Journal of Pure and Applied Algebra 7 (1976) 347–359

UNIQUE DECOMPOSITION AND ISOMORPHIC REFINEMENT THEOREMS IN ADDITIVE CATEGORIES

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Communicated by H. Bass
Received 3 September 1975

Under suitable hypotheses, if an object in an additive category is a direct sum of subobjects with local endomorphism rings, then any two such decompositions have the summands isomorphic in pairs, and any other decomposition has a refinement into a decomposition where the summands are of this sort. The purpose of this paper is to give careful proofs of such results, using what appear to be minimal hypotheses. Theorems of this sort for infinite direct sums in additive, non-Abelian categories, have been essential in recent work in Abelian group theory.

In 1909, MacLagan-Wedderburn [29] proved that if a finite group is expanded in two ways as a direct sum of indecomposable factors, then the summands are isomorphic in pairs. Remak [18] showed that the summands are actually centrally isomorphic. This theorem was extended by Krull [13] and Schmidt [19,20] to operator groups satisfying the double chain condition for admissible subgroups, and the numerous extensions of this famous theorem are most often referred to as Krull–Schmidt Theorems.

In this paper we are primarily interested in generalizations of such theorems concerning infinite direct sums of modules. The most famous such result is Azymaya's unique decomposition theorem, published in 1950: *Let M be a module which has a direct sum decomposition as a finite or infinite direct sum of indecomposable submodules M_i ($i \in I$), such that the endomorphism ring of each M_i ($i \in I$) is a local ring. Then any indecomposable summand of M is isomorphic to M_i for some $i \in I$, and if M is the direct sum of the indecomposable submodules N_j($j \in J$), then there is a bijective mapping $\phi : I \to J$ such that $M_i \cong N_{\phi(i)}$ for all $i \in I$.*

In 1964, Crawley and Jónsson [6] published some unique decomposition and isomorphic refinement theorem for general algebraic systems. A key condition in their work was the so-called exchange property (defined in Section 2 below). In

1969, Warfield [24] showed that an indecomposable module has this exchange property if and only if its endomorphism ring is a local ring. Using this, it turns out that the Crawley–Jónsson theorems for direct sums of indecomposable algebras include Azumaya's theorem, and also the following isomorphic refinement theorem: *Let M be a module which has a direct sum decomposition as a (finite or infinite) direct sum of indecomposable submodules* $M_i (i \in I)$ *such that each* M_i *is countably generated and the endomorphism ring of each* M_i *is a local ring. Then any two direct sum decompositions of M have isomorphic refinements, and, in particular if K is a summand of M, then there is a subset* $J \subseteq I$ *such that K is isomorphic to the direct sum of the modules* M_i, $i \in J$. (An economical proof of this theorem can be found in [1, Theorem 26.5]. An example of an earlier theorem implied by this one is Kaplansky's Theorem [12] that a projective module over a local ring is free.)

The first version of the Krull–Schmidt Theorem proved in the context of categories was Atiyah's Krull–Schmidt Theorem [2], published in 1956, for objects in an Abelian category satisfying a bichain condition. The result was applied to sheaf theory. In his thesis [8] Gabriel stated that Azumaya's theorem carries over immediately to Abelian categories with infinite direct sums and satisfying a Grothendieck condition, and used this to discuss the injective objects in a locally Noetherian category. Proofs of Azymaya's theorem in an Abelian Grothendieck category may be found in [5], [16], and [17].

In 1964, [23], Walker used the Azumaya Theorem for finite direct sums in an Abelian category to give a natural form to some interesting results discovered by B. Jónsson. Jónsson had shown (announced in 1945 [9], with details in 1957 [10]) that torsion-free Abelian groups of finite rank fail to satisfy the Krull–Schmidt Theorem. However (announced in 1945 [9], with details in 1959 [11]) he was able to obtain an approximate Krull–Schmidt Theorem involving an equivalence relation weaker than isomorphism (two such groups are quasi-isomorphic if each is isomorphic to a subgroup of finite index in the other). Walker showed that Jónsson's result could be viewed as an application of the Azumaya Theorem for finite sums in an Abelian category whose objects are Abelian groups but where the morphism group is defined differently.

Two other extensions of the Krull–Schmidt Theorem were given by Bass [4, pp. 18–20] and Warfield [25]. Bass extended the Azymaya theorem for finite sums to additive categories in which all idempotents split, and applied the result to K-theory. A similar result has been needed in recent work in the K-theory of Abelian groups of finite rank [14]. Warfield extended most of the Crawley–Jónsson results, including those for sums of indecomposables, to Abelian categories satisfying a Grothendieck condition, and applied the results to the structure of injective modules.

Recent work in Abelian group theory has made it essential to use unique decomposition and isomorphic refinement theorems in additive categories which are not Abelian. The first of the relevant examples arose in Warfield's work on the classification theory of modules over a discrete valuation ring ([26], [27], and [28]) in which a certain family of cardinal numbers associated to a certain class of modules were shown to be isomorphism invariants by the application of a version of Azumaya's theorem, not in the category of modules but in an associated, additive, non-Abelian category. The invariants were extended to summands of modules in the class by application of a version of the Crawley–Jónsson results in the same category. A very similar category, but one more easily described, was introduced by E.A. Walker. Let $\mathcal{M}$ be the category whose objects are modules over a discrete valuation ring R and whose morphisms are the sets $\mathrm{Hom}_R(A,B)/\mathrm{Hom}_R(A,tB)$, where tB denotes the torsion submodule of B. The modules A and B are isomorphic in $\mathcal{M}$ if and only if there are torsion modules S and T such that $A \oplus S \cong B \oplus T$. This category is an additive category with kernels and infinite direct sums, but it is not Abelian. The isomorphic refinement theorem in this category implies the following theorem, proved directly by Stanton [21]: If M is a module which is a summand of a direct sum of modules whose torsion-free rank is one, then there is a torsion module T such that $M \oplus T$ is a direct sum of modules of torsion-free rank one. (This had previously been known only for modules of finite rank over a complete discrete valuation ring.) The key is that modules whose torsion-free rank is one have a local endomorphism ring in $\mathcal{M}$.

In a different direction, Jónsson's Theorem on quasi-decompositions can be extended to groups of infinite rank using a notion of local quasi-isomorphism. (If B is a torsion-free group and A a subgroup, A is locally quasi-equal to B if for every finite rank subgroup F of B, there is a positive integer n such that $nF \subseteq A$. Groups G and H are locally quasi-isomorphic if they have subgroups G' and H', locally quasi-equal to G and H respectively, such that $G' \cong H'$.) In [7], Fuchs and Viljoen obtain unique decomposition and isomorphic refinement theorems in this context. A special case of their results yields the following: If $G = \Sigma_{i\in I} M_i = \Sigma_{j\in J} N_j$, where each M_i and N_j is torsion-free of finite rank and contains no decomposable subgroup of finite index, then there is a bijective map $\phi : I \to J$ such that M_i is isomorphic to a subgroup of finite index in $N_{\phi(i)}$. Further, if G has another decomposition $G = X \oplus Y$, then there is a subgroup B of X, locally quasi-equal to X, and a subset I' of I, such that $B \cong \Sigma_{i\in I'} M_i$.

Walker [22] put this theory in a natural setting by defining an additive category $\mathcal{L}$ whose objects are torsion-free groups and whose morphism groups $\mathcal{L}(A,B)$ are defined to be the direct limits of the groups $\mathrm{Hom}(X,B)$, where X ranges over the set of subgroups of A which are quasi-equal to A. This category is an additive category with infinite direct sums and with kernels.

The general theorem is that if G is isomorphic in $\mathcal{L}$ to a direct sum of groups of finite rank, then in any two such decompositions, the summands are quasi-isomorphic in pairs, and that any summand of G in the category $\mathcal{L}$ is $\mathcal{L}$-isomorphic to a direct sum of groups of finite rank.

These various applications made it essential to have a careful proof of the relevant theorems available. A number of difficulties arise in non-Abelian categories, so that at a number of key points the proofs do not simply carry over immediately from the Abelian category case. In the first section below we prove a Krull–Schmidt Theorem for finite decompositions in any additive category, which generalizes Bass's result [4] somewhat. In Sections 2 and 3 we prove unique decomposition and isomorphic refinement theorems for infinite direct sums in additive categories with infinite direct sums and kernels. Throughout this paper, we use the notation Σ for direct sums and we call a ring a local ring if the sum of two non-units is a non-unit (i.e. the ring has a unique maximal left ideal, but is not necessarily Noetherian).

1. A unique decomposition theorem for finite direct sums

Throughout this paper we work in an additive category satisfying the standard axioms [15, pp. 249–253]. Roughly speaking it is a category together with an Abelian group structure on each Hom(A,B) relative to which composition is bilinear. For every two objects A,B there is an object $A \oplus B$ together with maps $e_A : A \to A \oplus B$, $e_B : B \to A \oplus B$, $p_A : A \oplus B \to A$, $p_B : A \oplus B \to B$ satisfying $p_A e_A = 1_A$, $p_B e_B = 1_B$ and $e_A p_A + e_B p_B = 1_{A \oplus B}$.

The proof of the following lemma is straightforward.

Lemma 1. *Let $M \subset G = N \oplus R$, with p_N, p_R, e_N, e_R the projections and injections and $e_M : M \to G$ the inclusion. Suppose $p_N e_M : M \to N$ is an isomorphism. Then $G = M \oplus R$, the projections for this sum being $\pi_M = (p_N e_M)^{-1} p_N$ and $\pi_R = p_R(1_G - e_M \pi_M)$.*

Theorem 1. *Let $\mathcal{A}$ be any additive category and $M = \Sigma_{i=1}^{m} M_i$ where End(M_i) is a local ring for all i. Then any other direct sum decomposition of M refines to a decomposition into at most m indecomposable summands, and if $M = \Sigma_{i=1}^{n} N_i$ is another decomposition in which End(N_i) has no idempotents other than 0 and 1 ($1 \leqslant i \leqslant n$), then $m = n$ and, with renumbering, $M_i \cong N_i (1 \leqslant i \leqslant m)$.*

Proof. We prove the second statement first. Let $e_i : M_i \to M$ and $f_j : N_j \to M$ be the injection maps and $p_i : M \to M_i$ and $q_j : M \to N_j$ be the projection maps. Then $\Sigma_{j=1}^{n} p_1 f_j q_j e_1$ is the identity on M_1, and since End(M_1) is a local ring, one of the summands is a unit in the ring. Renumbering, we suppose that this is the element $\gamma = p_1 f_1 q_1 e_1$. We now look at the endomorphism of N_1 given by $q_1 e_1 \gamma^{-1} p_1 f_1$. Computation (using the definition of γ) shows that this is an idempotent, and hence either 0 or 1. Since it is a factor of $(\gamma^{-1} p_1 f_1 q_1 e_1)^2$, which is the identity of M_1, and $M_1 \neq 0$, this idempotent must be the identity of N_1. The map $p_1 f_1$ is therefore an isomorphism of N_1 onto M_1, with inverse $q_1 e_1 \gamma^{-1}$. Lemma 1 implies that

$$M = N_1 \oplus M_2 \oplus \ldots \oplus M_m .$$

An easier lemma, valid in any additive category, implies that complements to the same summand are isomorphic, whence

$$N_2 \oplus N_3 \oplus \ldots \oplus N_n \cong M_2 \oplus M_3 \oplus \ldots \oplus M_m .$$

The statements follows by induction.

Returning to the first statement of the theorem, we let R be the endomorphism ring of M and note that $R = \Sigma_{i=1}^{m} R(e_i p_i)$, where $e_i p_i$ is an idempotent in R, and this is a direct sum decomposition of R as a sum of left ideals. Further, we can identify $\mathrm{End}(M_i)$ with $\mathrm{End}(R(e_i p_i)) = (e_i p_i) R(e_i p_i)$, so these endomorphism rings are local. Our previously proved statement therefore applies to this decomposition of R. Since the category of R-modules is Abelian, any idempotent splits. An examination of the previous argument (or the well known results for modulus as in [24] or Proposition 2 below for example) show that if ${}_R R = \Sigma_{i=1}^{k} L_i$ is any direct sum decomposition of R as a sum of left ideals, then, renumbering if necessary, we can decompose $L_1 = L_1' \oplus L_1''$ such that $R = R(e_1 p_1) \oplus L_1'' \oplus (\Sigma_{i=2}^{k} L_i)$. By induction, we see that any decomposition of R as a module refines to one with at most m summands, all indecomposable. This means that any independent orthogonal set of idempotents in R has at most m members, which implies, in turn, that any direct sum decomposition of M has at most m members. This clearly implies that any decomposition of M refines to a decomposition with indecomposable summands. (We note that it does *not* imply that those indecomposable summands have endomorphism rings with no nontrivial idempotents, since we do not know that idempotents split in our category. If we knew this, we would have a more satisfactory result, like that proved in [4, pp. 18–20]).

2. A unique decomposition theorem for arbitrary direct sums

Definition. A *small* object is an object S such that every map into a sum $f: S \to \Sigma_{i \in I} M_i$ factors through a subsum $\Sigma_{i \in F} M_i$ with F some finite subset of I.

In a category of modules, finitely generated modules are small. The converse does not generally hold, but this observation provides an abundance of small objects. In particular, every module is the least upper bound of its small submodules. Utilizing the notion of small objects, this property is generalized to categories via the following definition.

Definition. An object M is *finitely approximable* if for any object L and map $f: L \to M$, f is an isomorphism if and only if (i) f is monic, and (ii) for any small object S and for any map $g: S \to M$ there is a map $h: S \to L$ such that $g = fh$ (i.e. g factors through f).

Lemma 2. *A small object is finitely approximable. If $M = \Sigma_{i \in J} M_i$ is finitely approximable, then each M_i is finitely approximable, and if the category has kernels, the converse holds also.*

Proof. The first assertion is clear. Let $M = A \oplus B$, and assume M is finitely approximable. Let $f : L \to A$ be monic. Let $i_A : A \to M$ and $p_A : M \to A$ be the injection and projection associated with the direct sum. Suppose for every $g : S \to A$ with S small, there is an $h : S \to L$ such that $fh = g$. Let $g : S \to A \oplus B$, with S small. There is an $h : S \to L$ such that $fh = p_A g$. There is a map $k : S \to L \oplus B$ such that $p_L k = h$ and $p'_B k = p_B g$, where p_L and p'_B are the projections relative to the sum $L \oplus B$, and p_B is the projection relative to the sum $A \oplus B$. Let $\bar{f} : L \oplus B \to A \oplus B$ be the map satisfying $p_A \bar{f} = f p_L$ and $p_B \bar{f} = p'_B$. Then $\bar{f}k = (i_A p_A + i_B p_B)\bar{f}k = i_A f p_L k + i_B p'_B k = i_A fh + i_B p_B g = i_A p_A g + i_B p_B g = g$. Thus every map $g : S \to A \oplus B$ with S small factors through $\bar{f}$, implying that $\bar{f}$ is an isomorphism. If γ is an inverse for $\bar{f}$, then we see easily that f is an isomorphism with inverse $p_L \gamma i_A$.

Now suppose that the category has kernels and that $M_i (i \in I)$ is finitely approximable. Let $f : L \to M = \Sigma_{i \in I} M_i$ be monic with the property that every map $S \to M$ with S small can be factored through f. Let p_i and e_i be the projection and injection maps associated with the sum $M = \Sigma_{i \in I} M_i$, and let $k_i : K_i \to L$ be a kernel for $(1 - e_i p_i)f$. Let $g_i = p_i f k_i$. Then $(1 - e_i p_i) f k_i = 0$ implies $e_i g_i = e_i p_i f k_i = f k_i$. Since both k_i and f are monic, g_i is also monic. Suppose $g : S \to M_i$ with S small. Then there is a map $h : S \to L$ with $fh = e_i g$. Now $(1 - e_i p_i) fh = (1 - e_i p_i) e_i g = 0$. Thus there is a map $h_i : S \to K_i$ with $k_i h_i = h$. Then using the fact that $fh = e_i g$ we have $g = p_i e_i g = p_i fh = p_i f k_i h_i = g_i h_i$. Thus $g_i : K_i \to M_i$ satisfies properties (i) and (ii) so that g_i is an isomorphism for each $i \in I$. Let $\alpha : M \to L$ be the unique map such that $\alpha e_i = k_i g_i^{-1} = e_i$, implying $f\alpha = 1_M$. Then $f(\alpha f) = (f\alpha) f = f$, with f monic, implies $\alpha f = 1_L$. Thus M is finitely approximable.

Lemma 3. *Let $\mathcal{A}$ be an additive category with kernels. The following hold.*

(1) *If $\pi \in \mathrm{End}(M)$ is an idempotent, then $M = \mathrm{Ker}\, \pi \oplus \mathrm{Ker}(1 - \pi)$.*

(2) *If $A \subset M$ is a kernel and $C \subset M$, then $A \cap C$ exists.*

(3) *If $A \subset S \subset A \oplus B \oplus C$, then $S \cap (A \oplus B) = A \oplus (S \cap B)$.*

(4) *If $A, B, C \subset M$ and B and C are kernels, then $A \cap (B \cap C)$ exists and equals $(A \cap B) \cap C$.*

Proof. (1) is well known. Suppose $e_A : A \to M$ is a kernel of $f : M \to N$, and let $e_C : C \to M$ be monic. Let $k : K \to C$ be a kernel for $f e_C : C \to N$. Then it is easy to show that $e_C k : K \to M$ is an intersection of the subobjects A and C of M. The rest of the lemma is straightforward. Note that (2) applies, in particular, when A is any direct summand of M, since a direct summand is a kernel.

Lemma 4. *Let $\mathcal{A}$ be an additive category with kernels. If $N \subset \Sigma_{i \in I} M_i$ and N is finitely approximable and nonzero, then there is a finite set $J \subset I$ such that $N \cap \Sigma_{i \in J} M_i \neq 0$.*

Proof. Since $0 \to N$ is not an isomorphism, but is monic, there is a small object S and a map $0 \neq f : S \to N$. Then $e_N f : S \to \Sigma_{i \in I} M_i$ factors through a finite sum $\Sigma_{i \in J} M_i$, and f induces a nonzero map $S \to N \cap \Sigma_{i \in J} M_i$. Thus the intersection is nonzero.

Definition. An object M in an additive category has the *exchange property* if for any object G, with $G = M \oplus N = \Sigma_{i\in I} G_i$, there are decompositions $G_i = G_i' \oplus H_i$ such that $G = M \oplus \Sigma_{i\in I} G_i'$. An object M has the *finite exchange property* if the above holds whenever the set I is finite.

Lemma 5. *If* $A \oplus B \oplus C = A \oplus \Sigma_{i\in I} D_i$ *and* B *has the exchange property, then* $A \oplus B \oplus C = A \oplus B \oplus \Sigma_{i\in I} D_i'$, *with* $D_i' \subset D_i (i \in I)$.

Proof. Injections and projections induce an isomorphism $B \oplus C \cong \Sigma_{i\in I} D_i$, and hence a decomposition $B' \oplus C' = \Sigma_{i\in I} D_i$ with $B' \cong B$. Thus $\Sigma_{i\in I} D_i = B' + \Sigma_{i\in I} D_i'$ with $D_i' \subset D_i$, and thus $A \oplus \Sigma_{i\in I} D_i = A \oplus B' \oplus \Sigma_{i\in I} D_i'$. Careful checking of the maps reveals that the isomorphism $B \cong B'$ is the map $p_{B'} e_B$, so by Lemma 1 we may replace B' by B in the direct sum.

Lemma 6. *If* $A_1, ..., A_n$ *have the exchange property then so does* $A_1 \oplus ... \oplus A_n$.

Proof. This follows easily from Lemma 5.

The converse of this lemma also holds. This fact is contained in [6]. The error in the proof, pointed out in [21], is in the nature of a misprint and can be corrected.

Proposition 1. *Let* $\mathcal{A}$ *be an additive category with kernels. If* M *has a local endomorphism ring, then* M *has the finite exchange property.*

Proof. Suppose $G = M \oplus C = \Sigma_{i=1}^n D_i$. Let $p_i : G \to D_i$ and $\pi_M : G \to M$ denote the projections, and $e_M : M \to G$ and $e_i : D_i \to G$ the injections. Then $1_M = \Sigma_{i=1}^n \pi_M e_i p_i e_M$ and since M has a local endomorphism ring, one of the summands, say $\gamma = \pi_M e_1 p_1 e_M$, must be an automorphism. Let $\alpha = e_1 p_1 e_M \gamma^{-1} \pi_M$. Then $\alpha^2 = \alpha$, so $G = \operatorname{Ker} \alpha \oplus \operatorname{Ker}(1 - \alpha)$. Let $k : \operatorname{Ker}(1 - \alpha) \to G$ be the injection. Then $0 = (1 - \alpha)k$ implies $k = \alpha k$, so $p_j \alpha = 0$ implies $p_j k = 0, j > 1$. Thus $\operatorname{Ker}(1 - \alpha) \subset D_1$. This implies $D_1 = \operatorname{Ker}(1 - \alpha) \oplus (D_1 \cap \operatorname{Ker} \alpha)$, and we have $G = \operatorname{Ker}(1 - \alpha) \oplus (D_1 \cap \operatorname{Ker} \alpha) \oplus \Sigma_{i=2}^n D_i$. The projection $G \to \operatorname{Ker}(1 - \alpha)$ relative to this direct sum is $\alpha' e_1 p_1$, where $k\alpha' = \alpha$. Now $\alpha' e_1 p_1 e_M : M \to \operatorname{Ker}(1 - \alpha)$ and $\gamma^{-1} \pi_M k : \operatorname{Ker}(1 - \alpha) \to M$ are inverses of each other. But $\alpha' e_1 p_1 e_M$ is the injection map of M into G followed by the projection onto the summand $\operatorname{Ker}(1 - \alpha)$. Thus by Lemma 4, $G = M \oplus (D_1 \cap \operatorname{Ker} \alpha) + \Sigma_{i=2}^n D_i$. Thus M has the finite exchange property.

Definition. An additive category satisfies a *weak Grothendieck condition* if for every index set I and every nonzero monic $A \to \Sigma_{i\in I} B_i$ there is a finite subset J of I and a commutative diagram

$$
\begin{array}{ccc}
C & \longrightarrow & \sum_{i\in J} B_i \\
\downarrow & & \downarrow \\
A & \longrightarrow & \sum_{i\in I} B_i
\end{array}
$$

with the map $C \to A$ nonzero. (Note that this is equivalent to saying $A \cap \Sigma_{i\in J} B_i \neq 0$, if the intersection exists, which it does, by Lemma 3, if the category has kernels.)

If every object is finitely approximable then the category satisfies a weak Grothendieck condition. (See the proof of Lemma 4.) This hypothesis occurs frequently for categories constructed from module categories. There are categories that satisfy the weak Grothendieck condition but fail to have enough small objects around for every object to be finitely approximable.

Proposition 2. *Let $\mathcal{A}$ be an additive category with kernels. If M is indecomposable with the finite exchange property and either* (i) *M is finitely approximable or* (ii) *the category satisfies a weak Grothendieck condition, then M has the exchange property.*

Proof. Let $G = M \oplus N = \Sigma_{i\in I} G_i$, and for $J \subset I$ let $G(J) = \Sigma_{i\in J} G_i$. Assuming either (i) or (ii), there is a finite subset J of I for which $M \cap \Sigma_{i\in J} G_i \neq 0$. Now by the finite exchange property, $G = M \oplus N = G(J) \oplus G(I \setminus J) = M \oplus \Sigma_{i\in J} G_i' \oplus E$, with $G_i' \subset G_i (i \in J)$ and $E \subset G(I \setminus J)$. Then $G_i = G_i' \oplus H_i$ and $G(I \setminus J) = E \oplus E'$, and $M \oplus (\Sigma_{i\in J} G_i' \oplus E) = (\Sigma_{i\in J} H_i \oplus E') \oplus (\Sigma_{i\in J} G_i' \oplus E)$ implies $M \cong \Sigma_{i\in J} H_i \oplus E'$. Since M is indecomposable, all but one of the H_i $(i \in J)$ and E' are zero. Suppose $E' \neq 0$ and $H_i = 0$ for all $i \in J$. Then $G_i' = G_i$ for all $i \in J$ implies $M \cap G(J) = 0$, contradicting the choice of J. Letting $G_i' = G_i$ for $i \in I \setminus J$, we have $G = M \oplus \Sigma_{i\in I} G_i'$, with $G_i' \subset G_i (i \in I)$. Thus M has the exchange property.

Proposition 3. *An indecomposable finitely approximable object in an additive category with kernels has the exchange property if and only if its endomorphism ring is local.*

Proof. Half of this proposition was proved in Propositions 1 and 2. For the other half, see the proof in [24] for modules.

Theorem 2. *Let $\mathcal{A}$ be an additive category with kernels, and let $M = \Sigma_{i\in I} M_i$, where the endomorphism ring of each M_i is a local ring. Suppose that either* (i) *each M_i is finitely approximable, or* (ii) *that $\mathcal{A}$ satisfies a weak Grothendieck condition. Then any indecomposable summand of M is isomorphic to one of the M_i.*

Proof. Let $M = N \oplus K = \Sigma_{i\in I} M_i$ with each endomorphism ring $\mathrm{End}(M_i)$ local, N nonzero and indecomposable. For $J \subset I$, let $M(J) = \Sigma_{i\in J} M_i$. If (i) holds (applying Lemmas 2 and 4) or if (ii) holds there is finite set $J \subset I$ such that $N \cap M(J) \neq 0$. By Propositions 1 and 2 and Lemma 6, $M(J)$ has the exchange property. Thus $M = M(J) \oplus N' \oplus K'$ with $N' \subset N$ and $K' \subset K$. Then $N = N' \oplus (N \cap (M(J) \oplus K'))$, $N \cap (M(J) \oplus K) \neq 0$, and N indecomposable, imply $N' = 0$. Thus $M = M(J) \oplus K'$, and $K = K' \oplus (K \cap M(J))$, and $M = N \oplus K' \oplus (K \cap M(J))$. This implies $M(J) \cong N \oplus (K \cap M(J))$, and we are reduced to the case I is finite. By Theorem 1, $K = \Sigma_{i=1}^{m} K_i$ where K_i is indecomposable. Since the category has kernels, Lemma 3 implies that in End(N) and End(K_i) there are no idempotents other than 0 or 1. By Theorem 1, $N \cong M_i$ for some i.

We now are ready for the main unique decomposition theorem of this paper.

Theorem 3. *Let $\mathcal{A}$ be an additive category with kernels and $M = \Sigma_{i\in I} M_i$ with the endomorphism ring of each M_i a local ring. Suppose that either* (i) *each M_i is finitely approximable or* (ii) *the category $\mathcal{A}$ satisfies a weak Grothendieck condition. Then if $M = \Sigma_{j\in J} N_j$ with each N_j indecomposable, there is a bijective map $\alpha : I \to J$ such that $M_i \cong N_{\alpha(i)}$ for each $i \in I$.*

Proof. By Theorem 2, each $N_j \cong M_i$ for some $i \in I$, and in particular, every M_i and every N_j has the exchange property. For $X \subset I$, $Y \subset J$ let $M(X) = \Sigma_{i\in X} M_i$ and $N(Y) = \Sigma_{j\in Y} N_j$. For $k \in I$, let $I_k = \{i \in I \mid M_i \cong M_k\}$. Then $\{I_k \mid k \in I\}$ partitions I. Let $J_k = \{j \in J \mid N_j \cong M_k\}$ for $k \in I$. Then $\{J_k \mid k \in I\}$ partitions J. Suppose I_k is finite. Then $M(I_k)$ has the exchange property, and $M = M(I_k) \oplus M(I \setminus I_k) = M(I_k) \oplus N(J')$, for some $J' \subset J$. And $M(I \setminus I_k) \cong N(J')$ implies that if $j \in J'$, then $N_j \not\cong M_k$ and thus $j \notin J_k$. Together with $M(I_k) \cong N(J \setminus J')$, this implies $J_k = J \setminus J'$. Thus $M(I_k) \cong N(J_k)$. From Theorem 1, it follows easily that $|I_k| = |J_k|$ whenever I_k is finite.

Now suppose I_k is infinite. Let $\Phi_k = \{j \in J \mid M = M_k \oplus N(J \setminus \{j\})\}$. There is a finite set $F \subset J$ such that $M_k \cap N(F) \neq 0$, by either (i) or (ii). If $j \notin F$ then $F \subset J \setminus \{j\}$ implies $M_k \cap N(J \setminus \{j\}) \neq 0$. Thus $\Phi_k \subset F$, so each Φ_k is finite. Now let $t \in J_k$. There is a finite subset G of I such that $N_t \cap M(G) \neq 0$, and $M = M(G) \oplus N(J \setminus H)$ for some finite subset H of J. Due to the nonzero intersection above, t must be in H. Then $M = N(H \setminus \{t\}) \oplus N_t \oplus N(J \setminus H) = M(G) \oplus N(J \setminus H) = N(H \setminus \{t\}) \oplus M_g \oplus N(J \setminus H)$ for some $g \in G$, since $N(H \setminus \{t\})$ has the exchange property (Lemma 5). Thus $t \in \Phi_g$. Also, $M_g \cong N_t \cong M_k$, so $g \in I_k$. Thus $J_k = \bigcup_{g\in I_k} \Phi_g$. Since I_k is infinite, and each Φ_g is finite and nonempty, it follows that $|J_k| \leqslant |I_k|$, and by symmetry, $|J_k| = |I_k|$.

3. An isomorphic refinement theorem in additive categories

Some countability hypothesis seems to be required for an isomorphic refinement theorem. The notion of countably small below replaces that of countably generated for modules.

We assume throughout this section that we have an additive category with kernels. For the isomorphic refinement theorem, we will find it necessary to also assume the category has infinite sums and satisfies a weak Grothendieck condition.

Definition. An object M is *countably small* if every homomorphism from M into a direct sum factors through a countable subsum.

Lemma. *Let $M = \Sigma_{i\in I} M_i = P \oplus Q$, with each M_i countably small and finitely approximable, and let $f : S \to M$ with S countably small. Then there is a countable subset J of I such that f factors through $M(J) = \Sigma_{i\in J} M_i \to M$ and such that $M(J) = (M(J) \cap P) \oplus (M(J) \cap Q)$.*

Proof. There is a countable subset $J_0 \subset I$ such that f factors through $M(J_0) \to M$. Let e_P and e_Q be the injections and π_P and π_Q the projections for the sum $M = P \oplus Q$, and let e_J and π_J be the corresponding maps for $M(J)$ relative to the sum $M = M(J) \oplus M(I \setminus J)$. There is a countable set $J_1 \subset I$ such that both $e_P \pi_P f$ and $e_Q \pi_Q f$ factor through $M(J_1) \to M$. There is a countable set $J_2 \subset I$ such that both $e_P \pi_P e_{J_1}$ and $e_Q \pi_Q e_{J_1}$ factor through $M(J_2) \to M$. Continuing in this fashion, get a countable set $J_{k+1} \subset I$ such that both $e_P \pi_P e_{J_k}$ and $e_Q \pi_Q e_{J_k}$ factor through $M(J_{k+1}) \to M$, $k = 1, 2, \ldots$, and let $J = \bigcup_{k=0}^{\infty} J_k$. Then J is countable and f factors through $M(J)$. Also $M(J) \supset (M(J) \cap P) \oplus (M(J) \cap Q)$. Let T be small and $g : T \to M(J)$. Then g factors through $M(J_k)$ for some k. There are commutative diagrams

$$
\begin{array}{ccc}
T & \xrightarrow{\ g\ } & M(J) \\
{\scriptstyle h}\downarrow & \nearrow{\scriptstyle e_k} & \downarrow{\scriptstyle e_J} \\
M(J_k) & \xrightarrow[e_{J_k}]{} & M
\end{array}
\qquad
\begin{array}{ccc}
M(J_k) & \xrightarrow{\ \pi_P e_{J_k}\ } & P \\
{\scriptstyle \alpha_k}\downarrow & & \downarrow{\scriptstyle e_P} \\
M(J_{k+1}) & \xrightarrow[e_{J_{k+1}}]{} & M
\end{array}
\qquad
\begin{array}{ccc}
M(J_k) & \xrightarrow{\ \pi_Q e_{J_k}\ } & Q \\
{\scriptstyle \beta_k}\downarrow & & \downarrow{\scriptstyle e_Q} \\
M(J_{k+1}) & \xrightarrow[e_{J_{k+1}}]{} & M
\end{array}\ .
$$

This gets $e_J g = e_J e_{k+1} \alpha_k h + e_J e_{k+1} \beta_k h$. But $e_{k+1} \alpha_k h$ factors through $P \cap M(J_{k+1}) \to M(J)$ and $e_{k+1} \beta_k h$ factors through $Q \cap M(J_{k+1}) \to M(J)$. Thus g factors through their sum. Since $M(J)$ is finitely approximable, this means $M(J) = (M(J) \cap P) \oplus (M(J) \cap Q)$.

The following is a version of Kaplansky's Theorem [12].

Theorem 4. *Let $\mathcal{A}$ be an additive category with kernels, and $M = \Sigma_{i\in I} M_i = P \oplus Q$, with each M_i countably small and finitely approximable. Then $P = \Sigma_{j\in J} P_j$ with each P_j isomorphic to a direct summand of a direct sum of a countable number of the M_i's.*

Proof. We construct a chain of subsets I_α of I such that

(i) $I = \bigcup_\alpha I_\alpha$.

(ii) If α is a limit, $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$.

(iii) $I_{\alpha+1} \setminus I_\alpha$ is countable and nonempty.

(iv) $M(I_\alpha) = (M(I_\alpha) \cap P) \oplus (M(I_\alpha) \cap Q)$.

Let $I_0 = \emptyset$. Assume $0 < \alpha$ and we have (ii)–(iv) for $\beta < \alpha$. If α is a limit, let $I_\alpha = \bigcup_{\beta<\alpha} I_\beta$. Let $M(I_\alpha) = \Sigma_{i\in I_\alpha} M_i$. Then $M(I_\alpha) \supset (M(I_\alpha) \cap P) \oplus M(I_\alpha \cap Q)$. If they are not equal, there is a map $f : S \to M(I_\alpha)$ with S small which does not factor through the sum. But f factors through $M(F)$ for some finite subset F of I_α, and $F \subset I_\beta$ for some $\beta < \alpha$. Therefore $M(I_\alpha) = (M(I_\alpha) \cap P) \oplus (M(I_\alpha) \cap Q)$. If $\alpha = \beta + 1$, and $M(I_\beta) \neq M$, let $f : S \to M$ be a map which does not factor through $M(I_\beta)$, with S small. There is a countable subset J of I such that f factors through $M(J) \to M$ and such that $M(J) = (M(J) \cap P) \oplus (M(J) \cap Q)$. Let $I_\alpha = I_\beta \cup J$. Then (ii)–(iv) are satisfied for α. Now from (iii), if $|\alpha| > |I|$ then $I_\alpha = I$. Thus $I = \bigcup_\alpha I_\alpha$. Now $M = M(I_\alpha) \oplus M(I \setminus I_\alpha) = (M(I_\alpha) \cap P) \oplus (M(I_\alpha) \cap Q) \oplus M(I \setminus I_\alpha)$ for each α. Since the I_α form a chain, $M(I_{\alpha+1}) \cap P = (M(I_\alpha) \cap P) \oplus P_\alpha$, where $P_\alpha = (M(I_{\alpha+1}) \cap P) \cap ((M(I_\alpha) \cap Q) \oplus M(I \setminus I_\alpha))$, for each α. Similarly, $M(I_{\alpha+1}) \cap Q = (M(I_\alpha) \cap Q) \oplus Q_\alpha$. Now $M(I_{\alpha+1}) = M(I_\alpha) + M(J_\alpha)$ with J_α countable ($J_\alpha = I_{\alpha+1} \setminus I_\alpha$). Also, $M(I_{\alpha+1}) = M(I_\alpha) \oplus P_\alpha \oplus Q_\alpha$, implying $P_\alpha \oplus Q_\alpha \cong M(J_\alpha)$. Thus P_α is isomorphic to a direct summand of a direct sum of a countable number of the M_i's.

It remains to show that $P = \Sigma_\alpha P_\alpha$. First, we show the map $e : \Sigma_\alpha P_\alpha \to P$ is monic. The map $P_0 \to P$ is monic. Suppose the map $e_\alpha : \Sigma_{\beta<\alpha} P_\beta \to P$ is monic. Then $P_\beta \subset (M(I_{\beta+1}) \cap P) \subset (M(I_\alpha) \cap P)$ all $\beta < \alpha$ implies e_α factors through $M(I_\alpha) \cap P$. Since $(M(I_\alpha) \cap P) \cap P_\alpha = 0$, we have $\Sigma_{\beta\leqslant\alpha} P_\beta \to P$ monic. Thus $\Sigma_\alpha P_\alpha \subset P$. To show they are equal, we show $M(I_\alpha) \cap P = \Sigma_{\beta<\alpha} P_\beta$. This works for $\alpha = 0$. Assume the equality holds for all $\alpha < \gamma$. If $\gamma = \alpha + 1$, then $M(I_\gamma) \cap P = (M(I_\alpha) \cap P) \oplus P_\alpha = \Sigma_{\beta<\gamma} P_\beta$. If γ is a limit ordinal, $I_\gamma = \bigcup_{\alpha<\gamma} I_\alpha$. Now $\Sigma_{\beta<\gamma} P_\beta \subset (M(I_\gamma) \cap P)$. Let $f : S \to (M(I_\gamma) \cap P)$ with S small. Then f factors through $M(F) \cap P \to M(I_\gamma) \cap P$ for some finite $F \subset I_\gamma$. But then $F \subset I_\alpha$ for some $\alpha < \gamma$, implying f factors through $M(I_\alpha) \cap P \to M(I_\gamma) \cap P$. Then $M(I_\alpha) \cap P = \Sigma_{\beta<\alpha} P_\beta \subset \Sigma_{\beta<\gamma} P_\beta$ implies that f factors through $\Sigma_{\beta<\gamma} P_\beta \to M(I_\gamma) \cap P$. Thus they are equal.

For the isomorphic refinement theorem, we require a somewhat stronger countability hypothesis.

Definition. An object M is *countably finitely approximable* if there is a countable family $\{f_i : S_i \to M\}_{i=1}^\infty$ of maps with each S_i small, with the property that a monic $f : L \to M$ is an isomorphism if and only if each f_i can be factored through f.

Proposition 4. *If M is countably finitely approximable, then M is countably small.*

Proof. Let $f : M \to A = \Sigma_{i\in I} A_i$. There is a countable family of maps $u_i : S_i \to M$ with S_i small, such that if $N \to M$ is monic and every u_i factors through $N \to M$, then $N \to M$ is an isomorphism. Let I_n be a finite subset of I such that fu_i factors through $A(I_n) = \Sigma_{i\in I_n} A_i \to A$. Let $I_0 = \bigcup_{i=1}^\infty I_i$. Let π_1, π_2, e_1, e_2 be the projections and injections for the direct sum $A = A(I_0) + A(I \setminus I_0)$. Let $N = \operatorname{Ker}(M \xrightarrow{f} A \xrightarrow{\pi_2} A(I \setminus I_0))$. Then every u_i factors through $N \to M$ since $\pi_2 f u_i = 0$ for all $i = 1, 2, \dots$. Thus $\pi_2 f = 0$, since this implies $N \to M$ is an isomorphism. Then $f = (e_1\pi_1 + e_2\pi_2)f = e_1\pi_1 f$ factors f through $A(I_0) \to A$. Therefore, M is countably small.

It is also true that a summand of a countably finitely approximable object is countably finitely approximable. The proof is essentially the same as that of Lemma

2. Also, by the same proof, a direct sum of a countable number of countably finitely approximable objects is countably finitely approximable.

Theorem 5. *Let $\mathcal{A}$ be an additive category with kernels and infinite sums which satisfies a weak Grothendieck condition. If $M = \Sigma_{i \in I} M_i = N \oplus K$, with each M_i countably finitely approximable, and with the endomorphism ring of each M_i a local ring, then N is isomorphic to a direct sum $\Sigma_{i \in J} M_i$, for some $J \subset I$. Consequently, any two direct decompositions of M have isomorphic refinements.*

Proof. By the previous theorem, $N = \Sigma_{\alpha \in X} N_\alpha$ with each N_α isomorphic to a direct summand of a direct sum of a countable number of the M_i's. Thus we may assume I is countable. If I is finite, the result follows from Theorems 1 and 2, so we may assume I is infinite and countable. Write $M = \Sigma_{i=1}^{\infty} M_i$, and let $M(k) = \Sigma_{i=1}^{k} M_i$. Let $\{f_k : S_k \to N\}_{k=1}^{\infty}$ be a family of maps which approximate N, with each S_k small. Let $N_0 = 0$. Assume $N_0, ..., N_k$ are given with $N_i \subset N$ for each i, the map $N(k) = \Sigma_{i=0}^{k} N_i \to N$ monic, $N(k)$ a direct summand of N, f_k factors through $N(k) \to N$, and $N(k)$ is isomorphic to a direct sum of a finite number of the M_i's. Then $M = N(k) \oplus \Sigma_{i=1}^{\infty} M_i'$ with $M_i' \subset M_i$. For some integer m, the composition $S_{k+1} \xrightarrow{f_{k+1}} N \to M$ factors through $N(k) \oplus M'(m) \to M$, where $M'(m) = \Sigma_{i=1}^{m} M_i'$. Thus f_{k+1} factors through $N \cap (N(k) + M'(m)) = N(k) \oplus (N \cap M'(m)) \to N$. Now $N = N(k) \oplus R_k$ for some R_k. Since $M'(m)$ has the exchange property, $M = N(k) \oplus M'(m) \oplus \Sigma_{i=m+1}^{\infty} M_i' = N(k) \oplus R_k \oplus K = N(k) \oplus M'(m) \oplus N' \oplus K'$, with $N' \subset R_k$ and $K' \subset K$. Then $N = N(k) \oplus N' \oplus (N \cap (M'(m) \oplus K'))$. Let $N_{k+1} = N \cap (M'(m) \oplus K')$ (using Lemma 5). Then f_{k+1} factors through $N(k) \oplus N_{k+1} = N(k+1) \to N$, the map $N(k+1) \to N$ is monic, and $N(k+1)$ is a direct summand of N. Now $K = K' \oplus H$, and $M = N(k) \oplus M'(m) \oplus N' \oplus K' = N(k) \oplus N_{k+1} \oplus N' \oplus K' \oplus H$ implies $M'(m) \cong N(k+1) \oplus H$. Thus N_{k+1} is isomorphic to a finite sum of M_i's. Thus by induction, we have a sequence $N_0, N_1, ...$ of subobjects of N satisfying the conditions above. Now $\Sigma_{i=0}^{\infty} N_i \to N$ is monic, for otherwise the kernel would have nonzero intersection with a finite sum, and hence it is an isomorphism since every f_k factors through it. Thus N is isomorphic to a direct sum of M_i's. It now follows from Theorem 3 that $N \cong \Sigma_{i \in J} M_i$ for some $J \subset I$.

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