Finite models of sketches

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Abstract

For every sketch with countable limit specifications and countable colimit specifications we prove that there exists a finitary sketch (i.e., one with finite limit and colimit specifications) with the same category of finite models. The sketch is even coherent, i.e., describable by the finitary first-order logic. Assuming the non-existence of measurable cardinals, we also prove that for every geometric sketch there exists a coherent sketch with the same category of finite models. The latter result is, in fact, equivalent to the assumption of non-existence of measurable cardinals. © 1997 Elsevier Science B.V.

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The present paper is devoted to the behavior of categories of models of a sketch in $\text{Set}_{\text{fin}}$, the category of finite sets and functions. Recall that a sketch $\mathcal{S} = (\mathcal{A}, L, C, \sigma)$ is a small category $\mathcal{A}$ together with a choice of "limit" diagrams $L$ and "colimit" diagrams $C$ and a map $\sigma$ assigning to each diagram $D$ in $L$ (or $C$) a cone (or cocone, respectively) $\sigma(D)$. A model of $\mathcal{S}$ in $\text{Set}_{\text{fin}}$ is a functor $F: \mathcal{A} \to \text{Set}_{\text{fin}}$ such that for each diagram $D$ in $L$ (or in $C$) the $F$-image of $\sigma(D)$ is a (co-)limit of the diagram $F \cdot D$. We denote by $\text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}})$ the category of all models and all natural transformations.

Example 1. For each set $I$ let $\mathcal{S}_I = (\mathcal{A}, L, C, \sigma)$ be the following sketch: let $\mathcal{A}$ consist of a single cone $(A \xrightarrow{a_i} A_i)_{i \in I}$, let $L$ consist of the single discrete diagram $\{A_i\}$ to
which \( \sigma \) assigns the given cone, and let \( C = \emptyset \). Then a model \( F \) consists of a finite set \( FA \) with \( FA = \prod_{i \in I} FA_i \). Thus, \( FA_i \) are sets such that

\[
(*) \quad \text{either all but finitely many are singleton sets, or some of them are empty.}
\]

Therefore, \( \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \) can be considered as the category of all \( I \)-tuples of finite sets satisfying \((*)\) (with \( I \)-tuples of functions as morphisms). We will see below that if \( I \) is countable, then this category can also be sketched by a sketch with finite limit and colimit specifications, but for \( I \) uncountable this is not the case.

**Example 2.** A sketch \( \mathcal{S}_\omega \) whose category of finite models is equivalent to \( \omega \) (linearly ordered). It consists of:

1. objects \( N \) and \( N^2 \) and morphisms
   \[ \pi_1, \pi_2 : N^2 \to N \quad \text{and} \quad \Delta : N \to N^2 \]

   specified to form a product and its diagonal, respectively;

2. an object \( 1 \), specified to be terminal, and a morphism \( \iota : 1 \to N \) [the first element];

3. a morphism \( r : R \to N^2 \) specified to be a monomorphism [the next-element relation], such that also \( \pi_1 r \) and \( \pi_2 r \) are specified to be monomorphisms;

4. a coproduct specification

\[
\begin{array}{c}
1 \\
\leftarrow \end{array}
\]

\[
\begin{array}{c}
R \\
\downarrow \pi_2 r \\
N \\
\downarrow \iota \\
1 \\
\end{array}
\]

5. morphisms \( r_n : R_n \to N^2, n \geq 1 \), specified to be monomorphisms which represent the composite of \( n \) copies of the relation \( R \) (composition of relations can obviously be sketched by means of pullback specifications and epi-specifications);

6. an object \( 0 \) specified to be initial;

7. a pullback specification for each \( n \geq 1 \):

\[
\begin{array}{c}
0 \\
\downarrow \pi_1 r \\
R_n \\
\downarrow \Delta \\
N \\
\end{array}
\]

A model \( F \) of \( \mathcal{S}_\omega \) consists of a finite set \( FN \) on which \( FR \) is a binary relation whose projections are injective, thus, \( FR \) consists of cycles and finite paths. Due to (5)–(7) there are no cycles. And due to (4) the path is unique. Thus, if \( FN \) has \( n \) elements
then $F \cong F_n$ where

$F_nN = \{0, 1, \ldots, n - 1\}$,

$F_nR = \{(i - 1, i); i = 1, \ldots, n - 1\}$,

$F_n1 = 1$ and $F_n\iota$ represents 0,

$F_nR_m = (F_nR) \circ \cdots \circ (F_nR)$.

It is clear that every morphism $f : F_n \to F_m$ preserves the first element and the next-element relation, thus, $n \leq m$ and $f$ is the inclusion map. Consequently,

$$\text{Mod} (\mathcal{S}_\omega, \text{Set}_\text{fin}) \cong \omega.$$ 

Following [3] we call a sketch finitary if it has only finite limit and finite colimit diagrams. If, moreover, the only colimit diagrams are either discrete, i.e., finite-coprocess specifications, or epi-specifications of some maps (via pushouts) the sketch is called coherent. Example: the above sketch $\mathcal{S}_\omega$ is coherent. Coherent sketches are precisely those describable in the first-order finitary logic, see [6].

More in general, $\lambda$-ary sketches are those where each diagrams in $\mathcal{L} \cup \mathcal{Z}$ has less than $\lambda$ morphisms. Recall also that geometric sketches are sketches with all limit diagrams finite. In [3] we have studied categories $\text{Mod} (\mathcal{S}, \text{Set})$ of models in $\text{Set}$ and we showed that if measurable cardinals exist, then some geometric sketches $\mathcal{S}$ have the property that $\text{Mod} (\mathcal{S}, \text{Set})$ cannot be sketched by a finitary sketch. In [1] we then proved that, conversely, if no measurable cardinals exist, then for each geometric sketch $\mathcal{S}$ there exists a finitary sketch $\mathcal{S}'$ such that $\text{Mod} (\mathcal{S}, \text{Set})$ and $\text{Mod} (\mathcal{S}', \text{Set})$ are equivalent categories. We will now prove the analogous result for models in $\text{Set}_\text{fin}$. But we will also prove that every $\omega_1$-ary sketch $\mathcal{S}$ has the property that $\text{Mod} (\mathcal{S}, \text{Set}_\text{fin})$ can be sketched by a finitary sketch. Let us remark that this has no analogy over $\text{Set}$: let $\mathcal{K}$ be the category of all lattices with countable joins and all homomorphisms preserving countable joins. This category is $\aleph_1$-accessible and therefore equivalent to $\text{Mod} (\mathcal{S}, \text{Set})$ for some countable-limit sketch $\mathcal{S}$. However, as proved by R. Paré, $\mathcal{K}$ cannot be sketched by a geometric sketch – see [4] for details.

**Definition.** Two sketches $\mathcal{S}$ and $\mathcal{S}'$ are said to be equivalent over $\text{Set}_\text{fin}$ provided that their categories of models, $\text{Mod} (\mathcal{S}, \text{Set}_\text{fin})$ and $\text{Mod} (\mathcal{S}', \text{Set}_\text{fin})$, are equivalent.

The purpose of our paper is to prove the following.

**Theorem.** (1) Every $\omega_1$-ary sketch is equivalent over $\text{Set}_\text{fin}$ to a coherent sketch.

(2) Assuming the non-existence of measurable cardinals, every geometric sketch is equivalent over $\text{Set}_\text{fin}$ to a coherent sketch.

**Remark 1.** We will prove a more general statement: for every cardinal $\lambda$ smaller than any measurable cardinal, each sketch with countable limit specifications and with
colimit specifications smaller or equal to $\lambda$ is equivalent to a coherent sketch over $\text{Set}_{\text{fin}}$.

During the International Conference on Category Theory CT 95 in Halifax, we announced that every sketch is equivalent to a finitary one over $\text{Set}_{\text{fin}}$. This was, unfortunately, wrong, as the following two counterexamples show:

**Example 3.** If $I$ is an uncountable set, then for the sketch $\mathcal{S}_1$ of Example 1 the category $\text{Mod}(\mathcal{S}_1, \text{Set}_{\text{fin}})$ cannot be sketched by a finitary sketch. In fact:

1. The category $\text{Mod}(\mathcal{S}_1, \text{Set}_{\text{fin}})$ does not have $\omega_1$-directed colimits. Consider the diagram consisting of the following models:

   $$F_M = (A_M, i)_{i \in I} \text{ for all } M \subseteq I \text{ countable},$$

where

$$A_M, i = \begin{cases} \{0, 1\} & \text{if } i \in M, \\ \emptyset & \text{else}, \end{cases}$$

and of the obvious morphisms $F_M \to F_{M'}$ for all $M \subseteq M' \subseteq I$. This diagram does not have a colimit in $\text{Mod}(\mathcal{S}_1, \text{Set}_{\text{fin}})$. In fact, suppose that, to the contrary, a colimit with a codomain $F^* = (A^*_i)$ exists. Then, obviously, $A^*_i \neq \emptyset$ for each $i$; consequently, there exists $i_0 \in I$ such that $A^*_{i_0}$ is a singleton set. Now consider the cocone $f_M : F_M \to F$ where $F = (A_i)$ with $A_{i_0} = \{0, 1\}$ and $A_i$ singleton set for all $i \neq i_0$, and the $i_0$-component of $f_M$ is $\text{id}_{\{0, 1\}}$ for all $M$ containing $i_0$. The latter cocone does not factorize through the purported colimit cocone.

2. Every category $\text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}})$, where $\mathcal{S}$ is finitary, has $\omega_1$-directed colimits. In fact, since $\text{Set}_{\text{fin}}$ is closed in $\text{Set}$ under $\omega_1$-directed colimits, and since $\omega_1$-directed colimits commute with finite limits (and finite colimits) in $\text{Set}$, the category $\text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}})$ is closed in $\text{Set}^{\mathcal{S}}$ under $\omega_1$-directed colimits.

**Example 4.** If $\lambda$ is a measurable cardinal, then its dual $\lambda^{\text{op}}$ (considered as a category) can be sketched over $\text{Set}_{\text{fin}}$ by a geometric sketch, but not by a finitary one.

In fact, let $\mathcal{S}$ be the sketch whose underlying category is $\lambda \cup \{T\}$, where $T$ is a largest element, whose limit specifications are as follows:

1. $T$ be a terminal object,
2. every morphism be a monomorphism (pullback specification)
3. the only colimit specification is $\text{colimi}_{\leq i} i = T$. 

For each $i \in \lambda$ we have a model $M_i$ given by

$$M_i(j) = \emptyset \text{ if } j < i, \quad M_i(j) = 1 \text{ if } j \geq i.$$ 

These are, up to isomorphism, all models of $\mathcal{S}$ in $\text{Set}_{\text{fin}}$, thus $\text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}})$ is equivalent to $\lambda^{\text{op}}$. 
We now prove that $\lambda^{op}$ cannot be sketched by a finitary sketch over $\text{Set}_{\text{fin}}$. If it could, it could also be axiomatized by a first-order theory $S$ in the logic $L_{\omega_1, \omega_1}$ allowing conjunctions and disjunctions over countably many formulas and quantification over countably many variables. (This is proved in [6] for models over $\text{Set}$, but this also holds for models over $\text{Set}_{\text{fin}}$, the proof is quite analogous.) In other words, the category $\text{Mod}_{\text{fin}}(S)$ of finite models of $S$ and homomorphisms, is equivalent to $\lambda^{op}$. We will prove that this cannot happen. Let $A_i (i \in \lambda)$ be finite models of $S$ such that they form skeleton of $\text{Mod}_{\text{fin}}(S)$, and let $a_{ij} : A_i \to A_j$ be the corresponding homomorphisms for $i \geq j$. Observe that if $i > j$, then $a_{ij}$ is not an elementary embedding: if it were, the number of elements of $A_i$ and $A_j$ in every sort would be the same (since this can be expressed by a formula in $L_{\omega_1, \omega_1}$) and thus $a_{ij}$ would be a bijection in every sort, from which the properties of elementary embeddings would imply that $a_{ij}$ is an isomorphism, but $\text{hom}(A_j, A_i) = \emptyset$. Further observe that $\text{Mod}_{\text{fin}}(S)$ is closed under $\mathcal{U}$-ultraproducts for all $\mathcal{U}$-complete ultrafilters $\mathcal{U}$: this is well known for models of theories of $L_{\omega_1, \omega_1}$ in $\text{Set}$, but since finiteness (in each sort) is a formula of $L_{\omega_1, \omega_1}$, it also holds for $\text{Mod}_{\text{fin}}(S)$. The cardinal $\lambda$ is measurable, thus, a $\lambda$-complete ultrafilter $\mathcal{U}$ containing only sets of cardinality $\lambda$ exists on the set $\lambda$. The ultraproduct $\prod_{\mathcal{U}} A_i$ is then isomorphic to $A_j$ for some $j \in \lambda$. We derive a contradiction by showing that $j \geq k$ for each $k \in \lambda$. The ultrapower $A_k^\mathcal{U}$ is isomorphic to $A_{k'}$ for some $k'$, and the existence of an elementary embedding of $A_k$ into $A_k^\mathcal{U}$ guarantees that $k' = k$. The set $\lambda - k = \{i \in \lambda; k \leq i\}$ belongs to $\mathcal{U}$ and for each $i \in \lambda - k$ we have a homomorphism $A_i \to A_k$ — thus there is a homomorphism from $\prod_{\mathcal{U}} A_i$ to $A_k^\mathcal{U} \cong A_k$, in other words, a homomorphism from $A_j$ to $A_k$, and this proves $j \geq k$, as required.

Remark 2. The following proof of the main theorem is divided into two parts:

(1) We prove (absolutely) that every $\lambda$-ary sketch with countable limit specifications is equivalent to a $\lambda$-ary geometric sketch over $\text{Set}_{\text{fin}}$.

Let us remark that for “co-geometric” sketches, i.e., sketches with arbitrary limit specifications and finite colimit specifications, the corresponding result is also absolutely true: every sketch is equivalent to a co-geometric sketch over $\text{Set}_{\text{fin}}$. In fact, this result, for equivalence over $\text{Set}$, is proved in [3], and the proof extends to $\text{Set}_{\text{fin}}$. (In contrast, we repeat, it is not true that every sketch is equivalent to a geometric sketch over $\text{Set}$!)

Next, we prove that geometric sketches are equivalent to coherent ones over $\text{Set}_{\text{fin}}$, assuming the non-existence of measurable cardinals. We actually show more:

(2) We prove that every $\lambda$-ary geometric sketch is equivalent to a coherent sketch over $\text{Set}_{\text{fin}}$, provided that no cardinal smaller than $\lambda$ is measurable.

Remark 3 (Decomposition of chains into mono-chains and epi-chains in $\text{Set}_{\text{fin}}$). Let $a_{n,m} : A_n \to A_m (n \geq m)$ be an $\omega^{op}$-chain in $\text{Set}_{\text{fin}}$ which has a limit. Then the limit can be computed via limits of mono-chains and epi-chains as follows; the idea of the following decomposition is due to Reiterman (see [5]).
(i) Let

\[ A_n \xrightarrow{a_{n,m}} A_m \]

\[ \downarrow e_{n,m} \quad \downarrow i_{n,m} \]

\[ B_{n,m} \]

\[ (n \geq m) \]

be an epi-mono factorization (with \( B_{n,n} = A_n, e_{n,n} = i_{n,n} = id \)).

(ii) Given \( k \geq n \geq m \), then \( e_{k,m} \) factorizes uniquely through \( e_{k,n} \) (via a map \( b_{k,n,m} \)) and \( i_{k,m} \) factorizes uniquely through \( i_{n,m} \) (via \( b_{k,n,m}^* \)):

Observe that \( b_{n,n,m} = e_{n,m} \) and \( b_{k,k,m}^* = i_{k,m} \).

(iii) In the resulting diagram:
each of the mono-chains
\[ B_m \equiv \left( B_{k,m} \xrightarrow{b_{k,n}} B_{n,m} \right)_{k \geq n \geq m} \]
has a limit
\[ \lim B_m = \left( C_m \xrightarrow{b_{n,m}} B_{n,m} \right)_{n \geq m}. \]
(Since intersections exist in \text{Set}_{\text{fin}}.)

(iv) For any \( n > m \) there is a unique
\[ c_{n,m} : C_n \to C_m \]
with \( \hat{b}_{r,m} \cdot c_{n,m} = \hat{b}_{r,n} \) for all \( r \geq n \) and this chain \( (c_{n,m})_{n \geq m} \) is an epi-chain.

Proof. For each \( m \) we can describe \( C_m \) as the intersection of the images of all \( a_{n,m}, n \geq m \), and \( c_{n,m} \) is the domain-codomain restrictions of \( a_{n,m} \). We shall prove that each \( x \in C_m \) lies in \( a_{n,m}(C_n) \). Assuming the contrary, for every \( y \in a_{n,m}^{-1}(x) \) there exists a \( k(y) \) such that \( y \) does not lie in the image of \( a_{k(y),n} \). Let \( k = \max\{ k(y) ; y \in a_{n,m}^{-1}(x) \} \), then \( a_{k,n}^{-1}(x) \) is disjoint with the image of \( a_{k,n} \). This, however, is a contradiction with \( x \in C_m \); there exists \( z \in A_k \) with \( x = a_{k,m}(z) \), and then \( y = a_{k,n}(z) \) lies in \( a_{n,m}^{-1}(x) \).

(v) The chains
\[ (a_{n,m}) \quad \text{and} \quad (c_{n,m}) \]
have the same limit. More precisely, if the latter chain has a limit
\[ (\hat{b}_{m,m} \cdot c_m : C \to A_m \ (m \in \omega)) \]
then the cone of all \( \hat{b}_{m,m} \cdot c_m : C \to A_m \ (m \in \omega) \) is a limit of the former chain. Conversely, if
\[ (a_m : a_{m} : A_m \to A_m \ (m \in \omega)) \]
is a limit of the chain \( (a_{n,m}) \), then for each \( m \), \( a_m \) uniquely factors through \( \hat{b}_{m,m} \)
and if
\[ a_m = \hat{b}_{m,m} \cdot c_m, \]
then the cone of all \( c_m \) is a limit of \( (c_{n,m}) \). \( \square \)

**Proposition 1.** Let \( \lambda \) be an uncountable cardinal. Every sketch with countable limit specifications and \( \lambda \)-ary colimit specifications is equivalent over \text{Set}_{\text{fin}} to a \( \lambda \)-ary geometric sketch.

Proof. (I) We show first that it is sufficient to consider sketches in which any infinite limit specification is a specification of a limit of an \( \omega^{op} \)-chain.

Let \( \mathcal{S} \) be the given sketch with underlying category \( \mathcal{A} \). We can assume that \( L \) has a unique infinite diagram, say, \( D : \mathcal{D} \to \mathcal{A} \) (since in case of more infinite diagrams we proceed in the same manner, removing step by step infinite limit specifications). We can assume that \( \mathcal{D} \) has no non-trivial compositions, i.e., if two morphisms compose,
then one of them is the identity map. (In fact, if $\mathcal{D}$ does not have this property, then we substitute it with the category $\mathcal{D}'$ whose objects are $\mathcal{D}_{\text{obj}} \times \{1,2\}$ and whose morphisms are the identity maps plus $f: (d,1) \rightarrow (d',2)$ for any $f: d \rightarrow d'$ in $\mathcal{A}$. We then replace the diagram $D: \mathcal{D} \rightarrow \mathcal{A}$ with $D': \mathcal{D}' \rightarrow \mathcal{A}$, where $D'(d,i) = Dd$ and $D'f = Df$.) Let $\{d_n\}_{n \in \omega}$ be the set of all objects of $\mathcal{D}$, and let $\mathcal{D}_n$ be the full subcategory of $\mathcal{D}$ with

$$\mathcal{D}_n^{\text{obj}} = \{d_0, d_1, \ldots, d_n\}.$$ 

Since $\mathcal{D}$ has no non-trivial compositions, $\mathcal{D}_n$ is finite.

We modify the given sketch $\mathcal{S}$ to a sketch $\mathcal{S}' = (\mathcal{A}', L', \sigma')$ as follows:

(a) Delete $D$ from $L$ and insert the finite diagrams $D_n$, $n \in \omega$.

(b) Add to $\mathcal{A}$ new objects $C_n$ ($n \in \omega$) and new cones for $D_n$:

$$\sigma'(D_n) = (C_n \xrightarrow{\gamma_{n,k}} d_k \downarrow k \leq n).$$

(c) Add new morphisms

$$c_{n,m}: C_n \rightarrow C_m \quad \text{for all } n \geq m,$$

forming a factorization of $\sigma'(D_n)$ through $\sigma'(D_m)$, i.e.,

$$\gamma_{n,k} = \gamma_{m,k} \cdot c_{n,m} \quad \text{for all } n \geq m \geq k. \quad (1)$$

(d) Add the $\omega^{\text{op}}$-chain $\mathcal{C} = (c_{n,m})$ to $L$. Add to $\mathcal{A}$ new morphisms

$$c_n: A \rightarrow C_n \quad (n \in \omega),$$

where $A$ is the domain of $\sigma(D) = (A \xrightarrow{\gamma_{n,k}} d_k \downarrow k \leq n)$, subject to the commutativity condition

$$\gamma_{n,k} c_n = \alpha_k \quad \text{for all } n \geq k. \quad (2)$$

Put

$$\sigma'({\mathcal{C}}) = (A \xrightarrow{\gamma_{n,k}} C_n)_{n \in \omega}.$$ 

The sketches $\mathcal{S}$ and $\mathcal{S}'$ are equivalent over $\text{Set}_{\text{fin}}$. In fact, for each model $M \in \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}})$ we have a corresponding model $M' \in \text{Mod}(\mathcal{S}', \text{Set}_{\text{fin}})$ which assigns to $\sigma'(D_n)$ a limit cone of $MD_n$, then $M'c_{n,m}$ are uniquely determined by (1) and consequently $M'C_n$ are uniquely determined by (2), and from the fact that $(MA \xrightarrow{\alpha_k} Md_k)$ is a limit of $MD$ it follows that $(MA \xrightarrow{\gamma_{n,k}} M'C_n)$ is a limit of $M'{\mathcal{C}}$. Thus, $M'$ is a model of $\mathcal{S}'$.

Next, any map $h: M_1 \rightarrow M_2$ in $\text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}})$ defines a unique map $h': M'_1 \rightarrow M'_2$ since $h'_{C_n}$ is uniquely determined by $M'_2 \gamma_{n,k} \cdot h'_{C_n} = h_{d_n} \cdot M'_1 \gamma_{n,k}$ for all $n \geq k$. In this way, we obtain a functor $\text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \rightarrow \text{Mod}(\mathcal{S}', \text{Set}_{\text{fin}})$, which is obviously full and faithful. To show that it is isomorphism-dense, i.e., an equivalence, observe that, conversely, every model $M' \in \text{Mod}(\mathcal{S}', \text{Set}_{\text{fin}})$ defines a model $M = M' / \mathcal{A}$ of $\mathcal{S}$, since (1) and (2) guarantee that $(MA \xrightarrow{\alpha_k} Md_k)$ is a limit of $MD$. 

Next we show that it is sufficient to consider sketches in which any infinite limit specification is either a limit of a mono-chain or of an epi-chain. More precisely, if \((a_n)_{n \in \omega}\) is an \(\omega^{op}\)-chain in \(L\), then either a mono-specification (via a pullback) of each \(a_n\) is a part of \(\mathcal{S}\), or an epi-specification (via a pushout) of each \(a_n\) is a part of \(\mathcal{S}\).

This follows from Remark 3. Suppose, again, that \(\mathcal{S}\) contains just one infinite limit diagram, and this is an \(\omega^{op}\)-chain (by (I), this is no loss of generality). Let \((a_{n,m} : A_n \to A_m)_{n \geq m}\) be that chain and let \((A \xrightarrow{\alpha} A_n)\) be its \(\sigma\)-cone. We modify \(\mathcal{S}\) as follows:

(a) Add to the underlying category of \(\mathcal{S}\) new objects \(B_{n,m}\) and \(C_{n,m}\) and new morphisms \(b_{k,n,m}, b_{k,n,m}^*, \hat{b}_{n,m}\) and \(c_{n,m}\) as in Remark 3.

(b) Add to \(L'_x\) pullbacks specifying that \(b_{k,n,m}^*\) be mono, and to \(C'_x\) pushouts specifying that \(c_{n,m}\) be epi.

(c) Add the chains \(B_m\) to \(L'_x\) and put

\[\sigma'(B_m) = (\hat{b}_{n,m})_{n \geq m}\quad \text{for all } m \in \omega.\]

(d) Add the chain \(\mathcal{C} = (c_{n,m})\) to \(L'_x\) and add a new cone \(\sigma'(\mathcal{C}) = (C \xrightarrow{c_n} C_n)_{n \in \omega}\) to the underlying category and add the commutativity conditions

\[\hat{b}_m \circ c_m = a_m \quad (m \in \omega).\]

It follows from Remark 3 that \(\text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \simeq \text{Mod}(\mathcal{S}', \text{Set}_{\text{fin}})\).

(III) We finally show how to modify a \(\lambda\)-ary sketch \(\mathcal{S}\), whose only infinite limit specifications are limits of \(\omega^{op}\)-chains formed either by specified monos or specified epis, to a geometric \(\lambda\)-ary sketch \(\mathcal{S}'\). We remove the \(\omega^{op}\)-chains from \(L\) one by one as follows:

(III.A) Modifications of limits of epi-chains to limits of mono-chains. We show how to change specifications of limits of \(\omega^{op}\)-chains of epimorphisms to limits of \(\omega^{op}\)-chains of monomorphisms. The idea is to observe that given an \(\omega^{op}\)-chain of epimorphisms \(b_{n,m} : B_n \to B_m\) and a compatible cone \(b_n : B \to B_n\) in \(\text{Set}_{\text{fin}}\), then that cone is a limit in \(\text{Set}_{\text{fin}}\) iff

(i) each \(b_n\) is an epimorphism

and

(ii) all \(b_n^*\)s are collectively monic (in other words, if \(C_n\) denotes the kernel equivalence of \(b_n\) then the mono-chain \(C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots\) has as an \(\omega^{op}\)-limit the diagonal relation of \(B\)).

Thus, assume that \(\mathcal{S}\) is a sketch with an \(\omega^{op}\)-chain \(\mathcal{B} = (b_{n,m} : B_n \to B_m)_{m \leq n < \omega}\) in \(L\) such that each \(b_{n,m}\) is specified to be an epimorphism in \(\mathcal{S}\). Let

\[\sigma(\mathcal{B}) = (b_n : B \to B_n)_{n < \omega}.\]

We modify \(\mathcal{S}\) to the following sketch \(\mathcal{S}' = (\mathcal{S}', L', \sigma')\) (in which \(\mathcal{B}\) is replaced by a mono-chain and which will be shown to be equivalent to \(\mathcal{S}\)):

(a) Delete the diagram \(\mathcal{B}\) from \(L'\).
(b) For each \( n < \omega \) add to \( \mathcal{A}' \) an object \( C_n \) and morphisms \( u_n, v_n : C_n \to B \) specified to form a kernel pair of \( b_n \) and \( d_n : B \to C_n \) specified to be the diagonal of \( u_n, v_n \). Specify \( b_n \) to be a coequalizer of \( u_n \) and \( v_n \).

(c) Add new morphisms \( c_{n,m} : C_n \to C_m \) \((m \leq n < \omega)\) with

\[
u_m c_{n,m} = u_n \quad \text{and} \quad v_m c_{n,m} = v_n.
\]

Specify each \( c_{n,m} \) to be a monomorphism and add the \( \omega \)-\( \mathbb{P} \)-chain \( \mathcal{E} \equiv (c_{n,m}) \) to \( \mathcal{L}' \) with

\[
\sigma'(\mathcal{E}) = (d_n : B \to C_n)_{n < \omega}.
\]

To show that \( \mathcal{S} \) and \( \mathcal{S}' \) are equivalent sketches, observe that for every model \( M \) of \( \mathcal{S} \) in \( \text{Set}_{\text{fin}} \) we have an (essentially unique) extension to a model \( M' \) of \( \mathcal{S}' \): we know that \( M b_n \) is an epimorphism and thus have to define \( M'u_n, M'v_n \) as a kernel pair of \( M b_n \) and \( M'd_n \) as a diagonal of that kernel pair. As remarked above, from the fact that \( M \) maps \( \sigma(\mathcal{B}) \) to \( \lim M \mathcal{B} \) it follows that \( M' \) maps \( \sigma'(\mathcal{E}) \) to \( \lim M' \mathcal{E} \). This extension of models obviously yields a functor from \( \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \) to \( \text{Mod}(\mathcal{S}', \text{Set}_{\text{fin}}) \), and this functor is full and faithful. To show that it is an equivalence of categories, it is sufficient to note that every model \( M' \) of \( \mathcal{S}' \) in \( \text{Set}_{\text{fin}} \) has, by the above remark, the property that the restriction of \( M' \) to \( \mathcal{A} \) is a model of \( \mathcal{S} \): we know that \( M'b_n \) are epis (see (b) above) and that the intersection of the kernels of \( M'b_n, n < \omega \), is the diagonal (see (c) above), thus, \( M' \) maps \( \sigma(\mathcal{B}) \) to \( \lim M' \mathcal{B} \).

(III.B) Modification of mono-chain limits. Given an \( \omega \)-\( \mathbb{P} \)-chain \( \mathcal{B} \equiv (b_{n,m} : B_n \to B_m)_{n \geq m} \) in \( \mathcal{L} \), where \( b_{n,m} \) are specified to be monomorphisms, with

\[
\sigma(\mathcal{B}) = (B \xrightarrow{b_n} B_n)_{n < \omega},
\]

we modify \( \mathcal{S} \) to the following sketch \( \mathcal{S}' \).

(a) Delete \( \mathcal{B} \) from \( \mathcal{L} \).

(b) Add mono-specification of each \( b_n \).

(c) Add a new object \( B^* \) and new morphisms \( b^*_n : B_n \to B^* \) \((n \in \omega)\) freely to \( \mathcal{A} \).

(d) Add the diagram \( \mathcal{B}' \equiv (B \xrightarrow{b_n} B_n)_{n < \omega} \) to \( \mathcal{C}' \) and put \( \sigma'(\mathcal{B}') = (B_n \xrightarrow{b^*_n} B^*) \).

To show that \( \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \cong \text{Mod}(\mathcal{S}', \text{Set}_{\text{fin}}) \), observe that given an \( \omega \)-\( \mathbb{P} \)-chain \( \mathcal{B} \equiv (b_{n,m} : B_n \to B_m)_{n \geq m} \) in \( \mathcal{L} \), then a cone of monos, compatible with the chain, is a limit of that chain iff the multiple pushout of that cone exists. In fact, any \( \omega \)-\( \mathbb{P} \)-chain stabilizes after \( k \) steps for some \( k \) finite, and the compatible mono-cone is a limit iff it is formed by isomorphisms starting from step \( k \); thus, iff its multiple pushout exists in \( \text{Set}_{\text{fin}} \). Consequently, given a model \( M \in \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \), we can extend it (essentially uniquely) to a model \( M' \in \text{Mod}(\mathcal{S}', \text{Set}_{\text{fin}}) \) by letting \( (M b_n \xrightarrow{M' b^*_n} M'B^*) \) be a multiple pushout of \( (M b_n)_{n < \omega} \). For any map \( h : M_1 \to M_2 \) in \( \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \) we then obtain a unique extension \( h' : M'_1 \to M'_2 \), and this defines a full and faithful functor from \( \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \) to \( \text{Mod}(\mathcal{S}', \text{Set}_{\text{fin}}) \). This functor is isomorphism-dense because for each model \( M' \in \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \), the restriction \( M = M' / \mathcal{A} \) is a model of \( \mathcal{S} \): the existence of the multiple pushout of all the monomorphisms \( M'b_n, n \in \omega \), guar-
Remark 4 (Reduction of colimits of chains to unions). Let $\mathcal{B} = (B_i \xrightarrow{b_{ij}} B_j)_{i \leq j < \alpha}$ be an $\alpha$-chain in $\text{Set}_{\text{fin}}$ and let

$$(B_i \xrightarrow{b_i} B)_{i < \alpha}$$

be a cocone of $\mathcal{B}$. We will show how the question of whether this is a colimit of $\mathcal{B}$ reduces to questions on unions of $\alpha$-chains of subsets:

(i) Let

\begin{center}
\begin{tikzcd}
B_i \arrow{r}{b_i} \arrow{d}[swap]{e_i} & B \arrow{d}{m_i} \\
C_i \arrow{r}[swap]{c_i} & C \\
\end{tikzcd}
\end{center}

(i$<\alpha$)

be an epi-mono factorization.

(ii) Given $i \leq j < \alpha$ define $c_{ij}$ by the commutativity of the following square:

\begin{center}
\begin{tikzcd}
B_i \arrow{r}{b_i} \arrow{d}[swap]{e_i} & B_j \arrow{d}{e_j} \\
C_i \arrow{r}[swap]{c_{ij}} & C_j \\
\end{tikzcd}
\end{center}

(iii) For each $k < \alpha$ denote by

$$D_k \subseteq B_k \times B_k \quad \text{and} \quad D_i^k \subseteq B_k \times B_k$$

the kernel equivalences of $b_k : B_k \to B$ and $b_{ki} : B_k \to B_i$ ($k \leq i$), respectively. Since $b_k$ factors through $b_{ki}$, we have $D_i^k \subseteq D^k$ and we denote by

$$d_i^k : D_i^k \to D^k \quad (i \geq k)$$

the inclusion map. Analogously, for $k \leq i \leq j$ let

$$d_{ij}^k : D_i^k \to D_j^k$$

denote the inclusion map. Then

(1) The $\alpha$-chains $\mathcal{C} = (c_{ij})$ and $\mathcal{D}^k = (d_{ij}^k)$, $k < \alpha$, are mono-chains and

(2) The given cone $(b_i)$ is a colimit of $\mathcal{B}$ if and only if both $(m_i)_{i < \alpha}$ is colimit of $\mathcal{C}$ and, for each $k < \alpha$, $(d_{ij}^k)_{k \leq i < j < \alpha}$ is a colimit of $(d_{ij}^k)_{k \leq i < j < \alpha}$. 
In fact, \((m_i) = \text{colim} \mathcal{G}\) holds iff the cone \((b_i)\) is collectively surjective, and \((d_k^i) = \text{colim} \mathcal{D}_k^i\) holds iff \(b_k\) merges only pairs merged by some \(d_{ki}\), \(i \geq k\). These two facts are equivalent to stating that \((b_i) = \text{colim} \mathcal{G}\) in \(\mathbf{Set}\). It is clear that \(\mathbf{Set}_{\text{fin}}\) is closed under existing colimits in \(\mathbf{Set}\).

**Proposition 2.** Every geometric sketch with countable colimit specifications is equivalent to a coherent sketch over \(\mathbf{Set}_{\text{fin}}\).

**Proof.** (1) We will first construct a sketch \(\mathcal{S}^\ast\) which models precisely all finite subsets of \(\omega\) in the following sense: \(\mathcal{S}^\ast\) has, among others, objects

\[ N \text{ and } C_n \ (n \in \omega) \]

and morphisms

\[ c_n : C_n \to N \ (n \in \omega) \]

such that (i) for each finite set \(A \subseteq N\) there exists a model \(F_A\) of \(\mathcal{S}^\ast\) with

\[ F_A N = A, \ F_A C_n = 1 \text{ if } n \in A \text{ and } F_A C_n = \emptyset \text{ if } n \notin A \]

and the image of \(F_{C_n}\) is \(\{n\}\) for every \(n \in A\), (ii) every model of \(\mathcal{S}^\ast\) is naturally isomorphic to \(F_A\) for some finite set \(A \subseteq N\) and (iii) for two finite sets \(A, B \subseteq N\) the set hom \((F_A, F_B)\) is either singleton, if \(A \subseteq B\), or \(\emptyset\) else.

The idea of the formal definition of \(\mathcal{S}^\ast\) below is to extend the sketch \(\mathcal{S}_{\omega}^\ast\) of Example 2 by

(a) a monomorphism \(c : C \to N\) disjoint with \(t : 1 \to N\) (the first element of \(N\) is only a "dummy" — e.g., the empty set is represented by a model \(F\) with \(FC = \emptyset\); however, \(FN\) has one point, viz. \(t\));

(b) morphisms \(t_n : T_n \to N\) which are "partial constants": we obtain them by relational composition of the graph of \(t\) with the next-element relation \(R\), thus, \(F_{T_n}\) represents \(n\) if \(n\) is contained in \(FN\), else, \(F_{T_n} = \emptyset\);

(c) a linear ordering \(L \hookrightarrow N^2\) containing \(R\) (i.e., \(L\) is the transitive-and-reflexive hull of \(R\));

(d) the relation \(L' = L \cap (N \times C)\) whose first projection is required to be epi-thus, \(FN\) is determined by \(FC\): if \(FC \subseteq \omega\) then \(FN = \{t, 0, 1, \ldots, n\}\) where \(n = \max FC\).

Formally, \(\mathcal{S}^\ast\) is obtained from \(\mathcal{S}_{\omega}^\ast\) by adding the following data:

1. An object \(C\) and a morphism \(c : C \to N\) specified to be mono.
2. An object \(N \times C\) and a morphism \(id \times c : N \times C \to N^2\), both specified to be products as indicated.
3. An object \(L\) and a morphism \(l : L \to N^2\) specified to be a monomorphism (relation) containing the relation \(\Delta : N \to N^2\) via a morphism \(d : N \to L\) with \(ld = \Delta\). [\(L\) is reflexive.]
4. A morphism \(l^{-1} : L \to N^2\) given by

\[ \pi_1 l = \pi_2 l^{-1} \text{ and } \pi_2 l = \pi_1 l^{-1} \]
and a pullback specification

\[
\begin{array}{ccc}
N & \xrightarrow{d} & L \\
\downarrow{d} & & \downarrow{l^{-1}} \\
L & \xrightarrow{l} & N^2
\end{array}
\]

\[L \text{ is antisymmetric.}\]

(5) A monomorphism \(\tilde{L} : \tilde{L} \rightarrow N^2\) specified to be a relational composite of \(l : L \rightarrow N^2\) with itself, and to be contained in \(l : L \rightarrow N^2\). \([L \text{ is transitive.}]\]

(6) An object \(L + L\) specified to be a coproduct as indicated and a morphism \(l^* : L + L \rightarrow N^2\), specified to be an epi and to have components \(l\) and \(l^{-1}\). \([L \text{ is a linear ordering.}]\]

(7) An object \(L'\) and morphisms \(c'\) and \(l'\) specified to form a pullback

\[
\begin{array}{ccc}
L' & \xrightarrow{l'} & N \times C \\
\downarrow{c'} & & \downarrow{id \times c} \\
L & \xrightarrow{l} & N^2
\end{array}
\]

Moreover, the composite of \(l'\) with the first projection is specified to be epi.

(8) A pullback specification

\[
\begin{array}{ccc}
O & \xrightarrow{t} & T \\
\downarrow{c} & & \downarrow{t} \\
C & \xrightarrow{c} & N
\end{array}
\]

(9) Relations (i.e., monomorphisms) \(\tau_n : T_n \rightarrow N^2\) specified to be as follows:

\(\tau_0\) is the relational composite of the graph of \(t : 1 \rightarrow N\) with \(r : R \rightarrow N^2\) and

\(\tau_{n+1}\) is the relational composite of \(\tau_n\) with \(R\).

We put

\[\tau_n = \pi_2 \cdot \tau_n : T_n \rightarrow N \quad (\text{for each } n \in \omega).\]
(10) Objects $C_n$ and morphisms $c_n$ and $c'_n$ with pullback specifications

\[
\begin{array}{ccc}
C_n & \xrightarrow{c'_n} & T'_n \\
\downarrow{c_n} & & \downarrow{t'_n} \\
C & \xrightarrow{c} & N
\end{array}
\]

(for each $n \in \omega$). For every finite set $A \subseteq \omega$ put $n = \max A$ if $A \neq \emptyset$ and $n = -1$ if $A = \emptyset$. Then we have a model $F_A$ of $\mathcal{S}^*$ with

- $F_A C = A$,
- $F_A N = \{t, 0, 1, \ldots, n\}$ (in particular, $F_0 N = \{t\}$),
- $F_A L$ is the linear ordering $t < 0 \leq 1 \ldots < n$,
- $F_A R$ is the relation $\{(t, 0), (0, 1), \ldots, (n - 1, n)\}$,
- $F_A t$ is the element $t$,
- $F_A T_m = \begin{cases} 
\emptyset & \text{if } m > n, \\
1 & \text{if } m \leq n,
\end{cases}$
- $F_A t_m$ is the element $m$ if $m \leq n$,

with the obvious interpretation of the other morphisms. We will show that every model $F$ is isomorphic to $F_A$ for some $A$. Without loss of generality, there is a number $n \in \omega \cup \{-1\}$ with $FN = \{t, 0, 1, \ldots, n\}$ and $FL, FR$ the above relations, and $FC \subseteq FN$. Put $A = FC$. Due to (7) above, every element of $FN$ is majorized (under the ordering $FL$) by an element of $FC$ thus, $n = \max A$ if $A \neq \emptyset$ and $n = -1$ if $A = \emptyset$. It is easy to see that $F \cong F_A$.

Let $f : F_A \to F_B$ be a morphism. Due to the partial constants $c_n : C_n \to C$ it follows that $A \subseteq B$ and $f$ is uniquely determined. Thus, $\text{Mod}(\mathcal{S}^*, \text{Set}_{\text{fin}})$ is isomorphic to the poset of all finite subsets of $\omega$.

(II) For every geometric sketch $\mathcal{S}$ with countable colimit specifications there exists a sketch $\mathcal{S}^\prime$, equivalent to $\mathcal{S}$ over $\text{Set}_{\text{fin}}$, which is $\sigma$-coherent, i.e., its colimit specifications are either countable-coproduct specifications or epi-specifications. To prove this, we can restrict our attention to sketches $\mathcal{S}$ whose colimit specifications are specifications of colimits of $\omega$-chains – this is the dual statement of the step (I) of the proof of Proposition 1 above.

(II.A) We will first show that we can, in fact, work solely with sketches specifying colimits of $\omega$-chains of specified monomorphisms. This follows from Remark 4, applied to $\mathcal{S} = \omega$ for a step-by-step modification; we thus assume, without loss of generality,
that $\mathbb{C}$ contains just one $\omega$-chain. Let $\mathcal{A} = (b_{ij} : B_i \to B_j)_{i \leq j < \omega}$ denote the $\omega$-chain in $\mathbb{C}$ with $\sigma(\mathcal{A}) = (b_i : B_i \to B)_{i < \omega}$. We denote by $\mathcal{S}'$ the following modification of $\mathcal{S}$:

(a) Delete $\mathcal{A}$ from $\mathbb{C}$.

(b) For each $i < \omega$ add a new object $B_i^2$ and new morphisms $\pi'_i, \pi''_i : B_i^2 \to B_i$ specified to form a product.

(c) For each $k \leq i < \omega$ add a new morphism $\delta^k : D^k \to B^k_i$, specified to form the kernel equivalence of $b_k$ and $\delta^k : D^k_i \to B^k_i$, specified to form the kernel equivalence of $b_k$ for all $k \leq i < \omega$. Add new morphisms $d_{ij}^k : D^k_i \to D^k_j$ ($k \leq i \leq j < \omega$) with the following commutativity condition:

\[ D^k_i \xrightarrow{d_{ij}^k} D^k_j \]

\[ \delta^k \]

\[ \delta^k \]

\[ (k \leq i \leq j < \omega) \]

(d) Add objects $C_i$ and morphisms $e_i, m_i, c_{ij}$ for all $i \leq j < \omega$ with the commutativity conditions (i) and (ii) of Remark 4 above.

(e) Add specifications that $e_i$ be epi and $m_i, c_{ij}$ and $d_{ij}^k$ be monos for all $k \leq i < j < \omega$.

(f) Add to $\mathbb{C}$ the following $\omega$-chains of specified monos:

$\mathfrak{C} = (c_{ij})_{i \leq j < \omega}$ with $\sigma'(\mathfrak{C}) = (m_i)_{i < \omega}$,

$\mathcal{D}^k = (d_{ij}^k)_{i \leq j < \omega}$ with $\sigma'(\mathcal{D}^k) = (d_i^k)_{i \leq j < \omega}$ for all $k < \omega$.

The fact that $\mathcal{S}'$ is equivalent to $\mathcal{S}$ over $\text{Set}_{\text{fin}}$ follows from Remark 4.

(II.B) We are now going to modify the sketch $\mathcal{S}'$, in which all colimit-specifications are either those of $\omega$-chains of specified monomorphisms or finite, to a $\sigma$-coherent sketch. We can perform the modification step by step, so we may assume that $\mathcal{S}'$ is $\sigma$-coherent except for one $\omega$-chain of monos (II.C) or except for one finite diagram in $\mathbb{C}$ (II.D).

(II.C) Let $\mathcal{S}$ be a sketch obtained from a $\sigma$-coherent sketch by the addition of one diagram

$\mathcal{A} = (b_{ij} : B_i \to B_j)_{i \leq j < \omega}$ in $\mathbb{C}$,

where $b_{ij}$ are specified monomorphisms and let

$\sigma(\mathcal{A}) = (B_i \xrightarrow{b_i} B)_{i < \omega}$.

We modify $\mathcal{S}$ to the following $\sigma$-coherent sketch $\mathcal{S}'$. The idea is to introduce $C_i = B - B_i$ and to require that the coproduct $C = \bigsqcup_{i < \omega} C_i$ exists; this implies that there is
\[ i \in \omega \text{ with } C_i = \emptyset, \text{ thus, } B = B_i, \] and this is equivalent (if \( b_i \) and \( b \) are monos) to \( \sigma(\mathcal{B}) \) forming a colimit of \( \mathcal{B} \). More precisely, we modify \( \mathcal{S} \) as follows:

(a) Delete \( \mathcal{B} \) from \( \mathcal{C} \).

(b) Add new objects \( C \) and \( C_i \) \((i \in \omega)\) and new morphisms:

\[
\begin{array}{ccc}
B_i & \xrightarrow{p_i} & B \\
\downarrow q_i & & \downarrow \\
C_i & \xrightarrow{c_i} & C
\end{array}
\]

and this is equivalent (if \( b_i \) and \( b \) are monos) to \( a(B) \) forming a colimit of \( \mathcal{B} \). More precisely, we modify \( \mathcal{S} \) as follows:

(c) Add the coproduct specifications

\[
B = B_i + C_i \quad \text{and} \quad C = \bigsqcup_{i < \omega} C_i
\]

to \( \mathcal{S}' \) (with the coproducts injections \( p_i, q_i \) and \( c_i \), respectively).

(d) Add mono-specifications for each \( b_i \).

To see that \( \mathcal{S} \) and \( \mathcal{S}' \) are equivalent over \( \text{Set}_{\text{fin}} \), consider \( M \in \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \). It has an essentially unique extension to a model \( M' \) of \( \mathcal{S}' \): we put (necessarily) \( M'B = MB_i + MC_i \) and \( M'C = \bigsqcup_{i < \omega} MC_i \), where the latter coproduct exists because from \((Mbi) = \text{colim } M\mathcal{B}\) it follows that \( MC_i = \emptyset \) for all but finitely many \( i < \omega \). Given a map \( h : M_1 \to M_2 \) in \( \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \), we have a unique extension \( h' : M'_1 \to M'_2 \), since \( h'_C \) is determined uniquely by \( h_B \) and \( h_{B'} \); that (for all \( i < \omega \)) determines \( h'_C \) uniquely. This yields a functor from \( \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \) to \( \text{Mod}(\mathcal{S}', \text{Set}_{\text{fin}}) \) which is, obviously, full and faithful. To show that it is isomorphism-dense, thus, an equivalence functor, observe that for each model \( M' \) of \( \mathcal{S}' \) the restriction \( M \) of \( M' \) to \( \mathcal{S} \) is a model of \( \mathcal{S} \).

In fact, from \( M'C = \bigsqcup_{i < \omega} M'C_i \) it follows that all \( M'C_i \) but finitely many are empty, thus, there exists \( i_0 \) such that \( M_{p_{i_0}} \) is an isomorphism, and this implies that \( (M_{bi}) \) is a colimit of \( M\mathcal{B} \).

(IID) Let \( \mathcal{S} \) be a sketch obtained from a \( \sigma \)-coherent sketch by the addition of a finite non-empty diagram \( D \). Without loss of generality, we assume that \( D \) is a parallel pair of morphisms, i.e., that the given colimit specification is a coequalizer specification. (If \( D \) is more general, we substitute the colimit-specification of \( D \) by two finite coproduct specifications and one coequalizer specification in the well-known way.)

Thus, we work with just one coequalizer specification:

\[
A \xrightarrow{f_1} B \xrightarrow{\varepsilon} C
\]

and show how to modify it \( \sigma \)-coherently. The sketch \( \mathcal{S}' \) is obtained from the \( \sigma \)-coherent sketch \( \mathcal{S}_0 \), which is the original sketch \( \mathcal{S} \) in which the above coequalizer
specification is ignored, by the extension described formally below. The idea is to introduce the relation \( r : R \to B^2 \) represented by the pair \( f_1, f_2 \) and relations

\[
R_0 = R \cup R^{-1} \cup \Delta, \\
R_{n+1} = R_0 \circ R_n \quad \text{(relational composite)}.
\]

Then \( c \) is a coequalizer iff it is epi, satisfies \( c \cdot f_1 = c \cdot f_2 \), and has \( \bigcup_{n \in \omega} R_n \) as the kernel relation.

Let \( \mathcal{S}' \) be the \( \sigma \)-coherent sketch obtained from \( \mathcal{S} \) by adding the following data:

1. An object \( B^2 \) and morphisms \( \pi_1, \pi_2 : B^2 \to B \) specified to form a product. A morphism \( f : A \to B^2 \) specified to have components \( f_1, f_2 \).
2. A relation (monomorphism) \( r : R \to B^2 \) and a morphism \( c : A \to R \) with \( f = rc \), specified to be an epimorphism.
3. A relation \( r_n : R_n \to B^2, n \in \omega \), specified to be as follows:

\[
r_0 = r \cup r^{-1} \cup \Delta, \\
r_{n+1} = r_0 \circ r_n.
\]

4. An object \( \coprod_{n \in \omega} R_n \), specified to be a coproduct of \( R_n \)'s, and a morphism \( \bar{r} : \coprod_{n \in \omega} R_n \to B^2 \) specified to have components \( r_n \).
5. A relation \( k : K \to B^2 \) specified to be the kernel relation of \( c \).
6. A morphism \( h : K \to \coprod_{n \in \omega} R_n \) with \( k = \bar{r} \cdot h \), and a morphism \( h' : R \to K \) with \( r = k \cdot h' \).
7. An epi-specification for \( c \).

If \( F \) is a model of \( \mathcal{S}' \), then the restriction \( F_0 \) to the original objects and morphisms is a model of \( \mathcal{S}_0 \), and \( F \) is determined by \( F_0 \) up to isomorphism of models. To prove that

\[
\text{Mod} (\mathcal{S}, \text{Set}_\infty) \cong \text{Mod} (\mathcal{S}', \text{Set}_\text{fin}),
\]

it is clearly sufficient to prove that \( F_0 \) satisfies the coequalizer specification above, i.e., \( F_0 c = \text{coeq}(F_0 f_1, F_0 f_2) \). In fact, by (1) and (2), \( FR \) is the relation of all pairs \((F_0 f_1(x), F_0 f_2(x))\). Due to \( h' \) in (6) we see that \( F_0 c \cdot F_0 f_1 = F_0 c \cdot F_0 f_2 \). Conversely, since the kernel relation of \( F_0 c \) is contained in the transitive hull of \( FR \cup (FR)^{-1} \cup \Delta \), due to \( h \) in (6), we conclude that \( F_0 c \) merges two points iff a coequalizer of \( F_0 f_1 \) and \( F_0 f_2 \) merges them. Since, by (7), \( F_0 c \) is surjective, this concludes the proof.

(III) For every \( \sigma \)-coherent sketch \( \mathcal{S} \) we want to find an equivalent coherent sketch. We assume that the infinite coproduct specifications of \( \mathcal{S} \) have the following form:

\[
D = (D_n \xrightarrow{d_{P,n}} A_D)_{n \in \omega}.
\]

We denote by \( \mathcal{S}' \) the coherent sketch obtained from \( \mathcal{S} \) by deleting each of these coproduct specifications. For each \( D \) as above we denote by \( \mathcal{S}'^{*D} \) a copy of the above sketch \( \mathcal{S}' \), see 1., with an upper index \( D \) on each object and each morphism for distinction. We denote by \( \tilde{\mathcal{S}} \) the coherent sketch obtained from the disjoint union
of $\mathcal{S}'$ and $\mathcal{S}^{*D}$ for all the infinite coproduct specifications $D$ of $\mathcal{S}$ by adding the following data:

(a) morphisms $f_D : A_D \rightarrow N^D$ and $t_{D,n} : D_n \rightarrow C^D_{n}$ for all $D$ and all $n$,

(b) pullback specifications

for all $D$ and all $n$ and

(c) epi-specifications for all $f_D$.

We will show that the sketches $\mathcal{S}$ and $\tilde{\mathcal{S}}$ are equivalent over $\text{Set}_{\text{fin}}$.

Let us first verify that every model $F$ of $\mathcal{S}$ has an essentially unique extension to a model $\tilde{F}$ of $\tilde{\mathcal{S}}$. First, let us denote, for each $D$, by $A(D)$ the set of all $n \in \omega$ with $FD_n \neq \emptyset$, and let $\tilde{F}$ be the extension of $F$ which coincides on $\mathcal{S}^{*D}$ with $F_{A(D)}$ (recall the models $F_A$ of $\mathcal{S}^{*}$ defined in Part I), with

\[ \tilde{F}f_D(x) = n \quad \text{for all } x \in FD_n, \]

\[ \tilde{F}t_{D,n} \text{ constant function} \quad \text{for all } n \in A(D). \]

It is obvious that $\tilde{F}$ is a model of $\tilde{\mathcal{S}}$. Conversely, if $\tilde{F}$ is a model of $\tilde{\mathcal{S}}$ extending $F$, then the above pullbacks (b) guarantee that $\tilde{F}f_D^{-1}(n) = \text{im}(\tilde{F}t_{D,n})$, and since $\tilde{F}f_D$ is an epimorphism, we conclude that

\[ \tilde{F}N^D = A(D) \quad \text{for each } D \]

and this, as proved in Part I, determines $\tilde{F}/\mathcal{S}^{*D}$ uniquely up to a natural isomorphism. Consequently, $F$ has an extension to a model of $\tilde{\mathcal{S}}$ unique up to a natural isomorphism.

Furthermore, for each morphism $f : F_1 \rightarrow F_2$ in $\text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}})$ there exists a unique extension to a morphism $\tilde{f} : \tilde{F}_1 \rightarrow \tilde{F}_2$ in $\text{Mod}(\tilde{\mathcal{S}}, \text{Set}_{\text{fin}})$: for each $D$ and $n$, $\tilde{F}_1C^D_{n} \neq \emptyset$ implies $\tilde{F}_2C^D_{n} \neq \emptyset$; it follows, as shown in Part I, that there exists a unique morphism from $\tilde{F}_1/\mathcal{S}^{*D}$ to $\tilde{F}_2/\mathcal{S}^{*D}$ for each $D$. It is obvious that the naturality condition of the corresponding $\tilde{f}$ is fulfilled also by the added morphisms $f_D$ and $t_{D,n}$.

In this manner we define a full and faithful functor $(\sim) : \text{Mod}(\mathcal{S}, \text{Set}_{\text{fin}}) \rightarrow \text{Mod}(\tilde{\mathcal{S}}, \text{Set}_{\text{fin}})$. To prove that this functor is an equivalence of categories, it is sufficient to show that if $\tilde{F}$ is a model of $\tilde{\mathcal{S}}$ then the restriction $F$ of $\tilde{F}$ to the underlying category of $\mathcal{S}$ is a model of $\mathcal{S}$. For each $D$ the maps $F_{dD,n}$ are one-to-one, since they lie opposite to the one-to-one maps $\tilde{F}c^D_{n}$ in a pullback, see (b) above. Those pullbacks also guarantee that the images of these maps $F_{dD,n}$ are pairwise disjoint and, since the maps $\tilde{F}c^D_{n}$, $n \in \omega$, are collectively onto, that $F_{dD,n}$ ($n \in \omega$) are collectively onto – thus, $FA_D = \bigsqcup FD_n$ with coproduct injections $F_{dD,n}$. 
This concludes the proof of equivalence of the sketches \( \mathcal{S} \) and \( \hat{\mathcal{S}} \) over \( \text{Set}_{\text{fin}} \). \( \square \)

**Notation.** If measurable cardinals exist, we denote by \( \mu \) the first measurable cardinal. If no measurable cardinals exist, \( \mu \) denotes a formal symbol larger than all cardinals.

**Proposition 3.** For every cardinal \( \lambda \leq \mu \), each \( \lambda \)-ary geometric sketch is equivalent to a coherent sketch over \( \text{Set}_{\text{fin}} \).

**Proof.** Let \( \mathcal{S} \) be a \( \lambda \)-ary geometric sketch. We will show that if \( \lambda \leq \mu \), then \( \mathcal{S} \) is equivalent to a geometric sketch with countable colimit specifications. This will conclude the proof due to Proposition 2.

(1) We first show that for each set \( I \) of cardinality smaller than \( \mu \) there exists a sketch \( \mathcal{S}_I^* \) modelling finite subsets of \( I \) in the analogous sense to \( \mathcal{S}_I^* \) in Part I of the proof of Proposition 2: \( \mathcal{S}_I^* \) has morphisms \( c_i : C_i \to I \quad (i \in I) \) such that for each finite set \( A \subseteq I \) there is a model \( F_A \) of \( \mathcal{S}_I^* \) with \( F_A(I) = I \), \( F_A C_i = 1 \) if \( i \in A \), \( = \emptyset \) if \( i \notin A \), and the image of \( Fc_i \) is \( \{i\} \) for each \( i \in A \). These models \( F_A \) are, up to natural isomorphism, the only models of \( \mathcal{S}_I^* \), and for two models \( F_A, F_B \) we either have exactly one morphism \( F_A \to F_B \), if \( A \subseteq B \), or none, if \( A \nsubseteq B \). The underlying category of \( \mathcal{S}_I^* \) is the poset \( \exp I \) of all subsets of \( I \) ordered by inclusion with a formal terminal object \( T \) (and morphisms \( ! : A \to T \) for all \( A \subseteq I \) ) added. Limit specifications of \( \mathcal{S}_I^* \) are:

1. mono-specification of all morphisms \( A \to B \) for \( A \subseteq B \subseteq I \) and \( ! : \{i\} \to T \) for each \( i \in I \);
2. a pullback specification

\[
\begin{tikzcd}
A \ar[dr] & A \cap B \ar[dl] \\
 & B
\end{tikzcd}
\]

for each pair \( A, B \in \exp I \); and colimit specifications are

3. \( \emptyset \) is an initial object; and
4. a coproduct specification \( (D_n \xrightarrow{d_n} I)_{n \in \omega} \) for each disjoint union \( I = \bigcup_{n \in \omega} D_n \).

For each finite subset \( A \subseteq I \) we have a model \( F_A \) of \( \mathcal{S}_I^* \) defined by

\[
F_A B = A \cap B \quad \text{for each } B \in \exp I, \quad F_A T = 1.
\]

Conversely, let \( F \) be a model of \( \mathcal{S}_I^* \) in \( \text{Set}_{\text{fin}} \), then we will show that \( F \) is naturally isomorphic to \( F_A \) for \( A = FI \). We know, due to (1), that \( F \{i\} \) has most one element for each \( i \in I \), and, due to (2) applied to \( A = \{i\} \) and \( B = \{j\} \) we know that the
images of $F(\{i\} \to I)$ are pairwise distinct. In order to prove that $F \cong F_A$, it is clearly sufficient to verify that every element $a \in A$ lies in the image of some $F(\{i\} \to I)$. For this sake, denote by $\mathcal{U}$ the set of all subsets $U \subseteq I$ such that $a$ lies in the image of $F(U \to I)$. We will prove that $\mathcal{U}$ is a $\sigma$-complete ultrafilter - since $\text{card} \ I < \mu$, this implies that $\mathcal{U}$ is a trivial ultrafilter, i.e., $\{i\} \in \mathcal{U}$ for some $i \in I$, which concludes the proof of $F \cong F_A$. Observe that

(a) if $U \subseteq U'$ and $U \in \mathcal{U}$, then $U' \in \mathcal{U}$: in fact, the morphism $U \to I$ factors through $U' \to I$;
(b) if $U, V \in \mathcal{U}$ then $U \cap V \in \mathcal{U}$: by (2) above, we have $F(U \cap V) = FU \cap FV$;
(c) $\emptyset \notin \mathcal{U}$ - see (3);
(d) for each disjoint union $I = \bigcup_{n \in \omega} D_n$ there exists $n \in \omega$ with $U_n \in \mathcal{U}$ - this follows from (4).

Consequently, $F \cong F_A$. The statement concerning morphisms of models of $\mathcal{S}^*$ is obvious.

(II) Let $\lambda \leq \mu$ and let $\mathcal{S}$ be a $\lambda$-ary geometric sketch. We proceed by transfinite induction: for every $\alpha$ with $\omega < \alpha < \lambda$ we will show how to reduce in $\mathcal{S}$ every colimit specification of a diagram with $\alpha$ morphisms to specifications of diagrams with less than $\alpha$ morphisms. By applying this procedure step-by-step we end up with a sketch with countable colimit-specifications, and by Proposition 2, this concludes the proof.

We can assume that our sketch $\mathcal{S}$ has just one diagram in $C$ with $\alpha$ morphisms. By the same argument as that in (II.A) of Proposition 2, we can assume that unique diagram is an $\alpha$-chain of specified monomorphisms. The argument in (II.C) reduces this case to that of a coproduct of $\alpha$ objects, say

$$(D_i \xrightarrow{d_i} A)_{i \in I}, \quad \text{card} \ I = \omega.$$  

We now remove this coproduct specification and add to our sketch a copy of the sketch $\mathcal{S}^*$ of (I) above. We modify our sketch as follows:

(a) Delete the above coproduct specification.
(b) Add a new morphism $f$ from $A$ (in $\mathcal{S}$) into $I$ (in $\mathcal{S}^*$) and morphisms $t_i$ from $D_i$ (in $\mathcal{S}$) to $\{i\}$ (in $\mathcal{S}^*$).
(c) Add epi-specification for $f$.
(d) Add the following pullback-specifications

\[
\begin{tikzcd}
D_i \ar{r}{d_i} \ar{d}{b_i} & A \ar{d}{f} \\
\{i\} \ar{r} & I
\end{tikzcd}
\]

The proof that the modified sketch is equivalent to the original one over $\text{Set}_{\text{fin}}$ is analogous to (III) in Proposition 2. □
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