *The set of all L–R polynomials of type* $(\lambda, \lambda'; \mu, \tau)$ *such that* $|\mu| = m_1$ *and* $|\tau| = m_2$ *is a* $\mathbb{Z}$*-linear basis of the relative invariants of* $\mathfrak{b}$ *in* R *of weight* $((m_1, m_2), \lambda, \lambda')$.

4. PROOF OF THE MAIN THEOREM

DEFINITION 4.1 (Transposed lexicographical order). Let λ and μ be dilagrams. We define a lexicographical order $>$ such that $\lambda > \mu$ means one of the following two conditions.

(a) $|\lambda| > |\mu|$. (b) $|\lambda| = |\mu|$ and there exists an index i such that $({}'\lambda)_j = ({}'\mu)_j$ for any $j < i$ and $({}'\lambda)_i > ({}'\mu)_i$.

If T is a standard tableau of shape λ and $i \leqslant n$, the inverse image of $\{1, 2, ..., i\}$ with respect to the map T is a subdiagram of λ. We denote this subdiagram by $T\langle i \rangle$.

DEFINITION 4.2. Let T and S be standard tableaux of shape λ and μ respectively. $T > S$ means that one of the following two conditions holds.

(a) $\lambda > \mu$. (b) $\lambda = \mu$ and there exists a symbol i such that $T\langle j \rangle = S\langle j \rangle$ for any $j < i$ and $T\langle i \rangle = S\langle i \rangle$.

Among standard bitableaux, we define a total order $>$ such that $(T_1, T_2) > (S_1, S_2)$ means (a) $T_1 > T_2$, or (b) $T_1 = T_2$ and $S_1 > S_2$.

By means of above ordering, we define a total order among the standard polynomials naturally. That is, $(T_1 | S_1)_1 \times (T_2 | S_2)_2 > (T_1' | S_1')_1 \times (T_2' | S_2')_2$ means (a) $(T_1, S_1) > (T_1', S_1')$ or (b) $(T_1, S_1) = (T_1', S_1')$ and $(T_2, S_2) > (T_2', S_2')$.

We prepare another (partial) order among the tableaux of a same fixed shape and of a same fixed content.

Notation 4.3. The sum of the number of occurrences of the symbol smaller than i in the rows from the top row to the rth row of a tableau T is denoted by $\# T\binom{i}{r}$.

DEFINITION 4.4. Let T and S be tableaux of a same shape. We denote $T \rightleftarrows S$ if T and S have a same content and $\# T(^i_j) \leqslant \# S(^i_j)$ for any i and r. $T \sim S$ means $T \rightleftarrows S$ and $S \rightleftarrows T$.

Remark. If T and S are standard tableaux and $T \rightleftarrows S$, then $T \geqslant S$.

Now we shall show the linear independence of the L–R polynomials. Let $F(T, S)$ be an L–R polynomial of type $(\lambda, \lambda'; \mu; \tau)$. Suppose we write $F(T, S)$ as a $\mathbb{Z}$-linear combination of standard polynomials.

Claim 4.5. *The highest nonzero term of above linear combination is* $(C_\mu | C_\mu)_1 \times (T^{*1} | S^{*1})_2$, *where* 1 *is the identity horizontal transformation. Further, its coefficient is* 1.

LEMMA A. *Let* (T, S) *be a bitableau of shape* λ. *When we write* $(T|S)_1$ *in a linear combination of standard polynomials of* R_0, *any nonzero term* $(T_0 | S_0)_1$ *satisfies one of the following conditions* (a) *or* (b).

(a) *The shape of* T_0 *is lower than* λ *with respect to the order* $>$.

(b) *The shape of* T_0 *is* λ *and both* $T \rightleftarrows T_0$ *and* $S \rightleftarrows S_0$.

This lemma is a direct consequence of the straightening law (Lemma S in Sect. 3, see also Lemma 1.5 in [6]). A similar statement holds for $(T|S)_2$.

Applying Lemma A to each term occurring in Definition 3.7 of L–R polynomials, it suffices to check the following (1), (2), and (3) to show Claim 4.5.

(1) For any $\alpha \in \mathfrak{S}_\lambda$, $C(1; \mu) \rightleftarrows C(\alpha; \mu)$.

(2) If $C(1; \mu) \sim C(\alpha; \mu)$, then $T^{*1} \rightleftarrows T^{*\alpha}$.

(3) $C(1; \mu) \sim C(\alpha; \mu)$ and $T^{*1} \sim T^{*\alpha}$ if and only if α stabilizes T^*.

It is a routine work to check (1), (2), and (3).

Claim 4.5 assures us that the highest terms of two arbitrary different L–R polynomials cannot coincide. So the linear independence of the L–R polynomials follows from the linear independence of the standard polynomials in R.

It remains to be shown that an arbitrary relative invariant of $\mathfrak{b}$ can be written as a $\mathbb{Z}$-linear combination of L–R polynomials. We shall show that the top term of any relative invariant must satisfy a typical condition. First, we prepare some facts about standard polynomials.

Let T be a standard tableau of shape λ and let p be a symbol of T. Suppose that we replace some p's from T by the same number of $p-1$'s to make a new tableau T_0. Then one of the following three cases occurs since there is at most one p in each row of T.

Case 1. T_0 is a standard tableau.

Case 2. There exists a suitable row of T_0 such that there are two $p-1$'s in it.

Case 3. T_0 is a normal tableau but not a standard tableau.

Let S be a standard tableau of shape λ. In Case 2, $(T_0 \mid S)_1 = (T_0 \mid S)_2 = 0$. The third case is most complicated. We define the operation $R(p, k \to k-1)$ on the set of tableau as follows. If there exist columns j of T such that $T(k, j) = p$ and $T(k-1, j) = p+1$, $R(p, k \to k-1)\, T$ is the tableau got by exchanging those p and $p+1$ for all such columns. Otherwise, $R(p, k \to k-1)\, T = T$.

We apply this operation to Case 3 as follows.

LEMMA. *Let T and T_0 be as above in Case* 3. *Let $m = ({}^t\lambda)_1$. Then $R(p-1, m \to m-1)\ R(p-1, m-1 \to m-2) \cdots R(p-1, 2 \to 1)\, T_0$ is a standard tableau. We denote $\bar{T}_0$ for this standard tableau.*

Proof of this lemma is easy since T is standard.

EXAMPLE. Let

$$T = \begin{matrix} 1 & 2 & 4 & 6 \\ 1 & 2 & 4 & \\ 2 & 4 & & \end{matrix} \qquad \text{and} \quad p = 4.$$

When we replace the symbol 4 in the second row of T by 3, the resulting tableau

$$T_0 = \begin{matrix} 1 & 2 & 4 & 6 \\ 1 & 2 & 3 & \\ 2 & 4 & & \end{matrix}$$

is a normal but not standard tableau, and

$$\bar{T}_0 = \begin{matrix} 1 & 2 & 3 & 6. \\ 1 & 2 & 4 & \\ 2 & 4 & & \end{matrix}$$

DEFINITION 4.6. $L_1(\lambda)$ is the $\mathbb{Z}$-submodule of R_1 spanned by all standard polynomials having lower shapes than λ with respect to $>$.

PROPOSITION 4.7. *In Case* 3, *using the notation above,* $(T_0 \mid S)_1 = (\bar{T}_0 \mid S)_1 + h$ *for a suitable element h of $L_1(\lambda)$.*

Proof. Using the straightening law for the $k-1$th row and kth row of $R(p-1, k-1 \to k-2) \cdots R(p-1, 2 \to 1)\, T_0$, we have

$$\begin{aligned}(R(p-1, k \to k-1)\, R(p-1, k-1 \to k-2) \cdots R(p-1, 2 \to 1)\, T_0 \mid S) \\ = (R(p-1, k-1 \to k-2)\, R(p-1, k-2 \to k-3) \cdots \\ \times R(p-1, 2 \to 1)\, T_0 \mid S)_1 + h_k\end{aligned}$$

where h_k is an element of $L_1(\lambda)$. Proposition 4.7 follows inductively because $h = h_1 + h_2 + \cdots + h_m \in L_1(\lambda)$.

Now let f be a relative invariant of $\mathfrak{b}$. We write

$$f = \sum_{T, T', S, S'} d(T, T'; S, S')\, (T \mid S)_1 \times (T' \mid S')_2 \cdots \qquad (*)$$

as a $\mathbb{Z}$-linear combination of standard polynomials.

Suppose that $(T_0 \mid S_0)_1 \times (A \mid B)_2$ is the highest nonzero term appearing in $(*)$. Since f is $\mathfrak{b}$-relative invariant, we see that $T_0 = S_0 = C_\mu$, a canonical tableau of a suitable type, from the argument in [4]. Here we must determine A and B. Let τ be the shape of A. We collect the terms $(C_\mu \mid C_\mu)_1 \times (T \mid S)_2$ in $(*)$ such that the shape of T is τ. We define

$$g := \sum_{T, S : (a)} d(C_\mu, C_\mu; T, S)(T \mid S)_2,$$

where the summation runs over bitableaux (T, S) satisfying the condition (a): the shape of T is τ. Then $f = (C_\mu \mid C_\mu)_1 \times g +$ lower terms.

CLAIM 1. *Let $k = ({}^t\mu)_i - ({}^t\mu)_{i+1}$. Then $(A_i)^{k+1} \cdot g = 0$ (see the proof of Theorem* 3.8 *for the definition of A_i).*

Proof. For abbreviation, we shall write $h^{(j)}$ for $(A_i)^j h$ for an element h of R. The ith column of the canonical tableau C_μ consists only of the symbol i, which occurs $({}^t\mu)_i$ times. Let C_μ^{ip} be the tableau got by exchanging the p pieces of i's with the same numbers of $i+1$'s in the ith column of C_μ from the bottom. Remark that C_μ^{ip} is a standard tableau if and only if p is smaller than or equal to k. We can prove the following lemma from the straightening law, Proposition 4.7, and Lemma A in 4.5.

LEMMA B. *Let $p \leqslant k-1$. Then $(C_\mu^{ip} \mid C_\mu)_1^{(1)} = p(C_\mu^{ip+1} \mid C_\mu)_1 +$ lower terms and there exists no other standard tableau T such that $(C_\mu^{ip+1} \mid C_\mu)_1$ occurs as a nonzero term in $(T \mid C_\mu)_1^{(1)}$.*

Considering the coefficient of $(C_\mu^{ik} \mid C_\mu)_1$ in $f^{(1)}$, Claim 1 is induced from

Lemma B. Further, it is simultaneously shown that the coefficient of $(C_\mu^{ip} | C_\mu)_1$ in f is $1/p! \times g^{(p)}$.

Now $(A | B)_2$ is the highest term in g. Then we define $g_0 = g_{A,B}$ by $g_0 := \sum_{T^{-1}(I) = A^{-1}(I)} d(C_\mu, C_\mu; T, B)(T | B)_2 \cdots$ (4.7.1) where $I = \{1, 2, ..., n\} - \{i, i+1\}$ and the summation runs over all standard tableaux T satisfying $T^{-1}(I) = A^{-1}(I)$.

CLAIM 2. *$g_0^{(k+1)}$ is a linear combination of standard polynomials $(T | S)$ either such that the shape of T is lower than τ or such that $T^{-1}(I) \neq A^{-1}(I)$.*

Proof. Because of the maximality of A, this claim follows from Lemma A in 4.5 and Claim 1.

We prepare a combinatorial lemma to complete the proof of theorem 3.13.

DEFINITION 4.8. Let Q be a sequence of two kinds of integers 1 and 2. For a nonnegative integer k, we say Q is a k-lattice permutation if 2 occurs in Q less than k times, or if the sequence Q' made from Q subtracted the first k pieces of 2 is a lattice permutation.

Let $K = (k_1, k_2, ..., k_{n-1})$ be a sequence of nonnegative integers. Let T be a standard tableau. We say that T satisfies the K-lattice permutation property if the subsequence of $w(T)$ consisting of all i's and $i+1$'s in $w(T)$ is a k_i-lattice permutation regarding i and $i+1$ as 1 and 2 respectively. (Note: $(0, 0, ..., 0)$-lattice permutation is the lattice permutation defined in 3.9.)

CLAIM 3. *Let $k_i = ({}^t\mu)_i - ({}^t\mu)_{i+1}$. Then both tableaux A and B occurring in the highest term $(A | B)_2$ of g satisfy the K-lattice permutation property.*

Let us prove the theorem assuming Claim 3. We construct an L–R bitableau (T_A, T_B) of type $(\lambda, \lambda'; \mu; \tau)$ where τ is the shape of A and λ (resp. λ') is determined by $({}^t\lambda)_i = ({}^t\mu)_i + (\nu_A)_i$ (resp. $({}^t\lambda')_i = ({}^t\mu)_i + (\nu_B)_i$). Here, ν_A and ν_B are contents of A and B respectively, and the K-lattice permutation property assures that $({}^t\mu)_i + (\nu_A)_i$ is nonincreasing with respect to i.

Suppose that the symbol p is located in the $i_1, i_2, ..., i_m$th rows of A, where $m = (\nu_A)_p$. Then we write the sequence $i_1, i_2, ..., i_m$ in the pth column of the skew diagram $\lambda - \mu$ column (strict) increasingly. After we do such operation starting from $p = 1$ to $p = n$, we get the tableau T_A. Since A satisfies the K-lattice permutation property, it follows that T_A is a standard tableau. Since A is a standard tableau, it follows that T_A has the L–R property. T_B is constructed similarly.

Now we have an L–R bitableau (T_A, T_B). Recall that the L–R polynomial $F(T_A, T_B)$ has the highest term $(C_\mu | C_\mu)_1 \times (A | B)_2$ (Claim 4.5).

Since both f and $F(T_A, T_B)$ are $\mathfrak{b}$-relative invariants of a same weight, $f - mF(T_A, T_B)$ is also a $\mathfrak{b}$-relative invariant for any constant m. Since the coefficient of the highest term in $F(T_A, T_B)$ is 1, we can choose an integer m such that the coefficient of $(C_\mu | C_\mu)_1 \times (A | B)_2$ in $h = f - mF(T_A, T_B)$ is zero. Then h is a relative invariant of $\mathfrak{b}$ such that its highest term is lower than that of f. So the theorem is proved by induction on the height of the highest terms with respect to the total order $>$ among the standard polynomials.

The rest to be shown is the proof of Claim 3.

Let $X = 1, 1, ..., 1, 2, 2, ..., 2, ..., n, ..., n$ be a nondecreasing sequence of positive integers. The number of occurrence of i in X is denoted by $m_X(i)$. We associate an indeterminant x_L for any subsequence L of X. For a natural number p, $X(p)$ denotes the set of all subsequences of length p of X. Let q be a natural number smaller then p. Let us consider an arbitrary map Ω from $X(p)$ into the set of real numbers. For any element M of $X(p)$, we define a polynomial $F_{q,\Omega}(M) := \sum_L C(L, M) \; \Omega(M) \, x_L$, where the summation runs over all subsequence L of M of length $p - q$. $C(L, M)$ denotes the repeated combination number to choose L out of M, i.e.,

$$C(L, M) = \prod_{i=1}^{n} \binom{m_M(i)}{m_L(i)}$$

(product of combination numbers).

LEMMA C. *Suppose* $\sum_{M \in X(p)} F_{q,\Omega}(M) = 0$ *for a nonzero map* Ω. *Then there exists an element* M *in* $X(p)$ *satisfying the following two conditions.*

(a) $\Omega(M) \neq 0$.

(b) *Let us denote the subsequence of* M *consisting of its last* $p - q$ *elements by* $\bar{M}$. *Let* $X - M$ *be the sequence made excluding the subsequence* M *from* X. *Then* $X - M > \bar{M}$, *where* $>$ *means both that the* i*th element of* $X - M$ *is smaller than the* i*th element of* $\bar{M}$ *for any* i *and that the length of* $X - M$ *is not shorter than that of* $\bar{M}$.

This lemma is an analogue of Lemma B, p. 362 in [16] and we can prove it similarly.

To apply this lemma to our problem, let T be a standard tableau and i be a symbol of T. If there exist both symbols i and $i+1$ in some row of T, we omit those rows. Then we get a standard tableau T' in which there exists no row containing both i and $i+1$. Suppose that i and $i+1$ occur k_r^1 and k_r^2 times in the rth column of T' respectively. We define a sequence $X(T)$ and subsequence $M(T)$ by $m_{X(T)}(r) = k_r^1 + k_r^2$ and $m_{M(T)}(r) = k_r^2$. Let us recall the definition of g_0. In the summation of the right hand of 4.7.1, we collected the tableau T such that $T^{-1}(I) = A^{-1}(I)$. It is easily seen that

$X(T)=X(A)$ for any T satisfying $T^{-1}(I)=A^{-1}(I)$. We set $X=X(A)$, p is the length of $M(A)$, $q=p-k-1$, and

$$\Omega(M(T)) := \begin{cases} d(C_\mu, C_\mu; T, B) & \text{if } T^{-1}(I)=A^{-1}(I) \\ 0 & \text{otherwise.} \end{cases}$$

Then we have $\sum_T F_{q,\Omega}(M(T))=0$ from Claim 2 and Proposition 4.7. Then Lemma C indicates that the subsequence of $w(A)$ consisting of all i's and $i+1$'s in $w(A)$ is a k_i-lattice permutation. Now we conclude that A satisfies the K-lattice permutation property. Similarly, B satisfies the K-lattice permutation property too. Q.E.D.

5. Littlewood–Richardson Rule

Let λ be a partition such that λ_1 is not longer than n. Then we can correspond an irreducible representation ρ_λ of $\mathfrak{gl}(n, \mathbb{C})$ to λ such that the highest weight of ρ_λ is ${}^t\lambda$. The $\mathbb{C}$-linear span of the standard polynomials $(T|C_\lambda)_1$ for all standard tableaux T is a representation space of ρ_λ with respect to the action of the subalgebra $\mathfrak{g}_0 := 0+\mathfrak{gl}(n, \mathbb{C})+0 \subset \mathfrak{h}_{\mathbb{C}}+\mathfrak{gl}(n, \mathbb{C})+\mathfrak{gl}(n, \mathbb{C})=\mathfrak{g}_{\mathbb{C}}$, where $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ are the complexification of $\mathfrak{g}$ and $\mathfrak{h}$ respectively (cf. [7]).

We denote V_λ for this representation space.

Now we shall study the branching of the tensor product representation space $V_\lambda \otimes V_\mu$ of $\mathfrak{gl}(n, \mathbb{C})$ embedded into the complexification of R.

Note. In many texts (e.g., [18]) they use a transposed diagram version of our notation to parametrize irreducible representations of $\mathfrak{gl}(n, \mathbb{C})$.

A representation space of $\rho_\lambda \otimes \rho_\mu$ is constructed in $R_{\mathbb{C}}$, the complexification of R. Indeed, the $\mathbb{C}$-linear span $V_{\lambda,\mu}$ of the set of standard polynomials $\{(T|C_\lambda)_1 \times (S|C_\mu)_2 \mid T \text{ and } S \text{ are standard tableaux of shape } \lambda \text{ and } \mu \text{ respectively}\}$ is a representation space of $\rho_\lambda \otimes \rho_\mu$. From the general theory on the representations of complex reductive Lie algebra (cf. [9]), this representation is completely reducible and the maximal weight vectors (i.e., relative invariants of a Borel subalgebra of $\mathfrak{g}_0$) of weight ${}^t\nu$ in this representation space become the highest weight vectors of the irreducible constituents ρ_ν of $\rho_\lambda \otimes \rho_\mu$. Let us study the space of maximal weight vectors of weight ν with respect to the Borel subalgebra $0+\mathfrak{b}_2+0$ of $\mathfrak{g}_0=0+\mathfrak{gl}(n, \mathbb{C})+0$. Since all vectors in $V_{\lambda,\mu}$ are $\mathfrak{h}+0+b_3$ relative invariant, it suffices to find all relative invariants of $\mathfrak{b}$ in $V_{\lambda,\mu}$. From the result of Section 4, a relative b-invariant in $V_{\lambda,\mu}$ is a linear combination of L–R polynomials $F(A, T)$ such that A has deficiency λ and content ${}^t\mu$, and T is a particular skewtableau of shape $\tau-\lambda$ defined by $T(i,j)=j$ for

$(i, j) \in \tau - \lambda$, where τ is the partition defined by $({}^t\tau)_i = ({}^t\lambda)_i + ({}^t\mu)_i$ for any i. The reason why T is such a tableau is the Claim 4.5. Conversely, from the definition of $F(A, T)$, such linear combination is of weight ν with respect to $\mathfrak{b}_2$ if and only if all A's appearing in it have shape $\nu - \lambda$. Then we have the following corollaries.

COROLLARY 5.1. *In the notations above, the space of the highest weight vectors corresponding to the irreducible constituent ρ_ν in $V_{\lambda,\nu} \approx V_\lambda \otimes V_\mu$ has a canonical basis*

$\{F(A, T) \mid$ L–R polynomials of type $(\nu, \tau; \lambda; {}^t\mu)$ such that T is the fixed skewtableau defined above$\}$.

COROLLARY 5.2 (Littlewood–Richardson rule). *Let ρ_λ and ρ_μ be irreducible representations of $\mathfrak{gl}(n, \mathbb{C})$. Then the multiplicity of ρ_ν in the tensor product representation $\rho_\lambda \otimes \rho_\mu$ equals the number of skewtableaux of shape $\nu - \lambda$ and the content ${}^t\mu$ satisfying L–R property.*

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