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The b-chromatic number of a graph

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Abstract

The achromatic number $\psi(G)$ of a graph $G=(V,E)$ is the maximum k such that V has a partition $V_1, V_2, \ldots, V_k$ into independent sets, the union of no pair of which is independent. Here we show that $\psi(G)$ can be viewed as the maximum over all minimal elements of a partial order defined on the set of all colourings of G. We introduce a natural refinement of this partial order, giving rise to a new parameter, which we call the *b-chromatic number*, $\varphi(G)$, of G. We prove that determining $\varphi(G)$ is NP-hard for general graphs, but polynomial-time solvable for trees. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

A *proper k-colouring* of a graph $G=(V,E)$ is a partition $P=\{V_1, V_2, \ldots, V_k\}$ of V into independent sets. The *chromatic number*, $\chi(G)$, of G is the minimum integer k such that G has a proper k-colouring. The parameter χ has been extensively studied with regard to algorithmic complexity (cf. [7, problem GT4]).

A related parameter, $\psi(G)$, may be defined (as in [7, problem GT5]) as the maximum k for which G has a proper colouring $\{V_1, V_2, \ldots, V_k\}$ that also satisfies the following property:

$$\forall 1 \leqslant i < j \leqslant k \bullet V_i \cup V_j \quad \text{is not independent.} \tag{1}$$

A proper colouring of a graph that also satisfies Property 1 is called a *complete* or *achromatic* colouring. The parameter ψ was first studied by Harary et al. [11] and was named the *achromatic number* by Harary and Hedetniemi [10]. The ACHROMATIC NUMBER problem of determining whether $\psi(G) \geqslant K$, for a given graph G and integer K, was shown to be NP-complete by Yannakakis and Gavril [16], even for the complements of bipartite graphs. Farber et al. [6] proved that ACHROMATIC NUMBER remains NP-complete

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for bipartite graphs, while Bodlaender [1] demonstrated NP-completeness for connected graphs that are simultaneously a cograph and an interval graph. Cairnie and Edwards [2] have recently shown that the problem remains NP-complete for trees. Chaudhary and Vishwanathan [3] obtained the first polynomial-time $o(n)$ approximation algorithm for ACHROMATIC NUMBER.

Many maximum and minimum graph parameters have 'minimum maximal' and 'maximum minimal' counterparts, respectively [9]. Much algorithmic activity has been focused on such parameters relating to domination [4], independence [14, 8] and irredundance [12]. However, the implicit partial order throughout is that of set inclusion. In other words, for a property P, a set S is maximal (minimal) if no proper superset (subset) of S also satisfies property P. But the concepts of maximality and minimality apply equally in cases where the partial order defined is other than set inclusion.

In this paper, we are concerned with two natural partial orders, $\prec_a^G$ and $\prec_b^G$ (a refinement of $\prec_a^G$), defined on the set of all partitions of the vertex set of a given graph G. We show that $\psi(G)$ has a natural interpretation as the maximum k such that $\{V_1, V_2, \ldots, V_k\}$ is a $\prec_a^G$-minimal partition into independent sets. Similarly, consideration of proper colourings that are minimal with respect to $\prec_b^G$ gives rise to a new parameter, $\varphi(G)$, which we call the *b-chromatic number* of G. We note, in passing, the considerable generality of this model, extending well beyond the realm of graph theory, in which a partial order is imposed on the feasible solutions of an optimisation problem, leading to many interesting 'minimaximal' and 'maximinimal' type problems [15].

The remainder of this paper is organised as follows. In Section 2, we show how to define the b-chromatic number of a graph. We then study the parameter with regard to algorithmic complexity: in Section 3 we prove that the problem of determining $\varphi(G)$ is NP-hard for arbitrary graphs, whilst in Section 4 we present a polynomial-time algorithm for trees.

2. Defining the b-chromatic number

We begin with some definitions that will be used in the rest of this paper. Let $G=(V,E)$ be an arbitrary graph. For a vertex $v \in V$, define the *open neighbourhood* of v to be $N(v)=\{w \in V\colon \{v,w\} \in E\}$. For a set $S \subseteq V$, the open neighbourhood of S is the set $N(S)=\bigcup_{v \in S} N(v)$. A *colouring* of G is *any* partition of V. A *proper colouring* of G is a partition of V into independent sets. We denote the set of all colourings of a graph G by $\mathscr{U}(G)$ (the *universal set*) and the set of all proper colourings by $\mathscr{F}(G)$ (the *feasible set*). If $v \in V$ then the *colour*, $c(v)$, assigned to v in any colouring $\{V_1, V_2, \ldots, V_k\}$ is the unique i such that $1 \leqslant i \leqslant k$ and $v \in V_i$. The degree of a vertex $v \in V$ is denoted $d(v)$.

For a graph $G=(V,E)$, consider the following relation, $\sqsubset_a^G$, defined on $\mathscr{U}(G)$:

$$\sqsubset_a^G = \left\{ (P,Q) \in \mathscr{U}(G) \times \mathscr{U}(G)\colon \begin{array}{c} P=\{U_1, U_2, \ldots, U_k\} \wedge \\ Q=\{V_1, V_2, \ldots, V_{k+1}\} \wedge \\ \forall 1 \leqslant i \leqslant k-1 \bullet U_i = V_i \end{array} \right\}.$$

Intuitively, for a graph G and two colourings $c_1, c_2 \in \mathscr{U}(G)$, $c_1 \sqsubset_a^G c_2$ if and only if (in order to produce colouring c_1) every vertex belonging to one of the colours i in c_2 is recoloured by one particular colour j chosen from the other colours, while every other vertex retains its original colour.

By setting $\prec_a^G$ equal to the transitive closure of $\sqsubset_a^G$, we obtain a strict partial order. Define a colouring $c \in \mathscr{U}(G)$ to be $\prec_a^G$-*minimal* if $c \in \mathscr{F}(G)$ and there is no $c' \in \mathscr{F}(G)$ such that $c' \prec_a^G c$. If we consider the restriction of the partial order $\prec_a^G$ to the set $\mathscr{F}(G)$, the proper colourings of G, it is easy to see that the minimal (with respect to the restricted partial order) proper colourings of G are exactly the achromatic colourings of G.

However, it is interesting to note that the same result holds for the partial order $\prec_a^G$, defined on $\mathscr{U}(G)$. The following lemma justifies the interpretation of the achromatic number parameter in terms of the concept of $\prec_a^G$-minimality.

Lemma 1. *A proper colouring* $c = \{V_1, V_2, \ldots, V_k\}$ *of a graph* G *is* $\prec_a^G$-*minimal if and only if*

$$\forall 1 \leqslant i < j \leqslant k \bullet V_i \cup V_j \quad \textit{is not independent.} \tag{2}$$

Proof. One direction is trivial. For the converse, suppose that $c_F \in \mathscr{F}(G)$ satisfies Property 2, and that some $c_U \in \mathscr{U}(G)$ satisfies $c_U \prec_a^G c_F$. Then there exists a chain

$$c_U = c_1 \sqsubset_a^G c_2 \sqsubset_a^G \cdots \sqsubset_a^G c_{n-1} \sqsubset_a^G c_n = c_F$$

for some $n > 1$, where $c_i \in \mathscr{U}(G)$ for $1 \leqslant i \leqslant n$. Clearly $c_{n-1} \notin \mathscr{F}(G)$, and hence there is some $V_{n-1} \in c_{n-1}$ such that V_{n-1} is not independent. But $c_1 \prec_a^G c_{n-1}$, and hence there is some $V_1 \in c_1$ such that $V_{n-1} \subseteq V_1$. Thus $c_1 = c_U \notin \mathscr{F}(G)$, as required. $\square$

By Lemma 1, we can make the definition

$$\psi(G) = \max\{|c| : c \in \mathscr{F}(G) \text{ is } \prec_a^G\text{-minimal}\}.$$

We now introduce a natural refinement of the partial order $\prec_a^G$, which gives rise to the *b-chromatic number* parameter. As described above, when $c_1 \sqsubset_a^G c_2$ for two colourings c_1 and c_2, intuitively there are two colours i and j of c_2 such that every vertex belonging to colour i in c_2 is recoloured by colour j, while every other vertex retains its original colour, in order to produce c_1. In fact it would be more flexible to allow the recolouring process to pick some colour i of c_2 and redistribute the vertices of colour i among the other colours of c_2, in order to produce c_1. More formally, we define the following relation, $\sqsubset_b^G$, on $\mathscr{U}(G)$, for a given graph G:

$$\sqsubset_b^G = \left\{ (P, Q) \in \mathscr{U}(G) \times \mathscr{U}(G) : \begin{array}{l} P = \{U_1, U_2, \ldots, U_k\} \wedge \\ Q = \{V_1, V_2, \ldots, V_{k+1}\} \wedge \\ \forall 1 \leqslant i \leqslant k \bullet V_i \subseteq U_i \end{array} \right\}.$$

By taking $\prec_b^G$ to be the transitive closure of $\sqsubset_b^G$, we obtain a strict partial order. Define a colouring $c \in \mathscr{U}(G)$ to be *$\prec_b^G$-minimal* if $c \in \mathscr{F}(G)$ and there is no $c' \in \mathscr{F}(G)$ such that $c' \prec_b^G c$.

Earlier, we saw a proper colouring c is minimal with respect to the partial order $\prec_a^G$ if and only if c is minimal with respect to the restriction of $\prec_a^G$ to the set $\mathscr{F}(G)$. Interestingly, the same result holds for the partial order $\prec_b^G$. For, it is possible that one proper colouring c' may be obtained from another proper colouring c, by making a series of redistributions using the relation $\sqsubset_b^G$, such that the intermediate colourings between c' and c are not proper. However, the following theorem shows that in fact we can always find intermediate *proper* colourings when obtaining c' from c by a series of redistributions. The two corollaries of the theorem establish a useful condition for $\prec_b^G$-minimality to be used in Sections 3 and 4.

Theorem 2. *For a graph G, let $c_{F_1}, c_{F_2} \in \mathscr{F}(G)$ be proper colourings such that $c_{F_1} \prec_b^G c_{F_2}$. Then, for some $n > 1$, there exists a chain*

$$c_{F_1} = c_1' \sqsubset_b^G c_2' \sqsubset_b^G \cdots \sqsubset_b^G c_{n-1}' \sqsubset_b^G c_n' = c_{F_2},$$

where $c_i' \in \mathscr{F}(G)$, for $1 \leqslant i \leqslant n$.

Proof. As $c_{F_1} \prec_b^G c_{F_2}$, there exists a chain

$$c_{F_1} = c_1 \sqsubset_b^G c_2 \sqsubset_b^G \cdots \sqsubset_b^G c_{n-1} \sqsubset_b^G c_n = c_{F_2}$$

for some $n > 1$, such that $c_i \in \mathscr{U}(G)$ for $1 \leqslant i \leqslant n$. For $2 \leqslant i \leqslant n-1$, define

$$V_i = \{v \in V\colon v \text{ has the same colour in } c_i \text{ as in } c_{F_2}\},$$

and for $v \in V$, define c_i' by

$$v \in V_i \Rightarrow v \quad \text{has the same colour in } c_i' \text{ as in } c_{F_2},$$

$$v \notin V_i \Rightarrow v \quad \text{has the same colour in } c_i' \text{ as in } c_{F_1}.$$

Letting $c_1' = c_{F_1}$ and $c_n' = c_{F_2}$, it is straightforward to verify that $c_i' \in \mathscr{F}(G)$ for $1 \leqslant i \leqslant n$, and

$$c_{F_1} = c_1' \sqsubset_b^G c_2' \sqsubset_b^G \cdots \sqsubset_b^G c_{n-1}' \sqsubset_b^G c_n' = c_{F_2}$$

which establishes the theorem. $\square$

Corollary 3. *For a graph G and a proper colouring $c \in \mathscr{F}(G)$, c is $\prec_b^G$-minimal if and only if there does not exist $c' \in \mathscr{F}(G)$ such that $c' \sqsubset_b^G c$.*

Corollary 3 implies that a proper colouring $\{V_1, V_2, \ldots, V_k\}$ is $\prec_b^G$-minimal if and only if it is not possible to redistribute the vertices of a colour i amongst the other colours $1, 2, \ldots, i-1, i+1, \ldots, k$, in order to obtain a proper colouring. This may easily be restated in the following form:

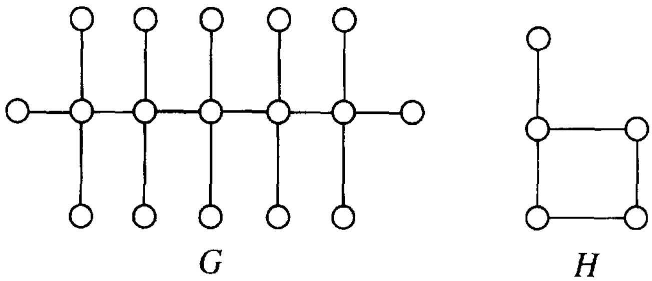

Fig. 1. Examples to show that $\Gamma(G)$ need not be an upper bound for $\varphi(G)$, or vice versa.

Corollary 4. *A proper colouring* $\{V_1, V_2, \ldots, V_k\}$ *of a graph* $G=(V,E)$ *is* $\prec_b^G$*-minimal if and only if*

$$\forall 1 \leqslant i \leqslant k \bullet \exists v_i \in V_i \bullet \forall 1 \leqslant j \neq i \leqslant k \bullet \exists w_j \in V_j \bullet \{v_i, w_j\} \in E. \tag{3}$$

Intuitively, a proper k-colouring is $\prec_b^G$-minimal if and only if each colour i contains at least one vertex v_i that is adjacent to a vertex of every colour j $(1 \leqslant j \neq i \leqslant k)$. We call such a vertex v_i a *b-chromatic vertex* for colour i. We call a proper colouring that satisfies Property 3 a *b-chromatic colouring*. We can now make the following definition:

Definition 5. The *b-chromatic number*, $\varphi(G)$, of a graph $G=(V,E)$ is defined by

$$\varphi(G) = \max\{|c| : c \in \mathscr{F}(G) \text{ is } \prec_b^G\text{-minimal}\}.$$

The B-CHROMATIC NUMBER problem is to determine whether $\varphi(G) \geqslant K$, for a given integer K and graph G.

Therefore, the b-chromatic number parameter of a graph G is the maximum number of colours for which G has a proper colouring such that every colour contains a vertex adjacent to a vertex of every other colour.

Hughes and MacGillivray [13] give an interpretation of $\psi(G)$ as being the largest number of colours in a proper colouring of G, 'which does not obviously use unnecessary colours'. The definition of the b-chromatic number therefore incorporates a partial order, $\prec_b^G$, that substantially strengthens this notion of not 'wasting' colours.

The parameter $\varphi(G)$ superficially resembles the *Grundy number*, $\Gamma(G)$, of G. The Grundy number (first named and studied by Christen and Selkow [5]) is the maximum number of colours k for which G has a *Grundy k-colouring*. A Grundy-k-colouring of G is a proper colouring of G using colours $0, 1, \ldots, k-1$ such that every vertex coloured i, for each $0 \leqslant i < k$, is adjacent to at least one vertex coloured j, for each $0 \leqslant j < i$. In general it is not the case that the Grundy number is an upper bound for the b-chromatic number, or vice versa, as is demonstrated in Fig. 1: here $\Gamma(G)=4$ while $\varphi(G)=5$, and $\Gamma(H)=3$ while $\varphi(H)=2$. Thus in general $\Gamma(G)$ and $\varphi(G)$ are distinct parameters, for a given graph G.

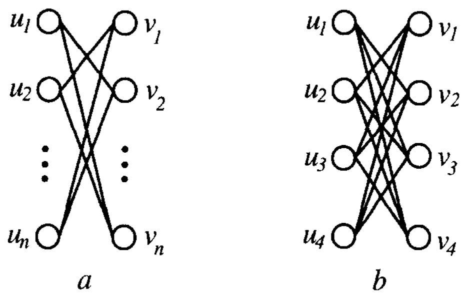

Fig. 2. Example to show that $\varphi(G)$ can be arbitrarily far from $\chi(G)$.

For a graph G, the partial order $\prec_b^G$ is a refinement of $\prec_a^G$, in that a proper colouring that is $\prec_b^G$-minimal is also $\prec_a^G$-minimal. Thus $\varphi(G) \leqslant \psi(G)$. An immediate lower bound for $\varphi(G)$ is $\chi(G)$, since any proper colouring of G using $\chi(G)$ colours must be b-chromatic. However, $\varphi(G)$ may be arbitrarily far away from $\chi(G)$: consider the graph G shown in Fig. 2(a), that is the complete bipartite graph $K_{n,n}$ minus a perfect matching. Letting $c(u_i)=c(v_i)=i$ for $1 \leqslant i \leqslant n$ gives a b-chromatic n-colouring. As each vertex has degree $n-1$, $\varphi(G)=n$, whereas $\chi(G)=2$.

Harary et al. [11] show that an arbitrary graph $G=(V,E)$ has achromatic colourings of any size between $\chi(G)$ and $\psi(G)$. Thus, in the terminology of Harary [9], ψ is an *interpolating invariant*, i.e. for any graph G, the set

$$S(G)=\{k \in \mathbb{Z}^+ \colon G \text{ has an achromatic colouring of size } k\}$$

is *convex*, that is, every n between $\min(S(G))$ and $\max(S(G))$ belongs to $S(G)$. It turns out that φ is not an interpolating invariant. This may be seen by considering the graph G of Fig. 2(a) with $n=4$, illustrated in Fig. 2(b). We saw previously that G has b-chromatic 2 and 4-colourings, but there is no 3-colouring of G that is b-chromatic, which may be seen as follows. Suppose that G *does* have a b-chromatic 3-colouring, and without loss of generality, suppose that $c(u_1)=1$ and $c(v_4)=2$. Suppose, again without loss of generality, that u_2 is a b-chromatic vertex for colour 3. Then $c(v_1)=1$ which in turn forces $c(u_3)=c(u_4)=3$ and $c(v_2)=c(v_3)=2$. Neither u_1 nor v_1 is b-chromatic, so we have a contradiction.

3. Complexity of B-CHROMATIC NUMBER

In this section we prove that determining $\varphi(G)$ for an arbitrary graph G is hard. It is clear that, for a graph G to have a b-chromatic colouring of k colours, G must contain at least k vertices, each of degree at least $k-1$. The following definition leads to a closely related, but stronger observation.

Definition 6. For a graph $G=(V,E)$, suppose that the vertices of G are ordered $v_1, v_2, \ldots, v_n$ so that $d(v_1) \geqslant d(v_2) \geqslant \cdots \geqslant d(v_n)$. Then the *m-degree*, $m(G)$, of G is defined by

$$m(G) = \max\{1 \leqslant i \leqslant n: d(v_i) \geqslant i - 1\}.$$

It turns out that $m(G)$ is an upper bound for $\varphi(G)$.

Lemma 7. *For any graph G, $\varphi(G) \leqslant m(G)$.*

Proof. The definition of the m-degree implies that there is some set of $m(G)$ vertices of G, each with degree $\geqslant m(G) - 1$, while the other $|V| - m(G)$ vertices of G each have degree $\leqslant m(G) - 1$. If $\varphi(G) > m(G)$ then in any b-chromatic colouring of size $\varphi(G)$, there is at least one colour c whose vertices all have degree $\leqslant m(G) - 1$. For, if not, then there are at least $\varphi(G) > m(G)$ vertices of degree $> m(G) - 1$, a contradiction. Hence all vertices of colour c have degree $< \varphi(G) - 1$, and none of these can be b-chromatic, a contradiction. □

This upper bound is tight: the graph of Fig. 2(b) satisfies $m(G) = 4$, and we have already seen that G has a b-chromatic 4-colouring. On the other hand, $\varphi(G)$ may be arbitrarily far from $m(G)$, as the example provided by the complete bipartite graph $K_{n,n}$ shows. Let $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$ be the vertices in the bipartition of the complete bipartite graph $K_{n,n}$. Then $m(G) = n + 1$, whereas $\varphi(G) = 2$, which may be seen as follows. We suppose that G has a b-chromatic colouring of size $\geqslant 3$ and without loss of generality, suppose that $c(u_1) = 1$, $c(w_1) = 2$ and w_2 is a b-chromatic vertex for colour 3. Then we have a contradiction, for w_2 cannot be adjacent to a vertex of colour 2.

We now prove that B-CHROMATIC NUMBER is NP-complete. The proof involves a transformation from the NP-complete problem EXACT COVER BY 3-SETS [7, p. 221], which may be defined as follows:

Name: EXACT COVER BY 3-SETS (X3C)
Instance: Set $S = \{s_1, s_2, \ldots, s_n\}$, where $n = 3k$ for some k, and a collection $T = \{T_1, T_2, \ldots, T_m\}$ of subsets of S, where $|T_i| = 3$ for each i.
Question: Does T contain an exact cover for S, i.e. is there a set T' ($T' \subseteq T$) of pairwise disjoint sets whose union is S?

Theorem 8. B-CHROMATIC NUMBER *is* NP-*complete.*

Proof. B-CHROMATIC NUMBER is certainly in NP, for, given a colouring of the vertices we can use the criterion of Corollary 4 to verify that the colouring is b-chromatic, in polynomial time. To prove NP-hardness, we provide a transformation from the X3C problem, as defined above. We suppose that $S = \{s_1, s_2, \ldots, s_n\}$ (where $n = 3k$ for some k), and $T = \{T_1, T_2, \ldots, T_m\}$ (where $T_i \subseteq S$ and $|T_i| = 3$, for each i) is some arbitrary instance of this problem. The X3C problem can easily be transformed to a restricted version of the problem, in which the instance satisfies the following two properties:

(1) $\bigcup_{1 \leqslant i \leqslant m} T_i = S$,
(2) $S \neq \emptyset$.

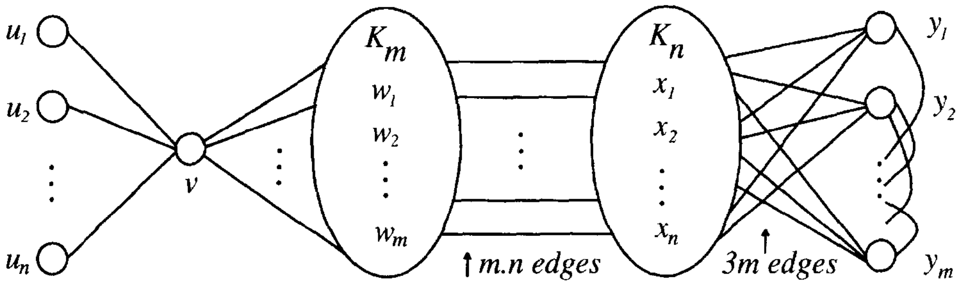

Fig. 3. Graph G derived from an instance of X3C.

We construct an instance of B-CHROMATIC NUMBER as follows. Let

$$V = \{u_1, \dots, u_n, v, w_1, \dots, w_m, x_1, \dots, x_n, y_1, \dots, y_m\},$$

and let E contain the elements

$\{u_i, v\}$ for $1 \leqslant i \leqslant n$,
$\{v, w_i\}$ for $1 \leqslant i \leqslant m$,
$\{w_i, w_j\}$ for $1 \leqslant i < j \leqslant m$,
$\{w_i, x_j\}$ for $1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n$,
$\{x_i, x_j\}$ for $1 \leqslant i < j \leqslant n$,
$\{x_i, y_j\}$ for $1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m \Leftrightarrow s_i \in T_j$,
$\{y_i, y_j\}$ for $1 \leqslant i < j \leqslant m \Leftrightarrow T_i \cap T_j \neq \emptyset$.

The resulting graph $G = (V, E)$ is shown in Fig. 3. We now find $m(G)$ in order to obtain an upper bound for $\varphi(G)$. It may be easily verified that:

- $d(u_i) = 1$ for $1 \leqslant i \leqslant n$.
- $d(v) = m + n$.
- $d(w_i) = m + n$ for $1 \leqslant i \leqslant m$.
- $d(x_i) \geqslant m + n$ for $1 \leqslant i \leqslant n$ (by Assumption 1 above).
- $d(y_i) \leqslant 3 + (m - 1) < m + n$ (since $n \geqslant 3$ by Assumption 2 above).

Therefore, $m + n + 1$ vertices of G have degree at least $m + n$ and all other vertices of G have degree less than $m + n$. Hence $m(G) = m + n + 1$ and by Lemma 7 we have that $\varphi(G) \leqslant m + n + 1$. The claim is that G has a b-chromatic colouring of size $m + n + 1$ if and only if T has an exact cover for S.

For, suppose that T has an exact cover $T_{i_1}, T_{i_2}, \dots, T_{i_r}$ for S, where $r \leqslant m$. We assign $m + n + 1$ colours to the vertices of G as follows:

- $c(u_i) = c(x_i) = i$ for $1 \leqslant i \leqslant n$,
- $c(v) = m + n + 1$,
- $c(w_i) = n + i$ for $1 \leqslant i \leqslant m$,
- $c(y_{i_j}) = m + n + 1$ for $1 \leqslant j \leqslant r$ and
- colour the remaining y_i (i.e. vertices $\{y_1, y_2, \dots, y_m\} \backslash \{y_{i_1}, y_{i_2}, \dots, y_{i_r}\}$) by colours $n + 1, n + 2, \dots, n + m - r$, respectively.

It remains to show that this colouring is b-chromatic. Certainly the colouring is proper, for the exact cover property gives us that $T_{i_j} \cap T_{i_k} = \emptyset$ for $1 \leqslant j < k \leqslant r$, so that $\{y_{i_j}, y_{i_k}\}$

$\notin E$. Also, $m-r<m+1$, so that no y_i such that T_i is not in the exact cover has colour $m+n+1$. We now check that Property 3 holds. Take each colour j in turn:

- If $j=m+n+1$ then v is a b-chromatic vertex for colour j.
- If $n+1 \leqslant j \leqslant n+m$ then w_{j-n} is a b-chromatic vertex for colour j.
- If $1 \leqslant j \leqslant n$ then x_j is adjacent to colours $1,\ldots,j-1,j+1,\ldots,n+m$, plus colour $m+n+1$ by the exact cover property of $T_{i_1},T_{i_2},\ldots,T_{i_r}$, so is a b-chromatic vertex for colour j.

Therefore this colouring is b-chromatic and has size $m+n+1$.

Conversely, suppose that G has a b-chromatic colouring of size $m+n+1$. Without loss of generality, we may assume that $c(x_i)=i$ for $1 \leqslant i \leqslant n$ and $c(w_i)=n+i$ for $1 \leqslant i \leqslant m$. There is only one remaining vertex of degree at least $m+n$, namely v, so the b-chromatic property forces $c(v)=m+n+1$, and also $u_1,u_2,\ldots,u_n$ must be coloured by some permutation of the colours $\{1,2,\ldots,n\}$. For each i $(1 \leqslant i \leqslant n)$, x_i is the b-chromatic vertex for colour i and hence is adjacent to some y_j such that $c(y_j)=m+n+1$. Thus there is a subcollection $T_{i_1},T_{i_2},\ldots,T_{i_r}$ for some r $(r \leqslant m)$ such that, for each j $(1 \leqslant j \leqslant n)$, $s_j \in T_{i_k}$ for some k $(1 \leqslant k \leqslant r)$. Moreover, the $y_{i_1},y_{i_2},\ldots,y_{i_r}$ are all coloured $m+n+1$ so that $\{y_{i_j},y_{i_k}\} \notin E$ for $1 \leqslant j<k \leqslant r$. Hence $T_{i_j} \cap T_{i_k}=\emptyset$ so that $T_{i_1},T_{i_2},\ldots,T_{i_r}$ forms an exact cover for S. □

The NP-completeness of B-CHROMATIC NUMBER for arbitrary graphs implies that approximation algorithms for the problem are of interest. The example of the complete bipartite graph $K_{n,n}$ (discussed earlier in this section) rules out the possibility of a good approximation algorithm for $\varphi(G)$ based on the value of $m(G)$. However, the existence of a polynomial-time approximation algorithm that always gives a b-chromatic colouring of G with at least $\varepsilon\varphi(G)$ colours (where $0<\varepsilon \leqslant 1$ is a constant) remains open.

4. Polynomial-time algorithm for trees

In contrast with the NP-completeness of ACHROMATIC NUMBER for trees [2], we show in this section that the b-chromatic number is polynomial-time computable for trees. In fact, apart from a very special class of exceptions, recognisable in polynomial time, the b-chromatic number of a tree T is equal to the upper bound $m=m(T)$.

Let us call a vertex v of T such that $d(v) \geqslant m-1$ a *dense* vertex of T. Our methods of finding b-chromatic colourings for trees hinge on colouring firstly vertices adjacent to those in a set $V'=\{v_1,v_2,\ldots,v_m\}$ of dense vertices of T. (For trees with more than m dense vertices, we shall demonstrate how V' is to be chosen.) We aim to establish a *partial b-chromatic m-colouring* of T, i.e., a partial proper colouring of T using m colours such that each v_i $(1 \leqslant i \leqslant m)$ has colour i and is adjacent to vertices of $m-1$ distinct colours. This approach is applicable for all trees except those satisfying the following criteria.

Definition 9. A tree $T=(V,E)$ is *pivoted* if T has exactly m dense vertices, and T contains a distinguished vertex v such that:
(1) v is not dense.
(2) Each dense vertex is adjacent either to v or to a dense vertex adjacent to v.
(3) Any dense vertex adjacent to v and to another dense vertex has degree $m-1$.
We call such a vertex v the *pivot* of T (clearly, a pivot is unique if it exists).

It is evident that we may test for a tree being pivoted in polynomial time. (In fact, such a test may be accomplished in linear time – see [15] for further details.) We now establish the b-chromatic number of pivoted trees.

Theorem 10. *If $T=(V,E)$ is a tree that is pivoted then $\varphi(T)=m(T)-1$.*

Proof. Denote by v the pivot of T. Let $V=\{v_1,v_2,\ldots,v_n\}$ be ordered so that $V'=\{v_1,v_2,\ldots,v_m\}$ is the set of dense vertices, $v_1,v_2,\ldots,v_p$ (for some $p\leqslant m$) are the dense vertices adjacent to v, and $v_1,v_2,\ldots,v_q$ (for some $q\leqslant p$) are the dense vertices adjacent to v each having at least one dense vertex as a neighbour. Since T has exactly m dense vertices, Properties 2 and 3 of Definition 9 give $p\geqslant 2$. Also $q\geqslant 1$, or else $p=m$ by Property 2 of Definition 9, so that v is itself a dense vertex, which is a contradiction.

Firstly, we show that $\varphi(T)<m(T)$. For, suppose that there is a b-chromatic colouring c of T, using m colours, where the dense vertices are coloured such that, without loss of generality, $c(v_i)=i$ $(1\leqslant i\leqslant m)$. As $d(v_j)=m-1$ for $1\leqslant j\leqslant q$, none of $v_1,v_2,\ldots,v_q$ can be adjacent to more than one vertex of any one colour. Between them, $v_1,v_2,\ldots,v_q$ are adjacent to dense vertices $v_{p+1},v_{p+2},\ldots,v_m$. Now v cannot have colour j for $1\leqslant j\leqslant p$, nor colour j for $p+1\leqslant j\leqslant m$, or else some v_k $(1\leqslant k\leqslant q)$ is adjacent to two vertices of that colour. Hence there is no available colour for v, a contradiction.

To establish equality, we construct a b-chromatic colouring c of T using $m-1$ colours. As $p\geqslant 2$ and $q\geqslant 1$, the dense vertices v_1,v_2 are adjacent to v, and for some r $(p+1\leqslant r\leqslant m)$, there is a dense vertex v_r adjacent to v_1. Set $c(v_i)=i$ for $2\leqslant i\leqslant m$, let $c(v)=r$ and assign $c(v_1)=2$. All other vertices of V are as yet uncoloured. We show how to extend this partial colouring into a b-chromatic $(m-1)$-colouring of T, namely a proper $(m-1)$-colouring of V, using colours $2,3,\ldots,m$, such that every vertex in $V'\backslash\{v_1\}$ is adjacent to vertices of $m-2$ distinct colours, as follows. For $2\leqslant i\leqslant m$, let $R_i=\{2,3,\ldots,m\}\backslash\{i\}$ (the *required colours* for surrounding v_i), let

$$C_i=\{c(v_j)\colon 1\leqslant j\leqslant n\wedge v_j\in N(v_i)\wedge v_j \text{ is coloured}\}$$

(the *existing colours* around v_i) and define

$$U_i=\{v_j\colon m+1\leqslant j\leqslant n\wedge v_j\in N(v_i)\wedge v_j \text{ is uncoloured}\}$$

(the *uncoloured vertices* adjacent to v_i). By construction, v_i is not adjacent to two vertices of the same colour. Hence

$$|C_i|+|U_i|=d(v_i)\geqslant m-1>m-2=|R_i|.$$

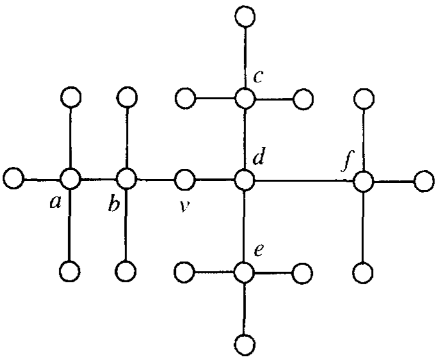

Fig. 4. Example tree T with $m(T) = \varphi(T) = 5$.

Hence, as $C_i \subseteq R_i$, it follows that $|U_i| \geqslant |R_i \backslash C_i|$. Thus if $R_i \backslash C_i = \{r_1^i, \ldots, r_{n_i'}\}$ (for some $n_i \geqslant 0$) then we may pick some $\{u_1^i, \ldots, u_{n_i'}\} \subseteq U_i$ and set $c(u_j^i) = r_j^i$ for $1 \leqslant j \leqslant n_i$. This process does not assign the same colour to any two adjacent vertices, since no two adjacent non-dense vertices are both adjacent to dense vertices. Nor does it assign more than one colour to any one vertex, since no two dense vertices have a common non-dense neighbour (except for v, which is already coloured).

For $m+1 \leqslant i \leqslant n$, suppose that v_i is uncoloured. As $d(v_i) < m-1$, not all of colours $2, 3, \ldots, m$ appear on neighbours of v_i. Hence there is some colour available for v_i. It follows that the constructed colouring is a b-chromatic $(m-1)$-colouring of T. □

We now show how to construct a b-chromatic colouring of an arbitrary non-pivoted tree, using $m = m(T)$ colours. For trees with more than m dense vertices, we need to be aware of a possible complication. For example, consider the tree T in Fig. 4: T satisfies $m(T) = 5$, but there are six dense vertices, namely a, b, c, d, e, f. It may be verified that no b-chromatic 5-colouring of T exists in which a, b, c, d, e are the b-chromatic vertices. However, a b-chromatic 5-colouring of T *does* exist, which may be achieved by taking either a, b, c, e, f or a, c, d, e, f to be the b-chromatic vertices.

Hence, as the example of Fig. 4 shows, a judicious choice of dense vertices may be required in order to achieve a partial b-chromatic m-colouring. In order to assist in making this selection, we formulate the following definition, which is closely related to the concept of a tree being pivoted.

Definition 11. Let $T = (V, E)$ be a tree, and let V' be the set of dense vertices of T. Suppose that V'' is a subset of V' of cardinality m. Then V'' *encircles* some vertex $v \in V \backslash V''$ if

(1) Each vertex in V'' is adjacent either to v or to some vertex in V'' adjacent to v.

(2) Any vertex in V'' adjacent to v and to another vertex in V'' has degree $m-1$.

We refer to v as an *encircled vertex* with respect to V''.

We now give an additional definition that incorporates this concept of encirclement.

Definition 12. Let $T=(V,E)$ be a tree, and let V' be the set of dense vertices of T. Suppose that V'' is a subset of V' of cardinality m. Then V'' is a *good set* with respect to T if

(a) V'' does not encircle any vertex in $V \backslash V''$.

(b) Any vertex $u \in V \backslash V''$ such that $d(u) \geqslant m$ is adjacent to some $v \in V''$ with $d(v)=m-1$.

In the example of Fig. 4, the set $\{a,b,c,d,e\}$ encircles vertex v. However, either of $\{a,b,c,e,f\}$ or $\{a,c,d,e,f\}$ is a good set with respect to T. In general, our aim is to build up a b-chromatic m-colouring by choosing a good set with respect to the given tree T. The following lemma describes how we make this choice, and also shows that such a choice is always possible in non-pivoted trees.

Lemma 13. *Let $T=(V,E)$ be a tree that is not pivoted. Then we may construct a good set for T.*

Proof. Let V' be the set of dense vertices of T. By the definition of $m(T)$, we may choose a subset V'' of V', with $|V''|=m$, so that every vertex in $V \backslash V''$ has degree less than m. Let $V=\{v_1,v_2,\ldots,v_n\}$ be ordered so that $V''=\{v_1,v_2,\ldots,v_m\}$. Suppose that V'' encircles some vertex $v \in V \backslash V''$ (for if not, we set $W=V''$ and we are done, since W satisfies Properties (a) and (b) of Definition 12). Without loss of generality, suppose that $v_1,v_2,\ldots,v_p$ (for some $p \leqslant m$) are the members of V'' adjacent to v, and $v_1,v_2,\ldots,v_q$ (for some $q \leqslant p$) are the members of V'' adjacent to v, each having at least one other member of V'' as a neighbour. Now $p \geqslant 2$, for otherwise $d(v_1) \geqslant m$ as each of $v_2,\ldots,v_m$ is adjacent to v_1 by Property 1 of Definition 11, contradicting Property 2 of Definition 11. Also $q \geqslant 1$, for otherwise $p=m$ by Property 1 of Definition 11, so $d(v) \geqslant m$, a contradiction to the choice of V''. Thus there is a vertex $v_r \in V''$, for some r $(p+1 \leqslant r \leqslant m)$, adjacent to v_1. We consider two cases.

Case (i): *v is dense*. Then $d(v)=m-1$ by the choice of V''. Let $W=(V'' \backslash \{v_2\}) \cup \{v\}$. Also by the choice of V'', the only vertex not in W that can have degree at least m is v_2. But v_2 is adjacent to $v \in W$, and $d(v)=m-1$, so that W satisfies Property (b) of Definition 12. Also, W satisfies Property (a) of Definition 12. For no vertex $w \in V \backslash (W \cup \{v_2\})$, adjacent to v_j, for some j $(p+1 \leqslant j \leqslant m)$, may be encircled by W, since v is at distance 3 from w. Also, no vertex $w \in V \backslash W$, adjacent to v_j, for some j $(1 \leqslant j \leqslant p)$, may be encircled by W, since there is some $v_k \in V''$ $(1 \leqslant k \leqslant p)$, adjacent to v, at distance 3 from w (as $p \geqslant 2$). Finally, no vertex w of $V \backslash W$, adjacent to v, may be encircled by W, since v_r is at distance 3 from w.

Case (ii): *v is not dense*. If $|V'|=m$ then T is pivoted at vertex v, a contradiction. Hence $|V'|>m$, so there is some $u \in V' \backslash V''$. Let $W=(V'' \backslash \{v_1\}) \cup \{u\}$. Now suppose that W encircles some vertex x. At most one vertex not in W lies on the path between any pair of non-adjacent vertices in W, namely x. But $v_1 \notin W$ and $v \notin W$ lie on the path between $v_2 \in W$ and $v_r \in W$. This contradiction implies that W satisfies Property (a) of Definition 12. Also, W satisfies Property (b) of Definition 12, since $d(v)<m-1$

and $d(v_1)=m-1$, and therefore every vertex outside W has degree less than m. □

We are now in a position to establish the b-chromatic number of trees that are not pivoted.

Theorem 14. *If $T=(V,E)$ is a tree that is not pivoted, then $\varphi(T)=m(T)$.*

Proof. Let $W=\{v_1,v_2,\ldots,v_m\}$ be a good set of m dense vertices of T. Such a choice is possible, by Lemma 13, since T is not pivoted. Attach colour i to vertex v_i $(1\leqslant i\leqslant m)$. We will show that this partial colouring can be extended to a partial b-chromatic m-colouring of T, in such a way that each v_i is a b-chromatic vertex, and then to a b-chromatic colouring with m colours.

Let $U=N(W)\backslash W$. We partition the set U into two subsets as follows: a vertex $u\in U$ is called *inner* if there are two vertices v_i,v_j of W, at distance at most 3 from each other, with u on the path between them; $u\in U$ is called *outer* otherwise. We first extend the colouring to the inner vertices, and then to the outer vertices and the vertices of $V\backslash U$.

Suppose, without loss of generality, that $v_1,v_2,\ldots,v_m$ are numbered so that, if $i<j$, then v_i has at least as many inner neighbours as v_j. Suppose that $v_1,\ldots,v_p$ have at least two inner neighbours, $v_{p+1},\ldots,v_q$ have one inner neighbour, and $v_{q+1},\ldots,v_m$ have no inner neighbours. Note that either or both of the sets $P=\{v_1,\ldots,v_p\}$ and $Q=\{v_{p+1},\ldots,v_q\}$ may be empty.

We begin by colouring the uncoloured inner neighbours of v_i, for each i in turn from 1 to q. Firstly, we deal with the inner neighbours of $v_1,\ldots,v_p$ (assuming that $P\neq\emptyset$). For an induction step, suppose that $i\leqslant p$, and that the inner neighbours of $v_1,\ldots,v_{i-1}$ have been coloured so that

(a) if an inner vertex u was assigned its current colour, say colour k, during the colouring of the inner neighbours of v_j, then the path from u to v_k passes through v_j;
(b) no two neighbours of any v_j have the same colour; and
(c) no two adjacent vertices have the same colour.

We show how to colour the uncoloured inner neighbours of v_i so that these three properties continue to hold.

Let the vertices in question be $x_1,\ldots,x_s$. For each j $(1\leqslant j\leqslant s)$, because x_j is inner, there is a vertex $v_{c_j}\in W$ with $c_j\neq i$, v_{c_j} at distance at most 2 from x_j, and x_j on the path from v_i to v_{c_j}. Further, the vertices $v_{c_1},\ldots,v_{c_s}$ are all distinct.

Case 1: $s>1$. Let $d_1,\ldots,d_s$ be a derangement of the colours $c_1,\ldots,c_s$, and apply colour d_j to vertex x_j $(1\leqslant j\leqslant s)$. Then it is straightforward to verify that Properties (a)–(c) continue to hold.

Case 2: $s=1$. Because v_i has at least two inner neighbours, there is some inner neighbour, y, already coloured, say colour d. Attach colour d to x_1 and colour c_1 to y. Then again it is not hard to verify that Properties (a)–(c) continue to hold.

Now we deal with the uncoloured inner neighbours of $v_{p+1},\ldots,v_q$ (assuming that $Q\neq\emptyset$). Let $z_1,\ldots z_k$ be the vertices in this category. Then no v_i can have more than one neighbour among the z_j, and so in assigning colours to the z_j at most one neighbour of

each v_i is coloured. For our purposes, it therefore suffices to ensure that, in assigning a colour to each z_j,
(i) the (partial) colouring remains proper, and
(ii) no v_i of degree exactly $m-1$ has two neighbours of the same colour.
To choose a colour for z_j, let $\mathscr{C}$ be the set of colours defined by

$$c \in \mathscr{C} \Leftrightarrow (v_c \in N(z_j)) \vee (\exists d \bullet v_c \in N(v_d) \wedge v_d \in N(z_j) \wedge d(v_d) = m-1),$$

where $1 \leqslant c, d \leqslant m$. Provided we choose a colour $c \notin \mathscr{C}$, Properties (i) and (ii) will be satisfied. If $\mathscr{C} = \{1, 2, \ldots, m\}$ then z_j is encircled by W, and we find that our choice of W as a good set is contradicted. Hence there is always a choice of colour available for z_j. Note that each v_r $(p+1 \leqslant r \leqslant q)$ has only one inner neighbour coloured in this way, so that if $d(v_r) \geqslant m$, then at most two vertices adjacent to v_r have the same colour. Having completed this step, all inner vertices are now coloured.

Now we deal with the remaining vertices of V, starting with a significant subset of the outer vertices of U. The outer neighbours of each v_i $(1 \leqslant i \leqslant m)$ can be coloured independently, since none is adjacent to any of the others, nor to any coloured vertex apart from the relevant v_i. If $d(v_i) = m-1$, then the inner neighbours of v_i all have different colours, and if $d(v_i) \geqslant m$, then at most two of the inner neighbours of v_i have the same colour. Hence we may form sets $R_i = \{1, 2, \ldots, m\} \backslash \{i\}$, C_i and U_i, as in the proof of Theorem 10, in order to choose colours for outer neighbours of v_i, to ensure that v_i is b-chromatic. This argument applies to each v_i $(1 \leqslant i \leqslant m)$ in turn.

We may extend this partial b-chromatic m-colouring to a b-chromatic m-colouring of T as follows. Any remaining uncoloured vertex v must satisfy $d(v) \leqslant m-1$. For, if $d(v) \geqslant m$, then by Property (b) of Definition 12, v is adjacent to some $w \in W$, where $d(w) = m-1$, so that v must already have been given a colour. A vertex v of degree less than m cannot have neighbours with all of the colours $1, 2, \ldots, m$. Hence there is some colour available for v. This completes the construction of a b-chromatic colouring of T using m colours. □

Since testing whether a tree is pivoted may be carried out in linear time, it is clear from the statements of Theorems 10 and 14 that we may compute the b-chromatic number of a tree in linear time. In addition, the proofs of Theorems 10 and 14 also imply polynomial-time algorithms for constructing maximum b-chromatic colourings, for pivoted and non-pivoted trees, respectively.

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