The two-way rewriting in action: Removing the mystery of Euler–Glaisher’s map

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Abstract

Starting with Euler’s bijection between the partitions into odd parts and the partitions into distinct parts, one basic activity in combinatorics is to establish partition identities by so-called ‘bijective proofs,’ which amounts to constructing explicit bijections between two classes of the partitions under consideration.

The aim of this paper is to give a global view on the Glaisher-type bijections and related rewriting maps.

Glaisher’s map is a bijection between partitions with no part divisible by $m$ and partitions with no parts repeated $m$ or more times, that uses basic number theoretic techniques. O’Hara’s rewriting map is also a bijection between those two sets (the map consists of repeated replacing any $m$ occurrences of a part, say $z$, by the number $mz$). It is remarkable that both of these bijections are identical.

Moreover, the bijections produced for many partition identities by the refine machinery developed by, for example, Remmel, Gordon, O’Hara, and Sellers, Sills, and Mullen, turn out to be the same bijections as the ones found by Euler and generalized by Glaisher.

Here we give a quite paradoxical answer to the question of why Euler–Glaisher’s bijections arise so persistently from their applications, namely: Whatever Euler-like partition identities we take, one and the same Euler–Glaisher’s map will be suited for all of them.

We prove this by giving an alternate description of the bijections using two-way rewriting bijections between any two equinumerous partition ideals of order 1, provided, as a partial case, by a general criterion from Kanovich [Finding direct partition bijections by two-directional rewriting techniques, Discrete Math. 285(1–3) (2004) 151–166]. The tricky part of the proof is that, generally speaking, Euler–Glaisher’s mapping differs from the rewriting mapping derived, but both mappings are proved to behave identically on all partitions that might have been involved as elements of some equinumerous ‘Euler pairs’.

We generalize Glaisher’s mapping by simply substituting mixed radix expansions for the base $m$ expansions in Glaisher’s original construction. With this direct generalization we extend the Euler–Glaisher’s phenomenon to any two equinumerous partition ideals of order 1, whenever one of the ideals consists of partitions into parts from a set. As a useful part of the proof, we develop a natural generalization of Andrews–Subbarao’s criterion [G.E. Andrews, Two theorems of Euler and a general partition theorem, Proc. Amer. Math. Soc. 20(2) (1969) 499–502; M.V. Subbarao, Partition theorems for Euler pairs, Proc. Amer. Math. Soc. 28(2) (1971) 330–336].

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1. Motivating examples and summary

An integer partition of \( n = m_1 + m_2 + \cdots + m_k \) can be identified as a multiset \( M \) consisting of positive integers \( m_1, m_2, \ldots, m_k \). We will represent this \( M \) as \( M = \{m_1, m_2, \ldots, m_k\} \), where the number of copies of some \( m \) shows the multiplicity of the \( m \) within \( M \). Each \( m_i \) is called a part of the partition. This sum \( m_1 + m_2 + \cdots + m_k \) will be also denoted by \( \| M \| \).

**Definition 1.1.** Two classes of partitions \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are called equinumerous if \( p(\mathcal{C}_1, n) = p(\mathcal{C}_2, n) \), for all \( n \). Here \( p(\mathcal{C}, n) \) stands for the number of partitions of \( n \) that belong to a given class \( \mathcal{C} \).

Starting with Euler’s bijection between the partitions into odd parts and the partitions into distinct parts, one basic activity in combinatorics is to establish partition identities by so-called ‘bijective proofs,’ which amounts to constructing explicit bijections between two classes of the partitions under consideration. A unified method for dealing with a large class of integer partition identities has been developed by Andrews, Garsia and Milne, Remmel, Gordon, Wilf, O’Hara, and others (see [2,18,10]). It is remarkable that the bijections produced for many partition identities by their refine machinery are the same bijections as the ones found by Euler and generalized by Glaisher [6] in pure number theoretic terms. This paper is inspired by the recent results of Sellers, Sills, and Mullen on Glaisher-type bijections [15], and based on techniques developed in Kanovich [10].

The aim of the paper is to show that a novel two-directional rewriting technique removes the mystery of certain known results and can establish new results in the theory of integer partitions. In particular, we prove that the appearance of Euler–Glaisher’s mapping in many contexts is not accidental and that Euler–Glaisher’s mapping is good for all Euler-type identities. In particular (see Corollary 5.1):

The same Euler–Glaisher’s original map always provides a bijection \( h \) between equinumerous partition ideals \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), 1 whenever \( \mathcal{C}_1 \) consists of all partitions into parts taken from some set \( S_1 \), and \( \mathcal{C}_2 \) consists of all partitions into parts from some set \( S_2 \) in which each part may occur at most \( m - 1 \) times (\( m \) is fixed).

To illustrate the basic features of our approach, consider Euler’s partition theorem and its numerous ‘relatives’:

**Example 1.1.** For any positive integer \( n \),

(a) Euler: The number of partitions of \( n \) into odd parts equals the number of partitions of \( n \) into distinct parts.

(b) Glaisher [6]: The number of partitions of \( n \) with no part divisible by \( m \) equals the number of partitions of \( n \) with no part repeated \( m \) or more times.

(c) Guy [8]: The number of partitions of \( n \) into odd parts greater than 1 equals the number of partitions of \( n \) into distinct parts which are not powers of 2.

(d) Schur [14]: The number of partitions of \( n \) into parts congruent to \( \pm 1 \pmod{6} \) equals the number of partitions of \( n \) into distinct parts congruent to \( \pm 1 \pmod{3} \).

(e) “1–2”: The number of partitions of \( n \) into ones equals the number of partitions of \( n \) into distinct powers of 2 (both numbers are equal to 1).

(f) ‘2-Euler’: The number of partitions of \( n \) into parts congruent to 2 mod 4 equals the number of partitions of \( n \) into distinct even parts.

It should be noted that Example 1.1 is a simple corollary of Andrews’ criterion for partition ideals of order 1 [2, Theorem 8.4], which was proved via generating functions.

However, to reveal the essence and get a broader understanding of the partition identities, bijective proofs are preferable. As for item (a), Euler himself established an explicit bijection between the partitions into odd parts and the partitions into distinct parts in pure number theoretic terms (see Comment 1.2): the basic ingredient of Euler’s map—that every number has a unique binary representation, is presented in item (e). Euler’s map was generalized by Glaisher [6] to cover item (b) in Example 1.1. Recently Sellers, Sills, and Mullen have shown Glaisher-type bijections for item (c) and its generalizations [15].

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1 We say “provides a bijection \( h \) between two classes of integer partitions \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \)” meaning that, for any \( n \), function \( h \) is a bijection between the partitions of \( n \) that belong to \( \mathcal{C}_1 \) and the partitions of \( n \) that belong to \( \mathcal{C}_2 \).
Our goal here is to explain, within one and the same transparent paradigm, why Euler–Glaisher’s bijections arose in all these particular cases, and, moreover, why one and the same Euler’s map can be served as a bijection for each of the above identities.

Example 1.2. For any positive integer \( n \),

(a) “4–2”: The number of partitions of \( n \) where each part is a power of 4 and may appear at most thrice equals the number of partitions of \( n \) into distinct powers of 2 (both numbers are equal to 1).

(b) “3–2”: The number of partitions of \( n \) where each part is a power of 3 and may appear at most twice equals the number of partitions of \( n \) into distinct powers of 2 (both numbers are equal to 1).

This simple Example 1.2 is an object of our interest because of the following likes and dislikes for Euler–Glaisher’s maps:

(a) To convert the quaternary representation of \( n \) into its binary equivalent, it suffices to substitute the corresponding binary digits:

\[
0_4 \rightarrow 00_2, \quad 1_4 \rightarrow 01_2, \quad 2_4 \rightarrow 10_2, \quad 3_4 \rightarrow 11_2.
\]

In terms of integer partitions, we apply here the multiset rewriting rules

\[
\{z, z\} \rightarrow \{2z\}
\]

with \( z \) being of the form \( 4^k \). As a result, the above bijection is directly computed by Euler’s original map (cf. Table 1).

(b) On the contrary, no Euler–Glaisher-type mapping is capable of direct converting from base 3 into base 2.

The reason is that the bijection \( h \) converting ternary numerals into their binary equivalents does not have the strong sub-partition property (cf. Comments 1.3 and 1.4).

E.g., \( h([3]) = \{2, 1\} \), and \( h([1, 1]) = \{2\} \).

but \( h([3, 1, 1]) = \{4, 1\} \neq h([3]) \cup h([1, 1]) \).

Table 1

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1.1. Glaisher’s map $G_m$

**Definition 1.2.** Given an integer $m \geq 2$, Glaisher’s map $G_m$ [6] from partitions $M$ into partitions $M'$ is defined as follows.

Suppose a partition $M$ of some $n$:

$$M = \{s_1, \ldots, s_1, \ldots, s_i, \ldots, s_i, \ldots, s_k, \ldots, s_k\},$$

where $s_i$ are distinct parts with multiplicity $b_i$, so that

$$n = b_1s_1 + b_2s_2 + \cdots + b_ks_k.$$ 

Write each $b_i$ in base $m$ notation:

$$b_i = \sum_{j} a_{ij}m^j,$$

with $0 \leq a_{ij} \leq m - 1$, for each $i$ and $j$.

Then

$$n = \sum_{i,j} a_{ij}m^j s_i = \sum_{i,j} a_{ij}(m^j s_i).$$

We now see the partition of $n$, say $M'$, into parts $m^j s_i$ repeated $a_{ij}$ times,

$$M' = \{m^0s_1, \ldots, m^0s_1, \ldots, m^1s_1, \ldots, m^1s_1, \ldots, m^2s_1, \ldots, m^2s_1, \ldots, m^js_1, \ldots, m^js_1, \ldots\},$$

and we set

$$G_m(M) = M'.$$

**Comment 1.1.** Glaisher’s map $G_m$ associates with every partition $M$ of $n$ with no part divisible by $m$ a unique partition $M'$ of $n$ with no part repeated $m$ or more times [6].

**Comment 1.2.** Taking $G_2$, we get exactly Euler’s original bijection between the partitions $M$ into odd parts and the partitions $M'$ into distinct parts.

**Comment 1.3.** Because of its construction, Glaisher’s map $G_m$ has the following **strong sub-partition property**:

For any disjoint partitions $M_1$ and $M_2$:

$$G_m(M_1 \uplus M_2) = G_m(M_1) \uplus G_m(M_2).$$

**Definition 1.3.** As for the ‘inverse map’, say $G_m^{-1}$, which provides a bijection from the partitions $M'$ with no part repeated $m$ or more times onto the partitions $M$ with no part divisible by $m$, it is explicitly defined as follows (see, for instance, [18]):

Given an $M'$ consisting of parts $d_1, d_2, \ldots, d_k$,

$$n = d_1 + d_2 + \cdots + d_k,$$

each integer $d_\ell$ is uniquely expressed as a power of $m$ times an integer not divisible by $m$: $d_\ell = m^{a_\ell}O_\ell$. 

---

2 By definition, $\{m_1, m_2, \ldots, m_k\} \uplus \{l_1, l_2, \ldots, l_s\} = \{m_1, m_2, \ldots, m_k, l_1, l_2, \ldots, l_s\}$. 

Thus

\[ n = m^{a_1}O_1 + m^{a_2}O_2 + m^{a_3}O_3 + \cdots + m^{a_k}O_k, \]

where each \( O_\ell \) is not divisible by \( m \).

If we now group together the identical \( O_\ell \)'s, we get an expression like

\[
\begin{align*}
\frac{n}{m} &= (m^{a_1} + m^{a_2} + \cdots) \cdot s_1 + (m^{b_1} + m^{b_2} + \cdots) \cdot s_2 + (m^{c_1} + m^{c_2} + \cdots) \cdot s_3 + \cdots \\
&= \mu_1 \cdot s_1 + \mu_2 \cdot s_2 + \mu_3 \cdot s_3 + \cdots,
\end{align*}
\]

where \( s_1, s_2, s_3, \ldots \) are distinct integers not divisible by \( m \).

We now can see the desired \( G_{m}^{-1}(M') \): it contains \( \mu_1 \) copies of \( s_1 \), \( \mu_2 \) copies of \( s_2 \), \( \mu_3 \) copies of \( s_3 \), etc.

Comment 1.4. The inverse Glaisher’s map \( G_{m}^{-1} \) has the strong sub-partition property (see Comment 1.3) as well.

1.2. The ‘rewriting’ map \( \mathcal{R}_m \) (see O’Hara [9], Wilf [18])

In this section we introduce another sort of mappings based on multiset rewriting rules.

Definition 1.4. Given an integer \( m \geq 2 \), let \( \Gamma_m \) consist of the following multiset rewriting rules:

\[
\gamma_z : \{z, z, \ldots, z\} \mapsto \{mz\}
\]

with \( z \) ranging over all positive integers.

For any integer partition \( M \), we set

\[
\mathcal{R}_m(M) = \tilde{M},
\]

where \( M \) is reducible to \( \tilde{M} \) by means of the above rules \( \gamma_z \), with \( \tilde{M} \) being \( \Gamma_m \)-irreducible.

E.g., for \( m = 2 \),

\[
\{1, 1, 1, 1, 1, 1\} \xrightarrow{\gamma_1} \{2, 1, 1, 1, 1\} \xrightarrow{\gamma_1} \{2, 2, 1, 1, 1\} \xrightarrow{\gamma_2} \{4, 1, 1\} \xrightarrow{\gamma_1} \{4, 2\},
\]

and, hence,

\[
\mathcal{R}_2([1, 1, 1, 1, 1]) = \{4, 2\}.
\]

Example 1.3 (continuing Examples 1.1 and 1.2). We illustrate peculiarities and subtleties of \( \mathcal{G}_2 \) and \( \mathcal{R}_2 \) by looking at \( n = 4, 6 \) in Table 1.

Notice the following special features of the table (cf. Corollary 5.1):

(i) No matter which particular \( S_1 \) and \( S_2 \) have been involved in, one and the same Euler’s map \( \mathcal{G}_2 \) provides a bijection between \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \)—from the “ODD-like” side to the “DISTINCT-like” side, for each of the equinumerous pairs of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) taken from Example 1.1, at least, for \( n = 4, 6 \). Though, as a total function, \( \mathcal{G}_2 \) is not a bijection at all.

(ii) As for the explicit inverse bijection from \( \mathcal{C}_2 \) onto \( \mathcal{C}_1 \),—the direction from the “DISTINCT-like” side to the “ODD-like” side, one and the same universal inverse Euler’s map \( \mathcal{G}_2^{-1} \) is suited for each of the pairs of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) taken from Example 1.1, except for item (f). Furthermore, these ‘standard’ cases can be easily distinguished as the cases where the partitions from the left-hand side of the corresponding identities contain no even parts.

(iii) Though mappings \( \mathcal{G}_2 \) and \( \mathcal{R}_2 \) are not identical: e.g.,

\[
\mathcal{G}_2([2, 1, 1]) \neq \mathcal{R}_2([2, 1, 1]),
\]

the ‘rewriting map’ \( \mathcal{R}_2 \) simulates Euler’s map \( \mathcal{G}_2 \) on all partitions we are interested in.
1.3. Summary

Section 2 of the paper deals with partition ideals of order 1 and their rewriting maps.

A general method from Kanovich [10] yields, in particular, rewriting bijections between any two equinumerous partition ideals of order 1, but it takes a certain effort to track through the rewriting machinery to express these bijections in more direct terms.

For this purpose, in Section 2 we introduce much more concrete rewriting maps $\mathcal{K}_\tau$, which are proved to be good for all equinumerous ideals of order 1, whenever one of the ideals consists of partitions into parts from some $S_1$ (see Theorem 2.2).

In addition to that, Theorem 2.2(b) provides a transparent and useful generalization of Andrews–Subbarao’s criterion [1,16]. (Andrews–Subbarao’s original criterion [1,16] says that, for any $n$, the number of partitions of $n$ into parts taken from some $S_1$ equals the number of partitions of $n$ into parts taken from $S_2$ with no part repeated more than $m − 1$ times if and only if $S_1 = S_2 − mS_2$, and $mS_2 \subseteq S_2$.)

Section 3 deals with Euler–Glaisher-type maps. Here we take advantage of a natural description of any ideal of order 1 in terms of a function $\tau$ controlling the number of repeated parts within its partitions.

Euler’s original map invokes the binary expansions of positive integers to handle the partitions into distinct parts. The step made by Glaisher [6] is to use the base $m$ expansions when the number of repeated parts is bounded by $m − 1$. The next natural step presented here is to invoke the mixed radix systems where the numerical base varies from position to position according to values of $\tau$.

Along these lines, following exactly the pattern of Euler–Glaisher’s original map, we define a generalized Euler–Glaisher’s map $\mathcal{E}_\tau$, and then prove that $\mathcal{E}_\tau$ and $\mathcal{K}_\tau$ are identical on all partitions we are interested in.

As a result, we show that these $\mathcal{E}_\tau$ provide bijections for any two equinumerous partition ideals $\mathcal{E}_1$ and $\mathcal{E}_2$, whenever $\mathcal{E}_1$ consists of partitions into parts from some $S_1$, and $\mathcal{E}_2$ is an arbitrary partition ideal of order 1.

As compared with other bijective proofs, we state that the generalized Euler–Glaisher’s mappings $\mathcal{E}_\tau$ are much more advantageous from the computational point of view (see Sections 3.2 and 3.3).

The aim of Section 4 is to show how a systematic and automated approach of Theorem 2.1, can be used to reveal the Euler–Glaisher’s bijections even for equinumerous partition ideals that are not within reach of the above general criterion given in Section 3.

2. Partition ideals

Let us recall the background material with which we are dealing (see, for instance, Andrews [2]).

**Definition 2.1 (Andrews [2]).** Generally, the classes $\mathcal{C}$ of partitions considered in the literature have the ‘local’ property that if $M$ is a partition in $\mathcal{C}$ and one or more parts are removed from $M$ to form a new partition $M'$, then $M'$ is also in $\mathcal{C}$. Such a class $\mathcal{C}$ is called a partition ideal, or an order ideal in terms of the lattice $\mathcal{P}$ of finite multisets of positive integers, ordered by $\subseteq$.

Dually, a class $\mathcal{F} \subseteq \mathcal{P}$ is an order filter if $M' \in \mathcal{F}$, whenever $M \in \mathcal{F}$ and $M \subseteq M'$.

It is readily seen that $\mathcal{C}$ is a partition ideal if and only if its complement $\overline{\mathcal{C}}$ is an order filter.

**Definition 2.2.** $M$ is minimal in an order filter $\mathcal{F}$, if $M \in \mathcal{F}$ and $M' = M$, for any multiset $M' \in \mathcal{F}$ such that $M' \subseteq M$. The support of $\mathcal{F}$ is defined as the set of all its minimal elements.

2.1. Partition ideals of order 1

**Definition 2.3 (Andrews [2]).** A partition $M$ is represented here as a sequence $\{f_i\}_{i=1}^{\infty}$, where $f_i$ is the number of occurrences of $i$ in $M$. A partition ideal $\mathcal{C}$ has order 1 if whenever $\{f_i\}_{i=1}^{\infty} \notin \mathcal{C}$, then there exists $i_0$ such that $\{f'_i\}_{i=1}^{\infty} \notin \mathcal{C}$.

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3 We say that $M' \subseteq M$ if $M'$ can be formed by removing a number of parts from $M$. E.g., $\{1, 5\} \subseteq \{1, 1, 3, 5, 5, 5\}$. 

where
\[ f'_i = \begin{cases} f_i & \text{for } i = i_0, \\ 0 & \text{otherwise.} \end{cases} \]

The partition ideals of order 1 are standard lattice ideals within the lattice \( \mathcal{P} \) of finite multisets of positive integers:

**Proposition 2.1** (Andrews [2]). A partition ideal \( \mathcal{C} \) has order 1 if and only if there exists a sequence \( \{d_i^\mathcal{C}\}_{i=1}^{\infty} \) (where each \( d_i^\mathcal{C} \) is a non-negative integer or \(+\infty\)) such that
\[
\mathcal{C} = \{ \{ f_i \}_{i=1}^{\infty} \mid f_i \leq d_i^\mathcal{C}, \text{ for all } i \}.
\]

Proposition 2.1 gives rise to the following description of partition ideals of order 1 in terms of how the number of repeated parts is being controlled.

**Proposition 2.2.** A partition ideal \( \mathcal{C} \) has order 1 if and only if one can find a partial mapping \( \tau \) from positive integers into positive integers so that for some set \( S \) of positive integers, this \( \mathcal{C} \) consists exactly of all partitions into parts taken from \( S \) in which each part \( z \) may occur no more than \( \tau(z) \) times. (For \( \tau(z) \) is undefined, the number of occurrences of the \( z \) is allowed to be unlimited.)

**Example 2.1.** The partitions from the right-hand side of all identities in Example 1.1 can be characterized with \( \tau(z) = 1 \), for all \( z \); or with \( \tau(z) = m - 1 \), for all \( z \).

**Example 2.2.** For the partitions where an odd part may appear at most twice, and an even part may appear at most once, the number of repeated parts is controlled by the following function \( \tau \):
\[
\tau(z) = \begin{cases} 1 & \text{if } z \text{ is even,} \\ 2 & \text{if } z \text{ is odd.} \end{cases}
\]

**Comment 2.1.** Proposition 2.2 allows us to reduce the complexity of the description of a partition ideal \( \mathcal{C} \) of order 1 by specifying it in two steps:

(a) First, with \( \tau \) we describe a general idea of what number of repeated parts is allowed. As a rule, this controlling function \( \tau \) is quite simple and easy to deal with.

(b) Then, we impose a more specific constraint on the above partitions with the help of \( S \) whose elements are allowed to be used as parts for the particular \( \mathcal{C} \).

2.2. The ‘bijective’ characterizations of any two equinumerous ideals of order 1

To sort out the problem why and when Euler–Glaisher’s bijections arise in many partition identities, we use the following general criterion from Kanovich [10].

**Theorem 2.1** (Kanovich [10]). Let \( \mathcal{C} \) and \( \mathcal{C}' \) be partition ideals such that the support of the complementary filter \( \overline{\mathcal{C}} \) is made of pairwise disjoint multisets, say \( C_1, C_2, \ldots, C_i, \ldots \), and the support of the complementary filter \( \overline{\mathcal{C}'} \) is made of pairwise disjoint multisets \( C'_1, C'_2, \ldots, C'_i, \ldots \).

Assume that these supports are sorted as lists so that the sequence of integers
\[
\|C_1\|, \|C_2\|, \ldots, \|C_i\|, \ldots
\]
is non-decreasing, and the sequence of integers
\[
\|C'_1\|, \|C'_2\|, \ldots, \|C'_i\|, \ldots
\]
is non-decreasing.
Then \( \mathcal{C} \) and \( \mathcal{C}' \) are equinumerous if and only if \( \|C_i\| = \|C'_i\| \), for all \( i \).

In addition to that, the system \( \Gamma \) consisting of the following multiset rewriting rules:

\[
\gamma_1 : C_1 \rightarrow C'_1, \quad \gamma_2 : C_2 \rightarrow C'_2, \quad \ldots, \gamma_i : C_i \rightarrow C'_i, \quad \ldots
\]

provides a bijection \( h \) between \( \mathcal{C}' \) and \( \mathcal{C} \).

Moreover, the inverse bijection \( h^{-1} \) from \( \mathcal{C}' \) onto \( \mathcal{C} \) is computed here with the help of the system \( \Gamma^{rev} \) consisting of the ‘reverse’ rewriting rules:

\[
\gamma_1^{-1} : C'_1 \rightarrow C_1, \quad \gamma_2^{-1} : C'_2 \rightarrow C_2, \quad \ldots, \quad \gamma_i^{-1} : C'_i \rightarrow C_i, \quad \ldots
\]

**Comment 2.2.** It is worthy of pointing out:

(a) Theorem 2.1 guarantees that every sequence of \( \Gamma \)-reductions must be terminated, whenever it started from a partition in \( \mathcal{C}' \), and every sequence of the reverse reductions must be also terminated, whenever it started from a partition in \( \mathcal{C} \).

(b) As for \( h \) itself, in addition to being a bijection between \( \mathcal{C}' \) and \( \mathcal{C} \), its image \( h(M) \) incorporates contributions made by \( M_1 \) and \( M_2 \), the sub-partitions of \( M \), so that \( h \) has the following *sub-partition property*:

(i) For any partitions \( M_1 \) and \( M_2 \) such that

\[
M_1 \uplus M_2 \in \mathcal{C}' \quad \text{and} \quad h(M_1) \uplus h(M_2) \in \mathcal{C},
\]

the following holds:

\[
h(M_1 \uplus M_2) = h(M_1) \uplus h(M_2).
\]

(ii) For any partitions \( M \) such that \( M \in \mathcal{C} \cap \mathcal{C}' \),

\[
h(M) = M.
\]

Note that the ideals of order 1 are a very specific subclass of the partition ideals whose complementary order filters are generated by pairwise disjoint minimal elements (see [10]), so that Theorem 2.1 has yielded the *first explicit bijective proof* [10] for Andrews’ criterion for ideals of order 1 [2, Theorem 8.4].

Moreover, for a number of identities we can track through the machinery to distill the corresponding rewriting systems and verbalize the related bijections in a nice way.

In this section we get much more concrete characterization in Theorem 2.2, but only if one of these ideals of order 1 consists of all partitions into parts taken from some \( S_1 \).

**Definition 2.4.** Given a partial mapping \( \tau \) from positive integers into positive integers, define \( \mathcal{K}_\tau \), a mapping from integer partitions into integer partitions, as follows:

Given the \( \tau \), let \( \mathcal{F} \) denote the function such that for all \( z \),

\[
\mathcal{F}(z) = z(1 + \tau(z)). \tag{1}
\]

Let \( \Gamma_\tau \) consist of the following multiset rewriting rules:

\[
\gamma_\tau : \{z, z, \ldots, z\} \rightarrow \{\mathcal{F}(z)\} \quad \text{with} \quad z \text{ ranging over the whole } \text{Dom}_\tau, \quad \text{the domain of } \tau.
\]

For any integer partition \( M \), we set

\[
\mathcal{K}_\tau(M) = \tilde{M},
\]

where \( M \) is reducible to \( \tilde{M} \) by means of the above rules \( \gamma_\tau \), with \( \tilde{M} \) being \( \Gamma_\tau \)-irreducible.

This \( \tilde{M} \) is guaranteed to be unique, because our \( \Gamma_\tau \) has the Church–Rosser property:

(a) the left-hand sides of the rules from \( \Gamma_\tau \) are pairwise disjoint multisets, which provides the confluence of \( \Gamma_\tau \), and
(b) each of the rewriting rules from \( \Gamma_z \) contracts the number of parts in a partition of \( n \), which provides the termination of \( \Gamma_z \)-reductions.

**Theorem 2.2.** Let \( \tau \) be a partial mapping from positive integers into positive integers.

Suppose that \( \mathcal{C}_1 \) consists of all partitions into parts taken from some set \( S_1 \), and \( \mathcal{C}_2 \) consists of all partitions into parts taken from some set \( S_2 \) in which each part \( z \) belonging to \( \text{Dom}_\tau \), the domain of \( \tau \), may occur no more than \( \tau(z) \) times.

Given the \( \tau \), let \( \mathcal{F} \) denote the function defined by (1) in Definition 2.4:

\[
\mathcal{F}(z) = z(\tau(z) + 1) \quad \text{for all } z.
\]

Then the following three statements are pairwise equivalent:

(a) \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) are equinumerous.

(b) \( \mathcal{F} \) is an injective map\(^4\) from \( S_2 \cap \text{Dom}_\tau \) into \( S_2 \), and

\[
S_1 = S_2 - \mathcal{F}(S_2 \cap \text{Dom}_\tau).
\]

(c) One and the same rewriting map \( \mathcal{K}_\tau \) provides a bijection \( h \) between \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \):

\[
\begin{align*}
(1) \quad & \text{For any } M \text{ from } \mathcal{C}_1, \mathcal{K}_\tau(M) \text{ belongs to } \mathcal{C}_2; \\
(2) \quad & \text{For any } M \text{ from } \mathcal{C}_2, \text{there is a unique } M \text{ from } \mathcal{C}_1 \text{ such that } \mathcal{K}_\tau(M) = \tilde{M}.
\end{align*}
\]

Moreover, the inverse bijection \( h^{-1} \) from \( \mathcal{C}_2 \) onto \( \mathcal{C}_1 \) is explicitly computed with the help of the system, say \( \mathcal{S}^{-\text{rev}} \), consisting of the ‘reverse’ rewriting rules

\[
\gamma^{-1}_\tau: \{\mathcal{F}(z)\} \rightarrow \begin{cases} \{z, z, \ldots, \underbrace{z}_{\tau(z)+1 \text{ copies}}\} \\
\end{cases}
\]

where \( z \) ranges only over \( S_2 \cap \text{Dom}_\tau \).

**Comment 2.3.** Notice that within Theorem 2.2 we use the same rewriting system \( \Gamma_z \) for each of the \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) under consideration.

**Proof of Theorem 2.2.** Following the general approach of Theorem 2.1, we take \( \mathcal{C}_2 \) as \( \mathcal{C} \), and \( \mathcal{C}_1 \) as \( \mathcal{C}' \), and investigate their complements, the order filters \( \overline{\mathcal{C}} \) and \( \overline{\mathcal{C}'} \).

The minimal elements of the first filter \( \overline{\mathcal{C}} \), the complement to \( \mathcal{C}_2 \), are of two kinds:

(i) Some are of the form \( \{y\} \), where \( y \) ranges over the complement to \( S_2 \).

(ii) Others are of the form \( \{z, z, \ldots, \underbrace{z}_{\tau(z)+1 \text{ copies}}\} \), where \( z \) ranges over \( S_2 \cap \text{Dom}_\tau \).

The minimal elements of the second filter \( \overline{\mathcal{C}'} \), the complement to \( \mathcal{C}_1 \), are of the form \( \{x\} \), where \( x \) ranges over the complement to \( S_1 \).

By Theorem 2.1, \( \mathcal{C} \) and \( \mathcal{C}' \) are equinumerous if and only if the support of filter \( \overline{\mathcal{C}} \) matches the support of filter \( \overline{\mathcal{C}'} \) in the sense that the following two sequences are merely reorderings of each other:

1. the sequence consisting of numbers \( \|\{y\}\| \)'s (with \( y \) ranging over \( S_2 \)) and of numbers \( \|\{z, z, \ldots, \underbrace{z}_{\tau(z)+1 \text{ copies}}\}\| \)'s (with \( z \) ranging over \( S_2 \cap \text{Dom}_\tau \));
2. the sequence of numbers \( \|\{x\}\| \)'s (with \( x \) ranging over \( S_1 \)).

\[^4\] That is, \( \mathcal{F}(S_2 \cap \text{Dom}_\tau) \subseteq S_2 \), and \( \mathcal{F}(z_1) \neq \mathcal{F}(z_2) \), for any distinct \( z_1 \) and \( z_2 \) from \( S_2 \cap \text{Dom}_\tau \).
Taking into account that \( \|y\| = y \), and \( \|\{z, z, \ldots, z\}\| = \mathcal{F}(z) \), and \( \|x\| = x \), we get the following:

\[
\begin{align*}
\tau(z) + 1 \text{ copies} \\
\overline{S}_2 \cup \mathcal{F}(\overline{S}_2 \cap \text{Dom}_z) & \xrightarrow{\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \ldots, \bar{x}_k, \ldots} \overline{S}_1
\end{align*}
\]

The support of filter \( \mathcal{C} \) matches the support of filter \( \mathcal{C}' \) if and only if:

- \( \overline{S}_2 \cup \mathcal{F}(\overline{S}_2 \cap \text{Dom}_z) = \overline{S}_1 \), and
- since all \( x \)'s are different, all \( y \)'s and \( \mathcal{F}(z) \)'s must be different; which yields that \( \overline{S}_2 \) and \( \mathcal{F}(\overline{S}_2 \cap \text{Dom}_z) \) are disjoint, and \( \mathcal{F}(z_1) \neq \mathcal{F}(z_2) \), for any distinct \( z_1 \) and \( z_2 \) from \( \overline{S}_2 \cap \text{Dom}_z \);

or, in other words,

- \( S_1 = S_2 - \mathcal{F}(S_2 \cap \text{Dom}_z) \), and
- \( \mathcal{F} \) provides an injective mapping from \( S_2 \cap \text{Dom}_z \) into \( S_2 \).

Thus, the above reasoning ensures the equivalence of items (a) and (b) in Theorem 2.2.

According to the ‘bijective’ part of Theorem 2.1, a bijection \( h \) between \( \mathcal{C}' \) and \( \mathcal{C} \) is provided by \( \Gamma_0 \) consisting of reduction rules of two kinds:

(i) the rules of the form \( \gamma_y : \{y\} \to \{y\} \), where \( y \) ranges over \( \overline{S}_2 \), and
(ii) the rules of the form \( \gamma_z : \{z, z, \ldots, z\} \to \{\mathcal{F}(z)\} \), where \( z \) ranges over \( S_2 \cap \text{Dom}_z \).

\[
\tau(z) + 1 \text{ copies}
\]

Now we take advantage of the ‘relative termination property’ (see Comment 2.2(a)) to distill the above rewriting systems.

The stuttering rules \( \gamma_y : \{y\} \to \{y\} \) have been originated here from the general matching machinery of Theorem 2.1.

From the termination point of view, the stuttering rules might have made troubles by causing infinite loops in reduction sequences. But Theorem 2.1 guarantees that every sequence of \( \Gamma_0 \)-reductions must be terminated, whenever it started from a partition in \( \mathcal{C}' \), and every sequence of the reverse reductions must also be terminated, whenever it started from a partition in \( \mathcal{C} \).

Hence, these \( \gamma_y \) cannot be enabled within any reductions we are dealing with. Therefore, such stuttering rules just can be left out entirely, and the ‘truncated’ \( \Gamma_{(3)} \) consisting only of the rules

\[
\gamma_z : \{z, z, \ldots, z\} \to \{\mathcal{F}(z)\}
\]

\[
\tau(z) + 1 \text{ copies}
\]

with \( z \) ranging over \( S_2 \cap \text{Dom}_z \), will produce the same bijection \( h \) between \( \mathcal{C}' \) and \( \mathcal{C} \).

In particular, the inverse bijection \( h^{-1} \) from \( \mathcal{C} \) onto \( \mathcal{C}' \) will be computed by the ‘reverse’ rewriting rules

\[
\gamma_z^{-1} : \{\mathcal{F}(z)\} \to \{z, z, \ldots, z\}
\]

\[
\tau(z) + 1 \text{ copies}
\]

where \( z \) ranges over \( S_2 \cap \text{Dom}_z \); as it is required in item (c) of Theorem 2.2. \( \square \)

To complete the proof of Theorem 2.2, it remains to show that the ‘universal’ system \( \Gamma_\tau \) and our possibly smaller system \( \Gamma_{(3)} \) specified for the given \( S_2 \), behave identically on all \( M \) taken from \( \mathcal{C}_1 \).

**Lemma 2.1.** Whatever sequence of \( \Gamma_\tau \)-reductions

\[
K_0 \xrightarrow{z_1} K_1 \xrightarrow{z_2} K_2 \xrightarrow{z_3} K_3 \xrightarrow{z_4} \ldots
\]

that started from a multiset \( M \) from \( \mathcal{C}_1 \) we take, each of the \( K_j \) contains only elements from \( S_2 \), and each of the rewriting rules \( z_1, z_2, z_3, \ldots \), turns out to be a rule from \( \Gamma_{(3)} \).
**Proof.** By induction. Suppose that a $I_{\tau}$'s rule of the form
\[
\gamma_z: \{z, z, \ldots, z\} \rightarrow \{\mathcal{I}(z)\}
\]
\[
\tau(z)+1 \text{ copies}
\]
has been applied to a multiset $K$ containing only elements from $S_2$, resulting in the multiset $K'$.

Then $z$ belongs to $S_2$, and thereby the rule can be thought of as a rule from the smaller $I_{(3)}$. According to item (b) in Theorem 2.2, $\mathcal{I}(z)$ must belong to $S_2$, which means that the resulting multiset $K'$ contains only elements from $S_2$, as well. \(\square\)

**Example 2.3 (continuing Example 1.3).** Taking $\tau(z) = m - 1$, for all $z$, and, respectively, $\mathcal{I}(z) = mz$, for all $z$, we conclude that one and same rewriting map $\mathcal{R}_m$ based on the rewriting rules ($z$ is an arbitrary positive integer)
\[
\gamma_z: \{z, z, \ldots, z\} \rightarrow \{mz\}
\]
\[
m \text{ copies}
\]
provides bijections for all identities taken from Example 1.1 (cf. Table 1).

For all items from Example 1.1 except (b), we are actually talking about bijections provided by one and same rewriting map $\mathcal{R}_2$ based on the rewriting rules
\[
\gamma_z: \{z, z\} \rightarrow \{2z\}.
\]

As for the inverse bijections, the following example illustrates the subtleties of how the limitations on the number of the reverse rules work:

(i) Within the first item of Example 1.1, the inverse image of a partition of the form $\{4, 2\}$ is computed in the following rewriting way:
\[
\{4, 2\} \xrightarrow[\gamma_1^{-1}]{} \{2, 2\} \xrightarrow[\gamma_1^{-1}]{} \{2, 2, 1, 1\} \xrightarrow[\gamma_1^{-1}]{} \{2, 1, 1, 1, 1\} \xrightarrow[\gamma_1^{-1}]{} \{1, 1, 1, 1, 1\},
\]
resulting in $\{1, 1, 1, 1, 1\}$, the correct result for the first item of Example 1.1.

(ii) But, within item (f) of Example 1.1, the inverse bijection is computed with the corresponding $I_{\tau, S_2}$, which invokes only rules of the form
\[
\gamma_z^{-1}: \{2z\} \rightarrow \{z, z\},
\]
where $z$ is even.

In particular, the rule $\gamma_1^{-1}: \{2\} \rightarrow \{1, 1\}$ is not allowed here, so that the corresponding reduction sequence started from the same partition $\{4, 2\}$ must terminate earlier:
\[
\{4, 2\} \xrightarrow[\gamma_2^{-1}]{} \{2, 2\},
\]
which yields $\{2, 2\}$, the correct inverse image but for item (f) of Example 1.1.

**Comment 2.4.**

(a) The ‘non-bijective’ part of Theorem 2.2 (the equivalence of items (a) and (b)) can be conceived of as an extreme generalization of Andrews–Subbarao’s criterion [1,16]. Andrews–Subbarao’s original criterion—that $S_1 = S_2 = mS_2$, and $mS_2 \subseteq S_2$, takes the simple case in which $\tau(z) = m - 1$, for all $z$, and, respectively, $\mathcal{I}(z) = mz$, for all $z$.

The generalized criterion can be also proved with generating functions (cf. Andrews [2]), but here, see (2), we get a clear ‘geometrical’ explanation of the raison d’être of the specific form of item (b).

(b) As for our main focus on—that the same $I_{\tau}$ computes explicit bijections $h$ for all $G$ and $G$ in question, one can develop a more direct proof, as well. For instance, it is readily seen that either of our $I_{\tau}$ and $I_{\tau, S_2}$ has the Church–Rosser property. It remains to check the correctness of the bijections provided by $I_{\tau}$ and $I_{\tau, S_2}$—but this requires a good deal of effort to prove it directly.
Here we take advantage of the general machinery of Theorem 2.1 in order to show that both parts, the ‘non-bijective’ and ‘bijective’ ones, of Theorem 2.2 can be covered synchronously in a unified and automated way (see also Section 4). On top of that, Theorem 2.2 yields a unique bijection \( h \) that respects the structure of the partition ideals and (the minimal elements of) order filters and, in particular, has the sub-partition property (see Comment 2.2).

Comment 2.5. Each of the rewriting rules from \( \Gamma_\tau \) contracts the number of parts in a partition of \( n \), and each of the rewriting rules from \( \Gamma_\tau^{rev} \) expands the number of parts in a partition of \( n \). Therefore, starting from any partition of \( n \), both \( \Gamma_\tau \)-reductions and \( \Gamma_\tau^{rev} \)-reductions must terminate in no more than \( n \) steps. The complexity effect is that, for a polytime computable

(a) The bijection \( h \) provided by Theorem 2.2 can be computed in polynomial time with respect to \( n \).

(b) For a polytime recognizable set \( S_2 \), the inverse bijection \( h^{-1} \) can be computed in polynomial time with respect to \( n \).

2.3. One and the same inverse rewriting mapping?

Within Theorem 2.2, any “ODD-like \( \Rightarrow \) DISTINCT-like” direction—that is from \( C_1 \) onto \( C_2 \), is provided by one and the same ‘universal’ \( K_\tau \) based on the rewriting system \( \Gamma_\tau \) that includes all rules of the form

\[
\gamma^\ast: \{ z, z, \ldots, z \} \rightarrow \{ T(z) \}
\]

with \( z \) ranging over the whole \( Dom_\tau \).

As for the inverse direction—that is from \( C_2 \) onto \( C_1 \), Theorem 2.2 suggests a smaller system \( \Gamma_\tau^{rev} \) adjusted to \( S_2 \), which consists of the ‘reverse’ rules

\[
\gamma^{-1}_\ast: \{ T(z) \} \rightarrow \{ z, z, \ldots, z \},
\]

where \( z \) ranges only over \( S_2 \cap Dom_\tau \).

In this section we show the numerous cases where the inverse directions can be also provided by one and the same ‘universal’ \( K_\tau^{rev} \), the full reverse version of \( K_\tau \).

Definition 2.5. Given a partial mapping \( \tau \) from positive integers into positive integers, let \( T \) denote the function defined by (1) in Definition 2.4:

\[
T(z) = z(\tau(z) + 1)
\]

for all \( z \).

Let \( \Gamma^{-1}_\tau \) consist of all reverse rewriting rules of the form

\[
\gamma^{-1}_\ast: \{ T(z) \} \rightarrow \{ z, z, \ldots, z \},
\]

where \( z \) ranges over the whole \( Dom_\tau \).

For any integer partition \( M' \), we set

\[
K_\tau^{rev}(M') = M,
\]

where \( M' \) is reducible to \( M \) by means of the rules from \( \Gamma^{-1}_\tau \), and \( M \) is \( \Gamma^{-1}_\tau \)-irreducible.

Proposition 2.3. For \( T \) being an injective map from \( Dom_\tau \) into \( \mathbb{N} \),\(^5\) the rewriting system \( \Gamma^{-1}_\tau \) has the Church–Rosser property, and, hence, our \( K_\tau^{rev} \) is a well-defined function.

Corollary 2.1. Let \( \tau \) be a partial mapping from positive integers into positive integers such that the related \( T \) (see Definition 2.4) is an injective map from \( Dom_\tau \) into \( \mathbb{N} \).

---

\(^5\) Here and, henceforth, by \( \mathbb{N} \) we denote the set of all positive integers; \( \mathbb{N} = \{ 1, 2, 3, 4, \ldots \} \).
Suppose that \( C_1 \) consists of all partitions into parts taken from some set \( S_1 \) such that
\[
S_1 \subseteq \mathbb{N} - \mathcal{F}(\text{Dom}_z),
\]
and \( C_2 \) consists of all partitions into parts taken from some set \( S_2 \) in which each part \( z \) belonging to \( \text{Dom}_z \) may occur no more than \( \tau(z) \) times.

Assume that \( C_1 \) and \( C_2 \) happen to be equinumerous.

Then there is a bijection \( h \) between \( C_1 \) and \( C_2 \) such that
1. For any \( \mathcal{M} \) from \( C_1 \), \( \mathcal{K}(\mathcal{M}) = h(\mathcal{M}) \).
2. For any \( \mathcal{M}' \) from \( C_2 \), \( \mathcal{K}_{\text{rev}}(\mathcal{M}') = h^{-1}(\mathcal{M}') \).

**Proof.** According to Theorem 2.2, there is a bijection \( h \) between \( C_1 \) and \( C_2 \) such that
1. For any \( \mathcal{M} \) from \( C_1 \), \( \mathcal{K}(\mathcal{M}) = h(\mathcal{M}) \).
2. The inverse bijection \( h^{-1} \) from \( C_2 \) onto \( C_1 \) is computed with the help of \( \mathcal{I}_{\tau, S_2}^{\text{rev}}, \) consisting of the ‘reverse’ rewriting rules
\[
\gamma^{-1}_z : \{ \tau(z) \} \rightarrow \{ z, z, \ldots, z \},
\]
where \( z \) ranges only over \( S_2 \cap \text{Dom}_z \).

It remains to show that for any \( \mathcal{M}' \) from \( C_2 \),
\[
\mathcal{K}_{\text{rev}}(\mathcal{M}') = h^{-1}(\mathcal{M}').
\]
Given an \( \mathcal{M}' \) from \( C_2 \), there is a sequence of \( \mathcal{I}_{\tau, S_2}^{\text{rev}} \)-reductions leading from \( \mathcal{M}' \) to \( \mathcal{M}'' \), where \( \mathcal{M}'' \) belong to \( C_1 \), and \( \mathcal{M}'' = h^{-1}(\mathcal{M}') \):
\[
\mathcal{M}' \xrightarrow{\gamma_1} K_1 \xrightarrow{\gamma_2} K_2 \xrightarrow{\gamma_3} K_3 \xrightarrow{\gamma_4} \ldots \xrightarrow{\gamma_k} \mathcal{M}''.
\]
Since \( \mathcal{M}'' \) belongs to \( C_1 \), and \( S_1 \subseteq \mathbb{N} - \mathcal{F}(\text{Dom}_z) \), the \( \mathcal{M}'' \) contains no part from \( \mathcal{F}(\text{Dom}_z) \). This means that even the full system \( \mathcal{I}_{\tau}^{-1} \) cannot continue the above reduction sequence. Hence,
\[
\mathcal{K}_{\tau}^{\text{rev}}(\mathcal{M}') = \mathcal{M}''.
\]
Bringing all together, we conclude that \( \mathcal{K}_{\tau}^{\text{rev}}(\mathcal{M}') = h^{-1}(\mathcal{M}') \). □

**Example 2.4 (continuing Example 2.3)**. Taking \( \tau(z) = m - 1 \), for all \( z \), and, respectively, \( \mathcal{F}(z) = mz \), for all \( z \), the inclusion condition (4) from Corollary 2.1 is rewritten as
\[
S_1 \subseteq \mathbb{N} - \mathcal{F}(\text{Dom}_z) = \{ x | x \text{ is not divisible by } m \}.
\]
Because of (5), one and the same rewriting map based on the rewriting rules \( (z \) is an arbitrary positive integer)
\[
\gamma^{-1}_z : \{ mz \} \rightarrow \{ z, z, \ldots, z \}
\]
is suited, as an inverse bijection, for each of the identities in Example 1.1, except item (f) (cf. Table 1).

**Example 2.5 (continuing Example 2.2)**. Let us combine ideas from Example 1.1:

(a) à la Euler–Schur: The number of partitions of \( n \) where each part is either congruent to \( 2 \mod 4 \) or congruent to \( \pm 1 \mod 6 \) equals the number of partitions of \( n \) where an even part may appear at most once, and an odd part may appear at most twice.
(b) à la Guy–Schur: The number of partitions of \( n \) into parts greater than 2 such that each part is either congruent to 2 mod 4 or congruent to ±1 mod 6 equals the number of partitions of \( n \) where an even part is not a power of 2 and may appear at most once, and an odd part is not a power of 3 and may appear at most twice.

(c) à la “1–2”: The number of partitions of \( n \) into ones and twos equals the number of partitions of \( n \) where an odd part is a power of 3 and may appear at most twice, and an even part is a power of 2 and may appear at most once.

To apply Theorem 2.2, we take

\[
\tau(z) = \begin{cases} 
1 & \text{if } z \text{ is even}, \\
2 & \text{if } z \text{ is odd}.
\end{cases} \quad \text{and, respectively,} \quad \mathcal{T}(z) = \begin{cases} 
2z & \text{if } z \text{ is even}, \\
3z & \text{if } z \text{ is odd}.
\end{cases}
\]

Theorem 2.2 shows that one and the same rewriting \( \mathcal{H} \), based on the rewriting rules of the form

\[
\gamma_{2^u}: \{2u, 2u\} \to \{4u\} \quad \text{and} \quad \gamma_{2^u+1}: \{2u + 1, 2u + 1, 2u + 1\} \to \{6u + 3\},
\]

can be served as a bijection \( h \) for each of the above identities in Example 2.5.

For instance,

\[
\{1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2\} \xrightarrow{\gamma_1} \{3, 1, 1, 1, 2, 2, 2, 2, 2\} \xrightarrow{\gamma_1} \ldots \xrightarrow{\gamma_1} \{3, 3, 4, 4\} \xrightarrow{\gamma_4} \{3, 3, 8, 4\}.
\]

The inclusion condition (4) from Corollary 2.1 is rewritten here as

\[
S_1 \subseteq \mathbb{N} \setminus \mathcal{T}(\mathbb{N}) = \{x \mid (x \equiv 2 \text{ (mod 4)}) \lor (x \equiv \pm1 \text{ (mod 6)})\}.
\]

(6)

For each of the above identities in Example 2.5, the corresponding \( S_1 \) satisfies this condition (6). Hence, according to Corollary 2.1, for each of the above identities in Example 2.5, the inverse \( h^{-1} \) can be computed by one and the same rewriting \( \mathcal{H}^{-\tau}_{\mathcal{T} \in \mathcal{V}} \) based on the rewriting rules of the form

\[
\gamma_{2^u}^{-1}: \{4u\} \to \{2u, 2u\} \quad \text{and} \quad \gamma_{2^u+1}^{-1}: \{6u + 3\} \to \{2u + 1, 2u + 1, 2u + 1\}.
\]

In particular,

\[
\{3, 3, 8, 4\} \xleftarrow{\gamma_2} \{3, 3, 8, 2, 2\} \xleftarrow{\gamma_1^{-1}} \{1, 1, 1, 3, 8, 2, 2\} \xleftarrow{\gamma_1^{-1}} \ldots \xleftarrow{\gamma_1^{-1}} \{1, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2\}.
\]

3. A generalized Euler–Glaisher’s map \( \mathcal{E}_\tau \)

Given a partial mapping \( \tau \) from positive integers into positive integers, which controls the number of the repeated parts, we develop a generalized Euler–Glaisher’s mapping \( \mathcal{E}_\tau \) following Euler–Glaisher’s original map in letter and in spirit.

We will show that \( \mathcal{E}_\tau \) coincides with \( \mathcal{H} \) first and then use this to show that \( \mathcal{E}_\tau \) is a bijection between the corresponding \( \mathcal{E}_1 \) and \( \mathcal{E}_2 \).

The core ingredient of Euler–Glaisher’s original map (see Definition 1.2) is the fact that every positive integer \( b \) can be uniquely represented in base \( m \) notation:

\[
b = \sum_j a_j m^j,
\]

with \( 0 \leq a_j \leq m - 1 \), for each \( j \).

Within this positional numeral system, the weights associated with the positions form a sequence

\[
1, \ m, \ m^2, \ m^3, \ldots
\]

(8)
starting from the least significant position, and each weight is related to the previous by a constant multiplier, namely, the base \( m \). Notice that Glaisher’s choice of this \( m \) follows the formula:

\[
m = \text{the number of repetitions allowed} + 1.
\]

(1) Euler’s original map exploits the binary expansions for partitions into distinct parts.

(2) The step made by Glaisher [6] is to use the numeral system with base \( m \) when the number of repeated parts is bounded by \( m - 1 \).

(3) Here we make the next natural step: to the mixed radix systems where the numerical base may vary from position to position [12]. Here the \( weights \) associated with the positions are intended to form a sequence where each weight is related to the previous by a multiplier, which is determined by the same token:

\[
\text{“ the multiplier = a value of } \tau + 1 \text{”}.
\]

(Recall that a part \( s \) can be repeated no more than \( \tau(s) \) times.)

**Definition 3.1.** Given a positive integer \( s \) as a ‘seed’, we specify the sequence of \( weights \)

\[
m_s^{(0)}, m_s^{(1)}, m_s^{(2)}, m_s^{(3)}, \ldots
\]

associated with the positions in a mixed radix system as follows (the sequence is terminated for the first \( j \) such that \( \tau(s(j)) \) is undefined):

\[
\begin{align*}
m_s^{(0)} &= 1, \\
m_s^{(1)} &= (\tau(s) + 1), \\
m_s^{(j+1)} &= (\tau(s(j)) + 1) \cdot m_s^{(j)}, \quad \text{where } s(j) = m_s^{(j)} s, \quad j = 0, 1, 2, \ldots.
\end{align*}
\]

**Proposition 3.1.** Every positive integer \( b \) can be uniquely represented as a sum with non-negative integer coefficients \( a_j \):

\[
b = \sum_j a_j m_s^{(j)},
\]

with \( a_j \leq \tau(s(j)) \) for each \( j \), whenever \( \tau(s(j)) \) is defined, here \( s(j) = m_s^{(j)} s \), according to our choice.

**Example 3.1.** Taking \( \tau(z) = m - 1 \), for all \( z \), we get the standard weights associated with the positions in the base \( m \) positional system:

\[
m_s^{(0)} = 1, \quad m_s^{(j+1)} = m \cdot m_s^{(j)} = m^{j+1}.
\]

**Example 3.2.** For \( \tau \) from Example 2.5:

\[
\tau(z) = \begin{cases} 
1 & \text{if } z \text{ is even,} \\
2 & \text{if } z \text{ is odd,}
\end{cases}
\]

we get the following weights associated with the positions in a mixed radix system:

\[
\begin{cases}
m_s^{(0)} = 1; \quad m_s^{(j+1)} = 2 \cdot m_s^{(j)} = 2^{j+1} & \text{for the seed } s \text{ being even,} \\
m_s^{(0)} = 1; \quad m_s^{(j+1)} = 3 \cdot m_s^{(j)} = 3^{j+1} & \text{for the seed } s \text{ being odd.}
\end{cases}
\]

**Example 3.3.** For the ‘swapped’ \( \tau \) defined as

\[
\tau(z) = \begin{cases} 
2 & \text{if } z \text{ is even,} \\
1 & \text{if } z \text{ is odd,}
\end{cases}
\]
we get the following weights associated with the positions in a mixed radix system:

\[
\begin{align*}
\{ m_1^{(0)} &= 1; m_s^{(j+1)} = 3^{j+1} \text{ for the seed } s \text{ being even,} \\
 m_1^{(0)} &= 1; m_s^{(j+1)} = 2 \cdot 3^j \text{ for the seed } s \text{ being odd.}
\end{align*}
\]

**Definition 3.2.** To define Euler–Glaisher’s map \( \vartheta \), we simply substitute the corresponding mixed radix system for the numeral system with base \( m \) in the original Definition 1.2.

Suppose a partition \( M \) of some \( n \):

\[
M = \{ s_1^{a_1}, \ldots, s_i^{a_i}, \ldots, s_k^{a_k} \},
\]

consists of distinct parts \( s_i \) with multiplicity \( b_i \), so that

\[
n = b_1 s_1 + b_2 s_2 + \cdots + b_k s_k.
\]

Write each \( b_i \) in a mixed radix notation as the following sum with non-negative integer coefficients \( a_{ij} \) (\( s_i \) is served here as the ‘seed’):

\[
b_i = \sum_j a_{ij} m_s^{(j)},
\]

where \( a_{ij} \leq \tau(s_i^{(j)}) \) for each \( i \) and \( j \), whenever \( \tau(s_i^{(j)}) \) is defined, here \( s_i^{(j)} = m_s^{(j)} s_i \).

Then

\[
n = \sum_{i,j} a_{ij} m_s^{(j)} s_i = \sum_{i,j} a_{ij} s_i^{(j)}.
\]

We now see the partition of \( n \), say \( M' \), into parts \( s_i^{(j)} \) repeated \( a_{ij} \) times,

\[
M' = \{ s_1^{(0)}, \ldots, s_1^{(a_{10})}, s_1^{(1)}, \ldots, s_1^{(a_{11})}, \ldots, s_i^{(0)}, \ldots, s_i^{(j)}, \ldots, s_i^{(a_{ij})}, \ldots \}.
\]

and we set

\[
\vartheta(M) = M'.
\]

**Comment 3.1.** While it is not even obvious that our \( \vartheta \) is injective it turns out to be a bijection in the interesting cases, which we will prove by showing that \( \vartheta \) acts exactly like \( \mathcal{H}_\tau \).

**Comment 3.2.** Taking \( \tau(z) = m - 1 \), for all \( z \), we get exactly Glaisher’s original map \( \mathcal{G}_m \).

**Example 3.4.** To illustrate this construction, take \( \tau \) from Example 3.2 and apply \( \vartheta \) to the partition of 18:

\[
M = \{ 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2, 2 \}.
\]

According to Example 3.2, the weights associated with the positions in our mixed radix systems are

\[
m_1^{(j)} = 3^j, \quad m_2^{(j)} = 2^j.
\]

Now we proceed as in the construction above

\[
18 = 6 \cdot 1 + 6 \cdot 2 \\
= (2 \cdot 3^1) \cdot 1 + (1 \cdot 2^1 + 1 \cdot 2^2) \cdot 2 \\
= 2 \cdot 3 + 3 \cdot 2 + 2 \cdot 2 \\
= 2 \cdot 3 + 1 \cdot 4 + 1 \cdot 8 = \text{two 3’s, one 4, and one 8,}
\]

which results in \( \{ 3, 3, 4, 8 \} \). (cf. Example 2.5.)
Comment 3.3. Mapping $E$ is very efficient from the computational point of view.
In the case of $\tau$ computable in polynomial time, $E$ runs on a partition $M$ of $n$ in polynomial time with respect to $\log_2 n$. (cf. Comment 2.5.)

Proof. To compute coefficients $a_j$ in the mixed radix expansions of Proposition 3.1:

$$b = \sum_j a_j m^{(j)}_s,$$

we can apply the standard fast procedure for the base $m$ expansions. Namely, the $a_j$ can be computed as the consecutive remainders in the process of repeated integer divisions by the corresponding $(\tau(s^{(j)}) + 1)$. It takes polynomial time with respect to $\log_2 b$. □

On the road of linking together our rewriting map $K$ and pure number theoretic map $E$, we will consider the following basic case:

Lemma 3.1. Given $s$, let $M_0$ be a partition of the form

$$M_0 = \{s, s, \ldots, s\} \quad \text{b copies} \quad (13)$$

Then

$$E(M_0) = K(M_0).$$

Besides, $E(M_0)$ is a partition into parts of the form $m^{(j)}_s s$, where $m^{(j)}_s$ is defined as in (10) of Definition 3.1.

Proof. By construction, $E(M_0)$ is a partition in which each part is of the form $s^{(j)}$ and may occur no more than $\tau(s^{(j)})$ times. Recall that $s^{(j)} = m^{(j)}_s s$.

Take two sets $S'$ and $S''$ so that $S' = \{s\}$, and $S''$ is the set of all integers of the form $s^{(j)}$.

Notice that $s^{(0)} = s$, and, by induction, for any $j$ (function $T$ is defined in Definition 2.4):

$$s^{(j+1)} = T(s^{(j)}).$$

This shows that $T(S'' \cap \text{Dom}_\tau) \subseteq S''$, and $S' = S'' - T(S'' \cap \text{Dom}_\tau)$, and $T$ is injective.

Let $C'$ consist of all partitions into parts taken from $S'$, and let $C''$ consist of all partitions into parts taken from $S''$ in which each part $z$ belonging to $\text{Dom}_\tau$ may occur no more than $\tau(z)$ times.

For the chosen $S'$ and $S''$, Theorem 2.2 guarantees that $C'$ and $C''$ are equinumerous, and $K$ provides a bijection between $C'$ and $C''$.

Take $M_0 = K(M_0)$, and $n_0 = bs$.

Since $n_0$ is uniquely expressed as the sum of $s$'s, the number of partitions of $n_0$ that belong to $C''$ is equal to 1, as well. This means that $M_0$, a partition of the form (12), is determined in a unique way, and by construction $E(M_0) = M_0$. □

Now we show that the generalized Euler's map $E$ coincides with the ‘rewriting’ map $K$ at least on all partitions $M$ taken from any good $C_1$.

Theorem 3.1. Let $\tau$ be a partial mapping from positive integers into positive integers.

Suppose that $C_1$ consists of all partitions into parts taken from some set $S_1$, and $C_2$ consists of all partitions into parts taken from some set $S_2$ in which each part $z$ belonging to $\text{Dom}_\tau$ may occur no more than $\tau(z)$ times.

If $C_1$ and $C_2$ happen to be equinumerous, then for any $M$ from $C_1$:

$$E(M) = K(M).$$
Proof. Given the $\tau$, let $\mathcal{F}$ denote the function defined by (1) in Definition 2.4:

$$\mathcal{F}(z) = z(\tau(z) + 1) \text{ for all } z.$$ 

According to Theorem 2.2

(1) $\mathcal{F}(S_2 \cap \text{Dom}_\tau) \subseteq S_2$,
(2) $\mathcal{F}(z_1) \neq \mathcal{F}(z_2)$, for any distinct $z_1$ and $z_2$ from $S_2 \cap \text{Dom}_\tau$, and
(3) $S_1 = S_2 - \mathcal{F}(S_2 \cap \text{Dom}_\tau)$.

Lemma 3.2. Given an $s$, let $s^{(j)}$ denote the number $m_s^{(j)}$, where $m_s^{(j)}$ is defined as in Definition 3.1. Then

(a) $s^{(j+1)} = \mathcal{F}(s^{(j)})$, for any $j$.
(b) If $s \in S_1$ then $s^{(j)} \in S_2$, for any $j$.
(c) If $y \in S_2$ then there is a unique $s \in S_1$ and a unique $j$ such that $y = s^{(j)}$.

Proof.

(a) By induction.
(b) It follows from the fact that $S_1 \subseteq S_2$ and $\mathcal{F}(S_2 \cap \text{Dom}_\tau) \subseteq S_2$.
(c) There are two items to be considered:

(c1) “The existence.” It is proved by infinite descent. Given a $y$, we construct a sequence $z_0, z_1, \ldots, z_k, z_{k+1}, \ldots$, as follows:

- $z_0 = y$.
- Suppose $z_k \in \mathcal{F}(S_2 \cap \text{Dom}_\tau)$. As $z_{k+1}$ we take a number $z$ from $S_2 \cap \text{Dom}_\tau$ such that $z_k = \mathcal{F}(z)$.
- If $z_k \notin \mathcal{F}(S_2 \cap \text{Dom}_\tau)$ then we terminate our sequence. According to Definition 2.4, $z_k = \mathcal{F}(z_{k+1}) \geq 2z_{k+1}$. Therefore, being strictly decreasing, our sequence of positive integers $z_0, z_1, \ldots, z_k, z_{k+1}, \ldots$, cannot be continued infinitely.

Let $s$ be the terminal term of the sequence. Then $s \notin \mathcal{F}(S_2 \cap \text{Dom}_\tau)$. Since all $z_k$ (including the $s$) belong to $S_2$ by construction, such an $s$ is an element of $S_2 - \mathcal{F}(S_2 \cap \text{Dom}_\tau)$, which means that $s \in S_1$.

By construction, each of the $z_k$ (including our initial $y$) is of the form $s^{(\ell)}$ for some $\ell$.

(c2) “The uniqueness.” Assume $y = s_1^{(j)} = s_2^{(k)}$, for some $s_1$ and $s_2$ from $S_1$, and $j \geq k$.

The fact that $\mathcal{F}$ is injective implies then $s_1^{(j-k)} = s_2$.

Since $s_2$ does not belong to $\mathcal{F}(S_2 \cap \text{Dom}_\tau)$, we conclude that $j - k = 0$, and, hence, $s_1 = s_2$.

Let $M$ from $C_1$ consists of distinct parts $s_i$ with multiplicity $b_i$:

$$M = \{s_1, s_2, \ldots, s_k\},$$

$$b_1 \text{ copies} \quad b_2 \text{ copies} \quad b_k \text{ copies}$$

We represent the $M$ as the sum of pairwise disjoint multiset

$$M = M_1 \uplus M_2 \uplus \cdots \uplus M_k,$$

where each of these $M_i$ is of the ‘singular’ form (13):

$$M_i = \{s_1, s_1, \ldots, s_i\},$$

$$b_i \text{ copies}$$

By construction,

$$\mathcal{E}_\tau(M) = \mathcal{E}_\tau(M_1) \uplus \mathcal{E}_\tau(M_2) \uplus \cdots \uplus \mathcal{E}_\tau(M_k),$$
and, applying Lemma 3.1,

$$E_\tau(M) = N_\tau(M_1) \cup N_\tau(M_2) \cup \cdots \cup N_\tau(M_k).$$

Each of the $N_\tau(M_i)$ is $\Gamma_\tau$-irreducible. Lemma 3.2 guarantees that all $N_\tau(M_i)$ are pairwise disjoint multisets. Therefore, no rule from $\Gamma_\tau$ can be applied to the whole $N_\tau(M_1) \cup N_\tau(M_2) \cup \cdots \cup N_\tau(M_k)$.

As a result,

$$N_\tau(M) = N_\tau(M_1) \cup N_\tau(M_2) \cup \cdots \cup N_\tau(M_k) = E_\tau(M),$$

which concludes the proof of Theorem 3.1. □

The universal nature of Euler–Glaisher’s mapping is fully revealed by the following:

**Corollary 3.1.** Let $\tau$ be a partial mapping from positive integers into positive integers.

Suppose that $\mathcal{C}_1$ consists of all partitions into parts taken from some set $S_1$, and $\mathcal{C}_2$ consists of all partitions into parts taken from some set $S_2$ in which each part $z$ belonging to $\text{Dom}_\tau$, the domain of $\tau$, may occur no more than $\tau(z)$ times.

If $\mathcal{C}_1$ and $\mathcal{C}_2$ happen to be equinumerous, then there is a bijection $h$ between $\mathcal{C}_1$ and $\mathcal{C}_2$ such that, for any $M$ from $\mathcal{C}_1$:

$$E_\tau(M) = h(M).$$

Moreover, the inverse bijection $h^{-1}$ from $\mathcal{C}_2$ onto $\mathcal{C}_1$, the direction from the “DISTINCT-like” side to the “ODD-like” side, can be explicitly computed in the following way:

Let $M'$ from $\mathcal{C}_2$ consist of parts $d_1, d_2, \ldots, d_\ell, \ldots, d_k$,

$$n = d_1 + d_2 + \cdots + d_\ell + \cdots + d_k, \quad (14)$$

Each integer $d_\ell$ is uniquely expressed as $m^{(a_\ell)}_{O_\ell} O_\ell$, where $O_\ell$ is an integer from $S_1$, and $m^{(a_\ell)}_{O_\ell}$ is a weight generated by (10) in Definition 3.1 with the seed $O_\ell$.

Thus

$$n = m^{(a_1)}_{O_1} O_1 + m^{(a_2)}_{O_2} O_2 + m^{(a_3)}_{O_3} O_3 + \cdots + m^{(a_\ell)}_{O_\ell} O_\ell + \cdots + m^{(a_k)}_{O_k} O_k, \quad (15)$$

where each $O_\ell$ is an integer from $S_1$.

If we now group together the identical $O_\ell$’s, we get an expression like

$$n = (m^{(z_1)}_{s_1} + m^{(z_2)}_{s_1} + \cdots) \cdot s_1 + (m^{(\beta_1)}_{s_1} + m^{(\beta_2)}_{s_1} + \cdots) \cdot s_2 + (m^{(\gamma_1)}_{s_1} + m^{(\gamma_2)}_{s_1} + \cdots) \cdot s_3 + \cdots$$

where $s_1, s_2, s_3, \ldots$ are distinct elements of $S_1$.

We now can see the desired $M = h^{-1}(M')$ contains $\mu_1$ copies of $s_1$, $\mu_2$ copies of $s_2$, $\mu_3$ copies of $s_3$, etc.

$$M = \{s_1, \ldots, s_1, \ldots, s_i, \ldots, s_i, \ldots, s_k, \ldots, s_k\}.$$

$$\mu_1 \text{ copies} \quad \mu_2 \text{ copies} \quad \mu_3 \text{ copies} \quad \mu_k \text{ copies} \quad (16)$$

**Proof.** Theorems 2.2 and 3.1 provide a bijection $h$ between $\mathcal{C}_1$ and $\mathcal{C}_2$ such that, for any $M$ from $\mathcal{C}_1$:

$$E_\tau(M) = N_\tau(M) = h(M).$$

To justify our explicit procedure to compute the $h^{-1}$, which is leading from $M'$ of the form (14) to $M$ of the form (16), we have to prove that for these $M'$ and $M$:

$$E_\tau(M) = M'.$$
Since $M'$ is from $C_2$,

(i) Each part $d_\ell$ is uniquely expressed in the form $m_{s_i}^{(j)}$ (see Lemma 3.2).

(ii) The same $d_\ell$ may occur more than once: let $a_{ij}$ denote the number of its repetitions. Whenever $d_\ell$ belongs to the domain of $\tau$, this $a_{ij}$ is bounded by $\tau(s_i^{(j)})$; recall that $d_\ell = m_{s_i}^{(j)} = s_i^{(j)}$.

Hence,

$$\mu_i = \sum_j a_{ij}m_{s_i}^{(j)},$$

where $a_{ij} \leq \tau(s_i^{(j)})$ for each $i$ and $j$, whenever $\tau(s_i^{(j)})$ is defined.

Running Definition 3.2, we conclude that $E(M) = M'$. □

**Example 3.5.** Taking $\tau(z) = m - 1$, for all $z$, we see that one and the same Euler–Glaisher’s original mapping $\mathcal{G}_m$ can be served as a bijection for each of the identities in Example 1.1. (Notice that mostly $m = 2$ there.)

To illustrate the inverse procedure suggested in Corollary 3.1, consider the partition \{4, 2\} from the right-hand side of the identity in the ‘controversial’ item (f) of Example 1.1 and find its bijective mate from the left-hand side of this identity.

Here $\tau(z) = 1$, for all $z$, and the weights $m_s^{(j)}$ generated with the seeds $s$, with which the inverse procedure is dealing, are the following (see Example 3.1):

$$m_2^{(j)} = 2^j, \quad m_6^{(j)} = 2^j, \quad m_{10}^{(j)} = 2^j, \ldots.$$ 

Now we proceed as follows:

$$6 = 4 + 2 = (2^1 \cdot 2) + (2^0 \cdot 2) = (2^1 + 2^0) \cdot 2 = 3 \cdot 2 = \text{three 2's},$$

which results in \{2, 2, 2\}. (cf. Example 2.3.)

### 3.1. One and the same inverse Euler–Glaisher’s mapping?

Within Corollary 3.1, any “ODD-like $\Rightarrow$ DISTINCT-like” direction—that is from $C_1$ onto $C_2$, is provided by one and the same ‘universal’ $E_\tau$, whereas the inverse procedure suggested in Corollary 3.1 is adjusted to the particular $S_1$ (see (15)).

Like Section 2.3, we show the numerous cases where even the inverse directions can be served with one and the same universal inverse Euler–Glaisher’s map $E_\tau^{inv}$. 

**Definition 3.3.** Given a partial mapping $\tau$ from positive integers into positive integers, let $\mathcal{F}$ denote the function defined by (1) in Definition 2.4:

$$\mathcal{F}(z) = z(\tau(z) + 1), \quad \text{for all } z.$$ 

Following the original Definition 1.3 to the letter, we define the ‘inverse’ $E_\tau^{inv}$ as follows: Let an arbitrary $M'$ consist of parts $d_1, d_2, \ldots, d_k$,

$$n = d_1 + d_2 + \cdots + d_k.$$ 

Each integer $d_\ell$ is expressed as $m_{O_\ell}^{(a_\ell)} O_\ell$, where $O_\ell$ does not belong to $\mathcal{F}(\text{Dom}_z)$, and $m_{O_\ell}^{(a_\ell)}$ is a weight generated by (10) in Definition 3.1 with the seed $O_\ell$. 
Thus
\[ n = m_{O_1}^{(a_1)} O_1 + m_{O_2}^{(a_2)} O_2 + m_{O_3}^{(a_3)} O_3 + \cdots + m_{O_k}^{(a_k)} O_k, \]
where each \( O_\ell \) does not belong to \( \mathcal{F}(\text{Dom}_\tau) \).

If we now group together the identical \( O_\ell \)'s, we get an expression like
\[
\begin{align*}
    n &= \left( m_{s_1}^{(y_1)} + m_{s_2}^{(y_2)} + \cdots \right) s_1 + \left( m_{s_1}^{(y_3)} + m_{s_1}^{(y_4)} + \cdots \right) s_2 + \left( m_{s_1}^{(y_5)} + m_{s_1}^{(y_6)} + \cdots \right) s_3 + \cdots \\
    &= \mu_1 \cdot s_1 + \mu_2 \cdot s_2 + \mu_3 \cdot s_3 + \cdots,
\end{align*}
\]
where \( s_1, s_2, s_3, \ldots \) are distinct elements from the complement to \( \mathcal{F}(\text{Dom}_\tau) \).

We now can see the resulting \( \delta_\tau^{\text{inv}}(M') \): it contains \( \mu_1 \) copies of \( s_1 \), \( \mu_2 \) copies of \( s_2 \), \( \mu_3 \) copies of \( s_3 \), etc.

**Example 3.6 (continuing Example 3.4).** Let us take \( \tau \) from Example 3.2 and apply \( \delta_\tau^{\text{inv}} \) to the following partition of 18:
\[ M' = \{3, 3, 4, 8\}. \]
Recall that
\[ \mathcal{F}(\mathbb{N}) = \{ x \mid (x \equiv 2 \pmod{4}) \lor (x \equiv \pm 1 \pmod{6})\}. \]
The weights \( m_s^{(j)} \) generated with the seeds \( s \), with which our \( \delta_\tau^{\text{inv}} \) is dealing, are the following (see Example 3.2):
\[
m_2^{(j)} = 2^j, \quad m_6^{(j)} = 2^j, \ldots, \quad m_1^{(j)} = 3^j, \quad m_5^{(j)} = 3^j, \quad m_7^{(j)} = 3^j, \ldots.
\]
Now we proceed as in Definition 3.3:
\[
18 = 3 + 3 + 4 + 8 \\
= 3^1 \cdot 1 + 3^1 \cdot 1 + 2^1 \cdot 2 + 2^2 \cdot 2 \\
= (3^1 + 3^1) \cdot 1 + (2^1 + 2^2) \cdot 2 = 6 \cdot 1 + 6 \cdot 2 = \text{six 1's, and six 2's},
\]
which results in \( \{1, 1, 1, 1, 1, 2, 2, 2, 2, 2, 2\} \). (cf. Example 2.5.)

**Corollary 3.2.** For \( \mathcal{F} \) being an injective map from \( \text{Dom}_\tau \) into \( \mathbb{N} \), the above \( \delta_\tau^{\text{inv}} \) is a well-defined mapping.

**Proof.** By taking \( S_2 = \mathbb{N} \) and \( S_1 = \mathcal{F}(\text{Dom}_\tau) = \mathbb{N} - \mathcal{F}(\text{Dom}_\tau) \) in Lemma 3.2, we verify that every integer \( d \) has a unique representation in the form \( m_s^{(j)} \) \( s \), where \( s \) belongs to \( \mathcal{F}(\text{Dom}_\tau) \), and \( m_s^{(j)} \) is a weight generated by (10) in Definition 3.1 with the seed \( s \). \( \square \)

**Corollary 3.3.** Let \( \tau \) be a partial mapping from positive integers into positive integers such that the related \( \mathcal{F} \) (see Definition 2.4) is an injective map from \( \text{Dom}_\tau \) into \( \mathbb{N} \).

Suppose that \( \mathcal{G}_1 \) consists of all partitions into parts taken from some set \( S_1 \) such that
\[
S_1 \subseteq \mathbb{N} - \mathcal{F}(\text{Dom}_\tau),
\]
and \( \mathcal{G}_2 \) consists of all partitions into parts taken from some set \( S_2 \) in which each part \( z \) belonging to \( \text{Dom}_\tau \) may occur no more than \( \tau(z) \) times.

Let \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) be equinumerous.

Then there is a bijection \( h \) between \( \mathcal{G}_1 \) and \( \mathcal{G}_2 \) such that
\[
\begin{align*}
(1) & \text{ For any } M \text{ from } \mathcal{G}_1, \delta_\tau(M) = h(M). \\
(2) & \text{ For any } M' \text{ from } \mathcal{G}_2, \delta_\tau^{\text{inv}}(M') = h^{-1}(M').
\end{align*}
\]
**Proof.** According to Corollary 3.1, there is a bijection \( h \) between \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) such that, for any \( M \) from \( \mathcal{C}_1 \):

\[
\mathcal{E}_\tau(M) = h(M).
\]

It remains to show that \( \mathcal{E}_\tau^{-1}(M') = h^{-1}(M') \), for any \( M' \) from \( \mathcal{C}_2 \).

Let \( M' \) from \( \mathcal{C}_2 \) consist of parts \( d_1, d_2, \ldots, d_k \),

\[
n = d_1 + d_2 + \cdots + d_k.
\]

Our \( \mathcal{E}_\tau^{-1}(M') \) runs in the following way:

First, each integer \( d_\ell \) is uniquely expressed as \( m(a_\ell)O_\ell \), where \( O_\ell \) belongs to \( \mathcal{T}(\text{Dom}_\tau) \). But Lemma 3.2 guarantees that this \( d_\ell \) has a unique representation in the form \( m(j)s \), where \( s \) is an element of \( S_1 \). The inclusion \( S_1 \subseteq \mathcal{T}(\text{Dom}_\tau) \) provides that \( O_\ell = s \), and, hence, \( O_\ell \) must be an element of \( S_1 \).

The effect is that \( \mathcal{E}_\tau^{-1}(M') \) will exactly mimic the inverse procedure described in Corollary 3.1, resulting in

\[
\mathcal{E}_\tau^{-1}(M') = h^{-1}(M').
\]

\( \square \)

**Example 3.7 (continuing Example 3.5).** Taking \( \tau(z) = m - 1 \), for all \( z \), and, respectively, \( \mathcal{F}(z) = mz \), for all \( z \), the inclusion condition (17) from Corollary 3.3 is rewritten as

\[
S_1 \subseteq \mathbb{N} - m\mathbb{N} = \{x \mid x \text{ is not divisible by } m\}.
\]

Because of (18), one and the same Euler–Glaisher’s original inverse map \( \mathcal{E}_m^{-1} \) is suited, as an inverse bijection, for each of the identities in Example 1.1, except item (f) (cf. Table 1).

See also a quite exotic Example 3.8, which takes advantage of a very simple \( \tau(z) = 1 \), for all \( z \), to have Euler’s original map as a bijection, even in the case of extremely complicated \( S_1 \) and \( S_2 \).

3.2. A comparison with the rewriting maps

Though the Euler–Glaisher’s mapping \( \mathcal{E}_\tau \) and the rewriting mapping \( \mathcal{K}_\tau \) simulates each other on all partitions of possible interest, the Euler–Glaisher’s mapping is exponentially advantageous from the computational point of view (cf. Comments 2.5 and 3.3):

In the case of \( \tau \) computable in polynomial time,

(a) whereas \( \mathcal{K}_\tau \) runs in polynomial time with respect to \( n \),
(b) \( \mathcal{E}_\tau \) runs in polynomial time but with respect to \( \log_2 n \).

3.3. A comparison with Sellers–Sills–Mullen’s bijections [15]

In [15, Theorem 3.6] Sellers, Sills, and Mullen have given an explicit bijection between \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), two equinumerous ideals of order 1, in the case when \( \mathcal{C}_1 \) consists of partitions into parts from some set \( S_1 \). Their bijection is defined with the help of an algorithm [15, Algorithm 3.5], the input of which are two sequences \( \{d_i^{(1)}\}_{i=1}^{\infty} \) and \( \{d_i^{(2)}\}_{i=1}^{\infty} \) assigned to \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), respectively, by Proposition 2.1.

Justifying the universal nature of the Euler–Glaisher’s mappings once again, one can show that our \( \mathcal{E}_\tau \), with an appropriate \( \tau \), incorporates their bijection \( h \) in the sense that, for any \( M \) from \( \mathcal{C}_1 \), \( \mathcal{E}_\tau(M) = h(M) \).

Notice that

(a) our \( \mathcal{E}_\tau \) is defined in pure number theoretic manner, carrying out the construction of Glaisher’s original map to the letter, and, moreover,
(b) one and the same \( \mathcal{E}_\tau \) can be used for a big variety of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) (see Corollaries 3.1 and 3.3).

On the contrary,

(a) Sellers–Sills–Mullen’s bijection \( h \) invokes algorithmically specified parameters, and
(b) their \( h \) is adapted to each particular pair of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \).
Plus, our $\delta_\tau$ are much more advantageous from the computational point of view.

With the following example we illustrate that our solution to the mystery of Euler–Glaisher’s bijections brings to our attention surprising computational features.

**Example 3.8.** Let $K$ be a set of positive integers, such that neither $K$ itself nor its complement is a recursively enumerable set.

Take two sets $S_1$ and $S_2$ such that

$$S_1 = \{2i + 1 | i \in K\} \quad \text{and} \quad S_2 = \{2j(2i + 1) | j = 0, 1, 2, \ldots; i \in K\}.$$  

Let $\mathcal{C}'$ be the set of all partitions into parts from $S_1$, and $\mathcal{D}'$ be the set of all partitions into distinct parts taken from $S_2$.

(a) Even though the generating functions of $\mathcal{C}'$ and $\mathcal{D}'$ are not computable, we can easily show that $\mathcal{C}'$ and $\mathcal{D}'$ are equinumerous by straightforward manipulation with their generating functions:

$$\prod_{i \in K, j \geq 0} (1 + q^{2i(2j+1)}) = \prod_{i \in K} \prod_{j \geq 0} (1 + (q^{2i+1})^{2j}) = \prod_{i \in K} \frac{1}{1 - q^{2i+1}}.$$  

(b) As for an explicit bijection between $\mathcal{C}'$ and $\mathcal{D}'$, the situation seems much more subtle. For instance, Sellers–Sills–Mullen’s bijection [15, Theorem 3.6] derived for this pair of $\mathcal{C}'$ and $\mathcal{D}'$ is not computable, because the respective input sequences $(d_i^\mathcal{C}')_{i=1}^\infty$ and $(d_i^\mathcal{D}')_{i=1}^\infty$ are not recursively enumerable.

In spite of this obstruction to finding computable bijections, Corollary 3.3 guarantees “for free” that Euler’s original map $\mathcal{D}_2$ together with Euler’s original inverse map $\mathcal{D}_2^{-1}$ form a bijection between $\mathcal{C}'$ and $\mathcal{D}'$.

Recall that both $\mathcal{D}_2$ and $\mathcal{D}_2^{-1}$ are computable in polynomial time with respect to $\log_2 n$.

### 4. Beyond Theorem 2.2 and Corollary 3.1

Assume that $\mathcal{C}$ and $\mathcal{C}'$ are equinumerous partition ideals of order 1.

Let $C_1, C_2, \ldots, C_i, \ldots,$ be a list of all minimal elements of the complementary filter $\overline{\mathcal{C}}$, and let $C_1', C_2', \ldots, C_i', \ldots,$ be a list of all minimal elements of the complementary filter $\overline{\mathcal{C}}'$.

Based on these two lists, Theorem 2.1 provides an explicit bijection $h$ between $\mathcal{C}$ and $\mathcal{C}'$, but it takes a significant effort to track through the machinery to express this rewriting bijection in more direct ‘one-step’ terms.

The situation where no repeated terms appear in the respective sequence of integers:

$$\|C_1\|, \|C_2\|, \ldots, \|C_i\|, \ldots$$  

is fully covered by Section 3.

Whenever one of the $\mathcal{C}$ and $\mathcal{C}'$ consists of partitions into parts from a set, we can directly apply Theorem 2.2 and Corollary 3.1.

Otherwise, one can take $S_0$ as the set of all $x$ such that $x$ does not appear in the above sequence (19) and invoke the ideal, say $\mathcal{C}_0$, that consists of partitions into parts from $S_0$. An explicit bijection between $\mathcal{C}$ and $\mathcal{C}'$ can be composed from the corresponding maps $\delta_\tau$, which provides a bijection between $\mathcal{C}_0$ and $\mathcal{C}$, and $\delta_\tau'$, which provides a bijection between $\mathcal{C}_0$ and $\mathcal{C}'$.

The difficulties of the most general case of partition ideals of order 1 are related to the situations when the corresponding sequence (19) is allowed to have repeated terms. In the latter case, the number of possible matching of the lists $C_1, C_2, \ldots, C_i, \ldots,$ and $C_1', C_2', \ldots, C_j', \ldots,$—that is invoked in Theorem 2.1, may have been infinite (even continual), which may have resulted in an infinite number of relevant bijections between the partition ideals.

The aim of Section 4 is to show how a systematic and automated approach of Theorem 2.1 can be used sometimes to reveal the Euler–Glaisher’s-type bijections even for equinumerous partition ideals that are not within reach of the general method given in Section 3.

**Example 4.1.** Let $\mathcal{C}_1$ consist of all partitions in which no part of the form $2 \cdot 3^k$ or of the form $12 \cdot 3^k$ occurs, each part of the form $3^k$ may occur at most thrice, and all other positive integers may appear without restriction, and let $\mathcal{C}_2$
consist of all partitions where each part of the form $3^k$ or of the form $2 \cdot 3^k$ may occur at most once, each part of the form $4 \cdot 3^k$ may occur at most twice, and all other positive integers may appear without restriction.

Our task is to figure out an Euler–Glaisher’s-type bijection for these $\mathcal{C}_1$ and $\mathcal{C}_2$.

The minimal elements of the first filter $\mathcal{F}_1$, the complement to $\mathcal{C}_1$, are the following:

$$\{1, 1, 1, 1\}, \{2\}, \{12\}, \{3, 3, 3, 3\}, \{6\}, \{36\}, \{9, 9, 9, 9\}, \{18\}, \{108\}, \ldots .$$

The minimal elements of the second filter $\mathcal{F}_2$, the complement to $\mathcal{C}_2$, are the following:

$$\{1, 1\}, \{2, 2\}, \{4, 4, 4\}, \{3, 3\}, \{6, 6\}, \{12, 12, 12\}, \{9, 9\}, \{18, 18\}, \{36, 36, 36\}, \ldots .$$

Notice that $\|\{4, 4, 4\}\| = \|\{6, 6\}\|$, $\|\{12, 12, 12\}\| = \|\{18, 18\}\|$, \ldots . which makes repetitions in the corresponding sequence of integers (19) and thereby violates item (b) in Theorem 2.2, so that Corollary 3.1 cannot be applied to Example 4.1.

The general ‘matching machinery’ of Theorem 2.1 provides a bijection $h$ between $\mathcal{C}_1$ and $\mathcal{C}_2$ computed by $\Gamma_{(20)-(22)}$ consisting of the following multiset rewriting rules:

$$\gamma_1: \{1, 1\} \to \{2\}, \gamma_3: \{3, 3\} \to \{6\}, \ldots , \gamma_z: \{z, z\} \to \{2z\}, \ldots \quad (\text{here } z \text{ is a power of } 3),$$

$$\gamma_2: \{2, 2\} \to \{1, 1, 1, 1\}, \ldots , \gamma_{2u}: \{2u, 2u\} \to \{u, u, u, u\}, \ldots \quad (\text{here } u \text{ is a power of } 3),$$

$$\gamma_4: \{4, 4, 4\} \to \{12\}, \ldots , \gamma_v: \{v, v, v\} \to \{3v\}, \ldots \quad (\text{here } v \text{ is of the form } 4 \cdot 3^k).$$

The inverse bijection $h^{-1}$ from $\mathcal{C}_2$ onto $\mathcal{C}_1$ is computed here by $\Gamma_{(23)-(25)}^{rev}$ consisting of the ‘reverse’ rewriting rules:

$$\gamma_z^{-1}: \{2z\} \to \{z, z\}, \ldots \quad (\text{where } z \text{ is a power of } 3),$$

$$\gamma_{2u}^{-1}: \{u, u, u, u\} \to \{2u, 2u\}, \ldots \quad (\text{where } u \text{ is a power of } 3),$$

$$\gamma_v^{-1}: \{3v\} \to \{v, v, v\}, \ldots \quad (\text{where } v \text{ is of the form } 4 \cdot 3^k).$$

Now we take advantage of the ‘relative termination property’ (see Comment 2.2(a)) to distill the system $\Gamma_{(20)-(22)}$ by discarding all rules of the form (21).

Assume that

$$K_0 \xrightarrow{\gamma_1} K_1 \xrightarrow{\gamma_2} K_2 \xrightarrow{\gamma_3} K_3 \xrightarrow{\gamma_4} \cdots$$

is a sequence of $\Gamma_{(20)-(22)}$-reductions that started from a multiset $M$ from $\mathcal{C}_1$.

Had two occurrences of a part of the form $2 \cdot 3^k$, for instance 2, happened within some $K_j$, it would have produced an infinite loop like:

$$\{2, 2, \ldots \} \xrightarrow{\gamma_2} \{1, 1, 1, 1, \ldots \} \xrightarrow{\gamma_1} \{1, 1, 2, \ldots \} \xrightarrow{\gamma_1} \{2, 2, \ldots \} \xrightarrow{\gamma_2} \ldots$$

which contradicts to Comment 2.2(a).

The effect is that the ‘truncated’ $\Gamma$ consisting only of the rules:

$$\gamma_z: \{z, z\} \to \{2z\}, \ldots \quad (\text{where } z \text{ is a power of } 3),$$

$$\gamma_v: \{v, v, v\} \to \{3v\}, \ldots \quad (\text{where } v \text{ is of the form } 4 \cdot 3^k).$$

will produce the same bijection $h$ between $\mathcal{C}_1$ and $\mathcal{C}_2$.

Take the following partial function $\tau$

$$\tau(z) = \begin{cases} 1 & \text{if } z \text{ is a power of } 3, \\ 2 & \text{if } z \text{ is of the form } 4 \cdot 3^k, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Then $\mathcal{X}_\tau$, the rewriting map introduced by Definition 2.4 for this $\tau$, computes this bijection $h$. 
Similarly, the \( h^{-1} \) will be computed by \( \mathcal{H}_{\tau}^{x \in Y} \) (see Definition 2.5) based on \( \Gamma_{\tau}^{-1} \) consisting only of the reverse rules:

- \( \gamma_{\tau}^{-1}: \{2z\} \rightarrow \{z, z\}, \ldots \), (where \( z \) is a power of 3),
- \( \gamma_{v}^{-1}: \{3v\} \rightarrow \{v, v, v\}, \ldots \), (where \( v \) is of the form \( 4 \cdot 3^k \)).

Because of the good correlations between \( \mathcal{H}_{\tau} \) and \( \mathcal{E}_{\tau} \) (see Theorem 3.1), we can conclude that the above bijection \( h \) between \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) turns out to be Euler–Glaisher’s:

1. For any \( M \) from \( \mathcal{C}_1 \), \( \mathcal{E}_{\tau}(M) = h(M) \).
2. For any \( M' \) from \( \mathcal{C}_2 \), \( \mathcal{E}_{\tau}^{-1}(M') = h^{-1}(M') \).

5. Concluding remarks

In this paper we have given a global view on the Glaisher-type bijections and related rewriting maps. The novelty of our approach to the combinatorics of partitions is in the use of two-way rewriting techniques [10] for sorting out the concrete problem that arises in combinatorics—that of why Euler–Glaisher’s map is so ubiquitous and is recovered within so many partition identities.

As compared with the traditional approach to bijective proofs of partition identities:

Given a pair of equinumerous \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \), find an explicit bijection \( h \) between \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \).

the framework of the paper is quite the opposite:

Given a nice mapping, say \( \mathcal{E} \), find a class, as wide as possible, of pairs of \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) such that one and the same \( \mathcal{E} \) is good, as a bijection, for all these pairs.

As for Euler–Glaisher’s classical mapping \( \mathcal{G}_m \), we have obtained a quite unexpected result—that one and the same \( \mathcal{G}_m \) incorporates bijections for all Euler-type pairs:

**Corollary 5.1.** Given an integer \( m \geq 2 \), let \( \mathcal{C}_1 \) consist of all partitions into parts taken from some set \( S_1 \), and let \( \mathcal{C}_2 \) consist of all partitions into parts taken from some set \( S_2 \) in which each part may occur at most \( m - 1 \) times.

If \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) happen to be equinumerous, then there is a bijection \( h \) between \( \mathcal{C}_1 \) and \( \mathcal{C}_2 \) such that, for any \( M \) from \( \mathcal{C}_1 \):

- \( \mathcal{G}_m(M) = h(M) \).

As for the explicit inverse bijection \( h^{-1} \) from \( \mathcal{C}_2 \) onto \( \mathcal{C}_1 \),

(i) For \( S_1 \) containing no numbers divisible by \( m \), the \( h^{-1} \) is computed by Glaisher’s original inverse map \( \mathcal{G}_m^{-1} \).
(ii) In any case, the \( h^{-1} \) can be computed with the help of the inverse procedure given in Corollary 3.1 adjusted to the particular \( S_1 \).

**Proof.** It follows from Corollaries 3.1 and 3.3 by taking \( \tau(z) = m - 1 \), for all \( z \).

To generalize the Euler–Glaisher’s construction exactly along the number theoretic lines of the original one, we take advantage of a transparent description of any partition ideal of order 1 in terms of a function \( \tau \) controlling the number of repeated parts within its partitions.

1. Euler’s original map invokes the binary expansions of positive integers to handle the partitions into distinct parts.
2. The step made by Glaisher [6] is to use the base \( m \) expansions when the number of repeated parts is bounded by \( m - 1 \).
3. The next natural step is to invoke the mixed radix systems where the numerical base varies from position to position according to values of \( \tau \). Thus, a generalized Euler–Glaisher’s map \( \mathcal{E}_{\tau} \) has been defined by simply substituting...
the corresponding mixed radix system for the numeral system with base \( m \) in Glaisher’s original definition (see Definition 3.2).

The evolution of the Euler–Glaisher’s construction here can be depicted as:

\[
\begin{align*}
\text{‘No repetitions’} & \quad \rightarrow \quad \text{base} = 2 \\
\downarrow & \\
\text{‘}d\text{ repetitions’} & \quad \rightarrow \quad \text{base} = d + 1 \\
\downarrow & \\
\text{‘}\tau(z)\text{ repetitions of } z\text{’} & \quad \rightarrow \quad \text{variable base} = \tau(s^{(j)}) + 1
\end{align*}
\]

Our \( \mathcal{E}_\tau \) incorporate bijections for any two equinumerous partition ideals of order 1, whenever one of the ideals consists of partitions into parts from a set.

We have proved the universal nature of the above construction in the partitions world (see Corollaries 3.1 and 3.3):

\[
\text{Our } \mathcal{E}_\tau \text{ incorporate bijections for any two equinumerous partition ideals of order 1, whenever one of the ideals consists of partitions into parts from a set.}
\]

In closing, these mappings \( \mathcal{E}_\tau \) turn out to be exponentially advantageous from the computational point of view, as compared with other bijective proofs (see Sections 3.2 and 3.3). □

6. Uncited references

[3,11,4,5,13,7,17].

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References