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Fixed point theorems for multi-valued contractions in complete metric spaces

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ABSTRACT

In this paper the concept of a contraction for multi-valued mappings in a metric space is introduced and the existence theorems for fixed points of such contractions in a complete metric space are proved. Presented results generalize and improve the recent results of Y. Feng, S. Liu [Y. Feng, S. Liu, Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings, J. Math. Anal. Appl. 317 (2006) 103–112], D. Klim, D. Wardowski [D. Klim, D. Wardowski, Fixed point theorems for set-valued contractions in complete metric spaces, J. Math. Anal. Appl. 334 (2007) 132–139] and several others. The method used in the proofs of our results is new and is simpler than methods used in the corresponding papers. Two examples are given to show that our results are genuine generalization of the results of Feng and Liu and Klim and Wardowski.

1. Introduction

Let (X,d) be a metric space and let $Cl(X)$, $CB(X)$ and $Comp(X)$ denote a collection of all non-empty closed, all non-empty closed and bounded and all non-empty compact subsets of X, respectively. Let $D(x,A)$ denote the distance from x to A and H the Hausdorff metric induced by d.

The Nadler's [8] fixed point theorem for multi-valued contractive mappings has been extended in many directions (cf. [1–12]). The following generalization of Nadler's result is given by Mizoguchi and Takahashi [7].

Theorem 1. *(See [7].) Let (X,d) be a complete metric space and let $T: X \to CB(X)$. If there exists a function $\varphi: (0,\infty) \to [0,1)$ such that*

$$\limsup_{r\to t+} \varphi(r) < 1 \quad \text{for each } t \in [0,\infty), \tag{1}$$

and if

$$H\big(T(x),T(y)\big) \leqslant \varphi\big(d(x,y)\big)d(x,y) \tag{2}$$

for all $x, y \in X$, then T has a fixed point.

An alternative proof of this theorem was given by Daffer and Kaneko [4, Theorem 2.1].

Recently some interesting results have been obtained by Feng and Liu [5]. They proved the following theorem.

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Theorem 2. *(See [5, Theorem 3.1].) Let (X, d) be a complete metric space and let $T : X \to Cl(X)$. If there exist constants $b, c \in (0, 1)$, $c < b$, such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions*:

$$bd(x, y) \leqslant D\big(x, T(x)\big) \tag{3}$$

and

$$D\big(y, T(y)\big) \leqslant cd(x, y), \tag{4}$$

then T has a fixed point in X provided a function $f(x) = D(x, T(x))$ is lower semi-continuous.

Very recently Klim and Wardowski [6] generalized Theorem 3.1 of Feng and Liu [5]. They proved the following two theorems.

Theorem 3. *(See [6, Theorem 2.1].) Let (X, d) be a complete metric space and let $T : X \to Cl(X)$. Assume that the following conditions hold*:

(i) *the map $f : X \to R$, defined by $f(x) = D(x, T(x))$, $x \in X$, is lower semi-continuous*;
(ii) *there exist a constant $b \in (0, 1)$ and a function $\varphi : [0, \infty) \to [0, b)$ satisfying*

$$\limsup_{r \to t+} \varphi(r) < b \quad \text{for each } t \in [0, \infty), \tag{5}$$

and for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:

$$bd(x, y) \leqslant D\big(x, T(x)\big) \tag{6}$$

and

$$D\big(y, T(y)\big) \leqslant \varphi\big(d(x, y)\big)d(x, y). \tag{7}$$

Then T has a fixed point.

Theorem 4. *(See [6, Theorem 2.2].) Let (X, d) be a complete metric space and let $T : X \to Comp(X)$. Assume that the following conditions hold*:

(i) *the map $f : X \to R$, defined by $f(x) = D(x, T(x))$, $x \in X$, is lower semi-continuous*;
(ii) *there exists $\varphi : [0, \infty) \to [0, 1)$ satisfying the condition*

$$\limsup_{r \to t+} \varphi(r) < 1 \quad \text{for each } t \in [0, \infty), \tag{8}$$

and such that for any $x \in X$ there is $y \in T(x)$ satisfying the condition

$$d(x, y) = D\big(x, T(x)\big) \tag{9}$$

and the condition (7).

Then T has a fixed point.

Theorem 3 generalizes Theorem 2, but not Theorem 1 of Mizoguchi and Takahashi [7], since the function φ in Theorem 3 need to satisfy the condition (5), which is stronger than (1), as $b < 1$. Also Theorem 4 does not generalize Theorem 1, as $T(x)$ in Theorem 4 need to be compact.

The aim of this paper is to present more general results which unify and generalize the corresponding results of Mizoguchi and Takahashi [7], Feng and Liu [5] and Klim and Wardowski [6]. Two examples are given to show that our results are genuine generalizations.

The method used in the proofs of our results is new and seems that is simpler than corresponding methods used by the cited authors.

2. Preliminaries

Let (X, d) be a metric space, $CB(X)$ a collection of all non-empty closed and bounded subsets of X and H the Hausdorff metric induced by d. Thus, for $A, B \in CB(X)$,

$$H(A, B) = \max\Big\{\sup_{x \in B} D(x, A), \sup_{x \in A} D(x, B)\Big\},$$

where $D(x, A) = \inf_{y \in A} d(x, y)$.

Definition 1. A function $f : X \to R$ is called lower semi-continuous, if for any $\{x_n\} \subseteq X$ and $x \in X$,

$$x_n \to x \quad \text{implies} \quad f(x) \leqslant \lim_{n\to\infty} \inf f(x_n).$$

Definition 2. Let X be a non-empty set. An element $x \in X$ is said to be a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in T(x)$. If $Tx = \{x\}$, then x is called a stationary point (or a strict fixed point) of T.

Definition 3. Let (X, d) be a metric space. A subset K is called *proximinal* if for each $x \in X$, there exists an element $k \in K$ such that

$$d(x,k) = D(x, K) = \inf\{d(x, y): \ y \in K\}.$$

It is well known that every closed convex subset of a uniformly convex Banach space is proximinal.

3. Main results

Now we shall prove a theorem which generalizes Theorem 1 of Mizoguchi and Takahashi [7].

Theorem 5. *Let (X, d) be a complete metric space and $T : X \to Cl(X)$ be a mapping of X into itself. If there exists a function $\varphi : [0,\infty) \to [0, 1)$ satisfying*

$$\lim_{r\to t+} \sup \varphi(r) < 1 \quad \textit{for each } t \in [0,\infty) \tag{10}$$

and is such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:

$$d(x, y) \leqslant \bigl(2 - \varphi\bigl(d(x, y)\bigr)\bigr) D\bigl(x, T(x)\bigr) \tag{11}$$

and

$$D\bigl(y, T(y)\bigr) \leqslant \varphi\bigl(d(x, y)\bigr) d(x, y), \tag{12}$$

then T has a fixed point in X provided $f(x) = D(x, T(x))$ is lower semi-continuous.

Proof. Since $\varphi(d(x, y)) < 1$ for all $x, y \in X$, it follows that $2 - \varphi(d(x, y)) > 1$ for all $x, y \in X$. Thus for any $x \in X$ there exists $y \in T(x)$ such that (11) holds.

Let $x_0 \in X$ be any initial point. Then there exists $x_1 \in X$ such that $x_1 \in T(x_0)$ and

$$d(x_0, x_1) \leqslant \bigl(2 - \varphi\bigl(d(x_0, x_1)\bigr)\bigr) D\bigl(x_0, T(x_0)\bigr). \tag{13}$$

Then from (12), with $x = x_0$ and $y = x_1$,

$$D\bigl(x_1, T(x_1)\bigr) \leqslant \varphi\bigl(d(x_0, x_1)\bigr) d(x_0, x_1). \tag{14}$$

From (13) and (14) we get

$$D\bigl(x_1, T(x_1)\bigr) \leqslant \varphi\bigl(d(x_0, x_1)\bigr)\bigl(2 - \varphi\bigl(d(x_0, x_1)\bigr)\bigr) D\bigl(x_0, T(x_0)\bigr). \tag{15}$$

Define a function $\psi : [0,\infty) \to [0, +\infty)$ by

$$\psi(t) = \varphi(t)\bigl(2 - \varphi(t)\bigr), \tag{16}$$

that is, by $\psi(t) = 1 - (1 - \varphi(t))^2$. Since $\varphi(t) < 1$ and $\lim_{r\to t+} \sup \varphi(r) < 1$ for each $t \in [0,\infty)$, it follows that

$$\psi(t) < 1 \tag{17}$$

and

$$\lim_{r\to t+} \sup \psi(r) < 1 \tag{18}$$

for each $t \in [0,\infty)$.

From (15) and (16),

$$D\bigl(x_1, T(x_1)\bigr) \leqslant \psi\bigl(d(x_0, x_1)\bigr) D\bigl(x_0, T(x_0)\bigr). \tag{19}$$

Now we choose $x_2 \in X$ such that $x_2 \in T(x_1)$ and

$$d(x_1, x_2) \leqslant \big(2 - \varphi\big(d(x_1, x_2)\big)\big) D\big(x_1, T(x_1)\big).$$

Then by (12) and (16) we get

$$D\big(x_2, T(x_2)\big) \leqslant \psi\big(d(x_1, x_2)\big) D\big(x_1, T(x_1)\big).$$

Continuing this process we can choose an iterative sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_{n+1} \in T(x_n)$,

$$d(x_n, x_{n+1}) \leqslant \big(2 - \varphi\big(d(x_n, x_{n+1})\big)\big) D\big(x_n, T(x_n)\big) \tag{20}$$

and

$$D\big(x_{n+1}, T(x_{n+1})\big) \leqslant \psi\big(d(x_n, x_{n+1})\big) D\big(x_n, T(x_n)\big), \quad n = 0, 1, 2, \ldots. \tag{21}$$

For simplicity denote $d_n = d(x_n, x_{n+1})$ and $D_n = D(x_n, T(x_n))$ for all $n \geqslant 0$. Then from (21),

$$D_{n+1} \leqslant \psi(d_n) D_n \quad \text{for all } n \geqslant 0. \tag{22}$$

If $D_n = D(x_n, T(x_n)) = 0$ for some n, then $x_n \in T(x_n)$, that is, x_n is a fixed point of T and the assertion of theorem is proved. So we shall assume that $D_n > 0$ for all $n \geqslant 0$.

From (22) and (17) we conclude that $\{D_n\}_{n=0}^{\infty}$ is a strictly decreasing sequence of non-negative reals. Therefore, there is some $\delta \geqslant 0$ such that

$$\lim_{n\to\infty} D_n = \delta. \tag{23}$$

Since $D(x_n, T(x_n)) \leqslant d(x_n, x_{n+1})$ for each $x_{n+1} \in T(x_n)$, and as $\varphi(t) < 1$ for all $t \geqslant 0$, from (20) we get

$$D_n \leqslant d_n < 2D_n. \tag{24}$$

Thus, the sequence $\{d_n\}_{n=0}^{\infty}$ is bounded and so there is some $d \geqslant \delta$ such that

$$\lim_{n\to\infty} \inf d_n = d. \tag{25}$$

Now we shall show that $d = \delta = 0$. Suppose, at first that $\delta = 0$. Then from (23) and (24) we have

$$\lim_{n\to\infty} d_n = 0.$$

Suppose now that $\delta > 0$. We shall show that $d = \delta$. Suppose, to the contrary, that $d > \delta$. Then $d - \delta > 0$ and so from (23) and (25) there is a positive integer n_0 such that

$$\delta \leqslant D_n \leqslant \delta + \frac{d-\delta}{4} \quad \text{for all } n \geqslant n_0 \tag{26}$$

and

$$d - \frac{d-\delta}{4} < d_n \quad \text{for all } n \geqslant n_0. \tag{27}$$

Then from (26), (27) and (20) we have

$$\delta + 3\frac{d-\delta}{4} = d - \frac{d-\delta}{4} < d_n \leqslant \big(2 - \varphi(d_n)\big) D_n \leqslant \big(2 - \varphi(d_n)\big)\left(\delta + \frac{d-\delta}{4}\right)$$

for all $n \geqslant n_0$. Hence we get

$$1 + \frac{2(d-\delta)}{3\delta + d} < 1 + \big(1 - \varphi(d_n)\big).$$

This inequality implies that

$$-\big(1 - \varphi(d_n)\big)^2 < -\left[\frac{2(d-\delta)}{3\delta + d}\right]^2.$$

Thus,

$$\psi(d_n) = 1 - \big(1 - \varphi(d_n)\big)^2 < 1 - \left[\frac{2(d-\delta)}{3\delta + d}\right]^2 \quad \text{for all } n \geqslant n_0.$$

Now from (22),

$$D_{n+1} \leqslant hD_n \quad \text{for all } n \geqslant n_0, \tag{28}$$

where $h = 1 - [2(d-\delta)/(3\delta+d)]^2$. Then $h < 1$, as $d > \delta$. Since $\delta > 0$, there is a positive integer k such that

$$h^k\left(\delta + \frac{d-\delta}{4}\right) < \delta.$$

Then from (28) and (26) we have

$$\delta \leqslant D_{n_0+k} \leqslant hD_{n_0+k-1} \leqslant h^2 D_{n_0+k-2} \leqslant \cdots \leqslant h^k D_{n_0} \leqslant h^k\left(\delta + \frac{d-\delta}{4}\right) < \delta,$$

a contradiction. Therefore, our assumption $d > \delta$ is wrong. Thus $d = \delta$. Since $\delta \leqslant D_n \leqslant d_n$, it follows that $\lim_{n\to\infty} \inf d_n = \delta+$. Hence we conclude that there exists a subsequence $\{d_{n_k}\}_{k=0}^{\infty}$ of $\{d_n\}$ such that

$$\lim_{k\to\infty} d_{n_k} = \delta+.$$

Then by (18),

$$\lim_{d_{n_k}\to\delta+} \sup \psi(d_{n_k}) < 1. \tag{29}$$

From (22) we have

$$D_{n_k+1} \leqslant \psi(d_{n_k}) D_{n_k}.$$

Thus by (23) we obtain

$$\delta = \lim_{k\to\infty} \sup D_{n_k+1} \leqslant \Big(\lim_{k\to+\infty} \sup \psi(d_{n_k})\Big)\Big(\lim_{k\to\infty} \sup D_{n_k}\Big) = \Big(\lim_{d_{n_k}\to\delta+} \sup \psi(d_{n_k})\Big)\delta.$$

If we suppose that $\delta > 0$, then from this inequality we have

$$1 \leqslant \lim_{d_{n_k}\to\delta+} \sup \psi(d_{n_k}),$$

a contradiction with (29). Thus $\delta = 0$. Then from (23) and (24) we have

$$\lim_{n\to\infty} d_n = 0.$$

Now we shall show that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Let

$$\alpha = \lim_{d_n\to 0+} \sup \psi(d_n).$$

Then by (18), $\alpha < 1$. Let q be such that $\alpha < q < 1$. Then there is some $n_1 \in N$ such that $\psi(d_n) < q$ for all $n \geqslant n_1$. So from (22) we have $D_{n+1} \leqslant qD_n$ for all $n \geqslant n_1$. Then by induction we get

$$D_n \leqslant q^{n-n_1} D_{n_1} \tag{30}$$

for all $n \geqslant n_1 + 1$. From (30) and (24) we get

$$d(x_n, x_{n+1}) \leqslant 2q^{n-n_1} D_{n_1}. \tag{31}$$

Now by (31), for all $m > n \geqslant n_1 + 1$, we have

$$\sum_{k=n_1}^{m} d(x_k, x_{k+1}) \leqslant 2\sum_{k=n_1}^{m} q^{k-n_1} D_{n_1} \leqslant 2\frac{1}{1-q} D_{n_1}.$$

Hence we conclude, as $q < 1$, that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence.

Since X is complete, there is some $z \in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{32}$$

We now show that z is a fixed point of T. Since $f(x) = D(x, T(x))$ is lower semi-continuous and $D(x_n, T(x_n)) = D_n \to 0$ as $n \to \infty$, we have

$$0 \leqslant D\big(z, T(z)\big) = f(z) \leqslant \lim_{n\to\infty} \sup f(x_n) = \lim_{n\to\infty} \sup D\big(x_n, T(x_n)\big) = 0.$$

Hence $D(z, T(z)) = 0$. This implies that $z \in T(z)$, as $T(z)$ is closed. Thus we proved that z is a fixed point of T. $\square$

Remark 1. Theorem 5 is a generalization of Theorem 1 of Mizoguchi and Takahashi [7], as $D(y, T(y)) \leqslant H(T(x), T(y))$ for each $y \in T(x)$. It is easily to construct examples in which Theorem 5 can be applied, but not Theorem 1.

Now we shall prove a theorem which is a different from Theorem 5 and is a generalization of Theorem 1 of Mizoguchi and Takahashi [7], Theorem 2 of Feng and Liu [5] and Theorem 3 of Klim and Wardowski [6]. We shall present a proof which seem to be simpler than the proof in [6].

Theorem 6. *Let (X, d) be a complete metric space and $T : X \to Cl(X)$ be a mapping of X into itself. If there exist a function $\varphi : [0, \infty) \to (0, 1)$ and a non-decreasing function $b : [0, \infty) \to [b, 1)$, $b > 0$, such that*

$$\varphi(t) < b(t) \tag{33}$$

and

$$\limsup_{t \to r+} \varphi(t) < \limsup_{t \to r+} b(t) \tag{34}$$

for all $t \in [0, \infty)$, and for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions:

$$b\big(d(x, y)\big)d(x, y) \leqslant D\big(x, T(x)\big) \tag{35}$$

and

$$D\big(y, T(y)\big) \leqslant \varphi\big(d(x, y)\big)d(x, y), \tag{36}$$

then T has a fixed point in X provided $f(x) = D(x, T(x))$ is lower semi-continuous.

Proof. Since $b(d(x, y)) < 1$ for all $x, y \in X$, it follows that for any $x \in X$ there exists some $y \in T(x)$ such that (35) holds. Let $x_0 \in X$ be arbitrary. Then we can choose $x_1 \in T(x_0)$ such that (35) and (36) hold, that is, such that

$$b\big(d(x_0, x_1)\big)d(x_0, x_1) \leqslant D\big(x_0, T(x_0)\big) \tag{37}$$

and

$$D\big(x_1, T(x_1)\big) \leqslant \varphi\big(d(x_0, x_1)\big)d(x_0, x_1). \tag{38}$$

From (37) and (38) we get

$$D\big(x_1, T(x_1)\big) \leqslant \frac{\varphi(d(x_0, x_1))}{b(d(x_0, x_1))} D\big(x_0, T(x_0)\big). \tag{39}$$

Define now a new function $\psi(t)$ on $[0, \infty)$ as follows:

$$\psi(t) = \frac{\varphi(t)}{b(t)} \quad \text{for all } t \in [0, \infty).$$

Then from (33) and (34),

$$\psi(t) < 1 \tag{40}$$

and

$$\limsup_{t \to r+} \psi(t) < 1 \tag{41}$$

for all $t \in [0, \infty)$. Thus from (39),

$$D\big(x_1, T(x_1)\big) \leqslant \psi\big(d(x_0, x_1)\big)D\big(x_0, T(x_0)\big).$$

Now we choose $x_2 \in X$ such that $x_2 \in T(x_1)$ and

$$b\big(d(x_1, x_2)\big)d(x_1, x_2) \leqslant D\big(x_1, T(x_1)\big)$$

and

$$D\big(x_2, T(x_2)\big) \leqslant \varphi\big(d(x_1, x_2)\big)d(x_1, x_2).$$

Then by definition of ψ we get

$$D\big(x_2, T(x_2)\big) \leqslant \psi\big(d(x_1, x_2)\big)D\big(x_1, T(x_1)\big).$$

Continuing this process and denoting $d_n = d(x_n, x_{n+1})$ and $D_n = D(x_n, T(x_n))$, we can choose an iterative sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_{n+1} \in T(x_n)$,

$$b(d_n)d_n \leqslant D_n \tag{42}$$

and

$$D_{n+1} \leqslant \varphi(d_n)d_n \tag{43}$$

for all $n \geqslant 0$. From (42) and (43) we have

$$D_{n+1} \leqslant \psi(d_n)D_n. \tag{44}$$

Again from (42) with $n = n+1$ and from (43),

$$d_{n+1} \leqslant \frac{\varphi(d_n)}{b(d_{n+1})}d_n. \tag{45}$$

If $D_n = 0$ for some n, then x_n is a fixed point of T and so we finished the proof. Thus we shall consider the case $D_n > 0$ for all $n \geqslant 0$. From (44) and (40) we have

$$D_{n+1} < D_n \quad \text{for all } n \geqslant 0.$$

Now we shall show that

$$d_{n+1} < d_n \quad \text{for all } n \geqslant 0. \tag{46}$$

Suppose, to the contrary, that $d_n \leqslant d_{n+1}$. Then $b(d_n) \leqslant b(d_{n+1})$, as $b(t)$ is a non-decreasing function. Now, by (45), we have

$$d_n \leqslant d_{n+1} \leqslant \frac{\varphi(d_n)}{b(d_{n+1})}d_n \leqslant \frac{\varphi(d_n)}{b(d_n)}d_n = \psi(d_n)d_n < d_n,$$

a contradiction. Thus we proved (46).

Since $\{D_n\}$ and $\{d_n\}$ are monotone, there exist $\delta \geqslant 0$ and $d \geqslant 0$ such that

$$\lim_{n\to\infty} D_n = \delta, \qquad \lim_{n\to\infty} d_n = d+.$$

Then from (44) we get

$$\delta \leqslant \Bigl(\lim_{n\to\infty} \sup \psi(d_n)\Bigr)\delta = \Bigl(\lim_{d_n\to d+} \sup \psi(d_n)\Bigr)\delta.$$

Hence by (41) we conclude that $\delta = 0$. Since $0 < b \leqslant b(t)$, from (42) we get $bd_n \leqslant b(d_n)d_n \leqslant D_n$ and hence

$$d_n \leqslant \frac{1}{b}D_n. \tag{47}$$

Since $\lim_{n\to\infty} D_n = 0$, we get

$$\lim_{n\to\infty} d_n = 0.$$

Let

$$\alpha = \lim_{d_n\to 0+} \sup \psi(d_n).$$

Then by (41), $\alpha < 1$. Let q be such that $\alpha < q < 1$. Then there is some $n_0 \in N$ such that $\psi(d_n) < q$ for all $n \geqslant n_0$. Thus from (44),

$$D_n \leqslant q^{n-n_0} D_{n_0} \quad \text{for each } n \geqslant n_0.$$

Then from (47),

$$d_n \leqslant \frac{1}{b}q^{n-n_0} D_{n_0} \quad \text{for each } n \geqslant n_0.$$

Proceeding as in the proof of Theorem 5 one can prove that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence and that its limit point is a fixed point of T. $\square$

Now we shall formulate a theorem, which is a generalization of Theorem 4 of Klim and Wardowski [6].

Theorem 7. *Let (X, d) be a complete metric space and T be a multi-valued mapping of X into a collection of all non-empty proximinal subsets of X. If there exists a function $\varphi : [0, \infty) \to [0, 1)$ satisfying (8) and such that for any $x \in X$ there is $y \in T(x)$ satisfying the following two conditions*:

$$d(x, y) = D\big(x, T(x)\big)$$

and

$$D\big(y, T(y)\big) \leqslant \varphi\big(d(x, y)\big)d(x, y),$$

then T has a fixed point in X provided $f(x) = D(x, T(x))$ is lower semi-continuous.

Remark 2. Theorem 6 is a genuine generalization of Theorem 3 of Klim and Wardowski [6]. Indeed, if in Theorem 6, $b(t) = b$-const. and $\varphi_1(t) = b\varphi(t)$, then $\varphi_1(t)$ satisfies (5) and (7) and therefore all hypotheses of Theorem 3 are satisfied. In the next section we shall construct and discuss two examples which show that Theorem 6 is strict generalization of Theorem 3.

Remark 3. In the next section we shall construct an example which shows that Theorem 7 is a strict generalization of Theorem 4.

4. Comparisons and examples

In this section we shall construct and discuss two examples which show that our results are genuine generalization of the results of Mizoguchi and Takahashi [7], Feng and Liu [5] and Klim and Wardowski [6].

The following example shows that there are mappings which satisfy all hypotheses in Theorem 5, but not in Theorem 3, and therefore in Theorem 2.

Example 1. Let $X = [0, 1]$ and $d : X \times X \to R$ be a standard metric. Let $T : X \to Cl(X)$ be defined as in Example 3.1 of Klim and Wardowski [6]:

$$T(x) = \begin{cases} \{\frac{1}{2}x^2\} & \text{for } x \in [0, \frac{15}{32}) \cup (\frac{15}{32}, 1], \\ \{\frac{17}{96}, \frac{1}{4}\} & \text{for } x = \frac{15}{32}. \end{cases}$$

Define now $\varphi : [0, \infty) \to [0, 1)$ as follows:

$$\varphi(t) = \begin{cases} \frac{8}{5}t & \text{for } t \in [0, \frac{7}{24}) \cup (\frac{7}{24}, \frac{1}{2}), \\ \frac{5}{8} & \text{for } t = \frac{7}{24}, \\ \frac{4}{5} & \text{for } t \in [\frac{1}{2}, \infty). \end{cases}$$

We shall show that T satisfies all hypotheses of our Theorem 5. It is easy to see that a function $f(x) = D(x, T(x))$ is lower semi-continuous. Moreover, for each $x \in [0, 15/32) \cup (15/32, 1]$ we have $T(x) = \{(1/2)x^2\}$ and therefore $y = (1/2)x^2$, $d(x, y) = D(x, T(x)) = x - (1/2)x^2$. Further,

$$\begin{aligned} D\big(y, T(y)\big) &= d\left(\frac{1}{2}x^2, \frac{1}{8}x^4\right) = \frac{1}{2}\left(x^2 - \left(\frac{1}{2}x^2\right)^2\right) = \frac{1}{2}\left(x + \frac{1}{2}x^2\right)\left(x - \frac{1}{2}x^2\right) \\ &= \frac{1}{2}\left(x + \frac{1}{2}x^2\right)d(x, y) \leqslant \frac{8}{5}\left(x - \frac{1}{2}x^2\right)d(x, y) = \varphi\big(d(x, y)\big)d(x, y). \end{aligned}$$

Thus, for $x \in [0, 1]$, $x \neq 15/32$, T satisfies (11) and the contractive condition (12) in Theorem 5.

Let now $x = 15/32$. Then for $y = 17/96 \in T(x)$ we have

$$d(x, y) = \frac{7}{24} < \left(2 - \frac{5}{8}\right)\frac{7}{32} = \big[2 - \varphi\big(d(x, y)\big)\big]D\big(x, T(x)\big)$$

and

$$D\big(y, T(y)\big) = d\left(\frac{17}{96}, \frac{1}{2}\frac{17^2}{96^2}\right) < \frac{17}{96} < \frac{5}{8}\frac{7}{24} = \varphi\big(d(x, y)\big)d(x, y).$$

Thus, T satisfies (11) and (12) for $x = 15/32$. Therefore, all assumptions of our Theorem 5 are satisfied and $Fix(T) = \{0\}$.

Now we shall show that a given map T does not satisfy hypotheses of Theorem 3 of Klim and Wardowski [6].

Let $b \in (0, 3/4]$. Then for any $\varphi : [0, \infty) \to [0, b)$, $3/4 > \varphi(d(x, y))$. Thus for $x = 1$ we have $T(x) = \{1/2\}$, $y = 1/2$, $T(y) = \{1/8\}$, $d(x, y) = 1/2$, $D(y, T(y)) = 3/8$ and, consequently,

$$D\big(y, T(y)\big) = \frac{3}{4} \cdot \frac{1}{2} = \frac{3}{4} d(x, y) > \varphi\big(d(x, y)\big) d(x, y).$$

Therefore, for $x = 1$ the inequality (7) in Theorem 3 is not satisfied.

Let now $b \in (3/4, 1)$ and let $x = 15/32$. Then

$$T(x) = \left\{\frac{17}{96}, \frac{1}{4}\right\} \quad \text{and} \quad d\big(x, T(x)\big) = \frac{7}{32}.$$

Consider at first the case $y = 17/96$. Then $d(x, y) = 7/24$ and so we have, as $b > 3/4$,

$$bd(x, y) > \frac{3}{4} d(x, y) = \frac{3}{4} \cdot \frac{7}{24} = \frac{7}{32} = d\big(x, T(x)\big).$$

Therefore, for $y = 17/96$ the inequality (6) is not satisfied.

Let now $y = 1/4$. Then $T(y) = \{1/32\}$, $d(x, y) = 7/32$, $D(y, T(y)) = 7/32$ and so we have, as for any $\varphi : [0, \infty) \to [0, b)$, $1 > b > \varphi(t)$,

$$D\big(y, T(y)\big) = \frac{7}{32} = d(x, y) > bd(x, y) > \varphi\big(d(x, y)\big) d(x, y).$$

Therefore, for $y = 1/4$ the inequality (7) is not satisfied. Thus, for $x = 15/32$ there is not $y \in T(x)$ which satisfies (6) and (7). So we showed that there do not exist $b \in (0, 1)$ and $\varphi : [0, \infty) \to [0, b)$ such that the mapping T satisfies hypotheses of Theorem 3.

Now we shall present an example which shows that Theorem 7 is a genuine generalization of Theorem 4.

Example 2. Let $X = [0, +\infty)$. Define $T : X \to Cl(X)$ and $\varphi : [0, \infty) \to [0, 1)$ as follows:

$$T(x) = \left\{\frac{x}{a}\right\} \cup \big[(1 + 2x), +\infty\big),$$
$$\varphi(t) = \frac{1}{a},$$

where $a > 1$. Then

$$f(x) = D\big(x, T(x)\big) = x - \frac{x}{a} = \frac{a - 1}{a} x$$

and so $f(x)$ is continuous. Further, for each $x \in X$ there exists $y = x/a \in T(x)$ such that $d(x, y) = D(x, T(x))$. Thus we have

$$D\big(y, T(y)\big) = \frac{x}{a} - \frac{x}{a^2} = \frac{1}{a} \frac{a - 1}{a} x = \varphi\big(d\big(x, T(x)\big)\big) d(x, y).$$

Therefore, all assumptions of our Theorem 7 are satisfied and $Fix(T) = \{0\}$. Clearly, T does not satisfy the hypotheses in Theorem 4, since $T(x)$ is not compact for all $x \in X$.

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