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Stochastic integrals for spde's: A comparison

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ABSTRACT

We present the Walsh theory of stochastic integrals with respect to martingale measures, and various extensions of this theory, alongside of the Da Prato and Zabczyk theory of stochastic integrals with respect to Hilbert-space-valued Wiener processes, and we explore the links between these theories. Somewhat surprisingly, the end results of both theories turn out to be essentially equivalent. We then show how each theory can be used to study stochastic partial differential equations, with an emphasis on the stochastic heat and wave equations driven by spatially homogeneous Gaussian noise that is white in time. We compare the solutions produced by the different theories.

1. Introduction

The theory of stochastic partial differential equations (spde's) developed on the one hand, from the work of Walsh [41], and on the other hand, through work on stochastic evolution equations in Hilbert spaces, such as in [13]. Important milestones in the latter approach are the books of Da Prato and Zabczyk [14] and Rozovskii [37] (see also Krylov and Rozovskii [23] and Krylov [22]).

These two approaches led to the development of two distinct schools of study for spde's, based on different theories of stochastic integration: the Walsh theory, which emphasizes integration with respect to worthy martingale measures, and a theory of integration with respect to Hilbert-space-valued processes, as expounded in [14]. A consequence of the presence of these separate theories is

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that the literature published by each of the two schools is difficult to access when one has been trained in the other school. This is unfortunate since both approaches have advantages and in some problems, using both approaches can be useful (one example of this is [10]).

The objective of this paper is to help create links between these two schools of study. It is addressed to researchers who have some familiarity with at least one of the two approaches. We develop both theories, and explore the links between the two. Then we show how each theory is used to study spde's. The Walsh theory emphasizes solutions that are random fields, while [14] centers around solutions in Hilbert spaces of functions. Each theory is presented rather succinctly, the main focus being on relationships between the theories. We show that these theories often (but not always) lead to the same solutions to various spde's.

It should be mentioned that the general theory of integration with respect to Hilbert-space-valued processes and its generalizations – such as the stochastic integral with respect to cylindrical processes – was well-developed several years before [41] and more than a decade before reference [14] appeared: see, for instance, the book of Métivier and Pellaumail [25]. This reference, and several others, are cited in [14] and [41]. However, Walsh preferred to develop his own integral, even though he realized that the two were related (see the Notes at the end of [41]).

Here, we present in Section 2.1 a modern formulation of the theory of stochastic integrals with respect to cylindrical Wiener processes, as developed in [25], as a unifying integral behind most of those that were introduced later on. This integral is briefly recalled in Section 2.1. In Section 2.2, we show how spatially homogeneous Gaussian noise that is white in time can be viewed as a cylindrical Wiener process on a particular Hilbert space. Emphasizing this type of noise is natural, since in recent years, following in particular the papers of Mueller [28], Dalang and Frangos [7], Dalang [6] and Peszat and Zabczyk [31,32], this type of noise has been used by several researchers. This is due in part to the fact that it leads to a theory of non-linear spde's in spatial dimensions greater than 1, while non-linear spde's driven by space-time white noise generally only have a solution in spatial dimension 1. In Section 2.3, we show (Proposition 2.6) that the Walsh stochastic integral and the extension presented by Dalang [6] and Nualart and Quer-Sardanyons [29] can be viewed as integrals as defined in Section 2.1. Section 2.4 gives a wide class of integrable processes. In Section 2.5, we discuss the relationship between this integral and the function-valued stochastic integral introduced by Dalang and Mueller in [9]. A further extension of real-valued integrals to Hilbert-space-valued stochastic integrals was developed in [29,35,38]; these extensions were motivated by the needs of Malliavin calculus: indeed, the so-called Malliavin derivative of the solution to an spde satisfies a stochastic integral equation which requires a Hilbert-space-valued integral. We give a unified presentation of these extensions in Section 2.6.

In Section 3, we sketch the construction of the infinite dimensional stochastic integral in the setup of Da Prato and Zabczyk [14]. We also make use of the more recent presentation of Prévôt and Röckner [33]. In Section 3.1, we recall some basic properties of Hilbert–Schmidt operators. Section 3.2 gives the relationship between a Hilbert-space-valued Wiener process and a cylindrical Brownian motion, in the case where the covariance operator has finite trace. Hilbert-space-valued stochastic integrals are defined in Section 3.3. In particular, we show in Proposition 3.4 how this infinite-dimensional stochastic integral can be written as a series of Itô stochastic integrals. This is used in Section 3.4 to show how the integrals of Section 2 can be interpreted in the infinite-dimensional context. The case of covariance operators with infinite-trace is discussed in Section 3.5. We do not discuss Banach-space-valued stochastic integrals, for which we refer to [2,40]. Finally, in Section 3.6, we establish the somewhat unexpected but interesting fact that the extension of the Walsh stochastic integral presented in Section 2.6 and the Da Prato and Zabczyk integral of Section 3.5 are in fact equivalent.

It is well-known that in certain cases, the Hilbert-space-valued integral is equivalent to a martingale-measure stochastic integral. For instance, it is pointed out in [14, Section 4.3] that when the random perturbation is space-time white noise, then Walsh's stochastic integral in [41] is equivalent to an infinite-dimensional stochastic integral as in [14] (see also [19]). Of course, space-time white noise is only a special case of spatially homogeneous noise, and we are interested in comparing solutions to spde's driven by this more general noise. The function-valued approach of [9] gives solutions to spde's for which it is not known if a random field solution exists, and the Hilbert-space approach is even more general. However, for a wide class of spde's that have solutions in two or more of these formulations, such as the stochastic heat equation ($d \geq 1$) and wave equation ($d \in \{1, 2, 3\}$) driven by spatially homogeneous noise, we will show that the solutions turn out to be equivalent. One does not expect this to be the case in all situations. Indeed, there are a few cases in which a solution exists with one approach and is known *not to exist* in one of the others. For instance, for noise concentrated on a hyperplane, as considered in [8], the authors establish existence of function-valued solutions and show that there is no random field solution.

In Section 4, we consider spde's driven by spatially homogeneous noise, with an emphasis on the stochastic heat and wave equations. In Sections 4.1–4.3, we discuss the random field approach, and we use the stochastic integral of Section 2.3 to extend the result of [6] to arbitrary initial conditions (Theorem 4.3). In Section 4.4, we discuss the Hilbert-space-valued approach to the study of the same equations, using the approach of [32]. In Section 4.5, we show that the mild random field solution of Theorem 4.3, when interpreted as a Hilbert-space-valued process, yields the solution given in [32]. This is achieved by identifying the multiplicative non-linearity with an appropriate Hilbert–Schmidt operator, and using the relationships between stochastic integrals identified in Section 3. Since the two solutions are defined using different Hilbert spaces, the embedding from one Hilbert space to the other has to be written explicitly. Finally, in Section 4.6, we compare the random field solution of the stochastic wave equation with the function-valued solution constructed in [9]. Again, in cases where both types of solutions are defined, that is, in spatial dimensions $d \in \{1, 2, 3\}$, we show that the random field solution yields the function-valued solution (Theorem 4.13). Overall, Section 4 unifies the existing literature on the stochastic heat and wave equations driven by spatially homogeneous noise, and clarifies the relationships between the various approaches.

2. Stochastic integrals with respect to a spatially homogeneous Gaussian noise

In this section, we recall in Section 2.1 the notion of cylindrical Wiener process and the stochastic integral with respect to such processes. In Section 2.2, we introduce a spatially homogeneous Gaussian noise that is white in time, and we show how to interpret this noise as a cylindrical Wiener process. Building on material presented in [29], we relate in Section 2.3 the stochastic integral with respect to this particular cylindrical Wiener process with Walsh's martingale measure stochastic integral and the extension given by Dalang in [6]. Some examples of integrands are given in Section 2.4. In Section 2.5, we discuss the function-valued extension given in Dalang and Mueller [9]. Finally, in Section 2.6, we give a unified presentation of the Hilbert-space-valued stochastic integral developed in [29,35,38].

2.1. Stochastic integration with respect to a cylindrical Wiener process

Fix a separable Hilbert space V with inner product $\langle \cdot, \cdot \rangle_V$. Following [18,25], we define the general notion of cylindrical Wiener process in V.

Definition 2.1. Let Q be a symmetric (self-adjoint) and non-negative definite bounded linear operator on V. A family of random variables $B = \{B_t(h),\ t \geq 0,\ h \in V\}$ is a *cylindrical Wiener process* on V if the following two conditions are fulfilled:

1. for any $h \in V$, $\{B_t(h),\ t \geq 0\}$ defines a Brownian motion with variance $t\langle Qh, h\rangle_V$;
2. for all $s, t \in \mathbb{R}_+$ and $h, g \in V$,

$$E\left(B_s(h)B_t(g)\right) = (s \wedge t)\langle Qh, g\rangle_V,$$

where $s \wedge t := \min(s, t)$. If $Q = \mathrm{Id}_V$ is the identity operator in V, then B will be called a *standard cylindrical Wiener process*. We will refer to Q as the *covariance* of B.

Let $\mathcal{F}_t$ be the σ-field generated by the random variables $\{B_s(h),\ h \in V,\ 0 \leq s \leq t\}$ and the P-null sets. We define the predictable σ-field as the σ-field in $[0, T] \times \Omega$ generated by the sets $\{(s, t] \times A,\ A \in \mathcal{F}_s,\ 0 \leq s < t \leq T\}$.

We denote by V_Q (the completion of) the Hilbert space V endowed with the inner-product

$$\langle h, g\rangle_{V_Q} := \langle Qh, g\rangle_V, \qquad h, g \in V.$$

We can now define the stochastic integral of any predictable square-integrable process with values in V_Q, as follows. Let $(v_j)_j$ be a complete orthonormal basis of the Hilbert space V_Q. For any predictable process $g \in L^2(\Omega \times [0, T]; V_Q)$, it turns out that the following series is convergent in $L^2(\Omega, \mathcal{F}, P)$ and the sum does not depend on the chosen orthonormal system:

$$g \cdot B := \sum_{j=1}^{\infty} \int_0^T \langle g_s, v_j\rangle_{V_Q} \, dB_s(v_j). \tag{2.1}$$

We notice that each summand in the above series is a classical Itô integral with respect to a standard Brownian motion, and the resulting stochastic integral is a real-valued random variable. The stochastic integral $g \cdot B$ is also denoted by $\int_0^T g_s \, dB_s$. The independence of the terms in the series (2.1) leads to the isometry property

$$E\left((g \cdot B)^2\right) = E\left(\left(\int_0^T g_s \, dB_s\right)^2\right) = E\left(\int_0^T \|g_s\|_{V_Q}^2 \, ds\right).$$

We note that there is an alternative way of defining this integral: one can start by defining the stochastic integral in (2.1) for a class of *simple* predictable V_Q-valued processes, and then use the isometry property to extend the integral to elements of $L^2(\Omega \times [0, T]; V_Q)$ by checking that these simple processes are dense in this set.

2.2. Spatially homogeneous noise as a cylindrical Wiener process

We now define the Gaussian random noise that will play a central role in this paper. On a complete probability space $(\Omega, \mathcal{F}, P)$, we consider a family of mean zero Gaussian random variables $W = \{W(\varphi), \varphi \in \mathcal{C}_0^\infty(\mathbb{R}^{d+1})\}$, where $\mathcal{C}_0^\infty(\mathbb{R}^{d+1})$ denotes the space of infinitely differentiable functions with compact support, with covariance

$$E(W(\varphi)W(\psi)) = \int_0^\infty dt \int_{\mathbb{R}^d} \Lambda(dx) \, (\varphi(t) * \tilde{\psi}(t))(x), \tag{2.2}$$

where "$*$" denotes convolution in the spatial variable and $\tilde{\psi}(t, x) := \psi(t, -x)$.

In the above, Λ is a non-negative and non-negative definite tempered measure on $\mathbb{R}^d$, it is therefore the Fourier transform of a non-negative tempered measure μ on $\mathbb{R}^d$. That is, by definition of the Fourier transform on the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions (see [39]), for all φ belonging to the space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing $\mathcal{C}^\infty$ functions,

$$\int_{\mathbb{R}^d} \varphi(x) \, \Lambda(dx) = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi) \, \mu(d\xi),$$

and there is an integer $m \geq 1$ such that

$$\int_{\mathbb{R}^d} (1 + |\xi|^2)^{-m} \mu(d\xi) < \infty. \tag{2.3}$$

We have denoted by $\mathcal{F}\varphi$ the Fourier transform of $\varphi \in \mathcal{S}(\mathbb{R}^d)$:

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} \varphi(x) e^{-2\pi i \xi \cdot x} \, dx.$$

The measure μ is called the *spectral measure* of W and is necessarily symmetric (see [39, Chap. VII, Théorème XVII]). The covariance (2.2) can also be written, using elementary properties of the Fourier transform, as

$$E(W(\varphi)W(\psi)) = \int_0^\infty dt \int_{\mathbb{R}^d} \mu(d\xi)\, \mathcal{F}\varphi(t)(\xi)\, \overline{\mathcal{F}\psi(t)(\xi)}.$$

Remark 2.2. In the case where the measure $\Lambda(dx)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$, with density f, formula (2.2) becomes

$$\int_0^\infty dt \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} dy\, \varphi(t,x) f(x-y)\, \psi(t,y),$$

which makes clear the spatially homogeneous character of the noise.

It is natural to associate a Hilbert space with W: let U the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ endowed with the semi-inner product

$$\langle \varphi, \psi \rangle_U = \int_{\mathbb{R}^d} \Lambda(dx)(\varphi * \tilde{\psi})(x) = \int_{\mathbb{R}^d} \mu(d\xi)\, \mathcal{F}\varphi(\xi)\, \overline{\mathcal{F}\psi(\xi)}, \tag{2.4}$$

$\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$, and associated semi-norm $\| \cdot \|_U$. Then U is a separable Hilbert space that may contain Schwartz distributions (see [6, Example 6]).

Remark 2.3. Let $\tilde{L}^2(\mathbb{R}^d, d\mu)$ be the subspace of $L^2(\mathbb{R}^d, d\mu)$ consisting of functions ϕ such that $\tilde{\phi} = \phi$. It is not difficult to check that one can identify U with the set $\{\Psi \in \mathcal{S}'(\mathbb{R}^d) : \Psi = \mathcal{F}^{-1}\phi$, where $\phi \in \tilde{L}^2(\mathbb{R}^d, d\mu)\}$, with inner product

$$\langle \mathcal{F}^{-1}\phi, \mathcal{F}^{-1}\varphi \rangle_U = \langle \phi, \varphi \rangle_{L^2(\mathbb{R}^d, d\mu)}, \qquad \phi, \varphi \in \tilde{L}^2(\mathbb{R}^d, d\mu).$$

We fix a time interval $[0, T]$ and we set $U_T := L^2([0, T]; U)$. This set is equipped with the norm given by

$$\|g\|_{U_T}^2 = \int_0^T \|g(s)\|_U^2\, ds.$$

We now associate a cylindrical Wiener process to W, as follows. A direct calculation using (2.2) shows that the generalized Gaussian random field $\{W(\varphi), \varphi \in \mathcal{C}_0^\infty([0, T] \times \mathbb{R}^d)\}$ is a random linear functional, in the sense that $W(a\varphi + b\psi) = aW(\varphi) + bW(\psi)$, and $\varphi \mapsto W(\varphi)$ is an isometry from $(\mathcal{C}_0^\infty([0, T] \times \mathbb{R}^d), \| \cdot \|_{U_T})$ into $L^2(\Omega, \mathcal{F}, P)$. The following lemma identifies the completion of $\mathcal{C}_0^\infty([0, T] \times \mathbb{R}^d)$ with respect to $\| \cdot \|_{U_T}$.

Lemma 2.4. *The space $\mathcal{C}_0^\infty([0, T] \times \mathbb{R}^d)$ is dense in $U_T = L^2([0, T]; U)$ for $\| \cdot \|_{U_T}$.*

Proof. Following [29], we will use the notation $\varphi_1(\,\cdot\,)$ to indicate that φ_1 is a function $t \mapsto \varphi_1(t)$ of the time-variable, and $\varphi_2(\,\star\,)$ to indicate that φ_2 is a function $x \mapsto \varphi_2(x)$ of the spatial variable.

Let C denote the closure of $\mathcal{C}_0^\infty([0, T] \times \mathbb{R}^d)$ in U_T for $\| \cdot \|_{U_T}$. Clearly, C is a subspace of U_T. The proof can be split into three parts.

Step 1. We show that elements of U_T of the form $\varphi_1(\,\cdot\,)\varphi_2(\,\star\,)$, where $\varphi_1 \in \mathcal{C}_0^\infty(\mathbb{R}_+; \mathbb{R})$ with support included in $[0, T]$ and $\varphi_2 \in \mathcal{S}(\mathbb{R}^d)$, belong to C. Using the fact that

$$\int_{\mathbb{R}^d} \Lambda(dx)\, (|\varphi_2| * |\tilde{\varphi}_2|)(x) < \infty$$

because Λ is a tempered measure and $|\varphi_2| * |\tilde{\varphi}_2|$ decreases rapidly, together with dominated convergence, one checks that there is a sequence $(\varphi_2^n)_n \subset \mathcal{C}_0^\infty(\mathbb{R}^d)$ such that $\lim_{n\to\infty} \|\varphi_2 - \varphi_2^n\|_U = 0$.

Then, by the very definition of the norm in U_T, one easily proves that $\lim_{n\to\infty}\|\varphi_1\varphi_2-\varphi_1\varphi_2^n\|_{U_T}=0$. Therefore, $\varphi_1(\,\cdot\,)\varphi_2(\,\star\,)\in U_T$.

Step 2. Suppose that we are given $\varphi_1\in L^2([0,T];\mathbb{R})$ and $\varphi_2\in\mathcal{S}(\mathbb{R}^d)$. We show that $\varphi_1(\,\cdot\,)\varphi_2(\,\star\,)\in C$. Indeed, let $(\varphi_1^n)_n\in\mathcal{C}_0^\infty(\mathbb{R}_+)$ be such that, for all n, the support of φ_1^n is contained in $[0,T]$ and $\varphi_1^n\to\varphi_1$ in $L^2([0,T];\mathbb{R})$. Then $\varphi_1^n\varphi_2\in C$ by Step 1, and one checks that $\varphi_1^n\varphi_2$ converges, as n tends to infinity, to $\varphi_1\varphi_2$ in U_T. Therefore, $\varphi_1(\,\cdot\,)\varphi_2(\,\star\,)\in C$.

Step 3. Suppose that $\varphi\in U_T$. We show that $\varphi\in C$. Indeed, let $(e_j)_j$ be a complete orthonormal basis of U with $e_j\in\mathcal{S}(\mathbb{R}^d)$, for all j. Then, since $\varphi(s)\in U$ for any $s\in[0,T]$,

$$\|\varphi\|_{U_T}^2=\int_0^T\|\varphi(s)\|_U^2\,ds=\sum_{j=1}^\infty\int_0^T\langle\varphi(s),e_j\rangle_U^2\,ds.$$

In particular, for any $j\geq 1$, the function $s\mapsto\langle\varphi(s),e_j\rangle_U$ belongs to $L^2([0,T];\mathbb{R})$. Thus, it follows from Step 2 that

$$\varphi^n(\cdot):=\sum_{j=1}^n\langle\varphi(\cdot),e_j\rangle_U\,e_j$$

belongs to C. Moreover, it is straightforward to verify that $\|\varphi-\varphi^n\|_{U_T}^2\to 0$ as $n\to\infty$. This shows that $\varphi\in C$. □

Therefore, taking into account the above lemma, $W(\varphi)$ can be defined for all $\varphi\in U_T$ following the standard method for extending an isometry. This establishes the following property.

Proposition 2.5. *For $t\geq 0$ and $\varphi\in U$, set $W_t(\varphi)=W(1_{[0,t]}(\,\cdot\,)\varphi(\,\star\,))$. Then the process $W=\{W_t(\varphi),\ t\geq 0,\ \varphi\in U\}$ is a cylindrical Wiener process as defined in* Section 2.1, *with V there replaced by U and $Q=\mathrm{Id}_U$. In particular, for any $\varphi\in U$, $\{W_t(\varphi),\ t\geq 0\}$ is a Brownian motion with variance $t\|\varphi\|_U$ and for all $s,t\geq 0$ and $\varphi,\psi\in U$, $E(W_t(\varphi)W_s(\psi))=(s\wedge t)\langle\varphi,\psi\rangle_U$.*

With this proposition, it becomes possible to use the stochastic integral defined in Section 2.1. This defines the stochastic integral $g\cdot W$ for all $g\in L^2(\Omega\times[0,T];U)\equiv L^2(\Omega;U_T)$. By definition of U, the complete orthonormal basis $(e_j)_j$ in the definition of $g\cdot W$ can be chosen such that $(e_j)_j\subset\mathcal{S}(\mathbb{R}^d)$.

Before discussing this further, we first relate the statement of Proposition 2.5 to Walsh's theory of stochastic integrals with respect to martingale measures. Let us recall that Walsh's theory of stochastic integration is based on the concept of *martingale measure*, which is a stochastic process of the form $\{M_t(A),\ \mathcal{F}_t,\ t\in[0,T],\ A\in\mathcal{B}_b(\mathbb{R}^d)\}$, where $\mathcal{B}_b(\mathbb{R}^d)$ denotes the set of bounded Borel sets of $\mathbb{R}^d$, and $(\mathcal{F}_t)_t$ is a filtration satisfying the *usual conditions*. For the precise definition of a martingale measure, we refer to [41, Chapter 2]. Hence, in order to use Walsh's construction, one has first to extend the generalized random field $\{W(\varphi),\ \varphi\in\mathcal{C}_0^\infty(\mathbb{R}_+\times\mathbb{R}^d)\}$ to a martingale measure. More precisely, using an approximation procedure similar to the one used in Lemma 2.4, one extends the definition of W to indicator functions of bounded Borel sets in $\mathbb{R}_+\times\mathbb{R}^d$ (for details see [7] or [34, p. 13]). Then one sets

$$M_t(A)=W(1_{[0,t]}(\cdot)1_A(\star)),\qquad t\in[0,T],\ A\in\mathcal{B}_b(\mathbb{R}^d).\tag{2.5}$$

Moreover, if we let $(\mathcal{F}_t)_t$ be the filtration generated by $\{M_t(A),\ A\in\mathcal{B}_b(\mathbb{R}^d)\}$ (completed and made right-continuous), then the process $\{M_t(A),\ \mathcal{F}_t,\ t\in[0,T],\ A\in\mathcal{B}_b(\mathbb{R}^d)\}$ defines a worthy martingale measure in the sense of Walsh [41]. Its *covariance measure* is determined by

$$\langle M(A),M(B)\rangle_t=t\int_{\mathbb{R}^d}\Lambda(dx)\,(1_A*\tilde{1}_B)(x),$$

$t\in[0,T]$, $A,B\in\mathcal{B}_b(\mathbb{R}^d)$, and its *dominating measure* coincides with the covariance measure (see [7]).

One easily checks that, for $\varphi \in \mathcal{S}(\mathbb{R}^d)$,

$$W_t(\varphi) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^d} 1_{[0,\,t]}(s)\, \varphi(x)\, M(ds, dx),$$

where the integral on the right-hand side is Walsh's stochastic integral.

2.3. The real-valued stochastic integral for spatially homogeneous noise

The aim of this section is to exhibit the relationship between the stochastic integral constructed in Section 2.1 and the random field approach of Walsh [41] and Dalang [6]. Recall that the stochastic integral with respect to M defined in [41] only allows function-valued integrands, and this theory was extended in [6] in order to cover more general integrands, such as certain processes with values in the space of (Schwartz) distributions. We are going to show that these two integrals can be interpreted in the context of Section 2.1.

Recall that Walsh's stochastic integral $g \cdot M$ is defined when $g \in \mathcal{P}_+$, where $\mathcal{P}_+$ is the set of predictable processes $(\omega, t, x) \mapsto g(t, x;\omega)$ such that

$$\|g\|_+^2 := E\left(\int_0^T dt \int_{\mathbb{R}^d} \Lambda(dx)\, (|g(t,\star)| * |\tilde{g}(t,\star)|)(x)\right) < \infty.$$

For $g \in \mathcal{P}_+$, we can consider that $g \in L^2(\Omega;U_T)$ and set

$$\|g\|_0^2 := E(\|g\|_{U_T}^2) = E\left(\int_0^T dt \int_{\mathbb{R}^d} \Lambda(dx)\, (g(t,\star) * \tilde{g}(t,\star))(x)\right). \tag{2.6}$$

In [6], Dalang considered (in the case $\Lambda(dx) = f(x)dx$) the set $\mathcal{P}_0$, which is the completion with respect to $\|\cdot\|_0$ of the subset $\mathcal{E}_0$ of $\mathcal{P}_+$ that consists of functions $g(s, x;\omega)$ such that $x \mapsto g(s, x;\omega) \in \mathcal{S}(\mathbb{R}^d)$, for all s and ω, and he defined the stochastic integral $g \cdot M$ for all $g \in \mathcal{P}_0$.

Finally, in order to use the stochastic integral of Section 2.1, let $(e_j)_j \subset \mathcal{S}(\mathbb{R}^d)$ be a complete orthonormal basis of U, and consider the cylindrical Wiener process $\{W_t(\varphi)\}$ defined in Proposition 2.5. For any predictable process $g \in L^2(\Omega \times [0, T];U)$, the stochastic integral of g with respect to W is

$$g \cdot W = \int_0^T g_s\, dW_s := \sum_{j=1}^{\infty} \int_0^T \langle g_s, e_j\rangle_U\, dW_s(e_j), \tag{2.7}$$

and the isometry property is given by

$$E\left((g \cdot W)^2\right) = E\left(\left(\int_0^T g_s\, dW_s\right)^2\right) = E\left(\int_0^T \|g_s\|_U^2\, ds\right). \tag{2.8}$$

We note that the right-hand side of (2.7) is essentially the definition of $W(\varphi)$ in [24,26,27]. We also use the notation

$$\int_0^T \int_{\mathbb{R}^d} g(s, y)\, W(ds, dy)$$

instead of $\int_0^T g_s\, dW_s$.

Proposition 2.6. *(a) If $g \in \mathcal{P}_+$, then $g \in L^2(\Omega \times [0, T];U)$ and $g \cdot M = g \cdot W$, where the left-hand side is a Walsh integral and the right-hand side is defined as in (2.7).*

(b) If $g \in \mathcal{P}_0$, then $g \in L^2(\Omega \times [0, T];U)$ and $g \cdot M = g \cdot W$, where the left-hand side is a Dalang integral and the right-hand side is defined as in (2.7).

Proof. Let us prove part (a) in the statement. We recall the inclusion $\mathcal{P}_+ \subset \mathcal{P}_0$, observed in [6], and we note that if $g \in \mathcal{P}_+$, then

$$\|g\|^2_{L^2(\Omega\times[0,T];\,U)} = E\left(\int_0^T dt \int_{\mathbb{R}^d} \Lambda(dx)\,(g(t,\star) * \tilde{g}(t,\star))(x)\right) \leq \|g\|^2_+ < +\infty, \tag{2.9}$$

and, in particular $g \in L^2(\Omega \times [0,T];U)$. Indeed, the equality in (2.9) holds when $g \in C_0^\infty([0,T]\times\mathbb{R}^d)$, and it also holds (by monotone approximation) for $g(s,x) = 1_{[a,b]}(s)\,1_A(x)$, when A is a product of open intervals, and finally, it holds for all $g \in \mathcal{P}_+$ via the Monotone Class Theorem (using arguments similar to those in the proof of [4, Theorem 2.6(b)]).

Secondly, in order to check the equality of the integrals, we use the fact that the set of elementary processes is dense in $(\mathcal{P}_+, \|\cdot\|_+)$ (see [41, Proposition 2.3]). Hence, by inequality (2.9), it suffices to show that both integrals coincide when g is an elementary process of the form

$$g(t,x;\omega) = 1_{(a,\,b]}(t)1_A(x)X(\omega), \tag{2.10}$$

where $0 \leq a < b \leq T$, $A \in \mathcal{B}_b(\mathbb{R}^d)$ and X is a bounded and $\mathcal{F}_a$-measurable random variable.

On one hand, when g has the particular form (2.10), according to [41] and (2.5),

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^d} g(t,x)\,M(dt,dx) &= [M_b(A) - M_a(A)]X = \left[W(1_{(0,\,b]}(\cdot)1_A(\star)) - W(1_{(0,\,a]}(\cdot)1_A(\star))\right]X\\ &= W(1_{(a,\,b]}(\cdot)1_A(\star))X.\end{aligned}$$

On the other hand, by the very definition of the integral (2.7),

$$\begin{aligned}\int_0^T g_t\,dW_t &= \sum_{j=1}^\infty \int_a^b X\langle 1_A, e_j\rangle_U\,dW_t(e_j) = X\sum_{j=1}^\infty \langle 1_A, e_j\rangle_U \left[W_b(e_j) - W_a(e_j)\right]\\ &= X\sum_{j=1}^\infty \langle 1_A, e_j\rangle_U\,W(1_{(a,\,b]}(\cdot)e_j) = XW(1_{(a,\,b]}(\cdot)1_A(\star)),\end{aligned}$$

which implies that

$$\int_0^T\int_{\mathbb{R}^d} g(t,x)M(dt,dx) = \int_0^T g_t\,dW_t,$$

for all g of the form (2.10). This concludes the first part of the proof.

Concerning part (b), let us point out that $\mathcal{P}_0$ is the completion of $\mathcal{E}_0$ with respect to $\|\cdot\|_0$ (see (2.6)), where the latter coincides with the norm in $L^2(\Omega\times[0,T];U)$ for smooth elements. Hence, since $\mathcal{E}_0 \subset \mathcal{P}_+ \subset L^2(\Omega\times[0,T];\,U)$, any $\|\cdot\|_0$-limit g of a sequence $(g_n)_n \subset \mathcal{E}_0$ will determine a well-defined element in $L^2(\Omega\times[0,T];U)$.

Moreover, as a consequence of this, we will only need to check the equality of the integrals for integrands g in $\mathcal{E}_0$. Since such elements are contained in $\mathcal{P}_+$, Dalang's integral of g with respect to the martingale measure M turns out to be a Walsh integral, so that we can conclude by using the first part of the proof. $\square$

Remark 2.7. According to Proposition 2.6, when one integrates an element of $\mathcal{P}_+$, it is possible to use either the Walsh integral or the integral with respect to a cylindrical Wiener process. However, the Walsh integral enjoys additional properties, in part because it is possible to make use of the dominating measure, which can be very useful in certain estimates. For example, establishing Hölder continuity of the solution to the 1-dimensional stochastic wave equation, in which a Walsh integral appears, is an easy exercise [41, Exercise 3.7], while for the 3-dimensional stochastic wave equation, this is quite involved [12].

2.4. Examples of integrands

In this section, we aim to provide useful examples of random distributions which belong to $L^2(\Omega \times [0,T];U)$, that is, for which we can define the stochastic integral (2.7) with respect to W.

Recall that an element $\Theta \in \mathcal{S}'(\mathbb{R}^d)$ is a *non-negative distribution with rapid decrease* if Θ is a non-negative measure and if

$$\int_{\mathbb{R}^d} (1+|x|^2)^{k/2}\, \Theta(dx) < +\infty,$$

for all $k > 0$ (see [39]).

Recall that μ is the spectral measure of W. We consider the following hypothesis.

Hypothesis 2.8. Let Γ be a function defined on $\mathbb{R}_+$ with values in $\mathcal{S}'(\mathbb{R}^d)$ such that, for all $t > 0$, $\Gamma(t)$ is a non-negative distribution with rapid decrease, and

$$\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t)(\xi)|^2 < \infty. \tag{2.11}$$

In addition, Γ is a non-negative measure of the form $\Gamma(t,dx)\,dt$ such that, for all $T > 0$,

$$\sup_{0\le t\le T} \Gamma(t,\mathbb{R}^d) < \infty.$$

The main examples of integrands are provided by the following proposition (see [29, Proposition 3.3 and Remark 3.4]). In comparison with the analogous result by Dalang [6, Theorem 2], Proposition 2.9 does not require that the stochastic process Z has a spatially homogeneous covariance (see Hypothesis A in [6]).

Proposition 2.9. *Assume that Γ satisfies Hypothesis 2.8. Let $Z = \{Z(t,x),\ (t,x) \in [0,T]\times\mathbb{R}^d\}$ be a predictable process such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|Z(t,x)|^p) < \infty, \tag{2.12}$$

for some $p \ge 2$. Then, the random measure $G = \{G(t,dx) = Z(t,x)\Gamma(t,dx),\ t\in[0,T]\}$ is a predictable process with values in $L^p(\Omega\times[0,T];U)$. Moreover,

$$E\left(\|G\|^2_{U_T}\right) = E\left[\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}(\Gamma(t)Z(t))(\xi)|^2\right]$$

and

$$E\left(|G\cdot W|^p\right) \le C \int_0^T dt \left(\sup_{x\in\mathbb{R}^d} E(|Z(t,x)|^p)\right) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t)(\xi)|^2.$$

The integral of $G = \{G(t,dx) = Z(t,x)\Gamma(t,dx),\ t\in[0,T]\}$ with respect to W will be also denoted by

$$G\cdot W = \int_0^T \int_{\mathbb{R}^d} \Gamma(s,y)Z(s,y)W(ds,dy). \tag{2.13}$$

It is worth pointing out two key steps in the proof of this proposition (see [29]): the first is to check that under Hypothesis 2.8, Γ belongs to $U_T = L^2([0,T];U)$; the second is to notice that if Γ and Z satisfy, respectively, Hypothesis 2.8 and condition (2.12), then $G(t) = Z(t,\star)\Gamma(t,\star)$ defines a distribution with rapid decrease, almost surely.

Remark 2.10. We note that [5] presents a further extension of Walsh's stochastic integral, with which it becomes possible to integrate certain random elements of the form $Z(t,\star)\Gamma(t,\star)$, where Γ is a tempered distribution which is not necessarily non-negative. This extension is useful for studying the stochastic wave equation in high spatial dimensions.

2.5. The Dalang–Mueller extension of the stochastic integral

We briefly summarize here the function-valued stochastic integral constructed in [9]. This is an extension of Walsh's stochastic integral, where one integrates processes that take values in $L^2(\mathbb{R}^d)$ (or a weighted L^2-space) and the value of the integral is in the same L^2-space.

Suppose that $s \mapsto \Gamma(s) \in \mathcal{S}'(\mathbb{R}^d)$ satisfies:

(1) For all $s \geq 0$, $\mathcal{F}\Gamma(s)$ is a function and

$$\int_0^T ds \sup_{\xi \in \mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\eta)|\mathcal{F}\Gamma(s)(\xi - \eta)|^2 < +\infty.$$

(2) For all $\phi \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\sup_{0 \leq s \leq T} \Gamma(s) * \phi$ is a bounded function on $\mathbb{R}^d$.

Suppose that $s \mapsto Z(s) \in L^2(\mathbb{R}^d)$ satisfies:

(3) For $0 \leq s \leq T$, $Z(s) \in L^2(\mathbb{R}^d)$ a.s., $Z(s)$ is $\mathcal{F}_s$-measurable, and $s \mapsto Z(s)$ is mean-square continuous from $[0, T]$ into $L^2(\mathbb{R}^d)$.

For such Γ and Z, one sets

$$I_{\Gamma,Z} := \int_0^T ds \int_{\mathbb{R}^d} d\xi\, E\left(|\mathcal{F}Z(s)(\xi)|^2\right) \int_{\mathbb{R}^d} \mu(d\eta)|\mathcal{F}\Gamma(s)(\xi - \eta)|^2 < +\infty. \tag{2.14}$$

Then the stochastic integral

$$v_{\Gamma,Z} = \int_0^T \int_{\mathbb{R}^d} \Gamma(s, \star - y)Z(s, y)\, M(ds, dy) \tag{2.15}$$

is defined as an element of $L^2(\Omega \times \mathbb{R}^d, dP \times dx)$, such that

$$E\left(\|v_{\Gamma,Z}\|^2_{L^2(\mathbb{R}^d)}\right) = I_{\Gamma,Z}. \tag{2.16}$$

This definition is obtained in three steps.

(a) If, in addition to (1), $\Gamma(s) \in \mathcal{C}^\infty(\mathbb{R}^d)$, for $0 \leq s \leq T$, and in addition to (3), $Z(s) \in \mathcal{C}_0^\infty(\mathbb{R}^d)$ and there is a compact $K \subset \mathbb{R}^d$ such that supp $Z(s) \subset K$, for $0 \leq s \leq T$, then

$$v_{\Gamma,Z}(x) = \int_0^T \int_{\mathbb{R}^d} \Gamma(s, x - y)Z(s, y)M(ds, dy),$$

where the right-hand side is a Walsh stochastic integral. Equality (2.16) is checked by direct calculation (see [9, Lemma 1]).

(b) If Γ is as in (a) and Z satisfies (3), then one checks that

$$\lim_{m\to\infty} \lim_{n\to\infty} I_{\Gamma, Z-(Z1_{[-m,m]})*\psi_n} = 0,$$

where $(\psi_n) \subset C_0^\infty(\mathbb{R}^d)$ is a sequence that converges to the Dirac distribution, and one sets

$$v_{\Gamma,Z} = \lim_{m\to\infty} \lim_{n\to\infty} v_{\Gamma,(Z1_{[-m,m]})*\psi_n},$$

where the limits are in $L^2(\Omega \times \mathbb{R}^d, dP \times dx)$.

(c) If Γ satisfies (1) and (2), and Z satisfies (3), then one checks that

$$\lim_{n\to\infty} I_{\Gamma - \Gamma * \psi_n, Z} = 0$$

and one sets

$$v_{\Gamma,Z} = \lim_{n\to\infty} v_{\Gamma*\psi_n,Z},$$

where the limit is in $L^2(\Omega \times \mathbb{R}^d, dP \times dx)$: see [9, Theorem 6].

In comparison with the stochastic integral of Section 2.3, we remark that the process Z verifies $\sup_{s\in[0,T]} E(\|Z(s)\|^2_{L^2(\mathbb{R}^d)}) < +\infty$, rather than (2.12), and the resulting integral $v_{\Gamma,Z}$, as a random function of x, belongs to $L^2(\Omega \times \mathbb{R}^d)$.

We now relate this stochastic integral to the one defined in Section 2.3.

Proposition 2.11. *Assume that Γ and Z satisfy conditions (1), (2) and (3) above. Then*:

(i) *For almost all $x \in \mathbb{R}^d$, the element $\Gamma(\,\cdot\,, x - \star\,)Z(\,\cdot\,,\,\star\,)$ belongs to $L^2(\Omega \times [0,T];U)$. Hence, as in (2.7), we can define the (real-valued) stochastic integral*

$$\mathcal{I}_{\Gamma,Z}(T,x) := \int_0^T \int_{\mathbb{R}^d} \Gamma(s, x-y)Z(s,y)\, W(ds,dy), \qquad \textit{for a.a. } x \in \mathbb{R}^d.$$

(ii) $\mathcal{I}_{\Gamma,Z}(T,\star) \in L^2(\Omega \times \mathbb{R}^d)$ *and* $\|\mathcal{I}_{\Gamma,Z}(T,\star)\|^2_{L^2(\Omega\times\mathbb{R}^d)} = I_{\Gamma,Z}$.

(iii) $\mathcal{I}_{\Gamma,Z}(T,\star) = v_{\Gamma,Z}$ *in* $L^2(\Omega \times \mathbb{R}^d)$.

Proof. We will split the proof in three steps, which essentially correspond to the construction of the Dalang–Mueller integral $v_{\Gamma,Z}$.

Step 1. Let us assume first that Γ and Z satisfy the hypotheses in (a) above. Then, as we pointed out there, for all $x \in \mathbb{R}^d$, the stochastic integral $v_{\Gamma,Z}(x)$ can be defined as a Walsh stochastic integral. Hence, by Proposition 2.6(a), the integrand $(s,y) \mapsto \Gamma(s, x-y)Z(s,y)$ defines an element in $L^2(\Omega \times [0,T];U)$ and, for all $x \in \mathbb{R}^d$, $v_{\Gamma,Z}(x) = \mathcal{I}_{\Gamma,Z}(T,x)$. Condition (ii) in the statement can be deduced from this latter equality and (2.16).

Step 2. Assume now that Γ is as in Step 1 and Z satisfies condition (3). Then, as in (b) above, there exists a sequence of processes $(Z_n)_n$ such that, for all $n \geq 1$, Z_n satisfies the hypotheses in (a) and I_{Γ,Z_n-Z} converges to zero as n tends to infinity. For this sequence,

$$v_{\Gamma,Z} := \lim_{n\to\infty} v_{\Gamma,Z_n} = \lim_{n\to\infty} \mathcal{I}_{\Gamma,Z_n}(T,\star) \tag{2.17}$$

by Step 1, where the limit is in $L^2(\Omega \times \mathbb{R}^d)$.

We now check property (i) in the statement of the proposition. Observe that, by Proposition 2.9,

$$\begin{aligned}&\int_{\mathbb{R}^d} dx\, \|\Gamma(\cdot, x-\star)[Z_n(\cdot,\star) - Z(\cdot,\star)]\|^2_{L^2(\Omega\times[0,T];\,U)}\\ &= \int_{\mathbb{R}^d} dx\, E\left(\int_0^T ds \int_{\mathbb{R}^d} \mu(d\eta) |\mathcal{F}(\Gamma(s, x-\star)[Z_n(s,\star) - Z(s,\star)])(\eta)|^2\right).\end{aligned} \tag{2.18}$$

Use the very last lines in the proof of [9, Lemma 1] to see that this is equal to I_{Γ,Z_n-Z}. Since this quantity converges to zero as $n \to \infty$, we deduce that there exists a subsequence $(n_j)_j$ such that, for almost all $x \in \mathbb{R}^d$,

$$\lim_{j\to\infty} \|\Gamma(\cdot, x-\star)Z_{n_j}(\cdot,\star) - \Gamma(\cdot, x-\star)Z(\cdot,\star)\|_{L^2(\Omega\times[0,T];\,U)} = 0.$$

This implies that, for almost all $x \in \mathbb{R}^d$, the element $(s, y) \mapsto \Gamma(s, x-y)Z(s, y)$ belongs to $L^2(\Omega \times [0, T]; U)$, and we can define the (real-valued) stochastic integral

$$\mathcal{I}_{\Gamma,Z}(T, x) := \int_0^T \int_{\mathbb{R}^d} \Gamma(s, x-y)Z(s, y)\, W(ds, dy), \tag{2.19}$$

and

$$\mathcal{I}_{\Gamma,Z}(T, x) = \lim_{j\to\infty} \mathcal{I}_{\Gamma,Z_{n_j}}(T, x) \qquad \text{in} \quad L^2(\Omega).$$

Notice that

$$\|\mathcal{I}_{\Gamma,Z_n}(T, \star) - \mathcal{I}_{\Gamma,Z}(T, \star)\|^2_{L^2(\Omega\times\mathbb{R}^d)} = \|\mathcal{I}_{\Gamma,Z_n-Z}(T, \star)\|^2_{L^2(\Omega\times\mathbb{R}^d)}. \tag{2.20}$$

By the isometry property (2.8), this is equal to (2.18), and therefore to I_{Γ,Z_n-Z}, which tends to 0 as $n \to \infty$. Therefore, using Step 1, we see that

$$\|\mathcal{I}_{\Gamma,Z}(T, \star)\|^2_{L^2(\Omega\times\mathbb{R}^d)} = \lim_{n\to\infty} \|\mathcal{I}_{\Gamma,Z_n}(T, \star)\|^2_{L^2(\Omega\times\mathbb{R}^d)} = \lim_{n\to\infty} I_{\Gamma,Z_n} = I_{\Gamma,Z},$$

which proves (ii). The arguments following (2.20) and (2.18) prove (iii).

Step 3. In this final part, we assume that Γ and Z satisfy conditions (1), (2) and (3). Then, it is a consequence of step (c) above that there exists $(\Gamma_n)_n$ such that, for all $n \geq 1$, Γ_n verifies the assumptions of the previous step and

$$\lim_{n\to\infty} I_{\Gamma_n-\Gamma,Z} = 0.$$

In order to prove parts (i), (ii) and (iii) for this case, one can follow exactly the same lines as we have done in Step 2. We omit the details. □

As we will explain in Section 4.6, for the particular case of the stochastic wave equation, it is useful to consider stochastic integrals of the form $v_{\Gamma,Z}$ which take values in some weighted L^2-space. We now describe this situation.

Fix $k > d$ and let $\theta : \mathbb{R}^d \to \mathbb{R}$ be a smooth function for which there are constants $0 < c < C$ such that

$$c(1 \wedge |x|^{-k}) \leq \theta(x) \leq C(1 \wedge |x|^{-k}).$$

The weighted L^2-space L^2_θ is the set of measurable $g : \mathbb{R}^d \to \mathbb{R}$ such that $\|g\|_\theta < +\infty$, where

$$\|g\|^2_\theta = \int_{\mathbb{R}^d} |g(x)|^2\, \theta(x)\, dx.$$

Consider a function $s \mapsto \Gamma(s) \in \mathcal{S}'(\mathbb{R}^d)$ that satisfies (1), (2) above, and, in addition,

(4) There is $R > 0$ such that for $s \in [0, T]$, supp $\Gamma(s) \subset B(0, R)$.

For a stochastic process Z, we consider the following hypothesis:

(5) For $0 \leq s \leq T$, $Z(s) \in L^2_\theta$ a.s., $Z(s)$ is $\mathcal{F}_s$-measurable, and $s \mapsto Z(s)$ is mean-square continuous from $[0, T]$ into L^2_θ.

Then the stochastic integral

$$v^\theta_{\Gamma,Z} = \int_0^T \int_{\mathbb{R}^d} \Gamma(s, \star - y)Z(s, y)M(ds, dy) \tag{2.21}$$

is defined as an element of $L^2(\Omega \times \mathbb{R}^d, dP \times \theta(x)dx)$, such that

$$E(\|v^{\theta}_{\Gamma,Z}\|^2_{L^2_\theta}) \leq I^{\theta}_{\Gamma,Z},$$

where

$$I^{\theta}_{\Gamma,Z} := \int_0^T ds\ E(\|Z(s,\star)\|^2_{L^2_\theta}) \sup_{\xi\in\mathbb{R}^d} \int_{\mathbb{R}^d} \mu(d\eta)|\mathcal{F}\Gamma(s)(\xi-\eta)|^2.$$

This definition is obtained by showing that $Z_n(s, \star) := Z(s, \star)1_{[-n,n]}(\star)$ also satisfies (5) as well as (3). Therefore, $v^{\theta}_{\Gamma,Z_n} = v_{\Gamma,Z_n}$ is defined as an element of $L^2(\Omega \times \mathbb{R}^d, dP \times dx)$, and one checks that this element also belongs to $L^2(\Omega \times \mathbb{R}^d, dP \times \theta(x)dx)$, and

$$\lim_{n\to\infty} I^{\theta}_{\Gamma,Z-Z_n} = 0,$$

provided that Γ satisfies (1), (2) and (4). Then one sets

$$v^{\theta}_{\Gamma,Z} = \lim_{n\to\infty} v_{\Gamma,Z_n},$$

where the limit is in $L^2(\Omega \times \mathbb{R}^d, dP \times \theta(x)dx)$: see [9, Theorem 12].

2.6. Hilbert-space-valued integrals and tensor products

In this section, we return to the general setting of Section 2.1 and we explain how the real-valued stochastic integral defined there can be naturally extended to a Hilbert-space-valued integral. In Section 3.6, we will show that this extended stochastic integral is equivalent to the stochastic integral of Da Prato and Zabczyk [14], that we will present in Section 3.3.

As far as we know, the stochastic integral that we present here does not appear explicitly in the literature. Nevertheless, in the particular case where the cylindrical Wiener process is given by the spatially homogeneous noise of Section 2.2, a definition of such Hilbert-space-valued integrals has been given in [35], in [38, Chapter 6], and in [29, Section 3] (for a particular form of the integrands, as in Proposition 2.12 below). For related papers where this type of integral has also been used, we refer the reader for instance to [24,26,36].

Let V and H be Hilbert spaces with inner product $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_H$, respectively, and $B = \{B_t(h),\ t \geq 0,\ h \in V\}$ be a cylindrical Wiener process on V with covariance Q (see Definition 2.1). Recall that V_Q denotes the Hilbert space V endowed with the inner product $\langle h, g\rangle_{V_Q} = \langle Qh, g\rangle_V$.

Let $V_Q \otimes H$ be the Hilbert space tensor product of V_Q and H. We recall that if $(v_j)_j$ and $(f_k)_k$ denote complete orthonormal bases of V_Q and H, respectively, then $(v_j \otimes f_k)_{j,k}$ defines a complete orthonormal basis of $V_Q \otimes H$ and any element $X \in V_Q \otimes H$ can be represented in the following forms (see e.g. [42, Section 3.4]):

$$X = \sum_{j,k=1}^{\infty} X^{j,k}\,(v_j \otimes f_k) = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} X^{j,k}\,v_j\right) \otimes f_k,$$

where $X^{j,k} \in \mathbb{R}$ and

$$\sum_{j,k=1}^{\infty} \left(X^{j,k}\right)^2 < +\infty, \tag{2.22}$$

so that $\|X\|^2_{V_Q\otimes H} = \sum_{j,k=1}^{\infty} (X^{j,k})^2$. This representation shows that the tensor product $V_Q \otimes H$ is isomorphic to the set of "matrices" $(X^{j,k})$ satisfying (2.22).

We will define an H-valued stochastic integral $g \cdot B$ of any predictable process $g \in L^2(\Omega \times [0,T]; V_Q \otimes H)$. More precisely, note first that if g is such a process, then for all $s \in [0,T]$, $g_s = \sum_{j,k=1}^{\infty} g_s^{j,k}(v_j \otimes f_k)$, and

$$E\left(\int_0^T \|g_s\|^2_{V_Q \otimes H}\, ds\right) = E\left(\int_0^T \sum_{j,k=1}^{\infty} \left(g_s^{j,k}\right)^2\, ds\right) < +\infty. \tag{2.23}$$

For any $k \in \mathbb{N}$, let g^k be the stochastic process

$$g_s^k := \sum_{j=1}^{\infty} g_s^{j,k} v_j, \qquad s \in [0,T].$$

Then g^k defines a predictable element in $L^2(\Omega \times [0,T]; V_Q)$. Indeed, for all s, we have that $g_s^k \in V_Q$ a.s., and by (2.23),

$$E\left(\int_0^T \|g_s^k\|^2_{V_Q}\, ds\right) = E\left(\int_0^T \sum_{j=1}^{\infty} \left(g_s^{j,k}\right)^2 ds\right) \le E\left(\int_0^T \sum_{j,k=1}^{\infty} \left(g_s^{j,k}\right)^2 ds\right) < +\infty.$$

As in (2.1), this implies that the real-valued stochastic integral $g^k \cdot B$ is well-defined and satisfies

$$E\left((g^k \cdot B)^2\right) = E\left(\int_0^T \|g_s^k\|^2_{V_Q}\, ds\right).$$

We now define the H-valued stochastic integral of g with respect to B as follows:

$$g \cdot B := \sum_{k=1}^{\infty} (g^k \cdot B) f_k. \tag{2.24}$$

We also use the notation

$$g \cdot B = \int_0^T g_s\, dB_s.$$

One easily verifies that the above series converges in $L^2(\Omega; H)$ and the isometry property in this case reads

$$E\left(\|g \cdot B\|_H^2\right) = E\left(\int_0^T \|g_s\|^2_{V_Q \otimes H}\, ds\right).$$

We now provide some examples of processes that can be integrated in the sense just defined. For this, we consider the spatially homogeneous Gaussian noise defined in Section 2.2, so that we use the Hilbert space U introduced in Section 2.2 and the standard cylindrical Wiener process $W = \{W_t(\varphi), t \ge 0, \varphi \in U\}$ defined in Proposition 2.5. We denote by $(e_j)_j$ a complete orthonormal basis of U. We have the following Hilbert-space-valued counterpart of Proposition 2.9. Its proof follows the same lines as the analogous result for real-valued integrals (see [29, Proposition 3.3 and (3.13)]).

Proposition 2.12. *Assume that Γ satisfies* Hypothesis 2.8. *Let $K = \{K(t,x), (t,x) \in [0,T] \times \mathbb{R}^d\}$ be a predictable process with values in H such that*

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} E(\|K(t,x)\|_H^p) < \infty,$$

for some $p \geq 2$. Then the element $G = \{G(t, dx) = K(t, x)\Gamma(t, dx),\ t \in [0, T]\}$ is a predictable process with values in $L^p(\Omega \times [0, T]; U \otimes H)$. Moreover,

$$E\left(\int_0^T \|G_t\|_{U\otimes H}^2\, dt\right) = \sum_{k=1}^{\infty} E\left[\int_0^T dt \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}(G^k(t))(\xi)|^2\right],$$

where $G^k \in L^2(\Omega \times [0,\ T];U)$ is the predictable process defined by $G^k{:=}\{G^k(t, dx) = K^k(t, x)\Gamma(t, dx),\ (t, x) \in [0, T] \times \mathbb{R}^d\}$, with $K^k(t, x) = \langle K(t, x), f_k\rangle_H$, and

$$E\left(\|G \cdot B\|_H^p\right) \leq C \int_0^T dt \left(\sup_{x\in\mathbb{R}^d} E(\|K(t, x)\|_H^p)\right) \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Gamma(t)(\xi)|^2.$$

3. Infinite-dimensional integration theory

In this section, we sketch in Sections 3.1–3.3 the construction of the infinite-dimensional stochastic integral in the setup of Da Prato and Zabczyk in [14]. For this, we will define the general concept of Hilbert-space-valued Q-Wiener process and study its relationship with the cylindrical Wiener process considered in Section 2.1. Then we will show in Sections 3.4 and 3.5 that the stochastic integral constructed in Section 2.1 can be inserted into this more abstract setting. In particular, we will treat specifically the case of the standard cylindrical Wiener process given by the spatially homogeneous noise described in Section 2.2. In Section 3.6, we establish the equivalence between the Hilbert-space-valued integral of Section 2.6 and the stochastic integral of Sections 3.3 and 3.5.

We begin by recalling some facts concerning nuclear and Hilbert–Schmidt operators on Hilbert spaces.

3.1. Nuclear and Hilbert–Schmidt operators

Let E, G be Banach spaces and let $L(E, G)$ be the vector space of all linear bounded operators from E into G. We denote by E^* and G^* the dual spaces of E and G, respectively.

An element $T \in L(E, G)$ is said to be a *nuclear* operator if there exist two sequences $(a_j)_j \subset G$ and $(\varphi_j)_j \subset E^*$ such that

$$T(x) = \sum_{j=1}^{\infty} a_j\, \varphi_j(x), \qquad \text{for all } x \in E,$$

and

$$\sum_{j=1}^{\infty} \|a_j\|_G\, \|\varphi_j\|_{E^*} < +\infty.$$

The space of all nuclear operators from E into G is denoted by $L_1(E, G)$. When endowed with the norm

$$\|T\|_1 = \inf\left\{\sum_{j=1}^{\infty} \|a_j\|_G\, \|\varphi_j\|_{E^*} \,:\, T(x) = \sum_{j=1}^{\infty} a_j\varphi(x),\ x \in E\right\},$$

it is a Banach space.

Let H be a separable Hilbert space and let $(e_j)_j$ be a complete orthonormal basis in H. For $T \in L_1(H, H)$, the *trace* of T is

$$\mathrm{Tr}\, T = \sum_{j=1}^{\infty} \langle T(e_j), e_j\rangle_H. \tag{3.1}$$

One proves that if $T \in L_1(H) := L_1(H, H)$, then Tr T is a well-defined real number and its value does not depend on the choice of the orthonormal basis (see, for instance, [14, Proposition C.1]). Further, according to [14, Proposition C.3], a non-negative definite operator $T \in L(H)$ is nuclear if and only if, for an orthonormal basis $(e_j)_j$ on H,

$$\sum_{j=1}^{\infty} \langle T(e_j), e_j \rangle_H < +\infty.$$

Moreover, in this case, $\mathrm{Tr}\, T = \|T\|_1$.

Let V and H be two separable Hilbert spaces and $(e_k)_k$ a complete orthonormal basis of V. A bounded linear operator $T : V \to H$ is said to be *Hilbert–Schmidt* if

$$\sum_{k=1}^{\infty} \|T(e_k)\|_H^2 < +\infty.$$

It turns out that the above property is independent of the choice of the basis in V. The set of Hilbert–Schmidt operators from V into H is denoted by $L_2(V, H)$. The norm in this space is defined by

$$\|T\|_2 = \left(\sum_{k=1}^{\infty} \|T(e_k)\|_H^2 \right)^{1/2}, \tag{3.2}$$

and defines a Hilbert space with inner product

$$\langle S, T \rangle_2 = \sum_{k=1}^{\infty} \langle S(e_k), T(e_k) \rangle_H. \tag{3.3}$$

Finally, let us point out that (3.1) and (3.2) imply that if $T \in L_2(V, H)$, then $TT^* \in L_1(H)$, where T^* is the adjoint operator of T, and

$$\|T\|_2^2 = \mathrm{Tr}\,(TT^*). \tag{3.4}$$

We conclude this section by recalling the definition and some properties of the pseudo-inverse of bounded linear operators (see, for instance, [33, Appendix C]).

Let $T \in L(V, H)$ and $\mathrm{Ker}\, T := \{x \in V : T(x) = 0\}$. The *pseudo-inverse* of the operator T is defined by

$$T^{-1} := \left(T_{|_{(\mathrm{Ker}\, T)^{\perp}}} \right)^{-1} : T(V) \to (\mathrm{Ker}\, T)^{\perp}.$$

Notice that T is one-to-one on $(\mathrm{Ker}\, T)^{\perp}$ (the orthogonal complement of $\mathrm{Ker}\, T$) and T^{-1} is linear and bijective.

If $T \in L(V)$ is a bounded linear operator defined on V and T^{-1} denotes the pseudo-inverse of T, then (see [33, Proposition C.0.3]):

1. $(T(V), \langle \cdot, \cdot \rangle_{T(V)})$ defines a Hilbert space, where

$$\langle x, y \rangle_{T(V)} := \langle T^{-1}(x), T^{-1}(y) \rangle_V, \qquad x, y \in T(V).$$

2. Let $(e_k)_k$ be an orthonormal basis of $(\mathrm{Ker}\, T)^{\perp}$. Then $(T(e_k))_k$ is an orthonormal basis of $(T(V), \langle \cdot, \cdot \rangle_{T(V)})$.

Finally, according to [33, Corollary C.0.6], if $T \in L(V, H)$ and we set $Q := TT^* \in L(H)$, then we have $\mathrm{Im}\, Q^{1/2} = \mathrm{Im}\, T$ and

$$\|Q^{-1/2}(x)\|_H = \|T^{-1}(x)\|_V, \qquad x \in \mathrm{Im}\, T,$$

where $Q^{-1/2}$ is the pseudo-inverse of $Q^{1/2}$.

3.2. Hilbert-space-valued Wiener processes

The stochastic integral presented in Da Prato and Zabczyk [14] is defined with respect to a class of Hilbert-space-valued processes, namely Q-Wiener processes, which we now introduce.

We consider a separable Hilbert space V and a linear, symmetric (self-adjoint) non-negative definite and bounded operator Q on V such that $\operatorname{Tr} Q < +\infty$.

Definition 3.1. A V-valued stochastic process $\{\mathcal{W}_t,\ t \geq 0\}$ is called a Q-Wiener process if (1) $\mathcal{W}_0 = 0$, (2) $\mathcal{W}$ has continuous trajectories, (3) $\mathcal{W}$ has independent increments, and (4) the law of $\mathcal{W}_t - \mathcal{W}_s$ is Gaussian with mean zero and covariance operator $(t-s)Q$, for all $0 \leq s \leq t$.

We recall that according to [14, Section 2.3.2], condition (4) above means that for any $h \in V$ and $0 \leq s \leq t$, the real-valued random variable $\langle \mathcal{W}_t - \mathcal{W}_s, h\rangle_V$ is Gaussian, with mean zero and variance $(t-s)\langle Qh, h\rangle_V$. In particular, using (3.1), we see that $E(\|\mathcal{W}_t\|_V^2) = t \operatorname{Tr} Q$, which is one reason why the assumption $\operatorname{Tr} Q < \infty$ is essential.

Let $(e_j)_j$ be an orthonormal basis of V that consists of eigenvectors of Q with corresponding eigenvalues $\lambda_j, j \in \mathbb{N}^*$. Let $(\beta_j)_j$ be a sequence of independent real-valued standard Brownian motions on a probability space $(\Omega, \mathcal{F}, P)$. Then the V-valued process

$$\mathcal{W}_t = \sum_{j=1}^{\infty} \sqrt{\lambda_j}\, \beta_j(t) e_j \tag{3.5}$$

(where the series converges in $L^2(\Omega; \mathcal{C}([0,T]; V))$), defines a Q-Wiener process on V (see (2.1.2) in [33]). We note that $\sqrt{\lambda_j}\, e_j = Q^{1/2}(e_j)$. In the special case where V is finite-dimensional, say $\dim V = n$, then Q can be identified with an $n \times n$-matrix which is the variance-covariance matrix of $\{\mathcal{W}_t\}$, and $\{\mathcal{W}_t\}$ has the same law as $\{Q^{1/2}\mathcal{W}_t^0\}$, where $\{\mathcal{W}_t^0\}$ is a standard Brownian motion with values in $\mathbb{R}^n$.

If $\{\mathcal{W}_t,\ t \geq 0\}$ is a Q-Wiener process on V, there is a natural way to associate to it a cylindrical Wiener process in the sense of Definition 2.1. Namely, for any $h \in V$ and $t \geq 0$, we set $W_t(h) := \langle \mathcal{W}_t, h\rangle_V$. Using polarization, one checks that $\{W_t(h),\ t \geq 0,\ h \in V\}$ is a cylindrical Wiener process on V with covariance operator Q. Note that in this case, $W_t(e_j) = \sqrt{\lambda_j}\, \beta_j(t)$, so the Brownian motions β_j in (3.5) are given by $\beta_j(t) = W_t(v_j)$, where

$$v_j = \lambda_j^{-1/2} e_j = Q^{-1/2}(e_j), \qquad \text{for } j \geq 1 \quad \text{with} \quad \lambda_j \neq 0. \tag{3.6}$$

In particular, $(v_j)_j$ is a complete orthonormal basis of the space V_Q of Section 2.1.

However, it is not true in general that any cylindrical Wiener process is associated to a Q-Wiener process on a Hilbert space. Indeed, we have the following result (see [25, p. 177]).

Theorem 3.2. *Let V be a separable Hilbert space and W a cylindrical Wiener process on V with covariance Q. Then, the following three conditions are equivalent:*

1. *W is associated to a V-valued Q-Wiener process $\mathcal{W}$, in the sense that $\langle \mathcal{W}_t, h\rangle_V = W_t(h)$, for all $h \in V$.*
2. *For any $t \geq 0$, $h \mapsto W_t(h)$ defines a Hilbert–Schmidt operator from V into $L^2(\Omega, \mathcal{F}, P)$.*
3. $\operatorname{Tr} Q < +\infty$.

If any one of the above conditions holds, then the norm of the Hilbert–Schmidt operator $h \mapsto W_t(h)$, as an element of $L_2(V, L^2(\Omega, \mathcal{F}, P))$, is given by

$$\|W_t\|_2 = E(\|\mathcal{W}_t\|_V^2) = t \operatorname{Tr} Q.$$

As a consequence of the above result, if $\dim V = +\infty$ and if W is a standard cylindrical Wiener process on V, that is $Q = \mathrm{Id}_V$, then there is no Q-Wiener process $\mathcal{W}$ associated to W. However, as we will explain in Section 3.5, it will be possible to find a Hilbert-space-valued Wiener process with values in a larger Hilbert space V_1 which will correspond to W in a certain sense.

3.3. H-valued stochastic integrals

We now sketch the construction of the infinite-dimensional stochastic integral of [14]. Let V and H be two separable Hilbert spaces and let $\{\mathcal{W}_t,\ t \geq 0\}$ be a Q-Wiener process defined on V. We denote by $(\mathcal{F}_t)_t$ the (completed) filtration generated by $\mathcal{W}$. In [14], the objective is to construct the H-valued stochastic integral

$$\int_0^t \Phi_s \, d\mathcal{W}_s, \qquad t \in [0, T],$$

where Φ is a process with values in the space of linear but not necessarily bounded operators from V into H.

Consider the subspace $V_0 := Q^{1/2}(V)$ of V which, endowed with the inner product

$$\langle h, g \rangle_0 := \langle Q^{-1/2} h, Q^{-1/2} g \rangle_V,$$

is a Hilbert space. Here $Q^{-1/2}$ denotes the pseudo-inverse of the operator $Q^{1/2}$ (see Section 3.1). Let us also set

$$L_2^0 := L_2(V_0, H),$$

which is the Hilbert space of all Hilbert–Schmidt operators from V_0 into H, equipped, as in (3.3), with the inner product

$$\langle \Phi, \Psi \rangle_{L_2^0} = \sum_{j=1}^{\infty} \langle \Phi \tilde{e}_j, \Psi \tilde{e}_j \rangle_H, \qquad \Phi, \Psi \in L_2^0, \tag{3.7}$$

where $(\tilde{e}_j)_j$ is any complete orthonormal basis of V_0. In particular, using the fact that we can take

$$\tilde{e}_j = \sqrt{\lambda_j}\, e_j = Q^{1/2}(e_j), \qquad j \geq 1,\ \lambda_j > 0, \tag{3.8}$$

where the $(e_j)_j$ are as in (3.5) (see condition 2. in the final part of Section 3.1) and applying (3.4), the norm of $\Psi \in L_2^0$ can be expressed as

$$\|\Psi\|_{L_2^0}^2 = \|\Psi \circ Q^{1/2}\|_{L_2(V,H)}^2 = \operatorname{Tr}(\Psi Q \Psi^*).$$

We note that in the case where $\dim V = n < +\infty$ and $\dim H = m < +\infty$, then it is natural to identify $\Psi \in L_2^0$ with an $m \times n$-matrix and Q with an $n \times n$-matrix. The norm of Ψ corresponds to a classical matrix norm of $\Psi Q^{1/2}$ (whose square is the sum of squares of entries of $\Psi Q^{1/2}$).

Let $\Phi = \{\Phi_t,\ t \in [0, T]\}$ be a measurable L_2^0-valued process. We define the norm of Φ by

$$\|\Phi\|_T := \left[E\left(\int_0^T \|\Phi_s\|_{L_2^0}^2 \, ds \right) \right]^{1/2}.$$

The aim of [14, Chapter 4], is to define the stochastic integral with respect to $\mathcal{W}$ of any L_2^0-valued predictable process Φ such that $\|\Phi\|_T < \infty$. More precisely, Da Prato and Zabczyk first consider *simple* processes, which are of the form $\Phi_t = \Phi_0 1_{(a,b]}(t)$, where Φ_0 is any $\mathcal{F}_a$-measurable $L(V, H)$-valued random variable and $0 \leq a < b \leq T$. For such processes, the stochastic integral takes values in H and is defined by the formula

$$\int_0^t \Phi_s \, d\mathcal{W}_s := \Phi_0(\mathcal{W}_{b\wedge t} - \mathcal{W}_{a \wedge t}), \qquad t \in [0, T]. \tag{3.9}$$

The map $\Phi \mapsto \int_0^{\cdot} \Phi_s d\mathcal{W}_s$ is an isometry between the set of simple processes and the space $\mathcal{M}_H$ of square-integrable H-valued $(\mathcal{F}_t)$-martingales $X = \{X_t,\ t \in [0, T]\}$ endowed with the norm $\|X\| =$

$[E(\|X_T\|_H^2)]^{1/2}$. Indeed, as it is proved in [14] (see also [33, Proposition 2.3.5]), the isometry property for simple processes reads

$$E\left(\left\|\int_0^T \Phi_t\, d\mathcal{W}_t\right\|_H^2\right) = \|\Phi\|_T^2. \tag{3.10}$$

Remark 3.3. The appearance of $\|\cdot\|_T$ can be understood by considering the case where $\Phi(t) = \Phi_0 1_{(a,b]}(t)$, where $\Phi_0 \in L(V,H)$ is deterministic and $0 \le a < b \le T$. Indeed, in this case, using (3.9) and the representation (3.5),

$$E\left(\left\|\int_0^T \Phi_t\, d\mathcal{W}_t\right\|_H^2\right) = E\left(\left\|\sum\nolimits_j \sqrt{\lambda_j}\,(\beta_j(b)-\beta_j(a))\,\Phi_0(e_j)\right\|_H^2\right),$$

and the right-hand side is equal to

$$\begin{aligned}\sum_j \lambda_j\,(b-a)\left\|\Phi_0(e_j)\right\|_H^2 &= (b-a)\sum_j \left\|\Phi_0(Q^{1/2}e_j)\right\|_H^2\\ &= (b-a)\left\|\Phi_0\circ Q^{1/2}\right\|_{L_2(V,H)}^2 = E\left(\int_0^T \|\Phi_s\|_{L_2^0}^2\, ds\right).\end{aligned}$$

Once the isometry property (3.10) is established, a completion argument is used to extend the above definition to all L_2^0-valued predictable processes Φ satisfying $\|\Phi\|_T < \infty$. The integral of Φ is denoted by

$$\Phi\cdot\mathcal{W} = \int_0^T \Phi_t\, d\mathcal{W}_t$$

and the isometry property (3.10) is preserved for such processes:

$$E(\|\Phi\cdot\mathcal{W}\|_H^2) = \|\Phi\|_T^2.$$

The details of this construction can be found in [14, Chapter 4].

Let us conclude this section by providing a representation of the stochastic integral $\Phi\cdot\mathcal{W}$ in terms of ordinary Itô integrals of real-valued processes. Indeed, observe first that the expansion (3.5) can be rewritten in the form

$$\mathcal{W}_t = \sum_{j=1}^{\infty} \beta_j(t)\tilde{e}_j, \tag{3.11}$$

where $(\tilde{e}_j)_j$ is defined in (3.8).

Proposition 3.4. *Let $(f_k)_k$ be a complete orthonormal basis in the Hilbert space H. Assume that $\Phi = \{\Phi_t,\ t\in[0,T]\}$ is any L_2^0-valued predictable process such that $\|\Phi\|_T < \infty$. Then*

$$\int_0^T \Phi_t\, d\mathcal{W}_t = \sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_0^T \langle \Phi_t(\tilde{e}_j), f_k\rangle_H\, d\beta_j(t)\right) f_k. \tag{3.12}$$

We note for future reference that this proposition remains valid even for cylindrical Wiener processes: see Section 3.5 below, and, in particular, Remark 3.9.

Proof of Proposition 3.4. First of all, we will prove that, under the standing hypotheses, the right-hand side of (3.12) is a well-defined element in $L^2(\Omega;H)$. For this, we will check that

$$E\left[\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_0^T \langle \Phi_t(\tilde{e}_j), f_k\rangle_H \, d\beta_j(t)\right)^2\right] = \|\Phi\|_T^2,$$

where the right-hand side is finite, by assumption.

Since $(\beta_j)_j$ is a family of independent standard Brownian motions,

$$E\left[\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_0^T \langle \Phi_t(\tilde{e}_j), f_k\rangle_H \, d\beta_j(t)\right)^2\right] = \sum_{k,j=1}^{\infty} E\left[\left(\int_0^T \langle \Phi_t(\tilde{e}_j), f_k\rangle_H \, d\beta_j(t)\right)^2\right],$$

and the right-hand side is equal to

$$\sum_{k,j=1}^{\infty}\int_0^T E\left[\langle \Phi_t(\tilde{e}_j), f_k\rangle_H^2\right] dt = E\left[\int_0^T \sum_{j=1}^{\infty}\|\Phi_t(\tilde{e}_j)\|_H^2 \, dt\right] = E\left[\int_0^T \|\Phi_t\|_{L_2^0}^2 \, dt\right],$$

and the last term is equal to $\|\Phi\|_T^2$. Hence, the series on the right-hand side of (3.12) defines an element in $L^2(\Omega;H)$ and its norm is given by $\|\Phi\|_T$. Therefore, by the isometry property of the stochastic integral (see (3.10)), in order to prove equality (3.12), we only need to check this equality for simple processes. Namely, assume that Φ is of the form $\Phi_t = \Phi_0 1_{(a,b]}(t)$, where Φ_0 is a $\mathcal{F}_a$-measurable $L(V, H)$-valued random variable and $0 \le a < b \le T$. Then, by (3.9),

$$\int_0^T \Phi_t \, d\mathcal{W}_t = \Phi_0(\mathcal{W}_b - \mathcal{W}_a).$$

On the other hand,

$$\sum_{k=1}^{\infty}\left(\sum_{j=1}^{\infty}\int_0^T \langle \Phi_t(\tilde{e}_j), f_k\rangle_H \, d\beta_j(t)\right) f_k = \sum_{k,j=1}^{\infty}\langle \Phi_0(\tilde{e}_j), f_k\rangle_H (\beta_j(b) - \beta_j(a)) f_k,$$

and the right-hand side is equal to

$$\sum_{j=1}^{\infty}(\beta_j(b) - \beta_j(a))\, \Phi_0(\tilde{e}_j) = \Phi_0\left(\sum_{j=1}^{\infty}(\beta_j(b) - \beta_j(a))\tilde{e}_j\right) = \Phi_0(\mathcal{W}_b - \mathcal{W}_a),$$

where the last equality follows from (3.11). The proof is complete. □

3.4. The case where $H = \mathbb{R}$

We consider a cylindrical Wiener process W on some separable Hilbert space V with covariance Q, such that $\mathrm{Tr}\, Q < +\infty$. By Theorem 3.2, W is associated to a V-valued Q-Wiener process $\mathcal{W}$. We shall check that the stochastic integral with respect to W, constructed in Section 2.1, is equal to an integral with respect to $\mathcal{W}$, constructed in [14] and sketched in Section 3.3, when the Hilbert space H in which the integral takes its values is $H = \mathbb{R}$.

In Section 2.1, we defined the Hilbert space V_Q and the stochastic integral

$$g \cdot W = \int_0^T g_s \, dW_s,$$

for any predictable stochastic process $g \in L^2(\Omega \times [0,T]; V_Q)$, with the isometry property

$$E\left((g \cdot W)^2\right) = E\left(\int_0^T \|g_s\|_{V_Q}^2 \, ds\right)^2.$$

For any $s \in [0,T]$ and $g \in L^2(\Omega \times [0,T]; V_Q)$, we define an operator $\Phi_s^g : V \to \mathbb{R}$ by

$$\Phi_s^g(\eta) := \langle g_s, \eta \rangle_V, \qquad \eta \in V. \tag{3.13}$$

We denote by L_2^0 the set $L_2(V_0, H)$, with $V_0 = Q^{1/2}(V)$ and $H = \mathbb{R}$.

Proposition 3.5. *Under the above assumptions, $\Phi^g = \{\Phi_s^g,\ s \in [0,T]\}$ defines a predictable process with values in $L_2^0 = L_2(V_0, \mathbb{R})$, such that*

$$E\left(\int_0^T \|\Phi_s^g\|_{L_2^0}^2 \, ds\right) = E\left(\int_0^T \|g_s\|_{V_Q}^2 \, ds\right). \tag{3.14}$$

Therefore, the stochastic integral of Φ^g with respect to $\mathcal{W}$ can be defined as in Section 3.3 *and in fact,*

$$\int_0^T \Phi_s^g \, d\mathcal{W}_s = \int_0^T g_s \, dW_s. \tag{3.15}$$

Proof. We first check (3.14). Let e_j be as in (3.5), $\tilde{e}_j$ be as in (3.8) and v_j be as in (3.6), so that $\tilde{e}_j = Q(v_j)$. By (3.7) with $H = \mathbb{R}$, and by (3.13),

$$\|\Phi_s^g\|_2^2 = \sum_{j=1}^{\infty} \langle g_s, \tilde{e}_j \rangle_V^2 = \sum_{j=1}^{\infty} \langle g_s, Q v_j \rangle_V^2 = \sum_{j=1}^{\infty} \langle g_s, v_j \rangle_{V_Q}^2 = \|g_s\|_{V_Q}^2.$$

We conclude that (3.14) holds. We note for later reference that this equality $\|\Phi_s^g\|_2 = \|g_s\|_{V_Q}$ remains valid even if $\mathrm{Tr}\, Q = +\infty$.

Since, by hypothesis, the right hand-side of (3.14) is finite, we deduce that Φ^g is a square integrable process with values in L_2^0 and the stochastic integral $\int_0^T \Phi_s^g \, d\mathcal{W}_s$ is well-defined.

It remains to prove (3.15). For this, we apply Proposition 3.4 in the following situation: $H = \mathbb{R}$, with one basis vector $f_k = 1$, Φ is defined in (3.13), and the sequence of independent standard Brownian motions in (3.12) is given by $\beta_j(t) = W_t(v_j)$. Therefore,

$$\int_0^T \Phi_t^g \, d\mathcal{W}_t = \sum_{j=1}^{\infty} \int_0^T \Phi_t^g(\tilde{e}_j) \, d\beta_j(t),$$

and the right-hand side is equal to

$$\sum_{j=1}^{\infty} \int_0^T \langle g_t, \tilde{e}_j \rangle_V \, dW_t(v_j) = \sum_{j=1}^{\infty} \int_0^T \langle g_t, v_j \rangle_{V_Q} \, dW_t(v_j) = \int_0^T g_t \, dW_t.$$

This completes the proof. □

3.5. *The case Tr $Q = +\infty$*

In Proposition 2.5, we showed that the covariance operator of the standard cylindrical Wiener process $\{W_t(g),\ t \geq 0,\ g \in U\}$ associated with the spatially homogeneous noise that we considered in Section 2.2 is $Q = \mathrm{Id}_U$, which implies that $\mathrm{Tr}\, Q = +\infty$. Therefore, we cannot make use of Proposition 3.5 since, in this case, there is no Q-*Wiener process* associated to W. However, there is the related notion of *cylindrical Q-Wiener process*, which we now define.

Let $(V, \|\cdot\|_V)$ be a Hilbert space. Let Q be a symmetric non-negative definite and bounded operator on V, possibly such that $\mathrm{Tr}\, Q = +\infty$. Let $(e_j)_j$ be an orthonormal basis of V that consists of eigenvectors of Q with corresponding eigenvalues $\lambda_j, j \in \mathbb{N}^*$. Define $V_0 = Q^{1/2}(V)$ as in Section 3.3.

It is always possible to find a Hilbert space V_1 and a bounded linear injective operator $J : (V, \|\cdot\|_V) \to (V_1, \|\cdot\|_{V_1})$ such that the restriction $J_0 = J_{|V_0} : (V_0, \|\cdot\|_{V_0}) \to (V_1, \|\cdot\|_{V_1})$ is Hilbert–Schmidt. Indeed, as explained in [33, Remark 2.5.1], we may choose $V_1 = V$, $\langle\cdot,\cdot\rangle_{V_1} = \langle\cdot,\cdot\rangle_V$, $\alpha_k \in (0, \infty)$ for all $k \geq 1$ such that $\sum_{k=1}^\infty \alpha_k^2 < +\infty$, and define $J : V \to V$ by

$$J(h) := \sum_{k=1}^\infty \alpha_k \langle h, e_k\rangle_V\, e_k, \qquad h \in V, \tag{3.16}$$

where $(e_k)_k$ is an orthonormal basis of V. Then, for $g \in V_0$, $g = \sum_{k=1}^\infty \langle g, \tilde{e}_k\rangle_{V_0} \tilde{e}_k$, where $\tilde{e}_k = Q^{1/2}(e_k)$, $k \geq 1$, we have

$$J_0(g) = \sum_{k=1}^\infty \alpha_k \langle g, \tilde{e}_k\rangle_{V_0} \sqrt{\lambda_k}\, e_k = \sum_{k=1}^\infty \alpha_k \langle g, \tilde{e}_k\rangle_{V_0}\, \tilde{e}_k,$$

and so $J_0 : (V_0, \|\cdot\|_{V_0}) \to (V, \|\cdot\|_V)$ is clearly Hilbert–Schmidt.

As an operator between Hilbert spaces, from V_0 to V_1, J_0 has an adjoint $J_0^* : V_1 \to V_0$. However, if we consider V_0 and V_1 as Banach spaces, it is more common to consider the adjoint $\tilde{J}_0^* : V_1^* \to V_0^*$.

Proposition 3.6 (*[14, Proposition 4.11] and [33, Proposition 2.5.2]*)**.**

1. Define $Q_1 = J_0 J_0^* : V_1 = \mathrm{Im}\, J_0 \to V_1$. Q_1 is symmetric (self-adjoint), non-negative definite and $\mathrm{Tr}\, Q_1 < +\infty$.
2. Let $\tilde{e}_j = Q^{1/2}(e_j)$, where $(e_j)_j$ is a complete orthonormal basis in V, and let $(\beta_j)_j$ be a family of independent real-valued standard Brownian motions. Then

$$\mathcal{W}_t := \sum_{j=1}^\infty \beta_j(t) J_0(\tilde{e}_j), \qquad t \geq 0, \tag{3.17}$$

   is a Q_1-Wiener process in V_1.
3. Let $I : V_0 \to V_0^*$ be the one-to-one mapping which identifies V_0 with its dual V_0^*, and consider the following diagram:

$$V_1^* \overset{\tilde{J}_0^*}{\to} V_0^* \overset{I^{-1}}{\to} V_0 \overset{J_0}{\to} V_1.$$

   Then, for all $s, t \geq 0$ and $h_1, h_2 \in V_1^*$,

$$E\left(\langle h_1, \mathcal{W}_s\rangle_1 \langle h_2, \mathcal{W}_t\rangle_1\right) = (t \wedge s)\langle (I^{-1} \circ \tilde{J}_0^*)(h_1), (I^{-1} \circ \tilde{J}_0^*)(h_2)\rangle_{V_0}, \tag{3.18}$$

   where $\langle\cdot,\cdot\rangle_1$ denotes the dual form on $V_1^* \times V_1$.
4. $\mathrm{Im}\, Q_1^{1/2} = \mathrm{Im}\, J_0$ and

$$\|h\|_0 = \|Q_1^{-1/2} J_0(h)\|_{V_1} = \|J_0(h)\|_{Q_1^{1/2}(V_1)}, \qquad h \in V_0,$$

   where $Q_1^{-1/2}$ denotes the pseudo-inverse of $Q_1^{1/2}$. Thus, $J_0 : V_0 \to Q_1^{1/2}(V_1)$ is an isometry.

Remark 3.7. (a) Part 3 in the Proposition's statement is commonly abbreviated in the following formal form (see, for instance, [31, Proposition 1.1]): for all $s, t \geq 0$ and $h_1, h_2 \in V_1^*$,

$$E\left(\langle h_1, \mathcal{W}_s\rangle_1 \langle h_2, \mathcal{W}_t\rangle_1\right) = (t \wedge s)\langle h_1, h_2\rangle_{V_0}.$$

(b) The Q_1-Wiener process $\{\mathcal{W}_t,\ t \geq 0\}$ obtained in Proposition 3.6 is usually also called a *cylindrical Q-Wiener process*. As it is pointed out in [14, p. 98], if $\operatorname{Tr} Q < +\infty$, then we can take $\alpha_k = 1$ in (3.16), so $V_1 = V$ and $J = \mathrm{Id}_V$, and we get the classical concept of Q-Wiener process. In this case, one can take $V_0^* = V_Q$, $I^{-1} = Q|_{V_Q}$ and the equality (3.18) reduces to

$$E\left(\langle h_1, \mathcal{W}_s\rangle_1 \langle h_2, \mathcal{W}_t\rangle_1\right) = (t \wedge s)\langle Qh_1, h_2\rangle_V.$$

Proof of Proposition 3.6. Statement 1. follows from (3.4) and the fact that J_0 is Hilbert–Schmidt. Concerning 2., we observe that for $h \in V_1$,

$$E\left(\langle \mathcal{W}_t, h\rangle_{V_1}^2\right) = E\left(\left(\sum_{j=1}^{\infty} \beta_j(t)\big\langle J_0(\tilde{e}_j), h\big\rangle_{V_1}\right)^2\right),$$

and the right-hand side is equal to

$$t\sum_{j=1}^{\infty}\big\langle J_0(\tilde{e}_j), h\big\rangle_{V_1}^2 = t\sum_{j=1}^{\infty}\big\langle \tilde{e}_j, J_0^*(h)\big\rangle_{V_0}^2 = t\,\|J_0^*(h)\|_{V_0}^2 = t\,\big\langle J_0^*(h), J_0^*(h)\big\rangle_{V_0} = t\,\big\langle J_0 J_0^*(h), h\big\rangle_{V_1}.$$

Let us prove now part 3. For the sake of clarity, we will prove the statement for $s = t$ and $h_1 = h_2$. Hence, let $t \geq 0$ and $h \in V_1^*$. We denote by $\langle\cdot,\cdot\rangle_0$ the dual form on $V_0^* \times V_0$. Then, by (3.17), the relation between J_0 and J_0^*, and the properties of I and the family $(\beta_j)_j$, we obtain

$$E\left(\langle h, \mathcal{W}_t\rangle_1^2\right) = E\left(\left\langle h, \sum_{j=1}^{\infty}\beta_j(t) J_0(\tilde{e}_j)\right\rangle_1^2\right),$$

and the right-hand side is equal to

$$t\sum_{j=1}^{\infty}\big\langle h, J_0(\tilde{e}_j)\big\rangle_1^2 = t\sum_{j=1}^{\infty}\left\langle \tilde{J}_0^*(h), \tilde{e}_j\right\rangle_0^2 = t\sum_{j=1}^{\infty}\left\langle (I^{-1}\circ \tilde{J}_0^*)(h), \tilde{e}_j\right\rangle_{V_0}^2 = t\,\|(I^{-1}\circ\tilde{J}_0^*)(h)\|_{V_0}^2.$$

For 4., we refer the reader to [33, Proposition 2.5.2]. □

Let $\{\mathcal{W}_t,\ t \geq 0\}$ be as in (3.17). A predictable stochastic process $\{\Phi_t,\ t \in [0, T]\}$ will be integrable with respect to $\mathcal{W}$ if it takes values in $L_2(Q_1^{1/2}(V_1), H)$ and

$$E\left(\int_0^T \|\Phi_t\|^2_{L_2(Q_1^{1/2}(V_1),H)}\,dt\right) < +\infty.$$

By part 4 of Proposition 3.6, we have

$$\Phi \in L_2^0 = L_2(V_0, H) \quad\Leftrightarrow\quad \Phi\circ J_0^{-1} \in L_2(Q_1^{1/2}(V_1), H).$$

Definition 3.8. For any square integrable predictable process Φ with values in L_2^0 such that

$$E\left(\int_0^T \|\Phi_t\|^2_{L_2^0}\,dt\right) < +\infty,$$

the H-valued stochastic integral $\Phi\cdot\mathcal{W}$ is defined by

$$\int_0^T \Phi_s\,d\mathcal{W}_s := \int_0^T \Phi_s\circ J_0^{-1}\,d\mathcal{W}_s.$$

Remark 3.9. We note that the class of integrable processes with respect to $\mathcal{W}$ does not depend on the choice of V_1, and one checks immediately that even though $\text{Tr}\, Q = +\infty$, formula (3.12) remains valid.

We now relate this notion of stochastic integral with the stochastic integral with respect to the cylindrical Wiener process of Section 2.1. Let $\{W_t,\ t \in [0,T]\}$ be a cylindrical Wiener process with covariance Q on the Hilbert space V, and let $g \in L^2(\Omega \times [0,T]; V_Q)$ be a predictable process, so that $g \cdot W$ is well defined as in Section 2.1. By Proposition 3.6, we can consider the cylindrical Q-Wiener process $\{\mathcal{W}_t,\ t \in [0,T]\}$ defined by

$$\mathcal{W}_t = \sum_{j=1}^{\infty} \beta_j(t) J_0(\tilde{e}_j) \tag{3.19}$$

as in formula (3.17) with $\beta_j(t) = W_t(v_j)$, where $v_j = Q^{-1/2}(e_j)$, $\tilde{e}_j = Q^{1/2}(e_j)$ and $(e_j)_j$ denotes a complete orthonormal basis in V consisting of eigenfunctions of Q, so that $(v_j)_j$ is a complete orthonormal basis in V_Q. This process takes values in some Hilbert space V_1.

For $g \in L^2(\Omega \times [0,T]; V_Q)$, we define, as in (3.13), the operator

$$\Phi_s^g(\eta) = \langle g_s, \eta \rangle_V, \qquad \eta \in V,$$

which takes values in $H = \mathbb{R}$. Recall that $V_0 = Q^{1/2}(V)$ and $V_Q = Q^{-1/2}(V)$.

Proposition 3.10. *The process $\{\Phi_s^g,\ s \in [0,T]\}$ defines a predictable process with values in $L_2(V_0, \mathbb{R})$, such that*

$$E\left(\int_0^T \|\Phi_s^g\|_2^2\, ds\right) = E\left(\int_0^T \|g_s\|_{V_Q}^2\, ds\right),$$

and

$$\int_0^T \Phi_s^g\, d\mathcal{W}_s = \int_0^T g_s\, dW_s.$$

Proof. First, we will prove that $\Phi_s^g \in L_2(V_0, \mathbb{R})$, for $s \in [0,T]$. As in the first part of the proof of Proposition 3.5, $\|\Phi_s^g\|_2 = \|g_s\|_{V_Q}$. This gives the equality of expectations in the statement of the proposition, and the right-hand side is finite by assumption.

Concerning the equality of integrals, we note that by definition,

$$\int_0^T \Phi_s\, d\mathcal{W}_s := \int_0^T \Phi_s \circ J_0^{-1}\, d\mathcal{W}_s,$$

where the right-hand side is defined using the finite-trace approach of Section 3.3. We note that by Proposition 3.6, part 4, $(J_0(\tilde{e}_j))_j$ is a complete orthonormal basis of $Q_1^{1/2}(V_1)$.

According to Proposition 3.4 with $H = \mathbb{R}$, a single basis element $f_k = 1$ of H, $\beta_j(s) = W_s(v_j)$, and $\tilde{e}_j$ there replaced by $J_0(\tilde{e}_j)$, formula (3.12) becomes

$$\int_0^T \Phi_s^g \circ J_0^{-1}\, d\mathcal{W}_s = \sum_{j=1}^{\infty} \int_0^T \Phi_s^g \circ J_0^{-1}(J_0(\tilde{e}_j))\, d\beta_j(s),$$

and the right-hand side is equal to

$$\sum_{j=1}^{\infty} \int_0^T \Phi_s^g(\tilde{e}_j)\, dW_s(v_j) = \sum_{j=1}^{\infty} \int_0^T \langle g_s, \tilde{e}_j \rangle_V\, dW_s(v_j) = \sum_{j=1}^{\infty} \int_0^T \langle g_s, v_j \rangle_{V_Q}\, dW_s(v_j).$$

The last expression is equal to $g \cdot W$. $\square$

Remark 3.11. Proposition 3.10 allows us in particular to associate the spatially homogeneous noise of Section 2.2, viewed as a cylindrical Wiener process with covariance Id_U in Proposition 2.5, with a cylindrical Q-Wiener process as in Proposition 3.6, with $Q = \mathrm{Id}_U$, on the Hilbert space U of Section 2.2, and to relate the associated stochastic integrals.

3.6. Equivalence of Hilbert-space-valued integrals

This section is devoted to proving that the Hilbert-space-valued stochastic integral introduced in Section 2.6 is in fact equivalent to the stochastic integral of Da Prato and Zabczyk described in Sections 3.3 and 3.5.

We consider a cylindrical Wiener process W on a separable Hilbert space V with covariance Q, such that $\mathrm{Tr}\, Q < +\infty$, and let $\mathcal{W}$ be the V-valued Q-Wiener process associated to W (see Theorem 3.2). We fix a separable Hilbert space H and a complete orthonormal basis $(f_k)_k$ of H. In Section 2.6, we defined the H-valued stochastic integral $g \cdot W = \int_0^T g_s\, dW_s$ of any predictable $g \in L^2(\Omega \times [0, T]; V_Q \otimes H)$.

The equivalence that we shall prove is based on the well-known generic fact that the tensor product $V_Q \otimes H$ is canonically isometric and isomorphic to the space $L_2(V_Q^*, H)$ of Hilbert–Schmidt operators defined on the dual space of V_Q and taking values in H (see e.g. [1, Section 12.3]). Notice that the space V_Q^* is non other than $V_0 = Q^{1/2}(V)$. Let $(e_j)_j$ be a complete orthonormal basis of V consisting of eigenvectors of Q. As we have already seen, $v_j = Q^{-1/2}(e_j)$ and $\tilde{e}_j = Q^{1/2}(e_j)$ define complete orthonormal bases of V_Q and V_0, respectively. If we are given an operator $\Psi \in L_2(V_0, H)$, then its associated element $X^\Psi \in V_Q \otimes H$ is given by

$$X^\Psi = \sum_{j,k=1}^{\infty} X_\Psi^{j,k}\, (v_j \otimes f_k) \qquad \text{with} \qquad X_\Psi^{j,k} = \left\langle \Psi(\tilde{e}_j), f_k \right\rangle_H.$$

Moreover, one easily checks that $\|\Psi\|_{L_2(V_0,H)} = \|X^\Psi\|_{V_Q \otimes H}$.

Proposition 3.12. *Let $\Phi = \{\Phi_t,\ t \in [0, T]\}$ be a predictable process in $L^2(\Omega \times [0, T]; L_2(V_0, H))$, so that we can define the H-valued stochastic integral $\int_0^T \Phi_t\, d\mathcal{W}_t$ in the sense of Da Prato and Zabczyk (see* Section 3.3*). Let $g^\Phi = \{g_t^\Phi,\ t \in [0, T]\}$ be the predictable process in $L^2(\Omega \times [0, T]; V_Q \otimes H)$ which is associated to Φ. That is, for all $t \in [0, T]$ we have*

$$g_t^\Phi = \sum_{j,k=1}^{\infty} (g_t^\Phi)^{j,k} (v_j \otimes f_k), \qquad \text{with} \qquad (g_s^\Phi)^{j,k} = \left\langle \Phi_s(\tilde{e}_j), f_k \right\rangle_H. \tag{3.20}$$

Then the H-valued stochastic integral $\int_0^T g_t^\Phi\, dW_t$ of Section 2.6 *is well-defined and*

$$\int_0^T g_t^\Phi\, dW_t = \int_0^T \Phi_t\, d\mathcal{W}_t.$$

Proof. First, note that Proposition 3.4 and the fact that $W_t(h) = \langle \mathcal{W}_t, h\rangle_V$, for all $h \in V$, yield

$$\int_0^T \Phi_t\, d\mathcal{W}_t = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \int_0^T \left\langle \Phi_t(\tilde{e}_j), f_k \right\rangle_H dW_t(v_j) \right) f_k.$$

On the other hand, by definition of the integral of g^Φ with respect to W (see (2.24)),

$$\int_0^T g_t^\Phi\, dW_t = \sum_{k=1}^{\infty} \left[\int_0^T \left(\sum_{j=1}^{\infty} (g_t^\Phi)^{j,k}\, v_j \right) dW_t \right] f_k.$$

Each of the integrals in the above series takes values in $\mathbb{R}$ and is defined by (2.1). Thus

$$\int_0^T g_t^{\Phi}\, dW_t = \sum_{k=1}^{\infty} \left(\sum_{j=1}^{\infty} \int_0^T (g_t^{\Phi})^{j,k} dW_t(v_j) \right) f_k.$$

By (3.20), the proposition is proved. □

Remark 3.13. Using Remark 3.9, we see that Proposition 3.12 remains valid in the case where W is a cylindrical Wiener process on V with covariance Q such that $\mathrm{Tr}\, Q = +\infty$.

4. Spde's driven by a spatially homogeneous noise

This section is devoted to presenting a class of spde's in $\mathbb{R}^d$ driven by a spatially homogeneous noise. In Section 4.1, we present the real-valued approach using the notion of a mild random field solution of the equation. Section 4.2 gives two examples: the stochastic heat equation in any spatial dimension and the stochastic wave dimension in spatial dimensions $d = 1, 2, 3$. In Section 4.3, we establish an existence and uniqueness result which extends a theorem of [6]. In Section 4.4, we present the infinite-dimensional formulation of these spde's. In Section 4.5, we examine the relationship between these two formulations, and conclude that they are equivalent (see Proposition 4.10). In Section 4.6, we examine the relationship with the approach of [9].

We are interested in the following class of non-linear spde's:

$$Lu(t,x) = \sigma(u(t,x))\dot{W}(t,x) + b(u(t,x)), \tag{4.1}$$

$t \geq 0$, $x \in \mathbb{R}^d$, where L denotes a general second order partial differential operator with constant coefficients, with appropriate initial conditions. The coefficients σ and b are real-valued functions and $\dot{W}(t,x)$ is the formal notation for the Gaussian random perturbation described at the beginning of Section 2.2.

If L is first order in time, such as the heat operator $L = \partial/\partial t - \Delta$, where Δ denotes the Laplacian operator on $\mathbb{R}^d$, then we impose initial conditions of the form

$$u(0,x) = u_0(x) \qquad x \in \mathbb{R}^d, \tag{4.2}$$

for some Borel function $u_0 : \mathbb{R}^d \to \mathbb{R}$. If L is second order in time, such as the wave operator $L = \partial^2/\partial t^2 - \Delta$, then we have to impose two initial conditions:

$$u(0,x) = u_0(x), \quad \frac{\partial u}{\partial t}(0,x) = v_0(x), \qquad x \in \mathbb{R}^d, \tag{4.3}$$

for some Borel functions $u_0, v_0 : \mathbb{R}^d \to \mathbb{R}$.

4.1. The random field approach

We now describe the notion of *mild random field* solution to Eq. (4.1). Recall that we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, where $(\mathcal{F}_t)_t$ is the filtration generated by the standard cylindrical Wiener process W of Proposition 2.5, and we fix a time horizon $T > 0$. A real-valued adapted stochastic process $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ is a *mild random field* solution of (4.1) if the following stochastic integral equation is satisfied:

$$\begin{aligned} u(t,x) = I_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))\, W(ds,dy) \\ &+ \int_0^t ds \int_{\mathbb{R}^d} \Gamma(s,dy)\, b(u(t-s,x-y)), \qquad a.s., \end{aligned} \tag{4.4}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$. In (4.4), Γ denotes the fundamental solution associated to L and $I_0(t,x)$ is the contribution of the initial conditions, which we define below. The stochastic integral on the right hand-side of (4.4) is as defined in Section 2.3. In particular, we need to assume that for any (t, x), the fundamental solution $\Gamma(t - \cdot, x - \star)$ satisfies Hypothesis 2.8, and to require that $s \mapsto \Gamma(t - s, x - \star)\sigma(u(s, \star))$, $s \in [0, t]$, defines a predictable process taking values in the space U of Section 2.2 such that

$$E\left(\int_0^t \|\Gamma(t-s, x-\star)\sigma(u(s,\star))\|_U^2 \, ds\right) < +\infty$$

(see Sections 2.2 and 2.4). As we will make explicit in Section 4.3, these assumptions will be satisfied under certain regularity assumptions on the coefficients b and σ (see Theorem 4.3).

The last integral on the right-hand side of (4.4) is considered in the pathwise sense, and we use the notation "$\Gamma(s, dy)$" because we will assume that $\Gamma(s)$ is a measure on $\mathbb{R}^d$. Concerning the term $I_0(t, x)$, if L is a parabolic-type operator and we consider the initial condition (4.2), then

$$I_0(t, x) = (\Gamma(t) * u_0)(x) = \int_{\mathbb{R}^d} u_0(x - y)\, \Gamma(t, dy). \tag{4.5}$$

On the other hand, in the case where L is second order in time with initial values (4.3),

$$\begin{aligned} I_0(t, x) &= (\Gamma(t) * v_0)(x) + \frac{\partial}{\partial t}(\Gamma(t) * u_0)(x) \\ &= \int_{\mathbb{R}^d} v_0(x - y)\, \Gamma(t, dy) + \frac{\partial}{\partial t}\left(\int_{\mathbb{R}^d} u_0(x - y)\, \Gamma(t, dy)\right). \end{aligned} \tag{4.6}$$

4.2. *Examples: stochastic heat and wave equations*

In the case of the stochastic heat equation in any space dimension $d \geq 1$ and the stochastic wave equation in dimensions $d = 1, 2, 3$, following [6, Section 3] (see also [29, Examples 4.2 and 4.3]), the fundamental solutions are well-known and the conditions in Hypothesis 2.8 can be made explicit.

Indeed, let Γ be the fundamental solution of the heat equation in $\mathbb{R}^d$, $d \geq 1$, so that

$$\Gamma(t, x) = (4\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{4t}\right).$$

In particular, we have $\mathcal{F}\Gamma(t)(\xi) = \exp(-4\pi^2 t|\xi|^2)$, $\xi \in \mathbb{R}^d$, and, because

$$\int_0^T \exp(-4\pi^2 t|\xi|^2)\, dt = \frac{1}{4\pi^2|\xi|^2}(1 - \exp(-4\pi^2 T|\xi|^2)),$$

we conclude that condition (2.11) in Hypothesis 2.8 holds if and only if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1 + |\xi|^2} < +\infty. \tag{4.7}$$

Now let Γ_d be the fundamental solution of the wave equation in $\mathbb{R}^d$, with $d = 1, 2, 3$. This restriction on the space dimension is due to the fact that the fundamental solution in $\mathbb{R}^d$ with $d > 3$ is no longer a non-negative distribution (for results on the stochastic wave equation in spatial dimension $d > 3$, we refer the reader to [5]: see Remark 2.10). It is well known (see [17, Chapter 5]) that

$$\Gamma_1(t, x) = \frac{1}{2} 1_{\{|x|<t\}}, \quad \Gamma_2(t, x) = \frac{1}{2\pi}(t^2 - |x|^2)_+^{-1/2}, \quad \Gamma_3(t)(dx) = \frac{1}{4\pi t}\sigma_t(dx),$$

where σ_t denotes the uniform surface measure on the three-dimensional sphere of radius t, with total mass $4\pi t^2$. This implies that, for each t, $\Gamma_d(t)$ has compact support. Furthermore, for all dimensions $d \geq 1$, the Fourier transform of $\Gamma_d(t)$ is

$$\mathcal{F}\Gamma_d(t)(\xi) = \frac{\sin(2\pi t|\xi|)}{2\pi|\xi|}.$$

Elementary estimates show that there are positive constants c_1 and c_2 depending on $T > 0$ such that

$$\frac{c_1}{1+|\xi|^2} \le \int_0^T \frac{\sin^2(2\pi t|\xi|)}{4\pi^2|\xi|^2}\, dt \le \frac{c_2}{1+|\xi|^2}.$$

Therefore, Γ_d satisfies condition (2.11) if and only if (4.7) holds.

For $d = 1$, $I_0(t, x)$ is given by the so-called d'Alembert's formula (see, for instance, [16, p. 68]):

$$I_0^1(t,x) = \frac{1}{2}\left[u_0(x+t) + u_0(x-t)\right] + \frac{1}{2}\int_{x-t}^{x+t} v_0(y)\, dy, \qquad x \in \mathbb{R}. \tag{4.8}$$

For $d = 2$ (see [16, p. 74]),

$$I_0^2(t,x) = \frac{1}{2\pi t}\int_{|x-y|<t} \frac{u_0(y+tv_0) + \nabla u_0(y)\cdot(x-y)}{(t^2-|x-y|^2)^{1/2}}\, dy, \qquad x \in \mathbb{R}^2.$$

Finally, for $d = 3$ (see [16, p. 77]), for $x \in \mathbb{R}^3$,

$$I_0^3(t,x) = \frac{1}{4\pi t^2}\int_{\mathbb{R}^3} \left(tv_0(x-y) + u_0(x-y) + \nabla u_0(x-y)\cdot y\right)\, \sigma_t(dy). \tag{4.9}$$

It is important to remark that in the above formulas, we have implicitly assumed that all integrals that appear are well defined. Indeed, in Lemma 4.2 below, we will exhibit sufficient conditions on u_0 and v_0 under which such integrals exist and are uniformly bounded with respect to t and x.

4.3. Random field solutions with arbitrary initial conditions

The aim of this section is to prove the existence and uniqueness of a mild random field solution to the stochastic integral equation (4.4).

We are interested in solutions that are L^p-bounded, as in (4.10) below, and L^2-continuous. This is only possible under certain assumptions on the initial conditions. In particular, the initial conditions will have to be such that the following hypothesis is satisfied.

Hypothesis 4.1. $(t, x) \mapsto I_0(t, x)$ is continuous and $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |I_0(t, x)| < +\infty$.

For the particular case of the heat equation in any spatial dimension and the wave equation with $d \in \{1, 2, 3\}$, sufficient conditions for Hypothesis 4.1 to hold are given in the next lemma.

Lemma 4.2. *Consider the following two sets of hypotheses:*

(i) *Heat equation.* $u_0 : \mathbb{R}^d \to \mathbb{R}$ *is measurable and bounded.*
(ii) *Wave equation. When* $d = 1$, u_0 *is bounded and continuous, and* v_0 *is bounded and measurable. When* $d = 2$, $u_0 \in C^1(\mathbb{R}^2)$ *and there is* $q_0 \in]2, \infty]$ *such that* $u_0, \nabla u_0, v_0$ *all belong to* $L^{q_0}(\mathbb{R}^2)$. *When* $d = 3$, $u_0 \in C^1(\mathbb{R}^3)$, u_0 *and* ∇u_0 *are bounded, and* v_0 *is bounded and continuous.*

Then under condition (*i*) *or* (*ii*), *Hypothesis 4.1 is satisfied.*

Proof. Assume first that L is the heat operator on $\mathbb{R}^d$, $d \ge 1$, with initial condition u_0 satisfying (i). Then, by (4.5),

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |I_0(t,x)| \le \|u_0\|_\infty \sup_{t\in]0,T[} \int_{\mathbb{R}^d} (2\pi t)^{-d/2} \exp\left(-\frac{|y|^2}{2t}\right) dy = \|u_0\|_\infty < +\infty.$$

Secondly, assume that L is the wave operator on $\mathbb{R}^d$, $d = 1, 2, 3$, and that condition (ii) is satisfied. We make explicit the dependence on the space dimension by denoting $I_0^d(t, x)$, $d = 1, 2, 3$, the term $I_0(t, x)$.

By (4.8), if $d = 1$ it is clear that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |I_0^1(t,x)| \le C(\|u_0\|_\infty + \|v_0\|_\infty).$$

To deal with the case $d = 2$, we refer to [26, pp. 808–809]. In this reference, the explicit formula $\Gamma_2(t,x) = \frac{1}{2\pi}(t^2 - |x|^2)_+^{-1/2}$ was used to show that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} |I_0^2(t,x)| \le C(\|u_0\|_\infty + \|\nabla u_0\|_\infty + \|v_0\|_\infty).$$

Finally, for the case $d = 3$ we have, by (4.9):

$$|I_0^3(t,x)| \le C(\|v_0\|_\infty + \|u_0\|_\infty + \|\nabla u_0\|_\infty) \sup_{s\in\,]0,T[} \frac{\sigma_s(\mathbb{R}^3)}{s^2},$$

where σ_s denotes the uniform surface measure on the three-dimensional sphere of radius s. In particular, the total mass of σ_s is proportional to s^2 and, therefore, $I_0^3(t,x)$ is uniformly bounded with respect to $t \in [0,T]$ and $x \in \mathbb{R}^3$.

Finally, the continuity property of $(t,x) \mapsto I_0(t,x)$ follows from the hypotheses and the explicit formulas for $I_0(t,x)$ given in Section 4.1. This concludes the proof. □

The next theorem discusses existence and uniqueness of mild random field solutions to equation (4.4). Since this theorem covers rather general initial conditions, it is an extension of Theorem 13 in [6]. Indeed, in this reference, only vanishing initial data could be considered, because of the spatially homogeneous covariance required for the process Z in the construction of the stochastic integral used there for the wave equation when $d = 3$ (see [6, p. 10 and Theorem 2]). Of course, in the case of the stochastic wave equation in spatial dimensions $d = 1, 2$, there are many results on existence and uniqueness of mild random field solutions with non-vanishing initial conditions: see for instance [3,7,26,28].

Theorem 4.3. *Assume that* Hypotheses 2.8 and 4.1 *are satisfied and that σ and b are Lipschitz functions. Then there exists a unique mild random field solution $\{u(t,x),\ (t,x) \in [0,T]\times\mathbb{R}^d\}$ of* Eq. (4.4). *Moreover, the process u is L^2-continuous and for all $p \ge 1$,*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u(t,x)|^p) < +\infty. \tag{4.10}$$

Proof. The proof is similar to those of [26, Theorem 1.2] and [6, Theorem 13]. We define the Picard iteration scheme

$$\begin{aligned} u^0(t,x) &= I_0(t,x),\\ u^{n+1}(t,x) &= u^0(t,x) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\sigma(u^n(s,y))\, W(ds,dy)\\ &\quad + \int_0^t \int_{\mathbb{R}^d} b(u^n(t-s,x-y))\, \Gamma(s,dy)\, ds, \end{aligned} \tag{4.11}$$

for $n \ge 0$. We prove by induction on n that the process $\{u^n(t,x),\ (t,x) \in [0,T]\times\mathbb{R}^d\}$ is well defined and, for $p \ge 1$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u^n(t,x)|^p) < +\infty, \tag{4.12}$$

for every $n \ge 0$.

Notice that by Hypothesis 4.1, the process u^0 is locally bounded, and the Lipschitz property on σ yields

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\sigma(u^0(t,x))|^p < +\infty.$$

By Proposition 2.9, this implies that the stochastic integral

$$\mathcal{I}^0(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\sigma(u^0(s,y))\, W(ds,dy)$$

is well-defined and

$$E(|\mathcal{I}^0(t,x)|^p) \leq C \int_0^t ds \sup_{z\in\mathbb{R}^d} \left(1+|u^0(s,z)|^p\right) \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Gamma(t-s)(\xi)|^2$$
$$\leq C \sup_{(s,z)\in[0,T]\times\mathbb{R}^d} (1+|u^0(s,z)|^p) \int_0^T ds J(s), \tag{4.13}$$

where

$$J(s) = \int_{\mathbb{R}^d} \mu(d\xi)|\mathcal{F}\Gamma(s)(\xi)|^2.$$

In order to deal with the pathwise integral

$$\mathcal{J}^0(t,x) = \int_0^t ds \int_{\mathbb{R}^d} \Gamma(s,dy)\, b(u^0(t-s,x-y)),$$

we apply Hölder's inequality with respect to the finite measure $\Gamma(s,dy)\,ds$ on $[0,T]\times\mathbb{R}^d$ and use the Lipschitz property of b:

$$|\mathcal{J}^0(t,x)|^p \leq C \int_0^t ds \int_{\mathbb{R}^d} \Gamma(s,dy) \left(1+|u^0(t-s,x-y)|^p\right). \tag{4.14}$$

The latter term is uniformly bounded with respect to t and x. Together with (4.13), this implies that $\{u^1(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^d\}$ is a well-defined measurable process. Further, by (4.13), (4.14) and Hypothesis 2.8,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u^1(t,x)|^p) < +\infty.$$

Consider now $n>1$ and assume that $\{u^n(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^d\}$ is a well-defined measurable process satisfying (4.12). Using the same arguments as above, one proves that the integrals $\mathcal{I}^{n+1}(t,x)$ and $\mathcal{J}^{n+1}(t,x)$ exist, so that the process u^{n+1} is well-defined and is uniformly bounded in $L^p(\Omega)$. This proves (4.12).

The next step consists in showing that the bound (4.12) is uniform with respect to n, that is

$$\sup_{n\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u^n(t,x)|^p) < +\infty. \tag{4.15}$$

Indeed, the same kind of estimates as in the first part of the proof show that for $n\geq 1$,

$$E(|u^{n+1}(t,x)|^p) \leq C\left(1+\int_0^t ds \left(1+\sup_{z\in\mathbb{R}^d} E(|u^n(s,z)|^p)\right)(J(t-s)+1)\right).$$

We conclude that (4.15) holds by the version of Gronwall's Lemma presented in [6, Lemma 15].

Now we show that the sequence $(u^n(t,x))_{n\geq 1}$ converges in $L^p(\Omega)$. Following the same lines as in the proof of [6, Theorem 13], let

$$M_n(t) := \sup_{(s,x)\in[0,t]\times\mathbb{R}^d} E(|u^{n+1}(s,x)-u^n(s,x)|^p).$$

Using the Lipschitz property of b and σ, and applying the same arguments as above, we obtain the estimate

$$M_n(t) \leq C\int_0^t ds\, M_{n-1}(s)(J(t-s)+1).$$

Hence, we apply again [6, Lemma 15] to conclude that $(u^n(t,x))_{n\geq 1}$ converges uniformly in $L^p(\Omega)$ to a limit $u(t,x)$. The process $\{u(t,x),\ (t,x)\in[0,T]\times\mathbb{R}^d\}$ has a measurable version that satisfies Eq. (4.4).

Indeed, let us sketch the calculations concerning the stochastic integral term $\mathcal{I}^n(t,x)$ of (4.11): we will prove that

$$\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|\mathcal{I}^n(t,x)-\mathcal{I}(t,x)|^p)=0.$$

By the Lipschitz property of σ, Proposition 2.9 and Hypothesis 2.8,

$$\begin{aligned} E(|\mathcal{I}^n(t,x)-\mathcal{I}(t,x)|^p) &\le E\left(\left|\int_0^t\int_{\mathbb{R}^d}\Gamma(t-s,x-y)[\sigma(u^{n-1}(s,y))-\sigma(u(s,y))]\,W(ds,dy)\right|^p\right)\\ &\le C\int_0^t ds\,\sup_{z\in\mathbb{R}^d}E(|u^{n-1}(s,z)-u(s,z)|^p)\int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}\Gamma(t-s)(\xi)|^2\\ &\le C\sup_{(s,z)\in[0,T]\times\mathbb{R}^d}E(|u^{n-1}(s,z)-u(s,z)|^p)\end{aligned}$$

and this last term converges to zero, as n tends to infinity. The pathwise integral term can be studied in a similar manner. Therefore, the process u solves (4.4). Finally, uniqueness of the solution can be checked by standard arguments. □

4.4. *Spatially homogeneous spde's in the infinite-dimensional setting*

Stochastic partial differential equations of the form (4.1) on $\mathbb{R}^d$ and driven by a spatially homogeneous Wiener process have been studied, in the context of Da Prato and Zabczyk [14], in a series of works: [15, Section 11.4], and [20,21,30–32]. The aim of this section is to sketch the formulation used in those papers, focusing mostly on the one used by Peszat and Zabczyk in [32]. Then, in Section 4.5, we will compare their solution with the mild random field solution of Section 4.1.

In [32], the stochastic wave equation with $d = 1, 2, 3$ and the stochastic heat equation in any space dimension are considered. This meshes well with the case considered in Section 4.3. However, we note that the stochastic wave equation in higher dimensions ($d > 3$) can also be formulated and solved in the infinite-dimensional setting, but using a slightly different formulation (see [30]).

4.4.1. *General framework*

We first recall the generic setup for evolution equations in infinite-dimensional spaces. These are usually written

$$\begin{cases} du(t)=(Au(t)+F(u(t)))\,dt+B(u(t))\,d\mathcal{W}_t, & t\in\,]0,T[,\\ u(0)=h. \end{cases} \tag{4.16}$$

In this equation, $(\mathcal{W}_t)$ is a cylindrical Q-Wiener process on a Hilbert space V, h is an element of a Hilbert space H, F is a mapping from H into H, and B is a mapping from H into $L(U,H)$. The operator $A: D(A)\subset H\to H$ is the infinitesimal generator of a (strongly) continuous semigroup $(S(t))_{t\in\mathbb{R}_+}$ (meaning, generically, that $S(0)=I_V$, $S(t+s)=S(t)S(s)$, and for $h\in D(A)$, $\frac{d}{dt}S(t)h=AS(t)h=S(t)Ah$, so that one sometimes writes $S(t)=e^{At}$).

The process $(\mathcal{W}_t)$ is assumed to be adapted to a filtration $\{\mathcal{F}_t,\ t\in\mathbb{R}_+\}$, such that for all $s,t\in\mathbb{R}_+$, $\mathcal{W}_{t+s}-\mathcal{W}_t$ is independent of $\mathcal{F}_t$.

An adapted H-valued process $\{u(t),\ t\in[0,T]\}$ is a *mild* solution of (4.16) provided for all $t\in[0,T]$, a.s.,

$$u(t)=S(t)h+\int_0^t S(t-s)F(u(s))\,ds+\int_0^t S(t-s)B(u(s))\,d\mathcal{W}_s. \tag{4.17}$$

Of course, $(S(t))$, F, B and $(\mathcal{W}_t)$ must satisfy certain conditions in order that this equation make sense, and further conditions in order to guarantee existence and uniqueness of the solution. It is also necessary to specify the meaning of the two integrals in (4.17): the first is essentially a Bochner integral, and for the second, one assumes that for all $h \in H$ and $t \in \mathbb{R}_+$, $S(t)B(h)$ belongs to $L_2(V_0, H)$, where $V_0 = Q^{1/2}(V)$. For existence and uniqueness, F and B must typically satisfy at least Lipschitz and linear growth conditions (see, for instance, [15, Theorem 5.3.1]).

In order to place the spde (4.1) in this framework, it is necessary to specify all the ingredients above (V, $\mathcal{W}$, H, A, S, F, and B). We begin with V and $\mathcal{W}$.

4.4.2. Spatially homogeneous noise

To begin with, we note that in [32]—and, indeed, in the above mentioned companion papers—a slightly more general spatially correlated noise than the one described in Section 2.2 has been used. More precisely, one considers a spatially homogeneous Wiener process $\{\mathcal{W}^*_t,\ t \geq 0\}$ with values in the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions. Denoting by $\langle \cdot, \cdot \rangle$ the usual duality action of $\mathcal{S}'(\mathbb{R}^d)$ on $\mathcal{S}(\mathbb{R}^d)$, this means that for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $\{\langle \mathcal{W}^*_t, \varphi\rangle,\ t \in \mathbb{R}_+\}$ is a centered Gaussian process and there exists $\Lambda \in \mathcal{S}'(\mathbb{R}^d)$ such that for all $\varphi, \psi \in \mathcal{S}(\mathbb{R}^d)$ and $s, t \in \mathbb{R}_+$,

$$E\left(\left\langle \mathcal{W}^*_s, \varphi\right\rangle \left\langle \mathcal{W}^*_t, \psi\right\rangle\right) = (s \wedge t)\langle \Lambda, \varphi * \tilde{\psi}\rangle,$$

where $\tilde{\psi}(x) = \psi(-x)$. The Schwartz distribution Λ must be the Fourier transform of a symmetric and non-negative tempered measure μ on $\mathbb{R}^d$.

Remark 4.4. In the particular case where Λ is a non-negative measure satisfying the conditions of Section 2.2, we recover the covariance operator of the cylindrical Wiener process W on the Hilbert space U defined in Proposition 2.5 (see (2.2)):

$$E\left(\left\langle \mathcal{W}^*_s, \varphi\right\rangle \left\langle \mathcal{W}^*_t, \psi\right\rangle\right) = (s \wedge t) \int_{\mathbb{R}^d} \Lambda(dx)\, (\varphi * \tilde{\psi})(x) = E\left(W_s(\varphi) W_t(\psi)\right). \tag{4.18}$$

In order to relate this general noise to the one defined in Section 2.2, let U be the Hilbert space defined in Section 2.2, and let U^* be the dual of U. The following characterization of U^* is given in [31, Proposition 1.2]. Recall that $\tilde{L}^2(\mathbb{R}^d, \mu)$ stands for the subspace of $L^2(\mathbb{R}^d, \mu)$ consisting of all functions ϕ such that $\tilde{\phi} = \phi$.

Lemma 4.5. *A distribution $g \in \mathcal{S}'(\mathbb{R}^d)$ belongs to U^* if and only if there is $\phi \in \tilde{L}^2(\mathbb{R}^d, \mu)$ such that $g = \mathcal{F}(\phi\mu)$. Moreover, if $g_1 = \mathcal{F}(\phi_1\mu)$ and $g_2 = \mathcal{F}(\phi_2\mu)$, with $\phi_1, \phi_2 \in \tilde{L}^2(\mathbb{R}^d, \mu)$, then*

$$\langle g_1, g_2\rangle_{U^*} = \langle \phi_1, \phi_2\rangle_{\tilde{L}^2(\mathbb{R}^d, \mu)}.$$

Remark 4.6. The previous lemma allows us to determine the explicit form of the isometry $I : U \to U^*$. More precisely, as stated in Remark 2.3, any element $g \in U$ can be written in the form $g = \mathcal{F}^{-1}\phi$, with $\phi \in \tilde{L}^2(\mathbb{R}^d; d\mu)$. Then, for such g, $I(g) \in U^*$ is defined by

$$I(g) = \mathcal{F}(\phi\mu).$$

Moreover, we have the following lemma whose proof is straightforward. In this lemma, $\tilde{\mathcal{S}}(\mathbb{R}^d)$ denotes the family of functions $\varphi \in \mathcal{S}(\mathbb{R}^d)$ such that $\tilde{\varphi} = \varphi$.

Lemma 4.7. *Let $\varphi \in U$ be such that $\varphi \in \tilde{\mathcal{S}}(\mathbb{R}^d)$. Then $I(\varphi) = \varphi * \Lambda$.*

As it has been explained in [31, p. 191] (see, in particular, Proposition 1.1 therein), $\mathcal{W}^*$ may be regarded as a U^*-valued cylindrical Q-Wiener process with $Q = \mathrm{Id}_{U^*}$ (so that the generic Hilbert space V used in (4.16) is $V = U^*$). More precisely, let U^*_1 be a Hilbert space such that there exists a dense Hilbert–Schmidt embedding $J^* : U^* \to U^*_1$ (see Proposition 3.6). Then

$$\mathcal{W}^*_t = \sum_{j=1}^{\infty} \beta_j(t) J^*(e^*_j), \tag{4.19}$$

where $(e^*_j)_j$ is a complete orthonormal basis in U^*, and the $\beta_j(t)$ are independent standard Brownian motions (note that $Q^{1/2} = Q^{-1/2} = \mathrm{Id}_{U^*}$). Therefore, we will be able to define Hilbert-space-valued stochastic integrals with respect to $\mathcal{W}^*$, as has been described in Section 3.5. Note that U^* is sometimes called the reproducing kernel Hilbert space associated to $\mathcal{W}^*$ (see, for instance, [14, Section 2.2.2] or [31, p. 191]).

4.4.3. The space H and the operators A, S, F and B

In [32], mild solutions to the formal Eq. (4.1) are considered in a Hilbert space H of the form $L^2_\vartheta = L^2(\mathbb{R}^d, \vartheta(x)dx)$, where $\vartheta \in C^\infty(\mathbb{R}^d)$ is a strictly positive even function such that $\vartheta(x) = e^{-|x|}$, for $|x| \geq 1$. Let us also denote by H^1_ϑ the weighted Sobolev space which is the completion of $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm

$$\|\psi\|_{H^1_\vartheta} = \left(\int_{\mathbb{R}^d} \left[|\psi(x)|^2 + |\nabla\psi(x)|^2\right] \vartheta(x)dx\right)^{1/2}.$$

The operator A is the Laplacian Δ, with domain the classical Sobolev space $H^{2,2}$. In the case of the heat equation, where $L = d/dt - \Delta$, the associated semigroup $S(t)$ is such that

$$S(t)\varphi(x) = \int_{\mathbb{R}^d} \Gamma(t, x-y)\varphi(y)\, dy, \quad \varphi \in \mathcal{S}(\mathbb{R}^d), \quad t \geq 0, \quad x \in \mathbb{R}^d, \tag{4.20}$$

where Γ is the fundamental solution of the heat equation in $\mathbb{R}^d$ (see Section 4.2), and it is shown in [31, Lemma 3.1] that S has a unique extension to a (holomorphic) semigroup on L^2_ϑ, which we still denote by S.

The operators F and B are the so-called *Nemitskii operators* associated respectively to the functions b and σ appearing in (4.1), and are defined by

$$F(v)(x) := b(v(x)), \qquad B(v)(x) := \sigma(v(x)), \qquad v \in L^2_\vartheta, \quad x \in \mathbb{R}^d.$$

With the choices just made, the mild formulation (4.17), with $\mathcal{W}$ replaced there by $\mathcal{W}^*$, is a formalization of (4.1) in the case where $L = \frac{d}{dt} - \Delta$ (the stochastic heat equation).

4.4.4. The case of the stochastic wave equation

In the case of the wave equation, $L = \partial^2/\partial t^2 - \Delta$, the formulations (4.16) and (4.17) are not immediately applicable, because L is not first-order in time. At least two approaches are possible: the second-order equation (in time) can be written as a system of first-order equations, or one can focus on the mild formulation (4.17), with $S(t)$ defined by

$$S(t)\varphi = \Gamma(t) * \varphi,$$

where $\Gamma(t)$ is now the fundamental solution of the wave equation, as in Section 4.2. In this case, $S(t)$ no longer defines a semigroup. Nevertheless, this approach was used in [32] in order to treat the heat and wave equations in a unified manner.

When $L = \frac{\partial^2}{\partial t^2} - \Delta$, we will restrict ourselves to spatial dimensions $d \in \{1, 2, 3\}$. Let Γ be the fundamental solution associated to L, let $u_0 \in H^1_\vartheta$, $v_0 \in L^2_\vartheta$, and fix a time horizon $T > 0$. By definition, a *mild L^2_ϑ-valued* solution of (4.1) with $L = \partial^2/\partial t^2 - \Delta$, is an $\mathcal{F}_t$-adapted process $\{u(t),\ t \in [0, T]\}$ with values in L^2_ϑ satisfying

$$u(t) = \frac{\partial}{\partial t}(\Gamma(t) * u_0) + \Gamma(t) * v_0 + \int_0^t \Gamma(t-s) * b(u(s))\, ds + \int_0^t \Gamma(t-s) * \sigma(u(s))\, d\mathcal{W}^*_s. \tag{4.21}$$

The stochastic integral on the right-hand side of (4.21) has to be defined. This requires interpreting the integrand $\Gamma(t-s) * \sigma(u(s))$ in the framework of Section 3.5.

Recall that, as in Section 3.5 and since $Q = \mathrm{Id}_{U^*}$ and so $U^* = (U^*)_0$, we will be able to define the stochastic integral with respect to $\mathcal{W}^*$ of any predictable process Φ taking values in the space $L_2(U^*, H)$, where $H = L^2_\vartheta$. Therefore, it is necessary to interpret $\Gamma(t-s) * \sigma(u(s))$ as an element of $L_2(U^*, H)$.

Let $U^{*,0}$ be the dense subspace of U^* consisting of all $g = \mathcal{F}(\phi\mu)$ with $\phi \in \tilde{\mathcal{S}}(\mathbb{R}^d)$. According to [32, p. 427], it holds that $U^{*,0} \subset \mathcal{C}_b(\mathbb{R}^d)$, the space of bounded and continuous functions on $\mathbb{R}^d$. For $u \in L^2_\vartheta$ and $t > 0$, define the following operator:

$$\mathcal{K}(t,u)(\eta) = \Gamma(t) * (u\eta), \qquad \eta \in U^{*,0}. \tag{4.22}$$

Then it is shown in Lemma 3.3 of [32] that, for all $t > 0$ and $u \in L^2_\vartheta$, $\mathcal{K}(t,u)$ has a unique extension to a Hilbert–Schmidt operator from U^* into L^2_ϑ. Thus extended, $\mathcal{K}(t,\cdot)$ becomes a bounded linear operator from L^2_ϑ into $L_2(U^*, L^2_\vartheta)$. Therefore, if u is an L^2_ϑ-valued adapted process, we can define the stochastic integral as follows:

$$\int_0^t (\Gamma(t-s) * \sigma(u(s)))\, d\mathcal{W}^*_s := \int_0^t \mathcal{K}(t-s, \sigma(u(s)))\, d\mathcal{W}^*_s. \tag{4.23}$$

In the formulation above, $\sigma(u(s))$ denotes the function $\sigma(u(s))(x) := \sigma(u(s,x))$, $x \in \mathbb{R}^d$, which belongs to L^2_ϑ.

This definition of the stochastic integral (4.23) is the one that is used in the mild formulation (4.21). The main result in [32] on existence and uniqueness of a solution to Eq. (4.21) is the following (see [32, Theorem 0.1]).

Theorem 4.8. *Assume that $d \in \{1,2,3\}$ and that the coefficients b and σ are Lipschitz functions. Suppose that there is $\kappa > 0$ such that $\Lambda + \kappa\, dx$ is a nonnegative measure (where dx denotes Lebesgue measure), and the spectral measure μ satisfies*

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+|\xi|^2} < +\infty. \tag{4.24}$$

Then, for arbitrary $u_0 \in H^1_\vartheta$ and $v_0 \in L^2_\vartheta$, there exists a unique L^2_ϑ-valued solution to Eq. (4.21).

4.4.5. The stochastic heat equation

We now return to the case of the stochastic heat equation, namely we consider Eq. (4.1) when $L = \partial/\partial t - \Delta$, with any spatial dimension $d \geq 1$.

Letting Γ be the fundamental solution of the heat equation in $\mathbb{R}^d$, and defining S as in (4.20), the formulation in (4.17) is equivalent (see [32, Section 5]) to the equation

$$u(t) = \Gamma(t) * u_0 + \int_0^t \Gamma(t-s) * b(u(s))\, ds + \int_0^t \Gamma(t-s) * \sigma(u(s))\, d\mathcal{W}^*_s. \tag{4.25}$$

Similar to (4.23), the stochastic integral here is defined by

$$\int_0^t \Gamma(t-s) * \sigma(u(s))\, d\mathcal{W}^*_s := \int_0^t \mathcal{P}(t-s, \sigma(u(s)))\, d\mathcal{W}^*_s,$$

where the operator $\mathcal{P}(t,u)$, for $u \in L^2_\vartheta$ and $t > 0$, is defined as in (4.22), but Γ is now the fundamental solution of the heat equation in $\mathbb{R}^d$. As explained in [32, Section 5], the equivalence between (4.25) and (4.17) can be understood through the equality

$$\mathcal{P}(t,u)(\eta) = S(t)(u\eta), \qquad u \in L^2_\vartheta, \quad \eta \in U^{*,0}.$$

Moreover, [32, Lemma 5.3] states that $\mathcal{P}(\cdot,u)$ defines a square-integrable process with values in $L_2(U^*, L^2_\vartheta)$, which implies that the above stochastic integral is well-defined.

The main result in [32] on existence and uniqueness of a solution to Eq. (4.25) is the following (see [32, Theorem 0.2]).

Theorem 4.9. *Assume that $d \geq 1$ and that the coefficients b and σ are Lipschitz functions. Suppose that there is $\kappa > 0$ such that $\Lambda + \kappa\, dx$ is a nonnegative measure, and the spectral measure μ satisfies* (4.24). *Then, for arbitrary $u_0 \in L^2_\vartheta$, there exists a unique L^2_ϑ-valued solution to* Eq. (4.25).

4.5. Relation with the random field approach

We now examine the relationship between the random field solution to equation (4.4) in the case of the stochastic wave and heat equations and the L^2_ϑ-valued solution to equations (4.21) and (4.25), respectively. For this, we assume that the cylindrical Wiener process W considered in the beginning of Section 4.1 and the cylindrical Q-Wiener process $\mathcal{W}^*$ (with $Q = \mathrm{Id}_{U^*}$) that appears in (4.21) are related as follows.

Let $(e_j)_j$ be a complete orthonormal basis of the Hilbert space U such that $e_j \in \tilde{\mathcal{S}}(\mathbb{R}^d)$, for all $j \geq 1$. Assume that the e_j^* and the $\beta_j(t)$ that appear in (4.19) are given by

$$e_j^* = I(e_j) \qquad \text{and} \qquad \beta_j(t) = W_t(e_j), \tag{4.26}$$

where I is the isometry described in Remark 4.6 (in particular, (e_j^*) is the dual basis of (e_j), that is, $\langle e_j^*, e_k \rangle_{U^*,U} = \delta_{j,k}$). Recall that $J^* : U^* \to U_1^*$ denotes a Hilbert–Schmidt embedding between U^* and a possibly larger Hilbert space U_1^*; moreover, by Proposition 3.6, $(J^*(e_j^*))_j$ defines a basis in U_1^*.

Let us consider $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$, the mild random field solution of (4.4) as given in Theorem 4.3, in the case where L is either the wave operator in spatial dimension $d \in \{1,2,3\}$ or the heat operator with $d \geq 1$ (so as to have a specific form for $I_0(t,x)$). Then for all $(t,x) \in [0,T] \times \mathbb{R}^d$,

$$\begin{aligned} u(t,x) = I_0(t,x) &+ \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y) \sigma(u(s,y))\, W(ds,dy) \\ &+ \int_0^t \int_{\mathbb{R}^d} b(u(t-s, x-y))\, \Gamma(s,dy)\, ds, \qquad a.s., \end{aligned} \tag{4.27}$$

Here, the expression for $I_0(t,x)$ is given either by (4.5) or (4.6), and we assume that the initial conditions satisfy the hypotheses specified in Lemma 4.2. The coefficients σ and b are Lipschitz functions. Recall that $(t,x) \mapsto u(t,x)$ is mean-square continuous and satisfies

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E(|u(t,x)|^2) < +\infty. \tag{4.28}$$

This section is devoted to proving the following result.

Proposition 4.10. *Let $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ be the mild random field solution of (4.27), where L is either the wave operator in spatial dimension $d \in \{1,2,3\}$ or the heat operator with $d \geq 1$. Let $u(t) = u(t, \star)$. Then $\{u(t),\ t \in [0,T]\}$ is the mild L^2_ϑ-valued solution of (4.21) or (4.25), respectively.*

Proof. This proof is written for the case of the stochastic wave equation in spatial dimension $d \in \{1, 2, 3\}$, but also applies to the stochastic heat equation with $d \geq 1$.

In view of the integral Eqs. (4.27) and (4.21), it is clear that the most delicate part in the proof corresponds to the analysis of the stochastic integral terms. Hence, we will start by assuming that both the initial conditions and the drift term b vanish. In this case, $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ solves the integral equation

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))\, W(ds,dy), \quad a.s.$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Let us use the following notation for the above stochastic integral:

$$\mathcal{I}(t,x) := \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, x-y)\sigma(u(s,y))\, W(ds,dy).$$

For any $(t,x) \in [0,T] \times \mathbb{R}^d$, the above integral is a real-valued random variable and it is well-defined because the integrand satisfies the hypotheses described in Section 2.4, that is, $\Gamma(t-\cdot\,, x-\star)$

verifies Hypothesis 2.8 and $\{\sigma(u(s,y)),\ (s,y) \in [0,t] \times \mathbb{R}^d\}$ is a predictable process such that

$$\sup_{(s,y)\in[0,T]\times\mathbb{R}^d} E(|\sigma(u(s,y))|^2) \leq C\left(1 + \sup_{(s,y)\in[0,t]\times\mathbb{R}^d} E(|u(s,y)|^2)\right) < +\infty. \tag{4.29}$$

Let $u(t) = u(t, \star)$, $t \in [0,T]$. We aim to prove that $\{u(t),\ t \in [0,T]\}$ defines a square-integrable stochastic process with values in the weighted space L^2_ϑ which satisfies

$$u(t) = \int_0^t \Gamma(t-s) * \sigma(u(s))\, d\mathcal{W}^*_s, \qquad t \in [0,T].$$

Hence, our objective is to prove that $\{\mathcal{I}(t,\star),\ t \in [0,T]\}$ defines an element in $L^2(\Omega \times [0,T]; L^2_\vartheta)$ and

$$\mathcal{I}(t,\star) = \int_0^t \Gamma(t-s) * \sigma(u(s))\, d\mathcal{W}^*_s.$$

In order to simplify the notation, we will write $Z(s,y) := \sigma(u(s,y))$ and let $Z(s)$ denote the function $Z(s)(y) = Z(s,y)$, $y \in \mathbb{R}^d$.

We will split the proof into several steps.

Step 1. We shall check that $\{\mathcal{I}(t,\star),\ t \in [0,T]\}$ belongs to $L^2(\Omega \times [0,T]; L^2_\vartheta)$ and that, for any fixed $(t,x) \in [0,T] \times \mathbb{R}^d$, the real-valued stochastic integral $\mathcal{I}(t,x)$ can be written as a stochastic integral with respect to a Hilbert-space-valued Wiener process.

Notice that the norm of $\mathcal{I}(\cdot,\star)$ in $L^2(\Omega \times [0,T]; L^2_\vartheta)$ coincides with the norm of $u(\,\cdot\,)$ in the same space, and the latter is given by

$$E\left(\int_0^T dt \int_{\mathbb{R}^d} dx\, \vartheta(x)|u(t,x)|^2\right).$$

By (4.28) and the fact that ϑ is integrable over $\mathbb{R}^d$, this quantity is finite. In particular, we also deduce that Z belongs to $L^2(\Omega \times [0,T]; L^2_\vartheta)$.

On the other hand, let us recall that $\mathcal{I}(t,x)$ is a stochastic integral with respect to the cylindrical Wiener process $\{W_s(h),\ s \in [0,T],\ h \in U\}$ (see Section 2.2) with covariance operator $Q = \mathrm{Id}_U$ and $s \mapsto \Gamma(t-s, x-\star)Z(s)$ is a predictable process in $L^2(\Omega \times [0,T], U)$ by Proposition 2.9. Hence, by Proposition 3.10, the stochastic integral $\mathcal{I}(t,x)$ may be written as

$$\mathcal{I}(t,x) = \int_0^t \Phi^{t,x}_s\, d\mathcal{W}_s, \tag{4.30}$$

where similar to (3.19),

$$\mathcal{W}_t = \sum_{j=1}^\infty W_t(e_j) J(e_j),$$

$J: U \to U_1$ is a Hilbert–Schmidt embedding from U into a possibly larger space U_1 (note that U_1 need not be the dual of U_1^* mentioned after (4.26)), and $\{\Phi^{t,x}_s, s \in [0,t]\}$ is the predictable and square integrable process with values in the space $L_2(U,\mathbb{R})$ of Hilbert–Schmidt operators from U into $\mathbb{R}$, given by

$$\Phi^{t,x}_s(h) = \langle \Gamma(t-s, x-\star)Z(s), h\rangle_U, \qquad h \in U.$$

Moreover,

$$E\left(\int_0^t \|\Phi_s^{t,x}\|^2_{L_2(U,\mathbb{R})}\,ds\right) = E\left(\int_0^t \|\Gamma(t-s,x-\star)Z(s)\|_U^2\,ds\right).$$

Step 2. Recall that we aim to prove that

$$\mathcal{I}(t,\star) = \int_0^t (\Gamma(t-s)*Z(s))\,d\mathcal{W}_s^*, \quad t\in[0,T], \tag{4.31}$$

where this equality must be understood in $L^2(\Omega\times[0,T];L^2_\vartheta)$.

Let $t\in[0,T]$ and $(f_k)_k$ be a complete orthonormal basis in L^2_ϑ. We will find a suitable expansion of $\mathcal{I}(t,\star)$ in terms of $(f_k)_k$. Indeed, by (4.30) and since $\mathcal{I}(t,\star)$ defines a square integrable L^2_ϑ-valued random variable, we have the following representation:

$$\mathcal{I}(t,\star) = \sum_{k=1}^\infty \left[\int_{\mathbb{R}^d} dx\,\vartheta(x)\left(\int_0^t \Phi_s^{t,x}\,d\mathcal{W}_s\right)\cdot f_k(x)\right] f_k. \tag{4.32}$$

Then, by definition of the stochastic integral with respect to $\mathcal{W}$ and using representation (3.12) in Proposition 3.4 (for $H=\mathbb{R}$), for all $x\in\mathbb{R}^d$,

$$\begin{aligned}\int_0^t \Phi_s^{t,x}\,d\mathcal{W}_s &= \int_0^t \Phi_s^{t,x}\circ J^{-1}\,d\mathcal{W}_s = \sum_{j=1}^\infty \int_0^t \Phi_s^{t,x}\circ J^{-1}(J(e_j))\,dW_s(e_j),\\ &= \sum_{j=1}^\infty \int_0^t \Phi_s^{t,x}(e_j)\,d\beta_j(s),\end{aligned} \tag{4.33}$$

where we have made use of (4.26). Hence, plugging (4.33) into (4.32), we see that

$$\mathcal{I}(t,\star) = \sum_{k=1}^\infty \left[\int_{\mathbb{R}^d} dx\,\vartheta(x)\left(\sum_{j=1}^\infty \int_0^t \Phi_s^{t,x}(e_j)\,d\beta_j(s)\right)\cdot f_k(x)\right] f_k. \tag{4.34}$$

Step 3. We now give an analogous representation for the stochastic integral on the right-hand side of (4.31). For this, we will again apply Proposition 3.4 directly to the right-hand side of (4.31); notice that here, $H=L^2_\vartheta$, and the J^* in (4.19) cancels with the $(J^*)^{-1}$ in the definition of the stochastic integral. Therefore, taking (4.26) into account, we see that

$$\int_0^t \Gamma(t-s)*Z(s)\,d\mathcal{W}^* = \sum_{k=1}^\infty \left(\sum_{j=1}^\infty \int_0^t \left\langle \Gamma(t-s)*\left(Z(s)I(e_j)\right), f_k\right\rangle_{L^2_\vartheta} d\beta_j(s)\right) f_k, \tag{4.35}$$

where $(f_k)_k$ and β_j are as in Step 2. Recall that, on the left-hand side of (4.35), $\Gamma(t-s)*Z(s)$ is the formal notation for the Hilbert–Schmidt operator defined on U^* and taking values in L^2_ϑ such that, for any $\eta\in U^{0,*}$, $(\Gamma(t-s)*Z(s))(\eta)=\mathcal{K}(s,Z(s))(\eta)=\Gamma(t-s)*(Z(s)\eta)$.

By Lemma 4.7, $I(e_j)=e_j*\Lambda$ (because $e_j\in\tilde{\mathcal{S}}(\mathbb{R}^d)$), so equality (4.35) can be written in the form

$$\begin{aligned}&\int_0^t \Gamma(t-s)*Z(s)\,d\mathcal{W}^*\\ &\quad= \sum_{k=1}^\infty \left(\sum_{j=1}^\infty \int_0^t \left(\int_{\mathbb{R}^d} dx\,\vartheta(x)\left[\Gamma(t-s)*\left(Z(s)(e_j*\Lambda)\right)\right](x)\cdot f_k(x)\right) d\beta_j(s)\right) f_k.\end{aligned}$$

Applying Fubini's Theorem and comparing the latter expression with (4.34), we observe that, in order to prove (4.31), it suffices to check that, for almost all $x \in \mathbb{R}^d$ and any $\varphi \in \tilde{\mathcal{S}}(\mathbb{R}^d)$,

$$\Phi_s^{t,x}(\varphi) = [\Gamma(t-s) * (Z(s)(\varphi * \Lambda))](x), \quad s \in [0,t].$$

By definition of the operator $\Phi_s^{t,x}$ and expanding the convolutions on the right-hand side above, this equality is equivalent to

$$\langle \Gamma(t-s, x-\star) Z(s), \varphi \rangle_U = \int_{\mathbb{R}^d} \Gamma(t-s, dz) Z(s, x-z) \int_{\mathbb{R}^d} \Lambda(dy)\, \varphi(x-z-y).$$

Notice that this is precisely the statement of Lemma 4.11 below. Therefore, we can conclude that (4.31) holds.

Step 4. Let us finally sketch the extension of what we have proved so far to the case of Eqs. (4.27) and (4.21). That is, we consider a general Lipschitz continuous drift b and initial conditions u_0, v_0 satisfying the hypotheses specified at the beginning of the section. Hence, $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ satisfies (4.27).

One proves that the process $\{u(t),\ t \in [0,T]\}$ belongs to $L^2(\Omega \times [0,T]; L^2_\vartheta)$ as we have done in Step 1. Indeed, an immediate consequence of the proof of Theorem 4.3 is that each term in Eq. (4.27) is bounded in $L^2(\Omega)$, uniformly with respect to $(t,x) \in [0,T] \times \mathbb{R}^d$. This clearly implies that each term in (4.27) defines an element in $L^2(\Omega \times [0,T]; L^2_\vartheta)$.

It follows that the stochastic integral $\int_0^t \Gamma(t-s) * \sigma(u(s))\, d\mathcal{W}^*$ is well-defined and, by Steps 2 and 3 above, we have

$$\int_0^t \Gamma(t-s) * \sigma(u(s))\, d\mathcal{W}^* = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,\ \star - y) \sigma(u(s,y))\, W(ds,dy),$$

where the $\star$ symbol on the right-hand side stands for the variable in L^2_ϑ.

Concerning the pathwise integral in (4.21), we have

$$\begin{aligned} \int_0^t \Gamma(t-s) * b(u(s))\, ds &= \int_0^t ds \int_{\mathbb{R}^d} \Gamma(t-s, dy)\, b(u(s,\ \star - y)) \\ &= \int_0^t ds \int_{\mathbb{R}^d} \Gamma(s, dy)\, b(u(t-s,\ \star - y)). \end{aligned}$$

It is also clear that the contributions of the initial conditions in Eqs. (4.27) and (4.21) coincide as elements in $L^2([0,T]; L^2_\vartheta)$. We have therefore proved that $\{u(t),\ t \in [0,T]\}$ is the mild solution of (4.21), which concludes the proof of Proposition 4.10. □

We now state and prove the following technical lemma, which was used in the proof of Proposition 4.10.

Lemma 4.11. *Fix $t \in [0,T]$. Then, for all $\varphi \in \tilde{\mathcal{S}}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, the stochastic process $\{\Phi_s^{t,x}(\varphi),\ s \in [0,t]\}$ given by*

$$\Phi_s^{t,x}(\varphi) = \langle \Gamma(t-s, x-\star) Z(s), \varphi \rangle_U$$

coincides, as an element in $L^2(\Omega \times [0,t])$, with $\{\mathcal{K}_s^{t,x}(\varphi),\ s \in [0,t]\}$, where

$$\mathcal{K}_s^{t,x}(\varphi) = \int_{\mathbb{R}^d} \Gamma(t-s, dz) Z(s, x-z) \int_{\mathbb{R}^d} \Lambda(dy)\, \varphi(x-z-y).$$

Proof. In order to prove the statement, we will first approximate $\{\Phi_s^{t,x}(\varphi),\ s \in [0,t]\}$ by a sequence of *smooth* processes.

More precisely, as it has been explained in [29, Proposition 3.3], for any $(s, x) \in [0, t] \times \mathbb{R}^d$, we can regularize the element $\Gamma(t-s, x-\star)Z(s)$ of U by means of an approximation of the identity $(\psi_n)_n \subset C_0^\infty(\mathbb{R}^d)$, and we can assume that ψ_n is symmetric, for all n, and $|\mathcal{F}\psi_n| \le 1$. Then, for any $s \in [0, t]$, set $J_n^{t,x}(s) := \psi_n * (\Gamma(t-s, x-\star)Z(s))$. Again by [29, Proposition 3.3], $J_n^{t,x}(s)$ belongs to $\mathcal{S}(\mathbb{R}^d)$ and, as $n \to \infty$, $J_n^{t,x}$ converges to $\Gamma(t-\cdot, x-\star)Z$ in $L^2([0, t] \times \Omega; U)$. Define

$$\Phi_{n,s}^{t,x}(h) := \langle J_n^{t,x}(s), h\rangle_U, \qquad h \in U.$$

This operator is well-defined because $J_n^{t,x}(s)$ is a smooth function and, in fact, it defines an element in $L^2([0, t] \times \Omega; L_2(U, \mathbb{R}))$.

Moreover, $\Phi_n^{t,x} \to \Phi^{t,x}$ in $L^2([0, t] \times \Omega; L_2(U, \mathbb{R}))$, as $n \to \infty$. Indeed, this is an immediate consequence of the fact that the norm of $\Phi_n^{t,x} - \Phi^{t,x}$ is given by

$$\begin{aligned} E\left(\int_0^t \left\|\Phi_n^{t,x} - \Phi^{t,x}\right\|_{L_2(U,\mathbb{R})}^2 ds\right) &= E\left(\int_0^t \sum_{j=1}^\infty \left|\langle J_n^{t,x}(s) - \Gamma(t-s, x-\star)Z(s), e_j\rangle_U\right|^2 ds\right) \\ &= E\left(\int_0^t \left\|J_n^{t,x}(s) - \Gamma(t-s, x-\star)Z(s)\right\|_U^2 ds\right), \end{aligned}$$

where $(e_j)_j$ is a complete orthonormal basis in U. The last term above tends to zero because, as mentioned before, $J_n^{t,x} \to \Gamma(t-\cdot, x-\star)Z$ in $L^2([0, t] \times \Omega; U)$.

Therefore, for any $\varphi \in \tilde{\mathcal{S}}(\mathbb{R}^d)$ (in fact, for any $\varphi \in U$), the sequence of real-valued processes $(\Phi_n^{t,x}(\varphi))_n$ converges to $\Phi^{t,x}(\varphi)$ in $L^2(\Omega \times [0, t])$. In particular, $\Phi_n^{t,x}(\varphi)$ converges weakly to $\Phi^{t,x}(\varphi)$, that is, for any $0 \le a < b \le t$ and $A \in \mathcal{F}$,

$$E\left(1_A \int_a^b ds\ \Phi_{n,s}^{t,x}(\varphi)\right) \to E\left(1_A \int_a^b ds\,\Phi_s^{t,x}(\varphi)\right). \tag{4.36}$$

We will conclude the proof by checking that the left-hand side of (4.36) also converges to

$$E\left(1_A \int_a^b ds\ \mathcal{K}_s^{t,x}(\varphi)\right). \tag{4.37}$$

For this, note that, by definition of $\Phi_{n,s}^{t,x}$, the left-hand side of (4.36) can be written as

$$E\left(1_A \int_a^b ds\ \langle J_n^{t,x}(s), \varphi\rangle_U\right).$$

Because $J_n^{t,x}(s)$ and φ are smooth functions, we can explicitly compute the inner product in the above expression:

$$\begin{aligned} \langle J_n^{t,x}(s), \varphi\rangle_U &= \int_{\mathbb{R}^d} \Lambda(dz) \int_{\mathbb{R}^d} dy\, J_n^{t,x}(s, y)\varphi(y-z) = \int_{\mathbb{R}^d} dy\, J_n^{t,x}(s, y)\,(\Lambda * \varphi)(y) \\ &= \int_{\mathbb{R}^d} dy \left(\int_{\mathbb{R}^d} \Gamma(t-s, dz)\,\psi_n(y-x+z)Z(s, x-z)\right)(\Lambda * \varphi)(y) \\ &= \int_{\mathbb{R}^d} \Gamma(t-s, dz)Z(s, x-z)\left(\int_{\mathbb{R}^d} dy\,\psi_n(y-x+z)\,(\Lambda * \varphi)(y)\right), \end{aligned}$$

and so the term on the left-hand side of (4.36) equals

$$E\left(1_A\int_a^b ds\int_{\mathbb{R}^d}\Gamma(t-s,dz)Z(s,x-z)\int_{\mathbb{R}^d}dy\,\psi_n(x-z-y)(\Lambda*\varphi)(y)\right). \tag{4.38}$$

Since $\varphi\in\tilde{\mathcal{S}}(\mathbb{R}^d)$, the function $y\mapsto(\Lambda*\varphi)(y)$ is continuous in $\mathbb{R}^d$ and $\lim_{|y|\to\infty}(\Lambda*\varphi)(y)=0$. This implies that, for any $x,z\in\mathbb{R}^d$,

$$\lim_{n\to\infty}\int_{\mathbb{R}^d}dy\,\psi_n(x-z-y)\,(\Lambda*\varphi)\,(y)=(\Lambda*\varphi)\,(x-z).$$

Moreover, because ψ_n and φ belong to $\tilde{\mathcal{S}}(\mathbb{R}^d)$, we can apply the definition of the Fourier transform of tempered distributions:

$$\begin{aligned}\left|\int_{\mathbb{R}^d}dy\,\psi_n(x-z-y)\,(\Lambda*\varphi)\,(y)\right| &= \left|\int_{\mathbb{R}^d}\mu(d\xi)\,\mathcal{F}\psi_n(x-z-\cdot)(\xi)\overline{\mathcal{F}\varphi(\xi)}\right|\\ &\le \int_{\mathbb{R}^d}\mu(d\xi)|\mathcal{F}\varphi(\xi)|\\ &< +\infty.\end{aligned}$$

Thus, in order to apply the Dominated Convergence Theorem in (4.38), it remains to prove that

$$E\left(1_A\int_a^b ds\int_{\mathbb{R}^d}\Gamma(t-s,dz)|Z(s,x-z)|\right)<+\infty,$$

and this follows from the hypothesis on Γ and the process Z. So we have proved that the limit of (4.38), as n goes to infinity, is

$$E\left(1_A\int_a^b ds\int_{\mathbb{R}^d}\Gamma(t-s,dz)\,Z(s,x-z)\int_{\mathbb{R}^d}\Lambda(dy)\,\varphi(x-z-y)\right).$$

This shows that the left-hand side of (4.36) converges to (4.37), which concludes the proof. □

4.6. *Relation with the Dalang–Mueller formulation*

In this section, we examine the relationship between the mild random field solution to equation (4.27) and the solution introduced by Dalang and Mueller in [9] (see also [11]), which is based on the L^2-valued stochastic integration framework that was summarized in Section 2.5. Let L^2_θ be the space defined in Section 2.5.

In [9], the authors consider solutions to the following stochastic wave equation in $\mathbb{R}^d$, for any $d\ge 1$:

$$\frac{\partial^2 u}{\partial t^2}(t,x)-\Delta u(t,x)=\sigma(u(t,x))\dot{W}(t,x), \tag{4.39}$$

with initial conditions

$$u(0,x)=u_0(x),\quad \frac{\partial u}{\partial t}(0,x)=v_0(x),\quad x\in\mathbb{R}^d,$$

where $u_0,v_0:\mathbb{R}^d\to\mathbb{R}$ are appropriate Borel functions. The noise $\dot{W}(t,x)$ corresponds to the spatially homogeneous Gaussian noise described in Section 2.2.

We denote by $H^{-1}(\mathbb{R}^d)$ the Sobolev space of distributions such that

$$\|v\|^2_{H^{-1}(\mathbb{R}^d)}:=\int_{\mathbb{R}^d}d\xi\,\frac{1}{1+|\xi|^2}|\mathcal{F}v(\xi)|^2<+\infty.$$

According to [9, Section 5], an adapted L^2_θ-valued process $\{u(t,\star),t\in[0,T]\}$ is a *mild L^2_θ-valued solution* to (4.39) if $t\mapsto u(t,\star)$ is mean-square continuous from $[0,T]$ into L^2_θ and the following L^2_θ-valued stochastic integral equation is satisfied:

$$u(t,\star) = \Gamma(t) * v_0 + \frac{\partial}{\partial t}(\Gamma(t) * u_0) + \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, \star - y)\sigma(u(s,y))M(ds,dy), \tag{4.40}$$

where Γ denotes the fundamental solution of the wave equation in $\mathbb{R}^d$. The stochastic integral in (4.40) takes values in L^2_θ and is defined in the final part of Section 2.5. The main result on existence and uniqueness of solutions to Eq. (4.40) is the following (see [9, Theorem 13]).

Theorem 4.12. *Assume that the spectral measure μ satisfies* (4.7), $u_0 \in L^2(\mathbb{R}^d)$, $v_0 \in H^{-1}(\mathbb{R}^d)$ *and σ is a Lipschitz function. Then* Eq. (4.40) *has a unique mild L^2_θ-valued solution.*

In order to be able to compare the solution of the above equation with the mild random field solution to (4.27), we consider space dimensions $d \in \{1, 2, 3\}$ and we set $b = 0$. The main result of this section is the following.

Theorem 4.13. *Let $d \in \{1,2,3\}$, and let $\{u(t,x),\ (t,x) \in [0,T]\times\mathbb{R}^d\}$ be the mild random field solution of* (4.27) *in the case of the stochastic wave equation ($d \in \{1, 2, 3\}$ and with $b = 0$). Let $u(t) = u(t, \star)$. Then $\{u(t),\ t \in [0,T]\}$ is the L^2_θ-valued solution of* (4.40).

Proof. For simplicity, we assume that the initial conditions vanish (the extension to the general case is straightforward). Recall that $d \in \{1, 2, 3\}$ and $\{u(t,x),\ (t,x) \in [0,T] \times \mathbb{R}^d\}$ satisfies the integral equation

$$u(t,x) = \mathcal{I}_{\Gamma,Z}(t,x), \quad a.s. \tag{4.41}$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$, where

$$\mathcal{I}_{\Gamma,Z}(t,x) := \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)Z(s,y)\,W(ds,dy)$$

and $Z(s,y) := \sigma(u(s,y))$. In order to prove that $\{u(t, \star),\ t \in [0,T]\}$ is the solution of (4.40), we observe that $u(t,\star) \in L^2_\theta$ a.s., since

$$E(\|u(t,\star)\|^2_{L^2_\theta}) = \int_{\mathbb{R}^d} E(u(t,x)^2)\,\theta(x)\,dx < \infty$$

by (4.28). Next, we note that $t \mapsto u(t, \star)$ from $[0,T]$ into L^2_θ is mean-square continuous, since

$$E(\|u(t,\star) - u(s,\star)\|^2_{L^2_\theta}) = \int_{\mathbb{R}^d} E((u(t,x) - u(s,x))^2)\,\theta(x)\,dx,$$

and we observe that as $s \to t$, by (4.28) and since $(t,x) \mapsto u(t,x)$ is $L^2(\Omega)$-continuous, the right-hand side converges to 0 by the Dominated Convergence Theorem.

For $t \in [0,T]$, define

$$v^\theta_{\Gamma,Z}(t,\star) = \int_0^t \int_{\mathbb{R}^d} \Gamma(t-s, \star - y)\sigma(u(s,y))\,M(ds,dy),$$

where the stochastic integral is defined as in (2.21). It remains to show that

$$\mathcal{I}_{\Gamma,Z}(t,\star) = v^\theta_{\Gamma,Z}(t,\star) \qquad \text{in} \quad L^2(\Omega \times \mathbb{R}^d, dP \times \theta(x)dx). \tag{4.42}$$

For this, set $Z_n(s,y) = Z(s,y)1_{[-n,n]^d}(y)$, so that, by definition,

$$v^\theta_{\Gamma,Z}(t,\star) = \lim_{n\to\infty} v^\theta_{\Gamma,Z_n}(t,\star) \qquad \text{in} \quad L^2(\Omega \times \mathbb{R}^d, dP \times \theta(x)dx),$$

where $v^{\theta}_{\Gamma,Z_n}(t,\star)$ is defined as in (2.15). By Proposition 2.11, $v^{\theta}_{\Gamma,Z_n}(t,\star) = \mathcal{I}_{\Gamma,Z_n}(t,\star)$ in $L^2(\Omega \times \mathbb{R}^d, dP \times dx)$, therefore also in $L^2(\Omega \times \mathbb{R}^d, dP \times \theta(x)dx)$. In order to establish (4.42), it suffices to show that

$$E(\|\mathcal{I}_{\Gamma,Z}(t,\star) - \mathcal{I}_{\Gamma,Z_n}(t,\star)\|^2_{L^2_\theta}) \to 0 \qquad \text{as } n \to \infty. \tag{4.43}$$

The expectation in (4.43) is equal to

$$\int_{\mathbb{R}^d} dx\, \theta(x)\, E\left(\left(\mathcal{I}_{\Gamma,Z_n-Z}(t,x)\right)^2\right)$$
$$= \int_{\mathbb{R}^d} dx\, \theta(x)\, E\left[\left(\int_0^t \int_{\mathbb{R}^d} \Gamma(t-s,x-y)\, (Z_n(s,y) - Z(s,y))\, W(ds,dy)\right)^2\right].$$

For $n \geq t$, let $D_{t,n} = [-n+t,\ n-t]^d$. As can be seen from the formulas given in Section 4.2, for $x \in D_{t,n}$, the support of $\Gamma(t-s, x-*)$ is contained in $[-n,n]^d$, and by definition, $Z_n(s,y) = Z(s,y)$ for $y \in [-n, n]^d$ and $s \in [0,t]$. Therefore, the above expression is equal to

$$\int_{\mathbb{R}^d \setminus D_{t,n}} dx\, \theta(x)\, E\left(\left(\mathcal{I}_{\Gamma,Z_n-Z}(t,x)\right)^2\right). \tag{4.44}$$

We notice that by Proposition 2.9,

$$E\left(\left(\mathcal{I}_{\Gamma,Z_n-Z}(t,x)\right)^2\right) \leq C \int_0^t ds \sup_{y\in\mathbb{R}^d} E\left((Z_n(s,y) - Z(s,y))^2\right) \int_{\mathbb{R}^d} \mu(d\xi)\, |\mathcal{F}\Gamma(t-s)(\xi)|^2,$$

and

$$\sup_{y\in\mathbb{R}^d} E\left((Z_n(s,y) - Z(s,y))^2\right) \leq \sup_{y\in\mathbb{R}^d} E((Z(t,x))^2) < \infty,$$

by (4.28) and the Lipschitz property of σ. Therefore, the expression in (4.44) converges to 0 as $n \to \infty$. This proves (4.43), and concludes the proof. □

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