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Global attractivity of the diffusive Nicholson blowflies equation with Neumann boundary condition: A non-monotone case ☆

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Abstract

In this paper, we establish the global attractivity of the positive steady state of the diffusive Nicholson's equation with homogeneous Neumann boundary value under a condition that makes the equation a non-monotone dynamical system. To achieve this, we develop a novel method: combining a dynamical systems argument with maximum principle and some subtle inequalities.
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1. Introduction

In order to describe the dynamics of the Nicholson's blowflies experiments [19], Gurney et al. [6] proposed the following delay differential equation model

$$\frac{\mathrm{d}u(t)}{\mathrm{d}t} = -\delta u(t) + pu(t-\tau)e^{-au(t-\tau)}, \tag{1.1}$$

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where $u(t)$ is the population of the adult flies at time t, p is the maximum per capita egg production rate; $1/a$ is the size at which the fly population reproduces at its maximum rate; δ is the per capita daily death rate; and τ is the maturation time. The model and its modifications have also been later used to describe population growth of other species (see, e.g., Cooke et al. [1] and the references therein), and thus, have been extensively and intensively studied.

If $0 < p/\delta \leqslant 1$, $u = 0$ is the only biologically meaningful equilibrium of (1.1) which attracts all non-negative solutions. When $p/\delta > 1$, $u = 0$ becomes unstable and there is a positive equilibrium $u^+ = \frac{1}{a} \ln \frac{p}{\delta}$. For u^+, it was proved in [23] that when $1 < p/\delta < e$, then u^+ is globally attractive, regardless of the magnitude of the delay τ (see, e.g., [1,10,12,23]). The global attractivity of u^+ was also established in Faria [2], Györi [7] and Kuang [11], when $e < p/\delta < e^2$ for all $\tau \geqslant 0$.

When the model is used to describe the population dynamics of a species in a non-laboratory habitat, spatial heterogeneity exists and spatial variables are needed. In this context, a diffusion term is needed to describe the random movement of individuals. In the case when the immature individuals do not diffuse but the matured ones do, the model (1.1) is naturally extended to the following delayed reaction diffusion equation

$$\frac{\partial u(t,x)}{\partial t} = d\Delta u(t,x) - \delta u(t,x) + pu(t-\tau,x)e^{-au(t-\tau,x)}, \quad x \in \Omega \subset R^m, \tag{1.2}$$

where $x = (x_1, \ldots, x_m)$ denotes the spatial variable vector in R^m, and Δ is the Laplacian operator in R^m. For a detailed derivation of (1.2), see So et al. [25] and Liang and Wu [14] for unbounded Ω, and Liang et al. [13,15] for bounded Ω.

For the diffusive Nicholson equation (1.2), depending on the situations of the spatial domain Ω, different problems may arise. If $\Omega = R^m$, traveling wave solutions are an important topic since such solutions may quite often determine the long term behavior of other solutions, and well describe the spatial invasion of the species. Existence and stability of traveling wave fronts of the delayed diffusive Nicholson equation have been investigated in Gourley [4], Mei et al. [18], So and Zou [24]. Gourley and Ruan [5] also explored the dynamics of a diffusive Nicholson equation with distributed delay when the spatial domain is the whole space.

When Ω is a bounded domain in R^m, various boundary conditions can be posed, among which are the typical homogeneous Dirichlet boundary value condition

$$u(t,x)|_{\partial\Omega} = 0 \tag{1.3}$$

and Neumann boundary value condition

$$\left.\frac{\partial u(t,x)}{\partial n}\right|_{\partial\Omega} = 0, \tag{1.4}$$

where $\frac{\partial u(t,x)}{\partial n}$ denotes the derivative along the outward normal direction on the boundary of Ω. Condition (1.3) describe a situation where the boundary is hostile to the species and condition (1.4) implies that the habitat Ω is isolated.

For (1.2), (1.3), So and Yang [22] systematically studied the solution behaviors. Addressed in [22] were the stability of the trivial steady state, existence and stability (local and global) of a positive steady state. Let λ_1 be the principal eigenvalue of $-\Delta$ associated with (1.3). So and Yang [22] proved that when $p/\delta - 1 < \lambda_1 d$, then the trivial steady state $u = 0$ attracts all non-negative solutions; when $p/\delta - 1 > \lambda_1 d$, then $u = 0$ becomes unstable and there appears a unique positive steady state $u^+(x)$ for (1.2), (1.3), which attracts all positive solutions provided $e < p/\delta \leqslant e^2$.

When Neumann boundary value condition (1.4) is considered, an equilibrium of (1.1) gives a steady state for (1.2) and (1.4). Yang and So [28] proved that when $0 < p/\delta \leqslant 1$, all positive solutions of (1.2) and (1.4) converge to $u = 0$; and when $1 < p/\delta \leqslant e$, all non-trivial solutions of (1.2) and (1.4) converge to $u^+ = \frac{1}{a} \ln \frac{p}{\delta}$, independent of $\tau \geqslant 0$. For $p/\delta > e^2$, Yang and So [28] also showed that u^+ may be unstable and Hopf bifurcation from u^+ may occur when the delay τ is increased. However, the dynamics of (1.2) and (1.4) for $e < p/\delta \leqslant e^2$ still remains an open problem. Motivated by the results in Faria [2], Györi [7], Kuang [11], and So and Yang [22], it is natural to conjecture that the solution u^+ remains globally attractive if $e < p/\delta \leqslant e^2$, regardless of the magnitude of the delay τ. In this paper, we give an affirmative answer to this conjecture.

We point out that if $1 < p/\delta \leqslant e$, the above mentioned results on convergence to the positive equilibrium or steady state for all cases can be easily established by the monotone method, since there is an interval which attracts all solutions and in which the delayed term is monotone (for details of monotone delay equations, see, e.g., Smith [21]). However, when $e < p/\delta \leqslant e^2$, the corresponding equations do not have such a monotonicity since the delayed term is not monotone any more on the interval $[0, u^+]$, and thus, are much harder to deal with. Therefore, alternative method is required. Here in this paper, we develop a new approach to obtain the global attractivity. More precisely, we combine a dynamical systems argument with maximum principle and some subtle inequalities to show that under $e < p/\delta \leqslant e^2$, the ω-limit set of every initial function that generates a positive solution of (1.2) and (1.4) is actually the singleton $\{u^+\}$, regardless of the magnitude of the delay $\tau \geqslant 0$. By this conclusion and the results in Yang and So [28], the global dynamics of the model (1.2) and (1.4) for $p/\delta \in (0, e^2)$ are completely determined now and are indeed independent of the time delay τ.

2. Preliminaries

For convenience, we rescale (1.2) and (1.4) to the following

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) = \Delta u(t,x) - \tau u(t,x) + \beta \tau u(t-1,x) e^{-u(t-1,x)} & \text{in } D \equiv (0,\infty) \times \Omega, \\ \dfrac{\partial u}{\partial n}(t,x) = 0 \quad \text{on } \Gamma \equiv (0,\infty) \times \partial\Omega, \end{cases} \tag{2.1}$$

where $\tau > 0$ and Ω is a bounded domain with smooth boundary $\partial\Omega$. As usual, we also need the following initial condition:

$$u(\theta, x) = \phi(\theta, x) \quad \text{for } (\theta, x) \in [-1, 0] \times \bar{\Omega}. \tag{2.2}$$

As we mentioned in the introduction, we are concerned with the situation of $e < p/\delta \leqslant e^2$, which, after rescaling, is transferred to $e < \beta \leqslant e^2$. This will be assumed in the rest of the paper.

Throughout this paper, denote by R (respectively R_+, R^+) the set of all (respectively non-negative, positive) real numbers. Let $C = C(\bar{\Omega}, R)$ and $X = C([-1,0] \times \bar{\Omega}, R)$, equipped with the usual supremum norm $\|\cdot\|$. Also, let $C_+ = C(\bar{\Omega}, R_+)$ and $X_+ = C([-1,0] \times \bar{\Omega}, R_+)$. For $a \in R$, $\hat{a} \in C$ is defined as $\hat{a}(x) = a$ for all $x \in \bar{\Omega}$. Similarly, $\hat{\hat{a}} \in X$ is defined as $\hat{\hat{a}}(\theta, x) = a$ for all $(\theta, x) \in [-1, 0] \times \bar{\Omega}$. For simplicity of notations, we shall write $a \triangleq \hat{a}$ and $a \triangleq \hat{\hat{a}}$. For a real interval I, let $I + [-1, 0] = \{t + \theta\colon t \in I \text{ and } \theta \in [-1, 0]\}$. For $u : (I + [-1, 0]) \times \bar{\Omega} \to R$ and

$t \in I$, we write $u_t(\cdot,\cdot)$ for the element of X defined by $u_t(\theta, x) = u(t+\theta, x)$, for $-1 \leqslant \theta \leqslant 0$ and $x \in \bar{\Omega}$.

Let $T(t)$ $(t \geqslant 0)$ be the strongly continuous semigroup of bounded linear operators on C generated by the Laplacian Δ under the Neumann boundary value condition. It is well known that $T(t)$ $(t \geqslant 0)$ is an analytic, compact and strongly positive semigroup on C. Moreover, there exist $M > 0$ and $w > 0$ such that $\|T(t)\| \equiv \sup\{\frac{\|T(t)\varphi\|}{\|\varphi\|}\colon \varphi \in C \text{ and } \varphi \neq 0\} \leqslant Me^{wt}$ for all $t \in R_+$ (see [20,21,27]). Define $F: X \to C$ by $F(\phi)(x) = -\tau\phi(0,x) + \beta\tau\phi(-1,x)e^{-\phi(-1,x)}$ for all $x \in \bar{\Omega}$. We consider the following integral equation with the given initial condition

$$\begin{cases} u(t) = T(t)\phi(0,\cdot) + \int_0^t T(t-s)F(u_s)\,\mathrm{d}s, & t \geqslant 0, \\ u_0 = \phi \in X. \end{cases} \tag{2.3}$$

By the standard theory (see [3,16,27]), for each $\phi \in X$, Eq. (2.3) admits a unique solution $u^\phi(t,\cdot)$ (with values in C) on its maximal interval $[0, \sigma_\phi)$. As is customary, $u^\phi(t,x)$ is also called a mild solution of (2.1), (2.2) (for details, see [16,17], or [27]).

Definition 2.1. For given real numbers t_1 and t_2 with $t_2 > t_1 + 1$, a continuous function $u: [t_1 - 1, t_2] \times \bar{\Omega} \to R$ is called a classical solution of (2.1) and (2.2) for $t \in [t_1, t_2)$ if all involved derivatives exist and for all $i, j \in \{1, \ldots, m\}$, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x_i \partial x_j}$ are continuous for $(t,x) \in (t_1,t_2) \times \Omega$, and $\frac{\partial u}{\partial x_i}$ is continuous for $(t,x) \in (t_1,t_2) \times \bar{\Omega}$, and u satisfies the relation (2.1) and (2.2) for $(t,x) \in (t_1,t_2) \times \bar{\Omega}$.

Lemma 2.1. *If $\phi \in X_+$, then we have the following results*:

(i) $(u^\phi)_t \in X_+$ *for all* $t \in [0, \sigma_\phi)$;
(ii) $\sigma_\phi = +\infty$;
(iii) $u^\phi(t,x)$ *is a classical solution of* (2.1) *for* $t \in (1, +\infty)$.

Proof. Without loss of generality, we may assume that $\phi \neq 0$. Note that $\psi(0) + hF(\psi) \in C_+$ for all $\psi \in X_+$ and $h \in [0, \frac{1}{\tau}]$. This implies

$$\lim_{h \to 0^+} \operatorname{dist}\big(\psi(0) + hF(\psi), C_+\big) = \lim_{h \to 0^+} \inf\big\{\big\|\psi(0) + hF(\psi) - \tilde{\psi}\big\|\colon \tilde{\psi} \in C_+\big\} = 0$$

for all $\psi \in X_+$. By Proposition 3 and Remark 2.4 in [16], we know that the statement (i) is true.

We now prove the statement (ii). Since $\sup\{ae^{-a}\colon a \in R\} = \frac{1}{e}$ and $\|T(t)\| \leqslant Me^{wt}$ for all $t \in R_+$, it follows from (2.3) that for all $t \in [0, \sigma_\phi)$, we have

$$\begin{aligned} \big\|u^\phi(t,\cdot)\big\| &= \left\|T(t)\phi(0,\cdot) + \int_0^t T(t-s)F\big(\big(u^\phi\big)_s\big)\,\mathrm{d}s\right\| \\ &\leqslant \big\|T(t)\phi(0,\cdot)\big\| + \left\|\int_0^t T(t-s)F\big(\big(u^\phi\big)_s\big)\,\mathrm{d}s\right\| \end{aligned}$$

$$\leqslant M\|\phi(0,\cdot)\|e^{wt}+\int_0^t \|T(t-s)F\big((u^\phi)_s\big)\|\,\mathrm{d}s$$

$$\leqslant M\|\phi(0,\cdot)\|e^{wt}+M\int_0^t \|F\big((u^\phi)_s\big)\|e^{w(t-s)}\,\mathrm{d}s$$

$$\leqslant M\|\phi(0,\cdot)\|e^{wt}+\tau Me^{wt}\int_0^t \|u^\phi(s,\cdot)\|e^{-ws}\,\mathrm{d}s+\frac{\tau\beta M}{e}\int_0^t e^{w(t-s)}\,\mathrm{d}s$$

$$=M\|\phi(0,\cdot)\|e^{wt}+\frac{\tau\beta M}{ew}\big(e^{wt}-1\big)+\tau Me^{wt}\int_0^t \|u^\phi(s,\cdot)\|e^{-ws}\,\mathrm{d}s.$$

Now by the Gronwall–Bellman inequality, we conclude that for every $t\in[0,\sigma_\phi)$,

$$\|u^\phi(t,\cdot)\|\leqslant a(t)+b(t)e^{\int_0^t b(s)c(s)\,\mathrm{d}s}\int_0^t \big[a(s)c(s)e^{-\int_0^s b(r)c(r)\,\mathrm{d}r}\big]\,\mathrm{d}s, \tag{2.4}$$

where $a(t)=M\|\phi(0,\cdot)\|e^{wt}+\frac{\tau\beta M}{ew}(e^{wt}-1)$, $b(t)=\tau Me^{wt}$ and $c(t)=e^{-wt}$ for all $t\in[0,\sigma_\phi)$. The inequality (2.4) implies $\sigma_\phi=+\infty$, since otherwise, $\sigma_\phi<+\infty$ would imply $\limsup_{t\to\sigma_\phi^-}\|u^\phi(t,\cdot)\|=+\infty$ (see, e.g., Theorem 2.2.2 in [27]), contradicting (2.4). This completes the proof of statement (ii).

Finally, the statement (iii) follows from (ii) and Theorem 2.2.6 in [27]. The proof of Lemma 2.1 is completed. □

Lemma 2.2. *If $\phi\in X_+\setminus\{0\}$, then we have the following results*:

(i) $(u^\phi)_t\in \mathrm{Int}(X_+)$ *for all* $t>3$;
(ii) *there exists* $K=K(\phi)>0$ *such that* $\|(u^\phi)_t\|\leqslant K$ *for all* $t\in R_+$.

Proof. Without causing confusion, we drop, in the proof, the ϕ in the notations by letting $u_t=(u^\phi)_t$ and $u(t,x)=u^\phi(t,x)$. According to Lemma 2.1(iii), we know $u(t,x)$ is a classical solution of (2.1) for $t>1$.

To prove the statement (i), we first claim that $u_1\in X_+\setminus\{0\}$. Suppose not, then Lemma 2.1(i) implies $u_1=0$, and thus $u(t,x)=0$ for $(t,x)\in[0,1]\times\bar{\Omega}$. By (2.3) and the assumption $u_1=0$, we have $\int_0^t T(t-s)F(u_s)\,\mathrm{d}s=0$ for all $t\in[0,1]$. Thus, $\int_0^1 T(t-s)[u(s-1,\cdot)e^{-u(s-1,\cdot)}]\,\mathrm{d}s=0$ for all $t\in[0,1]$. Since $T(\cdot)$ is a strongly positive semigroup (see Corollary 7.2.3 in [21]), we conclude that $u(s-1,\cdot)e^{-u(s-1,\cdot)}=0$ for all $s\in[0,1]$ and thus $\phi=u_0=0$, a contradiction. Similarly, we may prove $u_2\in X_+\setminus\{0\}$. Hence there exists $(t^*,x^*)\in(1,2)\times\bar{\Omega}$ such that $u(t^*,x^*)>0$. It follows from (2.1) and Lemma 2.1(i) that

$$\frac{\partial u}{\partial t}\geqslant \Delta u-\tau u \quad \text{in } (t^*,+\infty)\times\Omega,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } (t^*, +\infty) \times \partial\Omega \quad \text{and}$$
$$u(t^*, x) \geqslant 0 \quad \text{for all } x \in \Omega.$$

We now consider the following equation:

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) = \Delta v(t,x) - \tau v(t,x) & \text{in } (t^*, \infty) \times \Omega, \\ \dfrac{\partial v}{\partial n}(t,x) = 0 & \text{on } (t^*, \infty) \times \partial\Omega, \\ v(t^*, x) = u(t^*, x) & \text{for } x \in \bar{\Omega}. \end{cases}$$

By applying Theorem 7.3.4 in [21], we obtain that $u(t,x) \geqslant v(t,x)$ for all $(t,x) \in (t^*, +\infty) \times \bar{\Omega}$. On the other hand, by $v(t^*, x^*) > 0$ and Theorem 7.4.1 in [21], we have $v(t,x) > 0$ for all $(t,x) \in (t^*, +\infty) \times \bar{\Omega}$. Thus, $u(t,x) > 0$ for all $(t,x) \in (t^*, +\infty) \times \bar{\Omega}$, and the statement (i) holds (noting that $t^* \in (1,2)$).

Now we prove the statement (ii). Since $\sup\{ae^{-a}\colon a \in R\} = \frac{1}{e}$, it follows from (2.1) that

$$\frac{\partial u}{\partial t} \leqslant \Delta u - \tau u + \frac{\beta\tau}{e} \quad \text{in } (1, +\infty) \times \Omega,$$
$$\frac{\partial u}{\partial n} = 0 \quad \text{on } (1, +\infty) \times \partial\Omega \quad \text{and}$$
$$u(1, x) \geqslant 0 \quad \text{for all } x \in \Omega.$$

We now consider the following equation:

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) = \Delta w(t,x) - \tau w(t,x) + \dfrac{\beta\tau}{e} & \text{in } (1, \infty) \times \Omega, \\ \dfrac{\partial w}{\partial n}(t,x) = 0 & \text{on } (1, \infty) \times \partial\Omega, \\ w(1, x) = \|u(1, \cdot)\| & \text{for } x \in \bar{\Omega}. \end{cases}$$

It is easily seen that $w(t,x) = \frac{\beta}{e}[1 - e^{-\tau(t-1)}] + \|u(1,\cdot)\|e^{-\tau(t-1)}$ is a solution to the above problem for $(t,x) \in (1,\infty) \times \bar{\Omega}$. By Theorem 7.3.4 in [21], we conclude that $u(t,x) \leqslant w(t,x)$ for all $(t,x) \in (1,\infty) \times \bar{\Omega}$. Consequently, $u(t,x) \leqslant \frac{\beta}{e} + \|u(1,\cdot)\|e^{-\tau(t-1)}$ for all $(t,x) \in (1,\infty) \times \bar{\Omega}$. Let $K = K(\phi) = \frac{\beta}{e} + \sup\{\|u(t,\cdot)\|\colon t \in [-1,1]\}$. Then $|u(t,x)| \leqslant K$ for all $(t,x) \in (1,\infty) \times \bar{\Omega}$. This completes the proof of statement (ii). □

According to Lemma 2.1, we may define the map $U : R_+ \times X_+ \to X_+$ by $U(t,\phi) = (u^\phi)_t$ for $(t,\phi) \in R_+ \times X_+$. Then by an argument similar to Proposition 3.1 in [26], we obtain that U is a semiflow on X_+. Additionally, applying an argument similar to Proposition 2.4 in [26], we know that for a given $t > 1$, $U(t,\cdot) : X_+ \to X_+$ is completely continuous. More precisely, if $B \subset X_+$ is a bounded set B, then $U(t,\cdot)B$ is precompact for $t > 1$. For $\phi \in X_+$, let $O(\phi) = \{U(t,\phi)\colon t \geqslant 0\}$ and define $\omega(\phi) = \bigcap_{t \geqslant 0} \overline{O(U(t,\phi))}$. Given $\phi \in X_+$, by Lemma 2.2(ii), we know $\overline{O(\phi)}$ is compact, and hence $\omega(\phi)$ is nonempty, compact, connected and invariant. According to the invariance property of $\omega(\phi)$, for every $\psi \in \omega(\phi)$ there is a global solution $u: R \times \bar{\Omega} \to R_+$ with $u_0 = \psi$ and $u_t \in \omega(\phi)$ for all $t \in \mathbb{R}$ (see Hale [8]).

We now establish several important lemmas which are essential for proving our main result in Section 3.

Lemma 2.3. *Assume that $a \geqslant 0$ and $b \geqslant 0$. Then we have the following results.*

(i) *If $a - \ln\beta \geqslant |b - \ln\beta|$, then $-a + \beta b e^{-b} \leqslant 0$. Moreover, $-a + \beta b e^{-b} = 0$ if and only if $a = b = \ln\beta$.*
(ii) *If $\ln\beta - a \geqslant |b - \ln\beta|$, then $-a + \beta b e^{-b} \geqslant 0$. Moreover, $-a + \beta b e^{-b} = 0$ if and only if either $a = b = \ln\beta$ or $a = b = 0$.*

Proof. (i) We shall complete the proof by discussing three possible cases.

Case 1: $b \geqslant \ln\beta > 1$. In this case, we have $a - \ln\beta \geqslant |b - \ln\beta| = b - \ln\beta$ and thus $a \geqslant b \geqslant 0$. Since be^{-b} is strictly decreasing on $(1, \infty)$, $be^{-b} \leqslant \ln\beta e^{-\ln\beta} = \frac{\ln\beta}{\beta}$ and hence $-a + \beta b e^{-b} \leqslant -a + \ln\beta \leqslant 0$. In this case, we also notice that $a = b = \ln\beta$ if $-a + \beta b e^{-b} = 0$.

Case 2: $0 \leqslant b < \ln\beta$ and $a \geqslant \frac{\beta}{e}$. In this case, we have $a - \ln\beta \geqslant |b - \ln\beta| = \ln\beta - b$ and thus $a + b \geqslant 2\ln\beta$. Consider the function $z(\beta) = 2\ln\beta - 1 - \frac{\beta}{e}$. By the facts that $z(e) = 0$, $z(e^2) = 4 - 1 - e > 0$ and $\frac{\mathrm{d}z}{\mathrm{d}\beta} = \frac{2}{\beta} - \frac{1}{e}$, we conclude that $z(\beta) > 0$ for all $\beta \in (e, e^2]$. Thus, $2\ln\beta > 1 + \frac{\beta}{e}$ for all $\beta \in (e, e^2]$. Since $a + b \geqslant 2\ln\beta$, we have either $a \neq \frac{\beta}{e}$ or $b \neq 1$. This and the facts that $0 \leqslant b < \ln\beta$ and $a \geqslant \frac{\beta}{e}$ imply $-a + \beta b e^{-b} < -\frac{\beta}{e} + \frac{\beta}{e} = 0$.

Case 3: $0 \leqslant b < \ln\beta$ and $a < \frac{\beta}{e}$. In this case, we have $a - \ln\beta \geqslant |b - \ln\beta| = \ln\beta - b$ and thus $a + b \geqslant 2\ln\beta$. Let $h(b) = b + \beta b e^{-b}$. Then $h'(b) = 1 + \beta(1 - b)e^{-b}$ and $h''(b) = \beta(b - 2)e^{-b}$. From $\beta \leqslant e^2$ and $b < \ln\beta \leqslant 2$, we have $h''(b) < 0$ for $0 \leqslant b < \ln\beta$. Hence, $h'(b) > h'(\ln\beta) = 2 - \ln\beta \geqslant 0$ for all $b \in [0, \ln\beta)$, and hence $h(b) < h(\ln\beta) = 2\ln\beta$. Since $a + b \geqslant 2\ln\beta$, we have $-a + \beta b e^{-b} = -(a + b) + \beta b e^{-b} + b \leqslant -2\ln\beta + \beta b e^{-b} + b = -2\ln\beta + h(b) < 0$.

From the above three cases, we see that $a = b = \ln\beta$ if $-a + \beta b e^{-b} = 0$ (possible only in Case 1). On the other hand, $-a + \beta b e^{-b} = 0$ when $a = b = \ln\beta$. This completes the proof of (i).

The proof of (ii) is similar, and hence is omitted. This completes the proof of the lemma. □

Set $D \equiv \{\psi \in X_+ \colon\ \psi(\theta, x) < 1$ for all $(t, x) \in [-1, 0] \times \bar{\Omega}\}$. For $\phi \in D$, let $\eta_\phi = \sup\{t \in R_+ \colon\ u^\phi(s, x) < 1$ for all $(s, x) \in [0, t] \times \bar{\Omega}\}$. Also let $Y = \bigcup\{[0, \eta_\phi) \times \{\phi\} \colon\ \phi \in D\}$ and define the map U^Y by $U^Y(t, \phi) = U(t, \phi)$ for all $(t, \phi) \in Y$. Then U^Y is a dynamical system (or local semiflow) on Y.

Lemma 2.4. *We have the following results.*

(i) *U^Y is a* monotone *dynamical system on Y in the sense that if $\phi, \psi \in D$ with $\phi - \psi \in X_+$ and if $t \in [0, \min\{\eta_\phi, \eta_\psi\})$, then $U^Y(t, \phi) - U^Y(t, \psi) \in X_+$.*
(ii) *Let $v(t, x)$ and $w(t, x)$ be positive classical solutions of* (2.1), (2.2). *Suppose that there exists $T > 0$ such that $v(t, x) < 1$ and $w(t, x) < 1$ for all $(t, x) \in [-1, T] \times \bar{\Omega}$. If $v(\theta, x) \leqslant w(\theta, x)$ for all $(\theta, x) \in [-1, 0] \times \bar{\Omega}$, then $v(t, x) \leqslant w(t, x)$ for all $(t, x) \in [-1, T] \times \bar{\Omega}$.*

Proof. Since $\frac{\mathrm{d}(ae^{-a})}{\mathrm{d}a} = (1 - a)e^{-a} \geqslant 0$ for all $a \in [0, 1)$, Theorem 3.3 in [17] implies that the statement (i) is true. Thus, the statement (ii) follows. □

Next, consider the following scalar ordinary delay differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}t}(t) = -\tau y(t) + \beta\tau y(t-1)e^{-y(t-1)}, \tag{2.5}$$

where $\tau > 0$ and $e < \beta \leqslant e^2$. For a given $y_0 \in C([-1,0], R_+)$, by the theory in [9], (2.5) has a unique solution for $t \in R_+$, and by the theory in [21], this solution remains non-negative for $t \in R_+$. The following lemma gives more information on such a solution.

Lemma 2.5. *Let $y_0 \in C([-1,0], R_+) \setminus \{0\}$ be given and let $y : [-1,+\infty) \to R$ be the solution of* (2.5) *with this initial function. Then there exists $T_0 > 0$ such that $y(T_0) \geqslant 1$.*

Proof. Otherwise, $y(t) < 1$ for all $t > 0$. Let $u(t,x) = y(t)$ for all $(t,x) \in [-1,+\infty) \times \bar{\Omega}$. Then $u_0 \in X_+ \setminus \{0\}$ and $u(t,\cdot)$ also satisfies Eq. (2.3) with the initial value function $\phi = u_0$. Thus, $u_t = (u^{u_0})_t = U^Y(t,u_0)$ for all $t \in R_+$, where U^Y defined as in Lemma 2.4. By Lemma 2.2(i), we have $u_t \in \mathrm{Int}(X_+)$ for all $t > 3$. Hence there exist $t^* > 3$ and $\delta \in (0,1)$ such that $u_{t^*} > \delta$. Applying Lemma 2.4(i) and the facts $u_{t^*} > \delta$ and $u(t,x) < 1$ for all $(t,x) \in [-1,+\infty) \times \bar{\Omega}$, we obtain $U^Y(t+t^*,u_0) - U^Y(t,\delta) = U^Y(t, U^Y(t^*,u_0)) - U^Y(t,\delta) \in X_+$ for all $t \in [0,\eta_\delta)$. If $\eta_\delta < +\infty$, then there exists $x^* \in \bar{\Omega}$ such that $U(\eta_\delta,\delta)(0,x^*) = 1$, and thus $U^Y(\eta_\delta + t^*, u_0)(0,x^*) \geqslant U(\eta_\delta,\delta)(0,x^*) = 1$, a contradiction. So Lemma 2.5 follows.

Now we assume that $\eta_\delta = +\infty$. Let $\tilde{y} : [-1,+\infty) \to R$ be the solution of (2.5) with the initial value $\tilde{y}_0$ such that $\tilde{y}_0(\theta) = \delta$ for all $\theta \in [-1,0]$. Then $U(t,\delta)(\theta,x) = (u^\delta)_t(\theta,x) = \tilde{y}(t+\theta)$ for all $(\theta,t,x) \in [-1,0] \times R_+ \times \bar{\Omega}$. Thus, $y(t+t^*) \geqslant \tilde{y}(t)$ for all $t > 0$ and $\tilde{y}(t) < 1$ for all $t > 0$.

Since $\delta \in (0,1)$ and $\tilde{y}(t) = \delta$ for all $t \in [-1,0]$, it follows from (2.5) that

$$\begin{aligned}
\left.\frac{\mathrm{d}\tilde{y}(t)}{\mathrm{d}t}\right|_{t=0^+} &= \lim_{t\to 0^+} \frac{\mathrm{d}\tilde{y}(t)}{\mathrm{d}t} \\
&= -\tau\delta + \beta\tau\delta e^{-\delta} \\
&= \tau\delta\bigl(-1 + \beta e^{-\delta}\bigr) \\
&> \tau\delta\bigl(-1 + \beta e^{-1}\bigr) \\
&> 0.
\end{aligned}$$

Thus there exists $h^* \in (0,1)$ such that $\frac{\mathrm{d}\tilde{y}(t)}{\mathrm{d}t} > 0$ for all $t \in (0,h^*]$, which implies that $\tilde{y}$ is nondecreasing on $[-1,h^*]$.

We claim that $\tilde{y}$ is nondecreasing on $[-1,+\infty)$. Otherwise, there exists $h^* \leqslant t^* < +\infty$ such that $t^* = \sup\{t \in [0,+\infty)\colon \tilde{y} \text{ is nondecreasing on } [-1,t]\}$. Then $\tilde{y}$ is nondecreasing on $[-1,t^*]$. For any given $h \in [0,h^*]$, let $z(t) = \tilde{y}(t+h) - \tilde{y}(t)$ for all $t \in [0, 1+t^*-h]$. Then $z(0) \geqslant 0$. We next prove that $z(t) \geqslant 0$ for all $t \in [0,1+t^*-h]$. Obviously, we have $\tilde{y}(t+h-1) \geqslant \tilde{y}(t-1)$ for all $t \in [0,1+t^*-h]$ since $\tilde{y}$ is nondecreasing on $[-1,t^*]$. According to the fact that be^{-b} is increasing on $[0,1]$, we have $\beta\tau\tilde{y}(t+h-1)e^{-\tilde{y}(t+h-1)} - \beta\tau\tilde{y}(t-1)e^{-\tilde{y}(t-1)} \geqslant 0$ for all $t \in [0,1+t^*-h]$. It follows from (2.5) that for all $t \in [0,1+t^*-h]$,

$$\begin{aligned}
z'(t) &= -\tau z(t) + \beta\tau\tilde{y}(t+h-1)e^{-\tilde{y}(t+h-1)} - \beta\tau\tilde{y}(t-1)e^{-\tilde{y}(t-1)} \\
&\geqslant -\tau z(t).
\end{aligned}$$

Hence $z(t) \geqslant e^{-\tau t} z(0) \geqslant 0$ for all $t \in [0, 1 + t^* - h]$. This and the arbitrariness of $h \in [0, h^*]$ show that $\tilde{y}$ is nondecreasing on $[0, t^* + 1)$. So, $\tilde{y}$ is nondecreasing on $[-1, t^* + 1]$ which contradicts to the choice of t^*. Thus the claim holds, that is, $\tilde{y}$ is nondecreasing on $[-1, +\infty)$.

By the above claim and the fact that $\tilde{y}(t) < 1$ for all $t > 0$, we conclude that there exists $y^* \in (0, 1]$ such that $\tilde{y}(t) \to y^*$ as $t \to +\infty$ and $0 < y^* \leqslant 1 < \ln\beta$. From Eq. (2.5), we have $\lim_{t\to+\infty} \frac{\mathrm{d}\tilde{y}(t)}{\mathrm{d}t} = -\tau y^* + \tau\beta y^* e^{-y^*} > 0$. So, there exists $T^* > 1$ such that $\frac{\mathrm{d}\tilde{y}(t)}{\mathrm{d}t} \geqslant \frac{-\tau y^* + \tau\beta y^* e^{-y^*}}{2} > 0$ for all $t \geqslant T^*$. Thus for every $t \geqslant T^*$, we have $\tilde{y}(t) = \tilde{y}(T^*) + \int_{T^*}^{t} \frac{\mathrm{d}\tilde{y}(s)}{\mathrm{d}s}\,\mathrm{d}s \geqslant \tilde{y}(T^*) + \frac{-\tau y^* + \tau\beta y^* e^{-y^*}}{2}(t - T^*)$. This implies $\lim_{t\to+\infty} \tilde{y}(t) = +\infty$, a contradiction. Therefore, there exists $T_0 > 0$ such that $y(T_0) \geqslant 1$. This completes the proof. □

The following lemma is from [20].

Lemma 2.6. *Let $T > 0$ and $W \subseteq \bar{\Omega}$ be an open domain with a smooth boundary ∂W. Let $u(t, x)$ be a continuous function on $[0, T] \times \bar{\Omega}$ with derivatives $\frac{\partial u}{\partial x_i}$, $\frac{\partial^2 u}{\partial x_i \partial x_j}$ and $\frac{\partial u}{\partial t}$ existing and being continuous on $(0, T] \times \Omega$. Let $Lu(t, x) = \Delta u(t, x) - \frac{\partial u}{\partial t}(t, x)$. Then we have the following results.*

(i) *If $Lu(t, x) > 0$ for all $(t, x) \in (0, T) \times W$, then u cannot attain a local maximum in $(0, T) \times W$.*
(ii) *If $Lu(t, x) < 0$ for all $(t, x) \in (0, T) \times W$, then u cannot attain a local minimum in $(0, T) \times W$.*
(iii) *Suppose that the first derivatives of u with respect to the x_i exist and are continuous on $(0, T] \times \bar{\Omega}$. Let $Lu(t, x) \geqslant 0$ for all $(t, x) \in (0, T) \times W$. If there exist $(t^*, x^*) \in (0, T) \times \partial W$, $\varepsilon^* \in (0, T)$ and an open ball $S^* \subseteq W$ such that $S^* \cap \partial W = \{x^*\}$ and $u(t^*, x^*) > u(t, x)$ for all $(t, x) \in [t^* - \varepsilon^*, t^* + \varepsilon^*] \times S^*$ then $\frac{\partial u}{\partial n}|_{(t^*, x^*)} > 0$.*
(iv) *Suppose that the first derivatives of u with respect to the x_i exist and are continuous on $(0, T] \times \bar{\Omega}$. Let $Lu(t, x) \leqslant 0$ for all $(t, x) \in (0, T) \times W$. If there exist $(t^*, x^*) \in (0, T) \times \partial W$, $\varepsilon^* \in (0, T)$ and an open ball $S^* \subseteq W$ such that $S^* \cap \partial W = \{x^*\}$ and $u(t^*, x^*) < u(t, x)$ for all $(t, x) \in [t^* - \varepsilon^*, t^* + \varepsilon^*] \times S^*$ then $\frac{\partial u}{\partial n}|_{(t^*, x^*)} < 0$.*

3. Main result

For convenience of discussion, we let $h(a, b, \beta) = -a + \beta b e^{-b}$ throughout this section. Now we are in the position to state and prove our main result.

Theorem 3.1. *If $e < \beta \leqslant e^2$, then the positive steady state $u^+ \equiv \ln\beta$ attracts all positive solutions of* (2.1), (2.2).

Proof. Let $\phi_0 \in X_+$ be an initial value function corresponding to which, the classical solution $v(t, x)$ of (2.1) remains positive for all $t \geqslant 0$ and $x \in \bar{\Omega}$. We need to prove that $\omega(\phi_0) = \{\ln\beta\}$. Let $M_0 := \sup\{\|\phi - \ln\beta\|: \phi \in \omega(\phi_0)\} = \sup\{|\phi(\theta, x) - \ln\beta|: \theta \in [-1, 0],\ x \in \bar{\Omega},\ \phi \in \omega(\phi_0)\}$. We only need to show $M_0 = 0$. For the sake of contradiction, assume $M_0 > 0$. Note that by Lemma 2.2(i), $\omega(\phi_0) \setminus \{0\} \subseteq \mathrm{Int}(X_+)$ and $v_t \in \mathrm{Int}(X_+)$ for all $t > 3$. We have three possible cases.

Case 1: There is a $\phi \in \omega(\phi_0) \setminus \{0\}$ such that $\|\phi - \ln\beta\| = M_0$. Then, by the invariance property of $\omega(\phi_0)$ and Lemma 2.1(iii), we know that there is a global classical positive solution $u : R \times \bar{\Omega} \to R$ such that $u_t \in \omega(\phi_0) \cap \mathrm{Int}(X_+)$ for all $t \in R$ and $u_0 = \phi$. In this case, we claim that $|u(t,x) - \ln\beta| < M_0$ for all $(t,x) \in R \times \Omega$. Otherwise, there exists $(t_1, x_1) \in R \times \Omega$ such that $|u(t_1, x_1) - \ln\beta| = M_0 > 0$. If $u(t_1, x_1) - \ln\beta = M_0$, then $u(t_1, x_1) > \ln\beta$ and $u(t_1, x_1) = \sup\{u(t,x)\colon (t,x) \in R \times \bar{\Omega}\}$ (since $u(t_1,x_1) - u(t,x) = M_0 - [u(t,x) - \ln\beta] = M_0 - [u_t(x)(0) - \ln\beta] \geqslant 0$). By Lemma 2.3(i), we have $h(u(t_1,x_1), u(t_1-1,x_1), \beta) < 0$ and hence $\Delta u|_{(t_1,x_1)} - \frac{\partial u}{\partial t}|_{(t_1,x_1)} > 0$. It follows from Lemma 2.6(i) that u cannot attain a local maximum at (t_1, x_1), which yields a contradiction. If $u(t_1,x_1) - \ln\beta = -M_0$, then $0 < u(t_1,x_1) < \ln\beta$ and $u(t_1,x_1) = \inf\{u(t,x)\colon (t,x) \in R \times \bar{\Omega}\}$ (since $u(t_1,x_1) - u(t,x) = -[M_0 - (\ln\beta - u(t,x))] = -[M_0 - (\ln\beta - u_t(x)(0))] \leqslant 0$). By Lemma 2.3(ii), we have $h(u(t_1,x_1), u(t_1-1,x_1), \beta) > 0$ and hence $\Delta u|_{(t_1,x_1)} - \frac{\partial u}{\partial t}|_{(t_1,x_1)} < 0$. It follows from Lemma 2.6(ii) that u cannot attain a local minimum at (t_1, x_1), which yields a contradiction. Consequently, the above claim follows.

Since $\|u_0 - \ln\beta\| = M_0$, there exists $(t^*, x^*) \in [-1, 0] \times \partial\Omega$ such that $|u(t^*,x^*) - \ln\beta| = M_0 > 0$. If $u(t^*,x^*) - \ln\beta = M_0 > 0$, then $u(t^*,x^*) - \ln\beta = |u(t^*-1,x^*) - \ln\beta| + (M_0 - |u(t^*-1,x^*) - \ln\beta|) \geqslant |u(t^*-1,x^*) - \ln\beta|$. Hence by Lemma 2.3(i), we obtain that $h(u(t^*,x^*), u(t^*-1,x^*), \beta) < 0$. By the continuity of u and h, and the smoothness of $\partial\Omega$, there exist an $\varepsilon > 0$ and an open ball $S^* \subseteq \Omega$, such that $\partial S^* \cap \Omega = \{x^*\}$ and $h(u(t,x), u(t-1,x), \beta) \leqslant 0$ for $(t,x) \in [t^*-\varepsilon, t^*+\varepsilon] \times S^*$. From (2.1), we have $\Delta u(t,x) - \frac{\partial u}{\partial t}(t,x) \geqslant 0$ for $(t,x) \in [t^*-\varepsilon, t^*+\varepsilon] \times S^*$. Note that $u(t^*,x^*) = \ln\beta + M_0 > u(t,x)$ for all $(t,x) \in R \times \Omega$ since $|u(t,x) - \ln\beta| < M_0$ for all $(t,x) \in R \times \Omega$. Thus by Lemma 2.6(iii), we obtain $\frac{\partial u}{\partial n}|_{(t^*,x^*)} > 0$, a contradiction. If $0 < M_0 = \ln\beta - u(t^*,x^*)$, then $M_0 < \ln\beta$. By a similar argument to the above, we have $\frac{\partial u}{\partial n}|_{(t^*,x^*)} < 0$, a contradiction.

Case 2: $\omega(\phi_0) \setminus \{0\} \neq \emptyset$ and $M_0 > \|\phi - \ln\beta\|$ for all $\phi \in \omega(\phi_0) \setminus \{0\}$. In this case, by the definition of M_0, we have $0 \in \omega(\phi_0)$ and $M_0 = \ln\beta$. Let $\widetilde{M}_0 = \sup\{\phi(\theta,x) - \ln\beta\colon \phi \in \omega(\phi_0), \theta \in [-1,0] \text{ and } x \in \bar{\Omega}\}$. Then $-M_0 \leqslant \widetilde{M}_0 < M_0$.

Claim 1: There is an $s^* > 3$ such that $M_0 > M_0^* \equiv \sup\{v(t,x) - \ln\beta\colon (t,x) \in [s^*, +\infty) \times \bar{\Omega}\}$.

Indeed, by the definition of $\omega(\phi_0)$, there exists $s^* > 3$ such that $\inf\{\|v_t - \psi\|\colon \psi \in \omega(\phi_0)\} < \frac{M_0 - \widetilde{M}_0}{3}$ for all $t \in [s^*, +\infty)$. It follows from the compactness of $\omega(\phi_0)$ that for every $t \in [s^*, +\infty)$, there exists $\psi^t \in \omega(\phi_0)$ such that $\|v_t - \psi^t\| = \inf\{\|v_t - \psi\|\colon \psi \in \omega(\phi_0)\}$. Thus, according to the choice of ψ^t, we obtain that

$$
\begin{aligned}
M_0^* &= \sup\bigl\{v(t,x) - \ln\beta\colon (t,x) \in [s^*, +\infty) \times \bar{\Omega}\bigr\} \\
&\leqslant \sup\bigl\{v(t,x) - \psi^t(0,x)\colon (t,x) \in [s^*, +\infty) \times \bar{\Omega}\bigr\} \\
&\quad + \sup\bigl\{\psi^t(0,x) - \ln\beta\colon (t,x) \in [s^*, +\infty) \times \bar{\Omega}\bigr\} \\
&\leqslant \sup\bigl\{\bigl\|v_t - \psi^t\bigr\|\colon t \in [s^*, +\infty)\bigr\} + \sup\bigl\{\psi^t(0,x) - \ln\beta\colon (t,x) \in [s^*, +\infty) \times \bar{\Omega}\bigr\} \\
&\leqslant \frac{M_0 - \widetilde{M}_0}{3} + \widetilde{M}_0 \\
&< M_0.
\end{aligned}
$$

This completes the proof of Claim 1.

Since $\omega(\phi_0) \setminus \{0\} \neq \emptyset$ and $\omega(\phi_0) \setminus \{0\} \subseteq \mathrm{Int}(X_+)$, there exist $\varepsilon_0 \in (0, \frac{1}{8}\min\{1, M_0 - \widetilde{M}_0, M_0 - M_0^*\})$ and $\psi^* \in \omega(\phi_0)$ such that $\psi^* - \varepsilon_0 \in X_+$. In view of $\{0, \psi^*\} \subseteq \omega(\phi_0)$ and the definition of $\omega(\phi_0)$, there exist $s_1 > s^* + 1$, $s_2 > s^* + 1$ and $s_3 > s^* + 1$ such that $s_1 < s_2 - 1 < s_2 < s_3 - 1 < s_3$, $\|v_{s_2}\| < \frac{1}{8}\varepsilon_0$, $\|v_{s_1} - \psi^*\| < \frac{1}{8}\varepsilon_0$ and $\|v_{s_3} - \psi^*\| < \frac{1}{8}\varepsilon_0$. Let $M_1 = \sup\{|v(t,x) - \ln\beta|\colon (t,x) \in [s_1 - 1, s_3] \times \bar{\Omega}\}$. Then it follows from $M_0 > M_0^*$ and the definition of M_1 that

$M_0 = \ln\beta > M_1$. By the choice of s_i and the definition of M_1, we have $M_1 \geqslant \|v_{s_2} - \ln\beta\| \geqslant \|\ln\beta\| - \|v_{s_2}\| > M_0 - \frac{1}{8}\varepsilon_0$. In view of the choice of ε_0, we have $M_1 > \widetilde{M}_0$ and $M_1 > M_0^*$ and hence $M_0 = \ln\beta > M_1 > \max\{M_0^*, \widetilde{M}_0\}$.

Claim 2: $M_1 > |\ln\beta - v(t,x)|$ for all $(t,x) \in ([s_1 - 1, s_1] \cup [s_3 - 1, s_3]) \times \bar{\Omega}$. Indeed, from the choice of s_1, we have

$$\begin{aligned}\|v_{s_1} - \ln\beta\| &\leqslant \|\psi^* - \ln\beta\| + \|v_{s_1} - \psi^*\| \\ &= \max\bigl\{\sup\bigl\{\psi^*(\theta,x) - \ln\beta\colon (\theta,x) \in [-1,0] \times \bar{\Omega}\bigr\}, \\ &\quad \sup\bigl\{\ln\beta - \psi^*(\theta,x)\colon (\theta,x) \in [-1,0] \times \bar{\Omega}\bigr\}\bigr\} + \|v_{s_1} - \psi^*\| \\ &\leqslant \max\{\widetilde{M}_0, \ln\beta - \varepsilon_0\} + \|v_{s_1} - \psi^*\| \\ &< \max\{\widetilde{M}_0, \ln\beta - \varepsilon_0\} + \frac{1}{8}\varepsilon_0 \\ &= \max\left\{\widetilde{M}_0 + \frac{1}{8}\varepsilon_0, \ln\beta - \frac{7}{8}\varepsilon_0\right\}.\end{aligned}$$

According to the choice of ε_0, we have $\widetilde{M}_0 + \frac{1}{8}\varepsilon_0 < M_0 - \frac{1}{8}\varepsilon_0$ and hence $\|v_{s_1} - \ln\beta\| < M_0 - \frac{1}{8}\varepsilon_0 < M_1$. A similar argument shows that $\|v_{s_3} - \ln\beta\| < M_1$. So, $M_1 > \max\{\|v_{s_1} - \ln\beta\|, \|v_{s_3} - \ln\beta\|\}$, that is, Claim 2 holds.

Claim 3: $|v(t,x) - \ln\beta| < M_1$ for all $(t,x) \in [s_1 - 1, s_3] \times \Omega$. Otherwise, it follows from Claim 2 and $M_1 > M_0^*$ that there exists $(t_2, x_2) \in [s_1, s_3 - 1] \times \Omega$ such that $M_1 = \ln\beta - v(t_2, x_2) > 0$ and hence $0 < v(t_2, x_2) < \ln\beta$. By Lemma 2.3(ii), we have $h(v(t_2,x_2), v(t_2 - 1, x_2), \beta) > 0$ and hence $\Delta v|_{(t_2,x_2)} - \frac{\partial v}{\partial t}|_{(t_2,x_2)} < 0$. It follows from Lemma 2.6(ii) that v cannot attain a local minimum at (t_2, x_2), which yields a contradiction. Consequently, Claim 3 holds.

It follows from Claims 2, 3 and $M_1 > M_0^*$, that there exists $(t^{**}, x^{**}) \in (s_1, s_3) \times \partial\Omega$ such that $M_1 = \ln\beta - v(t^{**}, x^{**}) > 0$. Hence by Lemma 2.3(ii), we obtain that $h(v(t^{**}, x^{**}), v(t^{**} - 1, x^{**}), \beta) > 0$. By the continuity of v and h, and the smoothness of $\partial\Omega$, there exist an $\varepsilon > 0$ and an open ball $S^{**} \subseteq \Omega$, such that $\partial S^{**} \cap \Omega = \{x^{**}\}$ and $h(v(t,x), v(t-1,x), \beta) \geqslant 0$ for $(t,x) \in [t^{**} - \varepsilon, t^{**} + \varepsilon] \times S^{**}$. From (2.1), we have $\Delta v(t,x) - \frac{\partial v}{\partial t}(t,x) \leqslant 0$ for $(t,x) \in [t^{**} - \varepsilon, t^{**} + \varepsilon] \times S^{**}$. Note that by Claim 3, $v(t^{**}, x^{**}) = \ln\beta - M_1 < v(t,x)$ for all $(t,x) \in [s_1 - 1, s_3] \times \Omega$. Thus by Lemma 2.6(iv), we obtain $\frac{\partial v}{\partial n}|_{(t^{**},x^{**})} < 0$, a contradiction.

Case 3: $\omega(\phi_0) = \{0\}$. In this case, by the definition of $\omega(\phi_0)$, there exists $T_0 > 0$ such that $|v(t,x)| < 1$ for all $(t,x) \in [T_0, +\infty) \times \bar{\Omega}$. By Lemma 2.2(i), there exists $T_1 > T_0 + 3$ and $\varepsilon_1 > 0$ such that $v_{T_1} - \varepsilon_1 \in X_+$. Let $w(t,x)$ be a solution of (2.1) with $w_0 = \varepsilon_1$. Then $w(t,x)$ is independent of $x \in \bar{\Omega}$ and satisfies (2.5). Let $T_2 = \inf\{t > 0\colon\ w(t,x) = 1 \text{ for some } x \in \bar{\Omega}\}$. Thus Lemma 2.5 implies $T_2 \in R_+$. By Lemma 2.4, we have $1 > v(t,x) \geqslant w(t - T_1, x)$ for all $(t,x) \in [T_1, T_1 + T_2]$. This is a contradiction to the choice of $T_2 > 0$.

Summarizing the above Cases 1–3, we see that $M_0 > 0$ is impossible and hence $M_0 = 0$, leading to the conclusion $\omega(\phi_0) = \{\ln\beta\}$. This completes the proof of the theorem. □

Remark 3.1. According to Lemmas 2.1, 2.2 and Theorem 3.1, particularly for $\phi \in X_+ \setminus \{0\}$, $\omega(\phi) = \{u^+\}$ provided that $e < \beta \leqslant e^2$.

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