JOURNAL OF ALGEBRA **206,** 235–260 (1998)
ARTICLE NO. JA987435

Determining Group Structure from Sets of Irreducible Character Degrees

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Communicated by Walter Feit

Received October 17, 1997

1. INTRODUCTION

Let G be a (finite) group and write $\mathrm{Irr}(G)$ for the set of irreducible characters of G. We shall refer to the set $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ as the "character degrees" of G, instead of the more precise "irreducible character degrees."

In this paper, we consider the following problem: given the set $\mathrm{cd}(G)$, what can be said about the structure of G? The basic tools for studying this question can be found in Chapter 12 of [6], and more recently an expository article by Huppert outlined some of the results in this area (see [3]). In this paper, we will look at particular sets of character degrees and obtain surprisingly strong structural information about G. This is demonstrated in the following theorem.

THEOREM A. *Let p, q, and r be distinct primes. If G is a group such that $\mathrm{cd}(G) = \{1, p, q, r, pq, pr\}$, then $G = A \times B$, where $\mathrm{cd}(A) = \{1, p\}$ and $\mathrm{cd}(B) = \{1, q, r\}$.*

This result arose from [13]. In that paper, we proved that this set of character degrees was one possibility for the character degrees of groups that satisfy particular properties. We stated that direct products were one way to produce groups that have this set of character degrees, and we asked whether there were any other ways to produce groups that have this set of character degrees. In attempting to build examples of other groups having this set of character degrees, we proved that no other examples exist. For the groups studied in [13], Corollary 5.3 suffices to answer the

0021-8693/98 $25.00

question. Also, we can use one of the intermediate results for Theorem A (Theorem 5.2) to prove a similar result when we have four distinct primes. This is demonstrated in the following theorem.

THEOREM B. *Let p, q, r, and s be distinct primes. If G is a group such that* $\mathrm{cd}(G) = \{1, p, q, r, s, pr, ps, qr, qs\}$, *then* $G = A \times B$, *where* $\mathrm{cd}(A) = \{1, p, q\}$ *and* $\mathrm{cd}(B) = \{1, r, s\}$.

Our proofs of both Theorems A and B use a result of Huppert and Manz [4, Theorem 2.4] to reduce to the case where the group is solvable. Their result relies on the classification of finite simple groups. Therefore, we have implicitly used the classification to reduce to the case of solvable groups. Our contribution to the problem is to complete the solvable case. We believe that our results are surprising, since the similar result is not true when we have only two distinct primes. In particular, we will present an example where $\mathrm{cd}(G) = \{1, p, q, pq\}$ for distinct primes p and q, but where G is not a direct product.

One may also view this result as eliminating sets of integers as possibilities for being the set of character degrees for some group. In particular, the question is: given a set of integers, is there a group that has this set as its set of character degrees? Our results say that if we have distinct primes p, q, and r so that $\{1, p, q, r, pq, pr\}$ is the set of character degrees for some group, then $\{1, q, r\}$ must also be the set of character degrees for some group. This situation was studied in [9], and from that paper, we obtain some arithmetic restrictions between q and r. Similarly, if p, q, r, and s are primes so that $\{1, p, q, r, s, pr, ps, qr, qs\}$ is the character degree set of some group, then $\{1, p, q\}$ and $\{1, r, s\}$ are character degree sets. Hence, we have restrictions from [9] between p and q and between r and s.

2. REDUCING TO SOLVABLE GROUPS

Since the techniques used to reduce our problem to the solvable case do not affect the rest of the paper, we separate out this reduction and dispense with it first. The following lemma is an easy consequence of Theorem 2.8 of [4]. Note that the proof used by Huppert and Manz [4] is dependent upon the classification of finite simple groups.

LEMMA 2.1. *Let $\{p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m\}$ be a set of distinct primes. If G is a group where* $\mathrm{cd}(G) = \{1, p_i, q_j, p_i q_j \mid 1 \le i \le n,\ 1 \le j \le m\}$, *then G is solvable.*

Proof. Suppose that G is not solvable. Observe that all the integers lying in $\mathrm{cd}(G)$ are square-free. Thus, we are in the situation of Theorem

2.8 of [4]. From that theorem, we know that $G = A \times S$, where A is isomorphic to the alternating group on seven letters (Alt(7)) and S is solvable. It is well known that $\text{cd}(A) = \{1, 6, 10, 14, 15, 21, 35\}$. On the other hand, we observe that any prime that divides a character degree of a direct summand of G must in fact be a character degree of that direct summand. Since A has no primes as character degrees, A cannot be a direct summand of G. Therefore, G must be solvable. ∎

Suppose that a group G has a character degree set of $\{1, p_i, q_j, p_i q_j \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$, where $\{p_1, p_2, \ldots, p_n, q_1, q_2, \ldots, q_m\}$ is a set of distinct primes. We claim that $n \leq 2$ and $m \leq 2$. To prove this, we begin by using Lemma 2.1 to see that G must be solvable. For solvable groups, we know that if $\{p_1, p_2, \ldots, p_n\}$ is a set of distinct primes with $n \geq 3$, then there exist integers $i \neq j$ so that $p_i p_j$ divides some character degree of G [14, Theorem 18.7]. Since this does not happen in G, we conclude that $n \leq 2$. In similar manner, we can see that $m \leq 2$.

3. COPRIME ACTIONS

In this section, we prove two short results about coprime actions. These are situations where we have a group S act by automorphisms on a group G with $(|S|, |G|) = 1$. We begin by looking at a situation that was studied by Isaacs [7]. This is the case when S acts nontrivially on G, but where S stabilizes all the nonlinear irreducible characters of G. The version of Isaacs' result that we use can be found in [14] as Theorem 19.3(b)(ii). Our next result provides more information for the case when G is nilpotent.

LEMMA 3.1. *Let the group S act coprimely by automorphisms on the group G, where $G' \subseteq \mathbf{Z}(G)$, and assume that S fixes all nonlinear irreducible characters of G. Let H be a subgroup of G, where $G' \subseteq H$ and H admits the action of S, and suppose that $[H, G] < G'$. Then $H \subseteq \mathbf{C}_G(S)$.*

Proof. First, we show that the action of S on G' is trivial. If λ is any nonprincipal linear character of G', then λ is a constituent of $\chi_{G'}$, where χ is a nonlinear irreducible character of G. Since χ is S-invariant and $\chi_{G'} = \chi(1)\lambda$ (recall that G' is central), it follows that λ is S-invariant. Thus S fixes all the members of $\text{Irr}(G')$, and hence it acts trivially on G', as claimed. It suffices now to show that S acts trivially on H/G'.

Let ν be a nontrivial linear character of $G'/[H, G]$. Note that all the irreducible constituents of ν^H are linear and G-invariant since $H/[H, G]$ is a central factor of G. Fix μ as an irreducible constituent of ν^H and let δ be an irreducible constituent of μ^G, so that $\delta_H = \delta(1)\mu$. Because

$G' \nsubseteq \ker(\mu)$, we see that δ is not linear and δ is S-invariant. It follows that μ must be S-invariant. Therefore, all the irreducible constituents of ν^H are linear and S-invariant.

To show that S acts trivially on H/G' it suffices to show that every irreducible character α of H/G' is S-invariant. If μ is an irreducible constituent of ν^H, then $\alpha\mu$ is also an irreducible constituent of ν^H. By the previous paragraph, both μ and $\alpha\mu$ are S-invariant. Therefore, α must be S-invariant. ∎

The second result in this section looks at the case where S acts on G with the property that $[G, S] \subseteq \mathbf{Z}(G)$. In this case, we prove that G is a direct product. This lemma will be used on several occasions to find the direct products in Theorems A and B.

LEMMA 3.2. *Let S act on G so that $(|S|, |G|) = 1$. If $[G, S] \subseteq \mathbf{Z}(G)$, then $G = [G, S] \times \mathbf{C}_G(S)$.*

Proof. From general facts about coprime actions, we know that $G = [G, S]\mathbf{C}_G(S)$ and, since $[G, S] \subseteq \mathbf{Z}(G)$ by hypothesis, it follows that $\mathbf{C}_G(S)$ is normal in G. As $[G, S]$ is known to be normal in G automatically, all that remains is to show that $[G, S] \cap \mathbf{C}_G(S) = 1$. This is clear by Fitting's lemma applied to the action of S on $[G, S]$. Because $[[G, S], S] = [G, S]$, we conclude that $\mathbf{C}_{[G, S]}(S) = 1$. ∎

4. GROUPS WITH $\operatorname{cd}(G) = \{1, m, n\}$ WHERE m AND n ARE COPRIME

In this section, we consider a group G, where $\operatorname{cd}(G) = \{1, m, n\}$ for coprime integers m and n. The case where m and n are distinct primes was originally studied by Isaacs and Passman [9]. A more recent paper by Noritzsch [15] has taken their results and expanded them to the general case. While in this paper we only use the case where m and n are primes, the improvements that we make are true for the general case. Thus, we will refer to the results in [15]. These groups fall into two categories based on their Fitting heights. Using standard notation, we define the *Fitting subgroup* $\mathbf{F}(G)$ of G to be the largest normal nilpotent subgroup of G. Inductively, we define $F_0 = 1$ and $F_{i+1}/F_i = \mathbf{F}(G/F_i)$ for integers $i \geq 0$. When G is solvable, it is clear that there is some integer i so that $F_i = G$. We define the Fitting height of G to be the smallest integer i so that $F_i = G$. From a theorem of Garrison [6, Corollary 12.21], we know that the Fitting height of G is less than or equal to $|\operatorname{cd}(G)|$. When $\operatorname{cd}(G) = \{1, m, n\}$, it follows that the Fitting height of G is at most 3. Since $mn \notin \operatorname{cd}(G)$, we can see that G is not nilpotent and G has Fitting height at least 2. The groups of interest can be categorized as those having Fitting height 3 and those having Fitting height 2. The next result mostly is the content of Theorem 3.5 of [15], but it does include some small improvements. The parts that are new in this paper are (a)(ii), (b)(v)–(vii), and the nilpotence class in (b)(i). The other parts of (a) that do not precisely match the results in [15] follow from the results used to prove (a)(ii). In the next lemmas, we discuss fully ramified characters. Following [6], we say when N is a normal subgroup of G that the characters $\chi \in \mathrm{Irr}(G)$ and $\theta \in \mathrm{Irr}(N)$ are *fully ramified* with respect to G/N if θ is G-invariant and χ is the unique irreducible constituent of θ^G.

LEMMA 4.1. *Let G be a group with $\mathrm{cd}(G) = \{1, m, n\}$, where m and n are relatively prime integers. Write $F = \mathbf{F}(G)$ (the Fitting subgroup of G).*

(a) *Assume G has Fitting height 3. Write $E/F = \mathbf{F}(G/F)$ and $P = [E, F]$. Then the following are true:*

(i) *$F = P \times \mathbf{Z}(G)$ and $\mathrm{cd}(G) = \mathrm{cd}(G/\mathbf{Z}(G)) = \{1, |G:E|, |E:F|\}$.*

(ii) *P is a minimal normal subgroup of G and $P = E'$.*

(iii) *$\mathrm{cd}(E) = \{1, |E:F|\}$ and F is abelian.*

(iv) *$E/\mathbf{Z}(G)$ is a Frobenius group with kernel $F/\mathbf{Z}(G)$.*

(v) *$|G:E|$ is a prime number and E/F is a cyclic group.*

(vi) *$|P| = p^{a|G:E|}$ for some prime p and some positive integer a.*

(vii) $|E:F|/(|E:F|, p^a - 1) = (p^{a|G:E|} - 1)/(p^a - 1)$.

(b) *Assume G has Fitting height 2. Then the following are true:*

(i) *$F = P \times Z$, where P is Sylow p-subgroup of G for some prime p such that P has nilpotence class 2.*

(ii) *$Z \subseteq \mathbf{Z}(G)$ and $\mathrm{cd}(G) = \mathrm{cd}(G/Z)$.*

(iii) *$|G:F| \in \mathrm{cd}(G)$and $\mathrm{cd}(F) = \mathrm{cd}(G) \setminus \{|G:F|\}$.*

(iv) *G/F is cyclic.*

Let R be a p-complement for G and write $C = \mathbf{C}_P(R)$. Then:

(v) *$G' = [P, R]$ and $P' \subseteq C$.*

(vi) *$P/P' = C/P' \times G'/P'$ and $G/P' = C/P' \times G'R/P'$.*

(vii) *If $\delta \in \mathrm{Irr}(P)$ is a nonlinear character, then δ is fully ramified with respect to P/C.*

Proof. First, assume that G has Fitting height 3. By Theorem 3.5(1) of [15], we know that $F = P \times \mathbf{Z}(G)$, where P is an abelian p-group, $P/\Phi(P)$ is a chief factor for G, where $\Phi(P)$ is the Frattini subgroup of P, and $G/(\Phi(P)\mathrm{Z}(G))$ is what Noritzsch defined as an affine semilinear group with $\mathrm{cd}(G/(\Phi(P)\mathbf{Z}(G))) = \mathrm{cd}(G)$. From Theorem 2.1.1 of [15], we obtain

(1) $\mathrm{cd}(G/(\Phi(P)\mathbf{Z}(G))) = \{1, |G:E|, |E:F|\}$, (2) $\mathrm{cd}(E) = \{1, |E:F|\}$, (3) $|G:E|$ is prime, (4) $|P:\Phi(P)| = p^{a|G:E|}$ for some positive integer a, and (5) $|E:F|/(|E:F|, p^a - 1) = (p^{a|G:E|} - 1)/(p^a - 1)$. We determine that E/F is cyclic and $E/(\Phi(P)\mathbf{Z}(G))$ is a Frobenius group with kernel $F/(\Phi(P)\mathbf{Z}(G))$ from Theorem II.3.11 of [2]. Thus to complete the proof of (a) is suffices to show that $P = [E, F] = E'$ and $\Phi(P) = 1$. (Since $P/\Phi(P)$ is chief factor of G, $\Phi(P) = 1$ implies that P is a minimal normal subgroup of G.)

We begin by proving that $P = [E, F] = E'$. Let H be a subgroup of E so that $\mathbf{Z}(G) \subseteq H$ and $H/\mathbf{Z}(G)$ is a p-complement for $E/\mathbf{Z}(G)$. Since $E/(\Phi(P)\mathbf{Z}(G))$ is a Frobenius group with kernel $F/(\Phi(P)\mathbf{Z}(G))$, we have $E = PH$ and $F = [F, E]\Phi(P) \times \mathbf{Z}(G)$. Thus, $P = [F, E]\Phi(P)$. Because $\Phi(P)$ is the Frattini subgroup of P, we know that $P = [E, F]$. This implies that $P = [P, H]$ and by Fitting's lemma, $\mathbf{C}_P(H) = 1$. Since $F = P \times \mathbf{Z}(G)$ and E/F is cyclic, it follows that $E/\mathbf{Z}(G)$ is abelian and $E' \subseteq P$. On the other hand, recall that $P = [E, F] \subseteq E'$, which yields $P = E'$.

We will suppose that $\Phi(P) > 1$, and obtain a contradiction. Without loss of generality, we assume that $|G:E| = m$ and $|E:F| = n$. Since we are assuming that $\Phi(P) > 1$, there is a character $\gamma \in \mathrm{Irr}(P)$ so that $\Phi(P) \not\subseteq \ker(\gamma)$. Let T be the stabilizer of γ in G. Observe that T is also the stabilizer of the character $\gamma \times 1_{\mathbf{Z}(G)}$, and since E/F is cyclic, $\gamma \times 1_{\mathbf{Z}(G)}$ must extend to $T \cap E$. This implies that $|E:T \cap E| \in \mathrm{cd}(E)$. Because $P = E'$ and $\gamma \neq 1_P$, we conclude that $E \neq T \cap E$. The only other possibility is that $T \cap E = F$.

Observe that if θ is an irreducible constituent of γ^E, then $\theta(1) = |E:F| = n$ and θ must be invariant in G. By a Frattini argument, we have $G = TE$. Note that $T/F \cong TE/E = G/E$ is cyclic, so γ extends to T. Writing γ^p for the pth power of γ we know since $\Phi(P) \not\subseteq \ker(\gamma)$ that $\gamma^p \neq 1$. Also, it is easy to see that the stabilizer of γ^p contains T (anything that stabilizes γ must stabilize γ^p). Because $\mathbf{C}_P(H) = 1$, the stabilizer of γ^p in E is F, and T must be the stabilizer of γ^p in G.

Consider a character $\varphi \in \mathrm{Irr}(P/\Phi(P))$ and note that $\varphi^p = 1$. In a similar manner, we can prove that the stabilizer in G of $\gamma\varphi$ equals the stabilizer in G of $(\gamma\varphi)^p$. On the other hand, we observe that $(\gamma\varphi)^p = \gamma^p\varphi^p = \gamma^p$, which implies that T is the stabilizer of $\gamma\varphi$ in G. Because T stabilizes γ, it follows that T stabilizes φ. Therefore, T stabilizes every irreducible character of $P/\Phi(P)$. This implies that $T/\Phi(P)$ centralizes $P/\Phi(P)$. We know that $\mathbf{C}_{G/\Phi(P)}(P)$ is a normal subgroup of $G/\Phi(P)$ and that $\mathbf{C}_{G/\Phi(P)}(P) \cap E/\Phi(P) = F/\Phi(P)$. Therefore, we conclude that $\mathbf{C}_{G/\Phi(P)}(P) = T/\Phi(P)$, but we now have $G/F = E/F \times T/F$, which contradicts $E/F = \mathbf{F}(G/F)$.

Suppose now that G has Fitting height 2. Then by Theorem 3.5(2) of [15], we know that (1) cd(G) = $\mathrm{cd}(G/Z)$, (2) |G:F| $\in$ $\mathrm{cd}(G)$, (3) $\mathrm{cd}(F) =$

$\mathrm{cd}(G) \setminus \{|G:F|\}$, (4) G/F is cyclic, and (5) $F = P \times Z$, where $P \in \mathrm{Syl}_p(G)$ and $Z \subseteq \mathbf{Z}(G)$. Also, we have $\mathrm{cd}(P) = \mathrm{cd}(F)$ and see that P is not abelian. Furthermore, we know that every nonlinear irreducible character of P is G-invariant. Let R be a p-complement for G and observe that R acts nontrivially on P fixing every nonlinear irreducible character. By Theorem 19.3(a) of [14], we see that $[P, R]' = P'$ and P is nilpotent of class 2. Since F/P is central in G/P and G/F is cyclic, we determine that G/P is abelian and R is abelian. We can now conclude that $G' = [P, R]P'$, and because $P' \subseteq [P, R]$, we deduce that $G' = [P, R]$. Applying Lemma 3.1 again, we see that $P' \subseteq C$. We now use Fitting's lemma to determine that $P/P' = C/P' \times [P, R]/P' = C/P' \times G'/P'$, and with R centralizing B, we deduce that $G/P' = C/P' \times G'R/P'$. Taking $\delta \in \mathrm{Irr}(P)$ as a nonlinear character, we utilize the fact that P has nilpotence class 2 to show that δ vanishes on $P \setminus \mathbf{Z}(\delta)$ (see [6, Corollary 2.30 and Theorem 2.31]). Using Problem 6.3 of [6], this implies that δ is fully ramified with respect to $P/\mathbf{Z}(\delta)$. Observe that $P' \subseteq \mathbf{Z}(P) \subseteq \mathbf{Z}(\delta)$ and $[\mathbf{Z}(\delta), P] \subseteq \ker(\delta) \cap P' < P'$. Thus, we may apply Lemma 3.1 to determine that $\mathbf{Z}(\delta) \subseteq C$. Finally, we refer to Lemma 12.4 of [14] to say that δ is fully ramified with respect to P/C. ∎

We now look at a group G which has a normal subgroup K with $\mathrm{cd}(G/K) = \{1, m, n\}$ for coprime integers m and n and G/K has Fitting height 3. In our next result, we are interested in looking at the irreducible characters of K. The hypotheses of our result involve an oddness condition. After the lemma, we will present an example that shows that this oddness condition is necessary.

When N is a normal subgroup of G so that G/N is abelian and θ is a G-invariant irreducible character on N, it is well known that θ induces a nondegenerate bilinear form $\langle\langle \cdot , \cdot \rangle\rangle_\theta$ on G/N (see [5], for example). (A bilinear form on a group A is a map $\langle\langle \cdot , \cdot \rangle\rangle: A \times A \to \mathbf{C}^*$ that is a homomorphism in each coordinate. The form is nondegenerate if either $\langle\langle a, A \rangle\rangle = 1$ or $\langle\langle A, a \rangle\rangle = 1$ implies that $a = 1$. We also note that $\langle\langle \cdot , \cdot \rangle\rangle_\theta$ is also symplectic, but that is not important for the purposes of this paper, so we will not define it here.) We also use in this next lemma some machinery that was developed in Section 2 of our paper [12]. Let C be a cyclic group and take p to be a prime that does not divide $|C|$. Write k for the smallest positive integer so that $|C|$ divides $p^k - 1$ and let $\mathcal{F}$ denote the field having order p^k. Observe that the multiplicative group of $\mathcal{F}$ has a unique subgroup that is isomorphic to C and identify these two groups. If we take M to be the additive group of $\mathcal{F}$, then we obtain an action of C on M by multiplication in $\mathcal{F}$. A bilinear form $\langle\langle \cdot , \cdot \rangle\rangle$ on M is C-invariant if $\langle\langle m^c, n^c \rangle\rangle = \langle\langle m, n \rangle\rangle$ for all elements m, $n \in M$ and $c \in C$. We say that M is self-dual under the action of C if M has a nondegenerate C-invariant bilinear form. In Lemma 2.6 of [12], we prove that if M is self-dual under the action of C, then -1 is congruent to a power of $p \bmod |C|$.

LEMMA 4.2. *Suppose that K is a normal subgroup of G so that* $\mathrm{cd}(G/K) = \{1, m, n\}$ *for relatively prime integers m and n. Assume that G/K has Fitting height 3 and write $F/K = \mathbf{F}(G/K)$ and $E/F = \mathbf{F}(G/F)$. When $2 \in \mathrm{cd}(G)$, assume that $[E, F]K/K$ is not a 2-group. If the character $\theta \in \mathrm{Irr}(K)$ is E-invariant, then θ extends to $[E, F]K$.*

Proof. Write $P = [E, F]K$. From Lemma 4.1(a), we have (1) $\mathrm{cd}(G/K) = \{1, |G:E|, |E:F|\}$, (2) $|G:E|$ is a prime, and (3) E/F are cyclic. Without loss of generality, take $|G:E| = m$ and $|E:F| = n$. By Lemma 4.1(a), we know that P/K is a chief factor for G. Let p be the prime dividing $|P:K|$. From Lemma 4.1(a), we determine that $|P:K| = p^{am}$ for some positive integer a and that $n/(n, p^a - 1) = (p^{am} - 1)/(p^a - 1)$. Because E/F is cyclic, there is a subgroup A in E that contains F, where $|A:F| = (p^{am} - 1)/(p^a - 1)$. Thus, P/K is a chief factor for A and, in fact, we can view P/K as the field having order p^{am} and the action of A/F as multiplication in the field (this is Satz II.3.10 of [2]). In the notation stated before this lemma, P/K is isomorphic to M under the action of A/F. Since θ is E-invariant, either θ extends to P or θ is fully ramified with respect to P/K (this is Problem 6.12 of [6]). If θ is fully ramified with respect to P/K, then $\langle\langle \cdot, \cdot \rangle\rangle_\theta$ is an E-invariant nondegenerate bilinear form on P/K (this is proved in [5]). This implies that P/K is self-dual under the action of E/F. By Lemma 2.6 of [12], we determine that -1 is congruent to some power of $p \bmod (p^{am} - 1)/(p^a - 1) = p^{a(m-1)} + p^{a(m-2)} + \cdots + p^a + 1$. The only way that this can happen is if $m = 2$, and since m and n are coprime, n must be odd. Therefore, $p = 2$, but this contradicts the hypothesis that P/K is not a 2-group. ∎

Notice that $\mathrm{GL}(2, 3)$ shows that the oddness hypothesis in Lemma 4.2 is necessary. In particular, let $G = \mathrm{GL}(2, 3)$, and take Z to be the center of G. Let Q be the normal quaternion group in G and let S be the subgroup of G that is isomorphic to $\mathrm{SL}(2, 3)$. Note that $Q/Z = F(G/Z)$, that $S/Q = F(G/Q)$, and that $Q = [S, Q]Z$. Furthermore, it easy to see that $\mathrm{cd}(G/Z) = \{1, 2, 3\}$. Finally, if λ is the nonprincipal irreducible character of Z, then λ is G-invariant and is fully ramified with respect to Q/Z.

5. DIRECT PRODUCTS WHEN $\mathrm{cd}(K) = \{1, q, r\}$

In this section, we consider a group G which has a normal subgroup K, where $\mathrm{cd}(K) = \{1, q, r\}$ for distinct primes q and r. We also have a third prime p with the property that $\mathbf{O}^p(K)$, the smallest normal subgroup of K

whose quotient is a p-group, is K itself. If $P \in \mathrm{Syl}_p(G)$, then we prove under some additional technical hypotheses that PK is a direct product. This result is a key step in proving Theorems A and B. Before getting to the main result of this section, we need to prove the following easy lemma regarding the characters of normal cyclic subgroups. Note that the hypothesis of the last statement of this lemma will be fulfilled when A is the Fitting subgroup of B, and this last statement says that B/A is central in G/A.

LEMMA 5.1. *Let A be a cyclic normal subgroup in G and write $C = \mathbf{C}_G(A)$. Then $G' \subseteq C$. Furthermore, if the character α is a faithful irreducible character of A, then C is the stabilizer of α in G. Finally, if B is a normal subgroup in G with $C \cap B \subseteq A$, then $[G, B] \subseteq A$.*

Proof. Because A is normal, we see that C is normal in G and G/C is isomorphic to a subgroup of the automorphism group of A (this is the N/C theorem of Problem 3.9 in [8]). Also, since A is cyclic, the automorphism group of A is abelian (see [8, Problem 2.19]). Thus, G/C is abelian and $G' \subseteq C$, as required. Let T be the stabilizer of α in G. As C centralizes A, we certainly have $C \subseteq T$. On the other hand, it is easy to see that $[A, T] \subseteq \ker(\alpha)$. As α is faithful, T centralizes A, so $T \subseteq C$. Finally, we have $[G, B] \subseteq G' \cap B \subseteq C \cap B \subseteq A$. ∎

We are now ready to prove the main result in this section. As we stated earlier, we have a normal subgroup whose character degree set consists of the value 1 and two prime numbers. There also is information about a third prime and some technical hypotheses. The main consequence of this theorem is part (4), where we prove that we have a direct product. We also obtain a number of other interesting facts about this setup which we use in proving Theorem B.

THEOREM 5.2. *Let p, q, and r be distinct primes and write $K = \mathbf{O}^p(G)$. Assume that $\mathrm{cd}(K) = \{1, q, r\} \subseteq \mathrm{cd}(G)$. Then G splits over K and there is a complement N such that the following hold.*

(1) $[K, N] \subseteq \mathbf{Z}(K)$, *and* $\mathbf{Z}(K)$ *is a* p'-*group*.

(2) $G = \mathbf{C}_K(N) \times N[K, N]$.

(3) $\mathrm{cd}(K) = \mathrm{cd}(\mathbf{C}_K(N))$.

Proof. We begin by observing that K is solvable (this is Theorem 12.15 of [6]). Since p divides no character degree of K, we know that K has a normal abelian Sylow p-subgroup U (see [6, Corollary 12.34]) Letting R be a p-complement of K, we obtain $K = RU$. Observe that $[U, R]R$ is a normal subgroup of K having index that is a power of p. Since $K = \mathbf{O}^p(K)$, we conclude that $K = [U, R]R$ and $U = [U, R]$. By Fitting's lemma, this implies that $\mathbf{C}_U(R) = 1$ and thus $\mathbf{N}_K(R) = R$ and $\mathbf{Z}(K) \cap U = 1$. In particular, $\mathbf{Z}(K)$ is a p'-group (this is the second part of (1)). By a Frattini argument, we have $G = K\mathbf{N}_G(R)$. It follows that R is a normal p-complement of $\mathbf{N}_G(R)$. By choosing a Sylow p-subgroup N of $\mathbf{N}_G(R)$, then $K \cap N = 1$ and $G = KN$. Note that $P = UN$ is a Sylow p-subgroup of G.

Let F denote $\mathbf{F}(K)$. From the earlier discussion, we know that K has Fitting height 2 or 3. The proof now splits into two cases based on the Fitting height of K.

Case 1. K has Fitting height 3. Write $E/F = \mathbf{F}(K/F)$. From Lemma 4.1(a), we know that $F = S \times \mathbf{Z}(K)$, where $S = [E, F]$ is a minimal normal subgroup of K. Then S is an elementary abelian s-group for some prime s that does not divide $|E:F|$. Also, Lemma 4.1(a) yields the fact that $\text{cd}(K) = \text{cd}(K/\mathbf{Z}(K)) = \{1, |K:E|, |E:F|\} = \{1, q, r\}$. It follows that K/E and E/F are both cyclic groups of prime orders. Since K/F is not nilpotent, K/E having prime order different from that of E/F implies that K/F is a Frobenius group with kernel E/F. Without loss of generality, we take $|K:E| = r$, and it follows that $|E:F| = q$ and $s \neq q$. Applying Lemma 5.1, we have K/E central in G/E, and K/E being a p-complement for G/E yields the consequence that PE is a normal subgroup of G.

Consider a character $\chi \in \text{Irr}(G)$ such that $\chi(1) = r$. In view of Corollary 11.29 of [6], we see that $\chi_K \in \text{Irr}(K)$. From Lemma 4.1(a) we know that $\text{cd}(E) = \{1, q\}$. Let θ be an irreducible constituent of χ_E. Then $\theta(1)$ lies in $\text{cd}(E)$ and $\theta(1)$ divides r. The only way that these two statements can both be true is if $\theta(1) = 1$. By Frobenius reciprocity [6, Lemma 5.2], χ_K is a constituent of θ^K, and because $\chi(1) = r = |K:E| = \theta^K(1)$, we deduce that $\chi_K = \theta^K$. From Lemma 4.1(a), $S = E'$, and since $\theta(1) = 1$, this implies that $S \subseteq \ker(\theta)$ and $\theta_S = 1_S$. Now we have $\theta_F = 1_S \times \zeta$ for some character $\zeta \in \text{Irr}(\mathbf{Z}(K))$. Clearly, this character is K-invariant, and as E/F is cyclic, it extends to E. It is not difficult to prove that $1_S \times \zeta$ has a unique K-invariant extension $\hat{\zeta} \in \text{Irr}(E)$ (this is the content of Lemma 2.1 of [11]). By Gallagher's theorem [6, Corollary 6.17], there exists a character $\lambda \in \text{Irr}(E/F)$ so that $\theta = \lambda\hat{\zeta}$. Since K/E is cyclic and $\hat{\zeta}$ is K-invariant, it follows that $\hat{\zeta}$ extends to $\tilde{\zeta} \in \text{Irr}(K)$. Thus, we see that $\chi_K = \theta^K = (\lambda\hat{\zeta})^K = \lambda^K\tilde{\zeta}$ and hence $\lambda^K \in \text{Irr}(K/F)$. We conclude that $\gamma \neq 1$. Because $|E:F| = q$, we know that λ is a faithful character of E/F. By Lemma 5.1, we determine that C is the stabilizer of λ in G, where $C = \mathbf{C}_G(E/F)$.

Write T for the stabilizer of θ in G so that $T \cap K = E$ by Lemma 5.1. Since χ_K is G-invariant, we use a Frattini argument to decide that $G = TK$, and it follows that $|G:T| = |K:T \cap K| = |K:E| = r$. Observe that any element that stabilizes θ must stabilize $\theta_F = 1_S \times \zeta$. By the uniqueness of $\hat{\zeta}$, this element must stabilize $\hat{\zeta}$. Thus, T is contained in the stabilizer of $\hat{\zeta}$. Consider an element $t \in T$, so that we have

$$\lambda\hat{\zeta} = \theta = \theta^t = \lambda^t\hat{\zeta}^t = \lambda^t\hat{\zeta},$$

and by applying Gallagher's theorem [6, Corollary 6.17], we determine that t stabilizes λ. Thus, we conclude that $T \subseteq C$, since C is the stabilizer of λ. We have

$$r = |G:T| \geq |G:C| \geq |CK:C| = |K:E| = r,$$

which implies that $G = CK$ and $T = C$ since there is equality throughout this equation. Because C is normal in G and $|G:C| = r$, we determine that $P \subseteq C$ and so $PE \subseteq C$. Therefore, E/F is central in PE/F. Observe that E/F is a p-complement in PE/F, which implies that PF is normal in PE. Because PF/F is a Sylow subgroup of G/F and PE is normal in PK, it follows that PF is normal in G. Now we see that $[P, K] \subseteq [PF, K] \subseteq PF \cap K = F$ and $[N, R] \subseteq [P, K] \cap R \subseteq F \cap R$. We must deal with two possibilities: either $s \neq p$ or $s = p$.

First, assume that $s \neq p$. This implies that p does not divide $|S|$. Recall that $F = S \times \mathbf{Z}(K)$. We already know that p does not divide $|\mathbf{Z}(K)|$ and $|K:F|$. Thus, p does not divide $|F|$ nor does it divide $|K|$. It follows that $U = 1$, $P = N$, and $R = K$. Now, N acts coprimely on K and $[K, N] = [K, N, N] \subseteq [F, N] \subseteq [K, N]$. Therefore, we have $[K, N] = [F, N]$ and $[K, N]\mathbf{Z}(K)$ as a normal subgroup of K. Since $F/\mathbf{Z}(K) \cong S$ is a chief factor for K, either $F = [K, N]\mathbf{Z}(K)$ or $[K, N] \subseteq \mathbf{Z}(K)$. Consider a character $\chi \in \mathrm{Irr}(G)$ with $\chi(1) = q$, so by Corollary 11.29 of [6], $\chi_E \in \mathrm{Irr}(E)$. Because $\chi(1) > 1$, we obtain $S = E' \not\subseteq \mathit{ker}(\chi_E)$. When we take characters $\sigma \in \mathrm{Irr}(S)$ and $\zeta \in \mathrm{Irr}(\mathbf{Z}(K))$, where $\sigma \times \zeta$ is and irreducible constituent of χ_F, then $\sigma \neq 1_S$.

Since χ_E is N-invariant, N acts on the irreducible constituents of χ_F. We also know that E/F acts transitively on the irreducible constituents of χ_F. Because the action of N on E/F is coprime and central, we conclude that all of the irreducible constituents of χ_F are N-invariant (this is Corollary 13.9 of [6]). In particular, $\sigma \times \zeta$ is N-invariant. Note that $S = [E, F]$, where both E and F are characteristic in K. It follows that S must be characteristic in K, and N acts on S. Because $\sigma \times \zeta$ is N-invariant, σ and thus $\sigma \times 1_{\mathbf{Z}(K)}$ must themselves be N-invariant. We now know that $[F, N]\mathbf{Z}(K) \subseteq \ker(\sigma \times 1) < F$. Hence, we conclude that $[K, N] = [F, N] \subseteq \mathbf{Z}(K)$ (this proves (1)). From Lemma 3.2, we have $K = \mathbf{C}_K(N) \times [K, N]$, and since $[K, N]$ is abelian, we obtain $\mathrm{cd}(\mathbf{C}_K(N)) = \mathrm{cd}(K)$, which is conclusion (3). Therefore, $G = \mathbf{C}_K(N)[K, N]N$. Observe that $\mathbf{C}_K(N)$ centralizes both $[K, N]$ and N, so that both $\mathbf{C}_K(N)$ and $[K, N]N$ are normal in G. Also, it is easy to see that $\mathbf{C}_K(N) \cap [K, N]N = 1$, which yields $G = \mathbf{C}_K(N) \times [K, N]N$. This proves the result in this case.

Now, we consider the case where $s = p$. Since $\mathbf{Z}(K)$ and K/F are p'-groups, it follows that S is a Sylow p-subgroup of K and so $S = U$. Recall that $[PF, R] \subseteq [PF, K] \subseteq F$. This implies that R normalizes PF, so R acts on the quotient $PF/\mathbf{Z}(K)$. Also, we see that $PF = PS\mathbf{Z}(K) = P\mathbf{Z}(K)$ and that $PF/\mathbf{Z}(K) \cong P$, which implies that $PF/\mathbf{Z}(K)$ is a p-group. It follows that $\mathbf{Z}(P)\mathbf{Z}(K)/\mathbf{Z}(K) = \mathbf{Z}(P\mathbf{Z}(K)/\mathbf{Z}(K))$, and $\mathbf{Z}(P)\mathbf{Z}(K)$ is normalized by R. We know that $\mathbf{Z}(K) < F \cap \mathbf{Z}(P)\mathbf{Z}(K)$ is normal in K. Since $F/\mathbf{Z}(K)$ is a chief factor for K, we have $U \times \mathbf{Z}(K) = F \subseteq \mathbf{Z}(P)\mathbf{Z}(K)$. Because $U \subseteq P$, we conclude that $U \subseteq \mathbf{Z}(P)$. In particular, N centralizes U, so $[K, N] = [UR, N] = [R, N]$ and $\mathbf{C}_K(N) = U\mathbf{C}_R(N)$.

Observe that $\mathbf{Z}(K)$ is a p-complement for F. This implies that $F \cap R = \mathbf{Z}(K)$. Also, recall that $[R, N] \subseteq F \cap R = \mathbf{Z}(K) \subseteq \mathbf{Z}(R)$ (this finishes conclusion (1)). By Lemma 3.2, we have $R = \mathbf{C}_R(N) \times [R, N]$. Note that $K = RU = [R, N]\mathbf{C}_R(N)U = [K, N]\mathbf{C}_K(N)$, and it is easy to see that $\mathbf{C}_K(N) \cap [K, N] = 1$. Since $[N, K] \subseteq \mathbf{Z}(K)$, we conclude that $K = \mathbf{C}_K(N) \times [K, N]$, and as $[N, K]$ is abelian, we obtain $\mathrm{cd}(\mathbf{C}_K(N)) = \mathrm{cd}(K)$, which is conclusion (3). In addition, we determine that $G = \mathbf{C}_K(N)[K, N]N$. Note that $\mathbf{C}_K(N)$ centralizes both $[K, N]$ and N, so $\mathbf{C}_K(N)$ and $[K, N]N$ are both normal in G. It is easy to show $\mathbf{C}_K(N) \cap [K, N]N = 1$, and this yields $G = \mathbf{C}_K(N) \times [K, N]N$, which is conclusion (2). This completes the proof in this case.

Case 2. K has Fitting height 2. By Lemma 4.1(b), we know that $|K{:}F| \in \mathrm{cd}(K)$, and without loss of generality, we can take $|K{:}F| = r$. By making this assumption we obtain from Lemma 4.1(b) that $\mathrm{cd}(F) = \{1, q\}$, that $F = Q \times Z$, where $Q \in \mathrm{Syl}_q(K)$ has nilpotence class 2 (i.e. $Q' \subseteq \mathbf{Z}(Q)$), and that $Z \subseteq \mathbf{Z}(K)$. Also, we have $\mathrm{cd}(Q) = \mathrm{cd}(F) = \{1, q\}$ and $K' \subseteq Q$. Because p does not divide either $|Q|$ or $|K{:}K'|$, we conclude that p does not divide $|K|$. Thus, we deduce that $K = R$, that $U = 1$, and that $N = P$. By Glauberman's lemma [6, Lemma 13.8], we may choose a q-complement V for K so that $[V, N] \subseteq V$. Write $C = \mathbf{C}_Q(V)$. From Lemma 4.1(b), we know that $K' = [Q, V]$, $Q/Q' = K'/Q' \times C/Q'$, and $K/Q' = C/Q' \times K'V/Q'$. Note that $V \cap F = Z$ and $K = FV$. As $Q' \subseteq C$, we see that C is normal in Q.

Consider a character $\chi \in \mathrm{Irr}(G)$ so that $\chi(1) = r$. Observe that $\chi_K \in \mathrm{Irr}(K)$ (Corollary 11.29 of [6], again) and write $\theta = \chi_K$. Note that every irreducible constituent of χ_Q is linear and $Q' \subseteq ker(\theta)$. This implies that $\theta = \alpha \times \beta$ for characters $\alpha \in \mathrm{Irr}(C/Q')$ and $\beta \in \mathrm{Irr}([Q, V]V/Q')$. Since C/Q' is abelian, we have $\alpha(1) = 1$ and hence $\beta(1) = \theta(1) = r$. Because θ is N-invariant, it follows that $1_C \times \beta$ is N-invariant. Thus, N acts on the set Ω of irreducible constituents of $(1_C \times \beta)_Q$ and V acts transitively on Ω. By Glauberman's lemma [6, Lemma 13.8], there is an N-invariant irreducible constituent $\gamma \in \mathrm{Irr}(Q/C)$ of $(1_C \times \beta)_Q$. Observing that $(1_C \times$

$\beta)_Q = 1_C \times \beta_{[Q,V]}$, we determine that $\gamma_{[Q,V]}$ is irreducible and is a constituent of $\beta_{[Q,V]}$. Use $\beta(1) \neq 1$ to see that $[Q,V] = K' \nsubseteq \ker(\beta)$. This implies that $\gamma_{[Q,V]} \neq 1_{[Q,V]}$, and hence, $\gamma \neq 1_Q$. On the other hand, it is easy to show that $[Q,N] \subseteq \ker(\gamma)$. Considering $C \subseteq \ker(\gamma)$, we conclude that $[Q,N]C < Q$.

Fix an irreducible character $\psi \in \mathrm{Irr}(G)$, where $\psi(1) = q$. By Corollary 11.29 of [6], we know that $\gamma_K \in \mathrm{Irr}(K)$. Since q does not divide $|K:Q|$, we deduce that $\delta = \psi_Q \in \mathrm{Irr}(Q)$. Clearly, δ is N-invariant, and with N centralizing $Q/[Q,N]C$, it follows that every irreducible constituent of $\delta_{[Q,N]C}$ is N-invariant (see Corollary 13.9 of [6]). Because $\delta \in \mathrm{Irr}(Q)$ is nonlinear, δ must be V-invariant. From Lemma 4.1(b), we see that δ is fully ramified with respect to Q/C. As $\delta(1) = q$, we determine that $|Q:C| = q^2$, and using Lemma 12.4 of [14], we have that δ is fully ramified with respect to $Q/[Q,N]C$. When we recall that $C \subseteq [Q,N]C < Q$ and $|Q:C| = q^2$, the previous statements yields $[Q,N]C = C$ and $[Q,N] \subseteq C$. Thus, we have $[Q,V,N] \subseteq [Q,N] \subseteq C$ and $[N,Q,V] \subseteq [C,V] = 1$. Applying Corollary 8.28 of [8], we obtain $[[V,N],Q] \subseteq C$ and $[[V,N],Q] \subseteq [V,Q]$ since V was chosen with $[V,N] \subseteq V$. It follows that $[[V,N],Q] \subseteq C \cap [V,Q] = Q' < [V,Q]$ and $[V,N] < V$ since if $[V,N] = V$, then $[[V,N],Q] = [V,Q]$. We now utilize $|V:V \cap \mathbf{Z}(K)| = r$ to say that either $[V,N] \subseteq \mathbf{Z}(K)$ or $V = [V,N](V \cap \mathbf{Z}(K))$. If $V = [V,N](V \cap \mathbf{Z}(K))$, then $[V,Q] = [[V,N],Q]$, which we know does not happen. Thus, we determine that $[V,N] \subseteq \mathbf{Z}(K)$ and we conclude that $[K,N] \subseteq [QV,N] \subseteq [Q,N][V,N] \subseteq \mathbf{Z}(Q)\mathbf{Z}(K) \subseteq \mathbf{Z}(K)$, yielding (1). By Lemma 3.2, we obtain $K = [K,N] \times \mathbf{C}_K(N)$, and as $[K,N]$ is abelian, $\mathrm{cd}(K) = \mathrm{cd}(\mathbf{C}_K(N))$, which is (3). Finally, it is easy to see that $G = \mathbf{C}_K(N) \times [K,N]N$. This is the remaining portion of conclusion (2) and proves the theorem. ∎

We now obtain a corollary to Theorem 5.2. In this corollary, we assume that we have a group G that satisfies the hypothesis of Theorem A. We also assume that $\mathrm{cd}(\mathbf{O}^p(G)) = \{1, q, r\}$. Under this additional hypothesis, we prove the conclusion of Theorem A. After proving Theorem B, the remainder of this paper will be spent showing that $\mathbf{O}^p(G)$ has the character degree set $\{1, q, r\}$ (under the hypothesis of Theorem A).

COROLLARY 5.3. *Let p, q, and r be distinct primes. Suppose that* $\mathrm{cd}(G) = \{1, p, q, r, pq, pr\}$ *and* $\mathrm{cd}(K) = \{1, q, r\}$, *where* $K = \mathbf{O}^p(G)$. *Then* $G = A \times B$, *where* $\mathrm{cd}(A) = \{1, q, r\}$, *and* $\mathrm{cd}(B) = \{1, p\}$.

Proof. By Theorem 5.2, there is a complement N so that $G = A \times B$ with $\mathrm{cd}(A) = \mathrm{cd}(K) = \{1, q, r\}$, where $A = \mathbf{C}_K(N)$ and $B = [K,N]N$. It is easy to show $\mathrm{cd}(B) = \{1, p\}$, and we have the result. ∎

6. THE PROOF OF THEOREM B

In this section, we prove Theorem B. Before we proceed with its proof, we would like to introduce some more notation. It is standard given a subgroup $H \subseteq G$ and a character $\theta \in \mathrm{Irr}(H)$ to use $\mathrm{Irr}(G \mid \theta)$ to denote $\{\chi \in \mathrm{Irr}(G) \mid [\chi, \theta^G] \neq 0\}$. In this spirit, we write $\mathrm{cd}(G \mid \theta) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G \mid \chi)\}$. We also use the notation $\mathrm{cd}_H(G \mid a)$ to represent the union of all the sets $\mathrm{cd}(G \mid \theta)$ over all $\theta \in \mathrm{Irr}(H)$, where $\theta(1) = a$. We note that while this is not precisely the definition used for this notation in [10] and [13], it is consistent with the definition found in both of those papers. The following easy lemma is proved as Lemma 2.1 and Corollary 2.2 in [13].

LEMMA 6.1. *Let G be a finite group and suppose that K is a normal subgroup of G. Assume that the character degrees a, b, and f lie in* $\mathrm{cd}(K)$, $\mathrm{cd}(G)$, *and* $\mathrm{cd}(G/K)$, *respectively. If $a \in \mathrm{cd}_K(G \mid a)$, then $af \in \mathrm{cd}_K(G \mid a)$. Furthermore if $(b, |G{:}K|) = 1$, then $b \in \mathrm{cd}(K)$ and $bf \in \mathrm{cd}_K(G \mid b)$.*

The following lemma is an immediate consequence of Theorem 12.4 of [6].

LEMMA 6.2. *Let K be a normal subgroup of G such that G/K is a Frobenius group with kernel N/K an elementary abelian p-group for some prime p. Suppose that $a \in \mathrm{cd}(G)$ is relatively prime to $|G{:}N|$. If $a|G{:}N| \notin \mathrm{cd}(G)$, then p divides a.*

Proof. Consider a character $\chi \in \mathrm{Irr}(G)$ so that $\chi(1) = a$. Let θ be an irreducible constituent of χ_N. From Corollary 11.29 of [6], we know that $\chi(1)/\theta(1)$ divides $|G{:}N|$. Since $\chi(1)$ is relatively prime to $|G{:}N|$, we determine that $\chi(1) = \theta(1)$. By Theorem 12.4 of [6], we see that either $\theta(1)|G{:}N| \in \mathrm{cd}(G)$ or p divides $\theta(1)$. Because $\theta(1) = \chi(1) = a$ and $a|G{:}N| \notin \mathrm{cd}(G)$, we conclude that p divides a. ∎

We now have everything we need to prove Theorem B.

Proof of Theorem B. In view of Lemma 2.1, G is solvable. Now, let K be maximal in G with respect to the properties that K is normal in G and G/K is not abelian. There is an integer $f > 1$ so that $\mathrm{cd}(G/K) = \{1, f\}$ (see Chapter 12 of [6]). From Lemma 12.3 of [6], we know that G/K is either a t-group for some prime t or G/K is a Frobenius group. Suppose that G/K is a t-group for some prime t. It follows that f is a power of t and a character degree of G. Since p, q, r, and s are the only powers of a prime that are character degrees of G, it follows that $f \in \{p, q, r, s\}$ and $f = t$. Without loss of generality, we may take $t = p$. We now apply Lemma 6.1 to determine that $qp \in \mathrm{cd}(G)$, which is a contradiction. Thus, G/K must be a Frobenius group with elementary abelian t-group kernel L/K

for some prime t and cyclic complement (see Lemma 12.3(b) of [6]). We also utilize that result to see that $|G:L| = f$.

Suppose first that $f \in \{pr, ps, qr, qs\}$. Without loss of generality, we may assume that $f = pr$. By Lemma 6.2, L/K must be both a q-group and an s-group, which is not possible since $q \neq s$. Therefore, we are left with $f \in \{p, q, r, s\}$, and without loss of generality, we may take $f = p$. In light of Lemma 6.1, we obtain $\{1, q, r, s, qr, qs\} \subseteq \mathrm{cd}(L)$. Using Lemma 6.2, we see that L/K is a q-group. Suppose that we have degrees $a \in \mathrm{cd}(N)$ and $b \in \mathrm{cd}_K(G \mid a)$. Then by Theorem 12.4 of [6], either $pa \in \mathrm{cd}(G)$ or q divides a. If $pa \in \mathrm{cd}(G)$, it follows that $pa \in \{p, pr, ps\}$ and hence $a \in \{1, r, s\}$. If q divides a, then q divides b so that $b \in \{q, qr, qs\}$. Since q, r, and s are primes, it follows that $a \in \{q, r, s, qr, qs\}$. This implies that $\mathrm{cd}(L) = \{1, q, r, s, qr, qs\}$. Because of Lemma 6.1, we determine that $\{1, r, s\} \subseteq \mathrm{cd}(K)$. Given characters $\theta \in \mathrm{Irr}(K)$ with $\theta(1) \in \{q, qr, qs\}$ and $\chi \in \mathrm{Irr}(G \mid \theta)$, then since $\chi(1)/\theta(1)$ divides $|G:K|$ by Corollary 11.29 of [6], we must have that $\chi_K = \theta$. By Gallagher's theorem [6, Corollary 6.17], we have $p\chi(1) \in \mathrm{Irr}(G)$, which is a contradiction. Therefore, we conclude that $\mathrm{cd}(K) = \{1, r, s\}$.

Let $M = \mathbf{O}^{\{p,q\}}(G)$ so that $M \subseteq K$ and $\mathrm{cd}(M) = \{1, r, s\}$. Take R to be a $\{p, q\}$-complement for M and take U to be a Hall $\{p, q\}$-subgroup of M. Since neither p nor q divides any character degrees of M, we may apply Corollary 12.34 of [6] to see that U is a normal abelian subgroup of M. Thus, $[U, R]R$ is a normal subgroup of M having $\{p, q\}$-index. Because $M = \mathbf{O}^{\{p,q\}}(G) \subseteq \mathbf{O}^{\{p,q\}}(M)$, we have $M = [U, R]R$ and $U = [U, R]$. Apply Fitting's lemma, where R acts on U, to determine that $\mathbf{C}_U(R) = 1$ and hence $\mathbf{N}_M(R) = R$. Take N to be a Hall $\{p, q\}$-subgroup of $\mathbf{N}_G(R)$, so $\mathbf{N}_G(R) = RN$ and $N \cap M = 1$. Using a Frattini argument, we obtain $G = M\mathbf{N}_G(R) = MRN = MN$.

Let P be a Sylow p-subgroup and Q a Sylow q-subgroup of G. By Corollary 11.29 of [6] any irreducible character of G having degree of either p or q must restrict irreducibly to M, and hence to MP and MQ. Thus, $\{1, p, q\}$ is a subset of both $\mathrm{cd}(MP)$ and $\mathrm{cd}(MQ)$. Therefore, we may apply Theorem 5.2 to MP and MQ. By that theorem, there are subgroups N_p and N_q so that $MP = MN_p$, $MQ = MN_q$, $M \cap N_p = 1$, and $M \cap N_q = 1$. Also, we know that $[M, N_p] \subseteq \mathbf{Z}(M)$ and $[M, N_q] \subseteq \mathbf{Z}(M)$, where $\mathbf{Z}(M)$ is a $\{p, q\}'$-group. Observe that $[R, N_p] \subseteq [M, N_p] \subseteq \mathbf{Z}(M) \subseteq R$ and $[R, N_q] \subseteq \mathbf{Z}(M) \subseteq R$, so N_p and N_q are subgroups of $\mathbf{N}_G(R)$. Furthermore, we have $|N_p| = |MP:M| = |\mathbf{N}_G(R)|_p$ and similarly $|N_q| = |\mathbf{N}_G(R)|_q$. Thus, N_p and N_q are Sylow subgroups of $\mathbf{N}_G(R)$, and they must be conjugate to subgroups of N. Without loss of generality, we may assume that they are subgroups of N. Therefore, $N = N_p N_q$, and we determine that $[M, N] = [M, N_p][M, N_q] \subseteq \mathbf{Z}(M)$.

Recalling that U is a normal Hall subgroup of M, U is characteristic in M. This implies that $[U, N] \subseteq U$. Also, we have $[U, N] \subseteq [M, N] \subseteq \mathbf{Z}(M)$, which implies that $[U, N] \subseteq U \cap \mathbf{Z}(M) = 1$ since $\mathbf{Z}(M)$ is a $\{p, q\}'$-group. Therefore, we have $U \subseteq \mathbf{C}_M(N) = U\mathbf{C}_R(N)$ and $[R, N] = [M, N] \subseteq \mathbf{Z}(M) \subseteq \mathbf{Z}(R)$. By Lemma 3.2, $R = [R, N] \times \mathbf{C}_R(N) = [M, N] \times \mathbf{C}_R(N)$. We then obtain $M = UR = U\mathbf{C}_R(N) \times [M, N] = \mathbf{C}_M(N) \times [M, N]$ and so $\mathrm{cd}(\mathbf{C}_M(N)) = \mathrm{cd}(M) = \{1, p, q\}$ since $[M, N]$ is abelian. Finally, we have $G = MN = \mathbf{C}_M(N) \times [M, N]N$ because $\mathbf{C}_M(N)$ centralizes both $[M, N]$ and N. It is easy to compute $\mathrm{cd}([M, N]N) = \{1, r, s\}$. ∎

7. OBTAINING THE CHARACTER DEGREES OF $\mathbf{O}^p(G)$

In light of Corollary 5.3, to prove Theorem A it suffices to show that $\mathrm{cd}(\mathbf{O}^p(G)) = \{1, q, r\}$, and this is what we spend the remainder of this paper doing. Our strategy depends on the following. We take K to be a subgroup of G that is maximal with the property that K is normal in G and G/K is nonabelian. Since we may assume that G is solvable (from Lemma 2.1), we know that $\mathrm{cd}(G/K) = \{1, f\}$ for some character degree f (this is Lemma 12.3 of [6]). We break up our proof into different cases depending on the value of f. There are three different cases: $f = p$, $f \in \{q, r\}$, and $f \in \{pq, pr\}$. We know from Chapter 12 of [6] that G/K is either an s-group for some prime s or G/K is a Frobenius group. It is obvious that when $f \in \{pq, pr\}$ that G/K cannot be an s-group. It is not so obvious, but we will prove that G/K is not an s-group when $f \in \{q, r\}$. We first prove the result when $f = p$.

THEOREM 7.1. *Let p, q, and r be distinct primes. Suppose that G is solvable with* $\mathrm{cd}(G) = \{1, p, q, r, pq, pr\}$. *If K is a normal subgroup of G so that* $\mathrm{cd}(G/K) = \{1, p\}$, *then* $\mathrm{cd}(\mathbf{O}^p(G)) = \{1, q, r\}$.

Proof. Let M be a subgroup of G containing K and let M be maximal with respect to normality in G and G/M not abelian. Since $\mathrm{cd}(G/M) \subseteq \mathrm{cd}(G/K)$, we have $\mathrm{cd}(G/M) = \mathrm{cd}(G/K) = \{1, p\}$. From Chapter 12 of [6], we know that G/M is either a p-group or G/M is a Frobenius group. Suppose first that G/M is a p-group. Then $\mathbf{O}^p(G) \subseteq M$ and $q, r \in \mathrm{cd}(M)$ (this is Lemma 6.1). Consider character degrees $a \in \mathrm{cd}(M)$ and $b \in \mathrm{cd}_M(G \mid a)$. We use Corollary 11.29 of [6] to show that b/a divides $|G{:}M|$ and is thus a power of p. If $b > a$, then p divides b and $b \in \{p, pq, pr\}$. It follows that $a \in \{1, q, r\}$. It $b = a$, then by Lemma 6.1, $pa \in \mathrm{cd}(G)$. Again, we obtain $pa \in \{p, pq, pr\}$ and we deduce that $a \in \{1, q, r\}$. We conclude that $\mathrm{cd}(M) = \{1, q, r\}$, and since $|M{:}\mathbf{O}^p(G)|$ is a power of p, we have proved the result in this case.

We now assume that G/M is a Frobenius group with kernel N/M. By Chapter 12 of [6], we know that $p = |G:N|$ and that N/M is an elementary abelian s-group for some prime $s \neq p$. Observe that q and r both lie in $\mathrm{cd}(N)$ (again this is Lemma 6.1) and that $\mathbf{O}^p(G) \subseteq N$. It suffices to prove that $\mathrm{cd}(N) = \{1, q, r\}$ (using Lemma 6.1). Consider a character degree $a \in \mathrm{cd}(N)$. By Theorem 12.4 of [6], we know that either $pa \in \mathrm{cd}(G)$ or s divides a. When $pa \in \mathrm{cd}(G)$, then $pa \in \{p, pq, pr\}$ and $a \in \{1, q, r\}$. If $s \notin \{q, r\}$, then s divides no character degree of G, and hence it divides no character degree of N. Therefore, we determine that $\mathrm{cd}(N) = \{1, q, r\}$ to prove the result. We may assume that $s \in \{q, r\}$, and without loss generality we take $s = q$. Consider character degrees $a \in \mathrm{cd}(N)$ and $b \in \mathrm{cd}_N(G \mid a)$ so that q divides a. It follows that q divides b and $b \in \{q, pq\}$. Hence, we see that a is either q or pq, and we have shown that $\{1, q, r\} \subseteq \mathrm{cd}(N) \subseteq \{1, q, pq, r\}$. If $\mathrm{cd}(N) = \{1, q, r\}$, we are done. Therefore, we assume that $\mathrm{cd}(N) = \{1, q, pq, r\}$.

Since r is coprime to $|G:M|$, we apply Lemma 6.1 to see that $r \in \mathrm{cd}(M)$. If $pq \in \mathrm{cd}(M)$, then Lemma 6.1 implies that $p^2q \in \mathrm{cd}(G)$. Since this cannot happen, we have $pq \notin \mathrm{cd}(M)$ and $p \in \mathrm{cd}(M)$. Hence, the distinct character degrees of M are pairwise relatively prime, and thus $|\mathrm{cd}(M)| \leq 3$ (see Problem 12.3 of [6]). Now we know that $\mathrm{cd}(M) = \{1, p, r\}$. Write $Q = \mathbf{O}^q(N)$ and observe that $Q \subseteq M$ and $\mathrm{cd}(Q) = \{1, p, r\}$. Fix a character $\nu \in \mathrm{Irr}(N)$ having $\nu(1) = \mathrm{r}$. By Corollary 11.29 of [6], we know that $\nu_Q \in \mathrm{Irr}(Q)$. We see that $\mathrm{cd}(N \mid \nu_Q) = \{r\}$, and a theorem of Gallagher [6, Corollary 6.17], shows that $\mathrm{cd}(N|\nu_Q) = \{\nu(1)a \mid a \in \mathrm{cd}(N/Q)\}$. We determine that $\mathrm{cd}(N/Q) = \{1\}$, which forces N/Q to be abelian.

Consider a character $\mu \in \mathrm{Irr}(M)$ so that $\mu(1) = p$. Again, by Corollary 11.29 of [6], we have $\mu_Q \in \mathrm{Irr}(Q)$. Take T to be the stabilizer of μ_Q in G and observe that $M \subseteq T$. Since $Q = \mathbf{O}^q(Q)$, it follows that q does not divide $|Q:Q'|$. Thus, q does not divide the determinantal order of μ_Q. Because $|M:W|$ is a power of q, we may apply Corollary 6.28 to see that μ_Q has a canonical extension $\hat{\mu}$ on M so that T is the stabilizer of $\hat{\mu}$ in G. Since M was chosen to be maximal subject to being normal in G with G/M nonabelian, we know that N/M is a chief factor for G. Furthermore, if θ is an irreducible constituent of $\hat{\mu}^N$, then p divides $\theta(1)$ and $\theta(1) = pq$. This implies that θ must be G-invariant. Thus, we can use a Frattini argument to see that $G = TN$. We now know that $T \cap N$ is normal in G and $T \cap N = M$. By Clifford's theorem [6, Theorem 6.11], we determine that $\mu^N \in \mathrm{Irr}(N)$. We observe that $pq = \mu^N(1) = |N:M|\,\mu(1) = |N:M|p$ and $|N:M| = q$.

Since $\mathrm{cd}(M) = \{1, p, r\}$, we know that M has Fitting height of either 2 or 3. Assume first that M has a Fitting height of 3 and take $F = \mathbf{F}(M)$ and $E/F = \mathbf{F}(M/F)$. By Lemma 4.1(a), we know that $\mathrm{cd}(M) = \{1, |M:E|, |E:F|\}$. This implies that M/E and E/F are both cyclic groups of prime order. Furthermore, since M/F is not nilpotent and $|M:E|$ is a prime, it follow that M/F is a Frobenius group of order pr with kernel E/F. Let λ be a faithful character in $\mathrm{Irr}(E/F)$. Fix $C = \mathbf{C}_G(E/F)$, so that by Lemma 5.1, $G' \subseteq C$ and C is the stabilizer of λ in G. Because M/F is a Frobenius group with kernel E/F, observe that $C \cap M = E$, which implies that $\lambda^M \in \mathrm{Irr}(M)$. As $\lambda(1) = 1$, we have $\lambda^M(1) = |M:E| \in \mathrm{cd}(M)$.

Suppose first that $|M:E| = p$. If θ is an irreducible constituent of λ^N, then p divides $\theta(1)$ and $\theta(1) = pq = |N:E| = \lambda^N(1)$. It follows that $\lambda^N = \theta$, and since $C \cap N$ is the stabilizer of λ in N, we get $C \cap N = E$. On the other hand, θ is clearly G-invariant. From a Frattini argument, we obtain $G = CN$. Therefore, we may conclude that $G/E = C/E \times N/E$. Hence, we have $CM/M \in \mathrm{Syl}_p(G/M)$, but because CM is normal in G, this contradicts the fact that G/M is a Frobenius group.

Now we must suppose that $|M:E| = r$. Take P containing F so that $P/F \in \mathrm{Syl}_p(G/F)$. Observe that E/F is a normal p-group in G/F implying that $E \subseteq P$. Whereas $|P:F| = p^2$, we see that P/F is an abelian group, so $P \subseteq C$. Consider a character $\chi \in \mathrm{Irr}(G \mid \lambda)$ and note that λ^M is a constituent of χ_M. This implies that r divides $\chi(1)$ and thus $\chi(1) \in \{r, pr\}$. Since $|G:C|$ is greater than 1, divides $\chi(1)$, and is not divisible by p, we conclude that $|G:C| = r$. Hence, C/F is a normal r-complement for G/F. Let R be a subgroup of G containing F so that $R/F \in \mathrm{Syl}_r(G/F)$. Observe that $R \subseteq M$ and that $M = ER$. Thus, we have

$$[C, R] \subseteq [G, M] \subseteq G' \cap M \subseteq C \cap M = E.$$

In particular, R centralizes C/E. Since M/F is a Frobenius group with kernel E/F, we have $E = [E, R]F$. By Fitting's lemma, we have $C/F = D/F \times [C, R]F/F$, where $D/F = \mathbf{C}_{C/F}(R)$. Since R centralizes C/E, we see that $[C, R]F \subseteq E = [E, R]F \subseteq [C, R]F$. This implies $D \cap E = F$ and $C = DE$. Because E/F is in the center of C/F, we know that E normalizes D. Therefore, D is a normal subgroup of G and $G/D \cong M/F$. This, however, contradicts Lemma 6.2: G/D is a Frobenius group with kernel C/D, where C/D is an elementary abelian p-group and $r = |G:C|$, but $q \in \mathrm{cd}(G)$ and $qr \notin \mathrm{cd}(G)$. We conclude that M cannot have Fitting height 3.

We now must deal with the case where M has Fitting height 2. Again, take $F = \mathbf{F}(M)$. By Lemma 4.1(b), we know that M/F is cyclic and $|M:F| \in \mathrm{Irr}(M)$. Write $C/F = \mathbf{C}_{G/F}(M)$. By Lemma 5.1, we have $G' \subseteq C$. Using the fact that G/M is a Frobenius group with kernel N/M, we determine that $N = G'M$. Because both G' and M are contained in C, it follows that $N \subseteq C$. In particular, N/F is central-by-cyclic, so N/F is abelian. Now we have $N' \subseteq F \subseteq \mathbf{F}(N)$ and, in fact, it is easy to see that $\mathbf{F}(N) \cap M = F$. By Lemma 1.1 of [4], we determine that $|N:\mathbf{F}(N)| \in \mathrm{cd}(N)$.

Suppose that $\mathbf{F}(N)$ is not contained in M. Since $|N:M| = q$, it follows that $N = \mathbf{F}(N)M$ and that $|N:\mathbf{F}(N)| = |M:F| \in \mathrm{cd}(N) \cap \mathrm{cd}(M) = \{1, r\}$. Therefore, we have $|N:\mathbf{F}(N)| = |M:F| = r$. By Lemma 6.1, we know that q and pq lie in $\mathrm{cd}(\mathbf{F}(N))$. Furthermore, if we consider character degrees $a \in \mathrm{cd}(\mathbf{F}(N))$ and $b \in \mathrm{cd}_{\mathbf{F}(N)}(N \mid a)$, then b/a divides r. When $b > a$, then r divides b and $b = r$, implying that $a = 1$. Therefore $\{1, q, pq\} \subseteq \mathrm{cd}(\mathbf{F}(N)) \subseteq \mathrm{cd}(N) = \{1, q, pq, r\}$, but neither of these sets can be the character degree set for the nilpotent group $\mathbf{F}(N)$. This implies that $\mathbf{F}(N) \subseteq M$.

We are now in the situation where $F = \mathbf{F}(N)$. Applying the facts that $F < M$, that $|N:F| \in \mathrm{cd}(N)$, and that $q = |N:M|$, we determine that $|N:F| = pq$ and that $|M:F| = p$. It is not difficult to show that $\mathrm{cd}(F) = \{1, r\}$ (this follows from Lemma 4.1(b). Take P to be a subgroup of G containing F so that $P/F \in \mathrm{Syl}_p(G/F)$. Since M/F is a normal p-subgroup of G/F, we know that $M \subseteq P$, and as $|P:F| = p^2$, we conclude that P/F is an abelian group. Also, we know that N/F is abelian group. Since $G = PN$ and $M \subseteq N \cap P$, we deduce that $M/F \subseteq \mathbf{Z}(G/F)$ and, in particular, we obtain $[G, M] \subseteq F$. Let Q be a subgroup of N containing F so that $Q/F \in \mathrm{Syl}_q(N/F)$, which implies that $Q/F \in \mathrm{Syl}_q(G/F)$. Because $N = MQ$ and M/F is central in N/F, we see that Q/F is normal in N/F (and hence characteristic). It follows that Q is normal in G and $\mathbf{O}^p(G) \subseteq Q$. On the other hand, we have $|Q:F| = |N:M| = q$ and $\mathrm{cd}(F) = \{1, r\}$. Since qr divides no character degrees in Q, we obtain $\mathrm{cd}(Q) \subseteq \{1, q, r\}$. Since $|G:Q|$ is a p-power, we use Lemma 6.1 to get $\{1, q, r\} \subseteq \mathrm{cd}(Q)$. Therefore, $\mathrm{cd}(Q) = \{1, q, r\}$, which yields the desired result. ∎

We continue in the scenario outlined at the beginning of this section. Next, we prove the result when $f \in \{pq, pr\}$. As we stated earlier, we need only concern ourselves with the case where G/K is a Frobenius group.

THEOREM 7.2. *Let p, q, and r be distinct primes. Assume that G is a solvable group with* $\mathrm{cd}(G) = \{1, p, q, r, pq, pr\}$. *Suppose that K is a normal subgroup of G so that G/K is a Frobenius group with cyclic complement and elementary abelian s-group kernel N/K, where s is some prime. If* $|G:N| \in \{pq, pr\}$, *then* $\mathrm{cd}(\mathbf{O}^p(G)) = \{1, q, r\}$.

Proof. Without loss of generality, we may assume that $|G:N| = pq$, and from Lemma 6.2, we know that $s = r$. Also, we have $\mathrm{cd}(G/K) = \{1, pq\}$. Take $M = \mathbf{O}^p(G)N$ and observe that $\mathbf{O}^p(G/N) = M/N$ so that $|G:M| = p$ and $|M:N| = q$. Note that M/K is a Frobenius group with kernel N/K and $\mathrm{cd}(M/K) = \{1, q\}$. By Lemma 6.1, we know that $r \in \mathrm{cd}(M)$ and $r \in \mathrm{cd}(N)$. From Theorem 12.4 of [6], if one is given a character degree $a \in \mathrm{cd}(N)$, then either $apq \in \mathrm{cd}(G)$ or r divides a. Since $apq \in \mathrm{cd}(G)$ implies that $a = 1$, we see that r divides every nonlinear character degree of N. Suppose that $a > 1$ and consider a character degree $b \in \mathrm{cd}_N(G \mid a)$. Then r divides b and $b \in \{r, pr\}$. This implies that $a \in \{r, pr\}$, so, $\{1, r\} \subseteq \mathrm{cd}(N) \subseteq \{1, r, pr\}$ and $\{1, q, r\} \subseteq \mathrm{cd}(M) \subseteq \{1, q, r, pr\}$. If $\mathrm{cd}(N) = \{1, \mathrm{r}\}$, then $\mathrm{cd(M)} = \{1, \mathrm{q}, \mathrm{r}\}$ and we are done. Therefore, we assume that $\mathrm{cd}(N) = \{1, r, pr\}$, which implies that $\mathrm{cd}(M) = \{1, q, r, pr\}$.

Suppose that we have an irreducible character $\theta \in \mathrm{Irr}(N)$, so that $\theta(1) > 1$. Since $\theta(1)pq \notin \mathrm{cd}(G)$, Theorem 12.4(b) of [6] implies that $\mathbf{V}(\theta) \subseteq K$, where $\mathbf{V}(\theta)$ is the vanishing-off subgroup for θ. (This is the subgroup of N that is generated by all the elements where θ is not zero; see page 200 of [6].) Let α be an irreducible constituent of θ_K. Then we know that r divides $\theta(1)/\alpha(1)$, and we have $\mathrm{cd}(K) = \{1, p\}$.

Consider an irreducible character $\chi \in \mathrm{Irr}(M)$ so that $\chi(1) \in \{q, r\}$. Then the irreducible constituents of χ_K are all linear, and $K' \subseteq \ker(\chi)$. Thus, we have $\{1, q, r\} \subseteq \mathrm{cd}(M/K')$, but on the other hand, K/K' is a normal, abelian subgroup of M/K' having index of qr^n for some positive integer n. By Itô's theorem [6, Theorem 6.15], we know that the character degrees of M/K' divide qr^n. As qr and r^2 divide no character degrees of G, we conclude that $\mathrm{cd}(M/K') = \{1, q, r\}$.

Since $|\mathrm{cd}(K)| = 2$, we know that K has a Fitting height of 2 or 1. Suppose first that K is nilpotent (i.e., it has Fitting height 1). Then $K \subseteq \mathbf{F}(N)$ and $K = P \times Z$, where $P \in \mathrm{Syl}_p(K)$ and $Z \in \mathrm{Hall}_{p'}(K)$. Note that $\mathrm{cd}(P) = \mathrm{cd}(K) = \{1, p\}$. This implies that Z is abelian and $Z \subseteq \mathbf{Z}(K)$. Take $\pi \in \mathrm{Irr}(P)$, so that $\pi(1) = p$, and observe that $\pi \times 1_Z$ extends to its stabilizer in M (this is Corollary 6.28 of [6]). Let T be the stabilizer of $\pi \times 1_Z$ in G. Take θ to be an irreducible constituent of $(\pi \times 1_Z)^N$. Since p divides $\theta(1)$, we know that $\theta(1) = pr$. Because θ is induced by an extension of $\pi \times 1_Z$ on $T \cap N$, we deduce that $|N{:}T \cap N| = r$. Observe that θ extends to G, which shows θ is G-invariant. By Theorem 12.4(b) of [6], we know that $\mathbf{V}(\theta) \subseteq K$. With $\pi \times 1_Z$ extending to $T \cap N$, it follows that $K \subseteq T \cap N = \mathbf{V}(\theta)$ and $T \cap N = K$. This implies that $|N{:}K| = r$. Using a Frattini argument, we obtain $G = TN$. Now, focus on a character $\lambda \in \mathrm{Irr}(Z)$ and take S to be the stabilizer of λ in G. It is easy to see that the stabilizer of $\pi \times \lambda$ in G is $T \cap S$. Considering that $\pi \times \lambda \in \mathrm{Irr}(K)$ and $\pi \times \lambda(1) = p$, we have $|G{:}S \cap T| = r = |G{:}T|$. Therefore, $S \cap T = T$ and T stabilizes λ, and thus T must stabilize every irreducible character of Z. For any character $\lambda \in \mathrm{Irr}(Z)$ that is not G-invariant, T is the stabilizer of λ in G.

Consider an irreducible character $\chi \in \mathrm{Irr}(G)$, where $\chi(1) = q$, and let $\sigma \in \mathrm{Irr}(K)$ be a constituent of χ_K so that $\sigma(1) = 1$. Since $\mathrm{cd}(G/K) = \{1, pq\}$, we know that $\sigma \neq 1_K$. There exist characters $\alpha \in \mathrm{Irr}(P)$ and $\beta \in \mathrm{Irr}(Z)$ so that $\sigma = \alpha \times \beta$. Suppose that σ is G-invariant. Since N/K is cyclic, we know that σ extends to N. Also, G/N and $\mathrm{Irr}(N/K)$ both act on the set of extensions of σ to N. Thus, we may apply Glauberman's lemma [6, Lemma 13.8] to see that σ has a G-invariant extension to N. Because G/N is cyclic, this extension must extend to G, so σ extends to G. Using Gallagher's theorem [6, Corollary 6.17], we determine that $\mathrm{cd}(G \mid \sigma) = \{1, pq\}$, which is a contradiction. Therefore, σ is not G-invariant. Let X be the stabilizer of σ in G and observe that $|G:X| = q$. It is easy to see that X is the intersection of the stabilizer of α in G with the stabilizer of β in G. Thus, the stabilizer of β in G must contain both T and X. Since the indices of T and X in G are coprime, we have $G = TX$, and β is G-invariant. We now deduce that X is the stabilizer of α in G. Suppose that there is some character $\lambda \in \mathrm{Irr}(Z)$ that is not G-invariant. We decided in the last paragraph that the stabilizer of λ in G equals T which implies that the stabilizer of $\alpha \times \lambda$ in G is $X \cap T$. It is easy to see that $|G:X \cap T| = qr$, which is a contradiction since this index divides a character degree of G, and no character degree of G is divisible by qr. Therefore, every irreducible character of Z is G-invariant.

Observe that $\{1, q\} = \mathrm{cd}(M/K) \subseteq \mathrm{cd}(M/K'Z) \subseteq \mathrm{cd}(M/K') = \{1, q, r\}$, and if $\psi \in \mathrm{Irr}(M)$ is a character with degree r, then $\psi = (\eta \times \zeta)^N$ for $\eta \in \mathrm{Irr}(P)$ and $\zeta \in \mathrm{Irr}(Z)$. Since ζ is G-invariant, we have $(\eta \times 1_Z)^N \in \mathrm{Irr}(N/K'Z)$ with $(\eta \times 1_Z)^N(1) = \psi(1) = r$. Thus $\mathrm{cd}(N/K'Z) = \{1, r\}$ and $N/K'Z$ is not nilpotent. Therefore, $\mathrm{cd}(M/K'Z) = \{1, q, r\}$ and $M/K'Z$ has Fitting height 3. Define $Y/K'Z = \mathbf{Z}(M/K'Z)$. By Lemma 4.1(a), we know that K/Y is an elementary abelian p-group and that $\mathrm{cd}(M/Y) = \mathrm{cd}(M/K'Z)$. Also, from Lemma 4.1(a), we determine that N/Y is a Frobenius group with kernel K/Y. We see that G/N is cyclic of order pq and N/K is cyclic of order r. Moreover, since G/K is a Frobenius group with kernel N/K, we may apply Lemma 1.10 of [15] to see that $\mathrm{cd}(G/Y) \cup \{pqr\} = \{1, pq\} \cup \{r, pr, qr, pqr\}$. This contradicts the fact that $qr \notin \mathrm{cd}(G)$. Thus, K is not nilpotent.

We now suppose that K has Fitting height 2 and let $F = \mathbf{F}(K)$. By Lemma 1.1 of [4], we have $|K:F| \in \mathrm{cd}(K)$, so $|K:F| = p$. Also, we see from Lemma 1.6 of [15] that F is abelian. Write $C/F = \mathbf{C}_{G/F}(K)$ and take $\lambda \in \mathrm{Irr}(K/F)$ to be a faithful character. By Lemma 5.1, C is the stabilizer of λ in G. In addition, observe that C is normal in G and that $K \subseteq C$. Since G/K is a Frobenius group with kernel N/K, we use Satz V.8.16 of [2] to say that either $C < N$ or $N \subseteq C$, and as qr divides no character degree of G, we deduce that $N \subseteq C$. Hence, K/F is central in N/F, so the Sylow r-subgroup of N/F is normal in N/F. Because this subgroup is also abelian, we conclude that N/F is abelian and $N' \subseteq F \subseteq \mathbf{F}(N)$. In fact, it is easy to see that $F = N \cap \mathbf{F}(N)$. It follows that $p = |K:F|$ divides $|N:\mathbf{F}(N)|$. Applying Lemma 1.1 of [4], we obtain $|N:\mathbf{F}(N)| \in \mathrm{cd}(N)$. Considering that p divides this index, we must have $|N:\mathbf{F}(N)| = pr$. On the other hand, the fact that $|\mathbf{F}(N)K:\mathbf{F}(N)| = |K:F| = p$ implies that $|N:\mathbf{F}(N)K| = r$. Take S to be a subgroup containing $\mathbf{F}(N)$, where $S/\mathbf{F}(N)$

is a p-complement for $N/\mathbf{F}(N)$. As N/F is abelian, $S/\mathbf{F}(N)$ is a characteristic subgroup in $N/\mathbf{F}(N)$ and thus S is normal in G. Also, we observe that $\mathbf{F}(N)K/\mathbf{F}(N) \in \mathrm{Syl}_p(N/\mathbf{F}(N))$ and $N = S\mathbf{F}(N)K = SK$. Furthermore, we know that $|N:S| = p$, so by Lemma 6.1, $r \in \mathrm{cd}(S)$. In fact, it is not difficult to show that S/F is a p-complement for N/F, and since $K/F \in \mathrm{Syl}_p(N/F)$, it follows that $S \cap K = F$ and S/F is isomorphic to the r-group N/K. Because F is abelian, we can apply Itô's theorem [6, Theorem 6.15], to see that every element of $\mathrm{cd}(S)$ is a power of r, so $\mathrm{cd}(S) = \{1, r\}$.

Take $P/S \in \mathrm{Syl}_p(G/S)$. Because N/S is normal in G/S and is a p-group, we know that $N \subseteq P$. We have p normal in G since G/N is cyclic. Let $Q/S \in \mathrm{Syl}_q(G/S)$ and observe that $|G:S| = |G:N|\,|N:S| = pqp = p^2q$. We conclude that $G = PQ$ and $P \cap Q = S$. In addition, Q acts coprimely on the abelian group P/S. Writing $D/S = \mathbf{C}_{P/S}(Q)$, we use Fitting's lemma to obtain $P/S = [P, Q]S/S \times D/S$, and since G/N is cyclic, it follows that $[P, Q]S \subseteq N < P$. Also, note that D is a normal subgroup of G and that $\mathbf{O}^p(G)N = QN$. Thus, either $D < P$ or $D = P$. If $D < P$, then G/D is a Frobenius group with kernel P/D. This contradicts Lemma 6.2, since $|G:P| = q$ and $r \cap \mathrm{cd}(G)$, where P/D is not an r-group with $qr \notin \mathrm{cd}(G)$. We are left with $D = P$, which implies that Q is normal in G and hence $Q \subseteq \mathbf{O}^p(G)$. The fact that $\mathrm{cd}(S) = \{1, r\}$ and $|Q:S| = q$ yields $\mathrm{cd}(Q) \subseteq \{1, q, r\}$. By Lemma 6.1, we know that q, $r \in \mathrm{cd}(Q)$ and $\mathrm{cd}(Q) = \{1, q, r\}$. The desired conclusion is now proved. ∎

Despite appearances, we now proceed to the third prong of our strategy outlined at the beginning of this section. While on the surface it may not seem that this result deals with the case that $f \in \{q, r\}$, we will show that the hypotheses of Theorem 7.3 hold in this case. Note that in this theorem, we are able to use the more general hypothesis that m and n are coprime integers that are greater that 1 and not divisible by p. This is certainly satisfied by q and r when p, q, and r are distinct primes, which is the case when we want to use this result.

THEOREM 7.3. *Let p be a prime and let m and n be coprime integers that are greater than* 1 *and not divisible by p. Suppose that G is a solvable group and K is a normal subgroup of G, where G/K is a p'-group with* $\mathrm{cd}(G/K') = \{1, m, n\}$ *and* $\mathrm{cd}(K) = \{1, p\}$. *Assume that* $\mathrm{cd}(G) \subseteq \{1, p, m, n, pm, pn\}$. *If* $p \in \mathrm{cd}(G)$, *then* $\mathrm{cd}(\mathbf{O}^p(G)) = \{1, m, n\}$.

Proof. Without loss of generality, we may assume that $K = \mathbf{O}^{p'}(G)$. This implies that K/K' is a p-group. Take R to be a p-complement for G so that $G = RK$. We will work by induction on $|G|$. We have two cases to consider; either G/K' has Fitting height 3 or G/K' has Fitting height 2.

Case 1. G/K' has Fitting height 3. Let $F/K' = \mathbf{F}(G/K')$ and $E/F = \mathbf{F}(G/F)$. From Lemma 4.1(a), we know that F/K' is abelian, that G/E and E/F are cyclic, and that $\mathrm{cd}(G/K') = \{1, |G{:}E|, |E{:}F|\}$. Without loss of generality, we take $|G{:}E| = m$ and $|E{:}F| = n$. By Lemma 4.1(a), we have $F/K' = S/K' \times Z/K'$, where $S = [E, F]K'$ and $Z/K' = \mathbf{Z}(G/K')$. Furthermore, S/K' is a chief factor for G so that S/K' is elementary abelian for some prime s that does not divide n. We also obtain $\mathrm{cd}(G/Z) = \mathrm{cd}(G/K')$. Since K/K' is abelian, it follows that $K \subseteq F$, and we have two possibilities: either $s \neq p$ or $s = p$.

Assume that $s \neq p$. Since K/K' is a p-group and $|F{:}Z|$ is a power of s, we see that $K \subseteq Z$. Hence, we know that $[K, G] \subseteq K'$. Because $K' \subseteq K \subseteq Z$, we determine that $\mathrm{cd}(G/K) = \mathrm{cd}(G/K') = \mathrm{cd}(G/Z) = \{1, m, n\}$. Note that K/K' is a Sylow p-subgroup of G/K' and RK'/K' is a p-complement for G/K'. Considering that K/K' is central in G/K', it follows that RK' is normal in G having p-power index, and this implies that $\mathbf{O}^p(G) \subseteq RK'$. Observe that K' is a normal abelian subgroup of RK' having p-index. By Itô's theorem [6, Theorem 6.15], we determine that all of the character degrees of RK' are powers of p. From Lemma 6.1, we know that $\{1, m, n\} \subseteq \mathrm{cd}(RK')$. If we have character degrees $a \in \mathrm{cd}(RK')$ and $b \in \mathrm{cd}_{RK'}(G \mid a)$, then b/a divides $|G{:}RK'|$ (see Corollary 11.29 of [6]) and $a = b_p \in \{m, n\}$. Therefore, $\mathrm{cd}(RK') = \mathrm{cd}(\mathbf{O}^p(G)) = \{1, m, n\}$, which is the desired result in this case.

We now must deal with the possibility that $s = p$. Since $F = ZS$, we have $G = FR = Z(SR)$. Because Z/K' is central in G/K', it follows that SR is normal in G having index that is a p-power. Thus, we know that $\mathbf{O}^p(G) \subseteq SR$, and we use Lemma 6.1 to see that $\{m, n\} \subseteq \mathrm{cd}(SR/K')$. Since S/K' is an abelian subgroup of SR/K' having p'-index in SR/K', it follows from Itô's theorem [6, Theorem 6.15], that all the character degrees of SR/S' are not divisible by p. This implies that $\mathrm{cd}(SR/K') = \{1, m, n\}$. Also, note that $G = KR = K(SR)$, so $|G{:}SR| = |K{:}\ K \cap SR| = |K{:}S|$. Recall that S is a normal subgroup of K and that $\mathrm{cd}(K) = \{1, p\}$. Hence, the character degrees of S are contained in $\{1, \mathrm{p}\}$. If S is abelian, then $\mathrm{cd}(SR) = \{1, m, n\}$ by Itô's theorem and we get the desired result. If S is not abelian and $SR < G$, then we may apply induction to see that $\mathrm{cd}(\mathbf{O}^p(SR)) = \{1, m, n\}$. In combination with $\mathbf{O}^p(G) = \mathbf{O}^p(SR)$, this yields the desired conclusion, so we may assume that $G = SR$. This says that $S = K$.

Consider a character $\psi \in \mathrm{Irr}(G)$ so that $\psi(1) = p$. Since $K = \mathbf{O}^{p'}(G)$, we know by Corollary 11.29 of [6] that $\psi_K \in \mathrm{Irr}(K)$. Recall that $K = [E, F]K'$ and that K/K' is a chief factor of G. Let λ be an irreducible constituent of $\psi_{K'}$. Because K has two character degrees, we know that K' is abelian [6, Corollary 12.6]. This implies that $\lambda(1) = 1$ and so $\psi_{K'} \neq \lambda$. In view of Theorem 6.18 of [6], we see that the remaining possibilities are that

λ is fully ramified with respect to K/K' or $\lambda^K = \psi_K$. If λ is fully ramified with respect to K/K', then since ψ_K is G-invariant, ψ is G-invariant, and since p and r are not both 2, this violates Lemma 4.2. The remaining possibility is $\psi_k = \lambda^K$. By Lemma 4.1(a), there is a positive integer a so that $|K{:}K'| = p^{ar}$. We now have $\psi(1) = \lambda^K(1) = |K{:}K'| = p^{ar} > p = \psi(1)$, which is a contradiction. This completes the proof in this case.

Case 2. G/K' has Fitting height 2. Again, write $F/K' = \mathbf{F}(G/K')$. By Lemma 4.1(b), we know that $|G{:}F| \in \mathrm{cd}(G/K')$, and without loss of generality, we may suppose that $|G{:}F| = m$. This implies that $\mathrm{cd}(F/K') = \{1, n\}$ so that n is a power of some prime q. Also from Lemma 4.1(b), we know that $F/K' = QK'/K' \times Z/K'$, where $Q \in \mathrm{Syl}_q(G)$ and $Z/K' \subseteq \mathbf{Z}(G/K')$. It is easy to see that $\mathrm{cd}(QK'/K') = \mathrm{cd}(F/K')$. Since $K/K' \in \mathrm{Syl}_p(G/K')$, we must have $K \subseteq Z$ and $[K, G] \subseteq K'$. Because K/K' is central in G/K', it follows that RK' is normal in G having p-power index so that $\mathbf{O}^p(G) \subseteq RK'$. Since $|\mathrm{cd}(K)| = 2$, the derived length of K is 2 and K' is abelian. Using Itô's theorem [6, Theorem 6.15], and the fact that the index of K' in RK' is not divisible by p, the character degrees of RK' are also not divisible by p. In light of the character degrees of G, we determine that $\mathrm{cd}(RK') \subseteq \{1, m, n\}$. On the other hand, since $|G{:}RK'|$ is a p-power, we know that $\{m, n\} \subseteq \mathrm{cd}(RK')$, and this yields the desired conclusion. ∎

We now have assembled everything needed to prove Theorem A.

Proof of Theorem A. Because of Lemma 2.1, we know that G is solvable. In view of Corollary 5.3, it suffices to show that $\mathrm{cd}(\mathbf{O}^p(G)) = \{1, q, r\}$. Let K be maximal in G so that K is normal in G and G/K is not abelian. By Chapter 12 of [6], we know that $\mathrm{cd}(G/K) = \{1, f\}$ for some integer $f \in \mathrm{cd}(G)$. If $f = p$, then we are done by Theorem 7.1. If $f \in \{pq, pr\}$, then by Lemma 12.3 of [6], we know that G/K is a Frobenius group, and we are done by Theorem 7.2. The remaining possibility is that $f \in \{q, r\}$. Without loss of generality, we take $f = q$. Observe that if G/K were a q-group, then $r \in \mathrm{cd}(K)$ and $qr \in \mathrm{cd}(G)$ by Lemma 6.1. Hence, G/K is a Frobenius group with kernel N/K, where $|G{:}N| = f = q$, and by Lemma 6.2, we know that N/K is an r-group and $\mathbf{O}^{p'}(G) \subseteq K$. Consider a character degree $a \in \mathrm{cd}(N)$. From Theorem 12.4 of [6], we know that either $aq \in \mathrm{cd}(G)$ or r divides a. When $aq \in \mathrm{cd}(G)$, either $a = 1$ or $a = p$. If r divides a, then as a divides some character degree of G, we conclude that a is either r or pr, and so we see that $\mathrm{cd}(N) \subseteq \{1, p, r, pr\}$. On the other hand, we can apply Lemma 6.1 to show that $\{1, p, r, pr\} \subseteq \mathrm{cd}(N)$, so we have $\mathrm{cd}(N) = \{1, p, r, pr\}$. In another application of Lemma 6.1, we determine that $p \in \mathrm{cd}(K)$. On the other hand, given a character degree $c \in \mathrm{cd}(K)$ with $c \in \{r, pr\}$, we obtain $c \in \mathrm{cd}_K(G \mid c)$ by looking at the possible values in $\mathrm{cd}(G)$. Using Lemma 6.1 once more, we have $qc \in \mathrm{cd}(G)$. This is a contradiction because qc is divisible by qr, but no character degree of G is divisible by qr. Therefore, we can conclude that $\mathrm{cd}(K) = \{1, p\}$. It is easy to see that $\mathrm{cd}(G/K') = \{1, q, r\}$, and we are done by Theorem 7.3. ∎

8. AN EXAMPLE

In this section, we will present an example to show that, given any coprime integers m and n, there exists a group G with character degrees $\mathrm{cd}(G) = \{1, m, n, mn\}$, where G is not a direct product.

EXAMPLE 8.1. Let m and n be coprime integers and let p be a prime that does not divide either m or n. Write $d = \mathrm{ord}_{mn}(p)$, so that d is the smallest integer where $p^d \cong 1 \bmod mn$ (we say that d is the order of p with respect to mn). Take F to be the Galois field of order p^d and let K be a direct sum of three copies of the additive group of F. We know that mn divides $p^d - 1$. We write X for the unique subgroup of the multiplicative group of F having order mn. We now define an action of C on K by $(a, b, c)^x = (a * x, b * (x^n), c * (x^m))$ for all elements $a, b, c \in F$ and $x \in X$. It is easy to see that this action is well defined. We use G to denote the semidirect product of X acting on K. Observe that K is a normal abelian subgroup in G having index mn. Thus, by Itô's theorem [6, Theorem 6.15], we know that every character degree of G divides mn. Observe that every irreducible character of K having the form $1 \times \alpha \times 1$, where $\alpha \neq 1$, lies in an orbit of size m, and every one having the form $1 \times 1 \times \beta$, where $\beta \neq 1$, lies in an orbit of size n. The remaining nonprincipal irreducible characters of K lie in orbits of size mn. We conclude that $\mathrm{cd}(G) = \{1, m, n, mn\}$. On the other hand, it is easy to see that G is not a direct product.

9. QUESTIONS

We would like to conclude by proposing two open questions:

Question 9.1. Is it possible to relax the primeness hypothesis in Theorems A and B? By this we mean the following: suppose that a, b, c, and d are pairwise relatively prime positive integers. If G is a group with $\mathrm{cd}(G) = \{1, a, b, c, ab, ac\}$, must $G = A \times B$, where $\mathrm{cd}(A) = \{1, a\}$ and $\mathrm{cd}(B) = \{1, b, c\}$? Similarly, if $\mathrm{cd}(G) = \{1, a, b, c, d, ac, ad, bc, bd\}$, must $G = A \times B$, where $\mathrm{cd}(A) = \{1, a, b\}$ and $\mathrm{cd}(B) = \{1, c, d\}$?

There are several obstacles to extending our results to the more general question, not least of which is that our use of the classification depends on the fact that the character degrees are square-free. Also, in each of Theorems 5.2, 7.1, and 7.2, there are places where the fact that the character degrees are primes is used, and in many of these places, it appears that it is nontrivial to weaken this hypothesis.

Question 9.2. Can one classify those groups G that have $\mathrm{cd}(G) = \{1, m, n, mn\}$, where m and n are relatively prime integers?

This may be a difficult question. Besides the construction found in Section 8, we know of several other ways to construct such groups. Note that the construction in Section 8 has derived length and Fitting height that both equal 2. We know of examples where the Fitting height is 2 and the derived length is 3, and examples where the derived length and Fitting height are both 3. Is it possible in this case to prove that 3 is an upper bound for either the Fitting height or the derived length of these groups?

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