A guide to the graph labeling zoo

Joseph A. Gallian

Department of Mathematics and Statistics, University of Minnesota, Duluth, MN 55812, USA

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Abstract

In this paper we survey many of the variations of graceful and harmonious labeling methods that have been introduced and summarize much of what is known about each kind.

1. Introduction

A vertex labeling, valuation or numbering of a graph $G$ is an assignment $f$ of labels to the vertices of $G$ that induces for each edge $xy$ a label depending on the vertex labels $f(x)$ and $f(y)$. Most graph labeling methods trace their origin to one introduced by Rosa [Ga66] in 1967, or one given by Graham and Sloane [Ga36] in 1980. Rosa [Ga66] called a function $f$ a $\beta$-valuation of a graph $G$ with $q$ edges if $f$ is an injection from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that, when each edge $xy$ is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct. (Golomb [Ga32] subsequently called such labelings graceful and this term is now the popular one.) Rosa introduced $\beta$-valuations as well as a number of other valuations as tools for decomposing the complete graph into isomorphic subgraphs. In particular, $\beta$-valuations originated as a means of attacking the conjecture of Ringel [Ga64] that $K_{2n+1}$ can be decomposed into $2n + 1$ subgraphs that are all isomorphic to a given tree with $n$ edges. Harmonious graphs naturally arose in the study by Graham and Sloane [Ga36] of modular versions of additive bases problems stemming from error-correcting codes. They defined a graph $G$ with $q$ edges to be harmonious if there is an injection $f$ from the vertices of $G$ to the group of integers modulo $q$ such that, when each edge $xy$ is assigned the label $f(x) + f(y) \pmod{q}$, the resulting edge labels are distinct. When $G$ is a tree, exactly one label may be used on two vertices.

In the intervening years, close to 200 papers have spawned a bewildering array of graph labeling methods. Despite the unabated procession of papers, there are few

References whose number is preceded by Ga refer to an earlier survey by the author [25]. References denoted with a plain number are at the end of this paper.

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general results on graph labelings. Indeed, the papers focus on particular classes of graphs and methods, and feature ad hoc arguments. In this paper, we survey many of the variations of graceful and harmonious labeling methods that have been introduced and summarize much of what is known about each kind. Earlier surveys, restricted to one or two methods, include [Ga6, Ga11, Ga46, Ga78, 25]. The extension of graceful labelings to directed graphs arose in the characterization of finite neofields by Hsu and Keedwell [38, 39]. The relationship between graceful digraphs and a variety of algebraic structures including cyclic difference sets, sequenceable groups, generalized complete mappings, near-complete mappings and neofields is discussed in [Ga16]. The connection between graceful labelings and perfect systems of difference sets is given in [Ga9]. Other applications are detailed in [Ga12, Ga13]. Terms and notation not defined below follow that used in [Ga5, 25].

2. Variations of graceful labelings

Graceful labelings were covered in depth in [25]. Here we discuss variations of graceful labelings not mentioned in [25] and update the results presented in [25] on graceful labelings.

2.1. \(a\)-labelings

In [Ga66] Rosa defined an \(a\)-labeling to be a graceful labeling with the additional property that there exists an integer \(k\) so that for each edge \(xy\) either \(f(x) \leq k < f(y)\) or \(f(y) \leq k < f(x)\). (Other names for such labelings are \(balanced\) and \(interlaced\).) It follows that such a \(k\) must be the smaller of the two vertex labels that yield the edge labeled 1. Also, a graph with an \(a\)-labeling is necessarily bipartite and therefore cannot contain a cycle of odd length.

A common theme in graph labeling papers is to build up graphs that have desired labelings from pieces with particular properties. In these situations, starting with a graph that possesses an \(a\)-labeling is a common approach. (See [Ga21, Ga34, Ga41].) Moreover, Jungreis and Reid [Ga41] showed how sequential labelings of graphs (see Section 3.1) can often be obtained by modifying \(a\)-labelings of the graphs.

Graphs with \(a\)-labelings have often proved useful in the development of the theory of graph decompositions. Rosa [Ga66], for instance, has shown that if \(G\) is a graph with \(q\) edges and has an \(a\)-labeling, then for every natural number \(p\), the complete graph \(K_{2q^2+1}\) can be decomposed into copies of \(G\).

As to which graphs have \(a\)-labelings, Rosa [Ga66] observed that the \(n\)-cycle \(C_n\) has an \(a\)-labeling if and only if \(n \equiv 0 \pmod{4}\) while the path with \(n\) vertices \(P_n\) always has an \(a\)-labeling. Other familiar graphs that have \(a\)-labelings include caterpillars [Ga66], the \(n\)-cube [Ga66], \(B_{4n+1}\) (i.e., books with \(4n + 1\) pages) [Ga29], \(C_{2m} \cup C_{2m}\) [Ga49], and \(C_{4m} \cup C_{4m} \cup C_{4m}\) for all \(m > 1\) [Ga49]. Kotzig [Ga49] has also shown that \(C_{2m+1} \cup C_{2m+1}\) and \(C_4 \cup C_4 \cup C_4\) do not have \(a\)-labelings. He asked if \(n = 3\) is the only integer such that the disjoint union of \(n\) copies of \(C_4\) does not have an \(a\)-labeling and proved that for \(4 \leq n \leq 10\), the disjoint union of \(n\) copies has an \(a\)-labeling. Abrham
and Kotzig [1] have shown that for every positive integer \( n \) the disjoint union of \( n^2 \) or \( n^2 + n \) copies of \( C_4 \) has an \( x \)-labeling. Jungreis and Reid [Ga41] investigated the existence of \( x \)-labelings for graphs of the form \( P_m \times P_n, C_m \times P_n \) and \( C_m \times C_n \) (see also [26]). Of course, the cases involving \( C_m \) with \( m \) odd are not bipartite, so there is no \( x \)-labeling. The only unresolved cases among these three families are \( C_{4m+2} \times P_{2n+1} \) and \( C_{4m+2} \times C_{4m+2} \). All other cases result in \( x \)-labelings. Huang and Skiena [40] have shown that \( C_{4m+2} \times P_{2n+1} \) have graceful labelings. Snevily [70] has shown that all prisms of the form \( P_2 \times P_2 \times \cdots \times P_2 \times C_{4m} \) and the cycles \( C_{4m} \) with the path \( P_n \) adjoined at each vertex have \( x \)-labelings. He has also shown [71] that compositions of the form \( G[K_m] \) have an \( x \)-labeling whenever \( G \) does. Balakrishnan and Kumar [6] have shown that all graphs of the form \( P_2 \times P_2 \times \cdots \times P_2 \times G \) where \( G \) is \( K_{3,3} \), \( K_{4,4} \), or \( P_n \) have an \( x \)-labeling. Rosa [Ga66] also has shown that \( K_{p,q} \) has an \( x \)-labeling.

Although a proof of Ringel’s conjecture that every tree has a graceful labeling has withstood many attempts, examples of trees that do not have \( x \)-labelings are easy to construct (see [Ga66]).

Given two bipartite graphs \( G_1 = (H_1, L_1, E) \) and \( G_2 = (H_2, L_2, E) \), Snevily [70] defines their weak tensor product \( G_1 \otimes G_2 \) as the bipartite graph with vertex set \( (H_1 \times H_2, L_1 \times L_2) \) and with edge \((h_1, h_2)(l_1, l_2)\) if and only if \( h_1 l_1 \in E(G_1) \) and \( h_2 l_2 \in E(G_2) \). He proves that if \( G_1 \) and \( G_2 \) have \( x \)-labelings then so does \( G_1 \otimes G_2 \). This result considerably enlarges the class of graphs known to have \( x \)-labelings.

The sequential join \( G_1 + G_2 + \cdots + G_n \) of graphs \( G_1, G_2, \ldots, G_n \) is formed from \( G_1 \cup G_2 \cup \cdots \cup G_n \) by adding edges joining each vertex of \( G_i \) with each vertex of \( G_{i+1} \) for \( 1 \leq i \leq n - 1 \). Lee and Wang [52] have shown that for all \( n \geq 2 \) and any positive integers \( a_1, a_2, \ldots, a_n \) the graph \( K_{a_1} + K_{a_2} + \cdots + K_{a_n} \) has an \( x \)-labeling.

In [Ga30] Gallian weakened the condition for a \( x \)-labeling somewhat by defining a weakly \( x \)-labeling as a graceful labeling for which there is an integer \( k \) so that for each edge \( xy \) either \( f(x) \leq k \leq f(y) \) or \( f(y) \leq k \leq f(x) \). This condition allows the graph to have an odd cycle, but still places a severe restriction on the structure of the graph. Namely, that the vertex with the label \( k \) must be on every odd cycle. Gallian et al. [Ga30] showed that the prisms \( C_n \times P_2 \) with a vertex deleted have \( x \)-labelings. The same paper reveals that \( C_n \times P_2 \) with an edge deleted from a cycle has an \( x \)-labeling when \( n \) is even and a weakly \( x \)-labeling when \( n > 3 \).

A special case of \( x \)-labeling called strongly graceful was introduced by Maheo [Ga59] in 1980. As corollaries of his results, Maheo proved that \( (P_n \times K_2) \times K_2 \) and \( B_{2n} \times K_2 \) have \( x \)-labelings.

### 2.2. \( k \)-graceful labelings

A natural generalization of graceful graphs is the notion of \( k \)-graceful graphs introduced independently by Slater [69] in 1981 and by Maheo and Thuillier [Ga60] in 1982. A graph \( G \) with \( q \) edges is \( k \)-graceful if there is labeling \( f \) from the vertices of \( G \) to \( \{0, 1, 2, \ldots, q + k - 1\} \) such that the set of edge labels induced by the absolute value of the difference of the labels of adjacent vertices is \( \{k, k + 1, \ldots, q + k - 1\} \). Obviously, 1-graceful is graceful and it is readily shown that any graph that has an \( x \)-labeling is \( k \)-graceful for all \( k \). Graphs that are \( k \)-graceful for all \( k \) are sometimes
called arbitrarily graceful. Ng [63] has shown that there are graphs that are $k$-graceful for all $k$ but do not have a $z$-labeling.

Results of Maheo and Thuillier [Ga60] together with those of Slater [69] show that: $C_n$ is $k$-graceful if and only if

\[ n \equiv 0 \text{ or } 1 \pmod{4} \] with $k$ even and $k \leq \frac{1}{2}(n - 1)$

or

\[ n \equiv 3 \pmod{4} \] with $k$ odd and $k \leq \frac{1}{2}(n^2 - 1)$.

Acharya (see [2]) has shown that a $k$-graceful Eulerian graph with $q$ edges must satisfy one of the following conditions: $q \equiv 0 \pmod{4}$, $q \equiv 1 \pmod{4}$ if $k$ is even, or $q \equiv 3 \pmod{4}$ if $k$ is odd.

Maheo and Thuillier [Ga60] also proved that the wheel $W_{2k+1}$ is $k$-graceful and conjectured that $W_{2k}$ is $k$-graceful when $k \neq 3$ or $k \neq 4$. Kang [41] proved that $P_m \times C_4$ is $k$-graceful for all $k$, Lee and Wang [53] showed that all pyramids, lotuses and diamonds are $k$-graceful for all $k$.

Several authors have investigated the $k$-gracefulness of various classes of subgraphs of grid graphs. Acharya [Ga1] proved that all 2-dimensional polyminoes that are convex and Eulerian are $k$-graceful for all $k$; Lee [Ga33] showed that Mongolian tents and Mongolian villages are $k$-graceful for all $k$ (see [25] for definitions); Lee and Ng [Ga54] proved that all Young tableaus are $k$-graceful for all $k$ (see [25] for definitions). Lee and Ng [47] subsequently generalized their results on Young tableaus to a wider class of planar graphs.

Slater [Ga76] has extended the definition of $k$-graceful graphs to countable infinite graphs in a natural way. He proved that all countably infinite trees, the complete graph with countably many vertices and the countably infinite Dutch windmill are $k$-graceful for all $k$. Acharya and Hegde [3] generalized $k$-graceful to $(k, d)$-graceful labelings by permitting the vertex labels to belong to $\{0, 1, 2, \ldots, k + (q - 1)d\}$ and requiring the set of edge labels induced by the absolute value of the difference of labels of adjacent vertices to be $\{k, k + d, k + 2d, \ldots, k + (q - 1)d\}$. They also introduce an analog of $z$-labelings in the obvious way.

More specialized results on $k$-graceful labelings can be found in [Ga53, Ga54, Ga74, 47].

2.3. Skolem-graceful

A number of authors have invented analogues of graceful graphs by modifying the permissible vertex labels. For instance, Lee (see [50]) calls a graph $G$ with $p$ vertices and $q$ edges Skolem-graceful if there is an injection from the set of vertices of $G$ to $\{1, 2, \ldots, p\}$ such that the edge labels induced by $|f(x) - f(y)|$ for each edge $xy$ are $1, 2, \ldots, q$. A necessary condition for a graph to be Skolem-graceful is that $p \geq q + 1$. Lee and Wu [54] have shown that a connected graph is Skolem-graceful if and only if it is a graceful tree. They also prove that the disjoint union of 2 or 3 stars is Skolem graceful if and only if at least one star has an even size. Denham et al. [18] proved that the disjoint union of any four stars is Skolem-graceful. Lee et al. [48] showed that the disjoint union of the path $P_n$ and the star of size $m$ is Skolem-graceful if and only if $n = 2$ and $m$ is even or $n \geq 3$ and $m \geq 1$. It follows from the work of
Skolem [68] that \( n \cdot P_2 \), the disjoint union of \( n \) copies of \( P_2 \), is Skolem-graceful if and only if \( n \equiv 0 \) or \( 1 \pmod{4} \). Harary and Hsu [33] studied Skolem-graceful graphs under the name node-graceful. Frucht [22] has shown that \( P_m \cup P_n \) is Skolem-graceful when \( m + n \geq 5 \).

2.4. Odd graceful labelings

Gnanajothi [28, p. 182] defined a graph \( G \) with \( q \) edges to be odd graceful if there is an injection \( f \) from \( V(G) \) to \( \{0, 1, 2, \ldots, 2q - 1\} \) such that, when each edge \( xy \) is assigned the label \( |f(x) - f(y)| \), the resulting edge labels are \( \{1, 3, \ldots, 2q - 1\} \). She proved that the class of odd graceful graphs lies between the class of graphs with \( \alpha \)-labelings and the class of bipartite graphs by showing that every graph with an \( \alpha \)-labeling has an odd graceful labeling and every graph with an odd cycle is not odd graceful. She also proved that the following graphs are odd graceful: \( P_n; C_n \) if and only if \( n \) is even; \( K_m \); \( P_4 \cup K_1 \); books; crowns \( C_n \cup K_1 \) if and only if \( n \) is even; \( mC_4 \), the disjoint union of \( m \) copies of \( C_4 \); \( C_n \times K_2 \) if and only if \( n \) is even; caterpillars; rooted trees of height \( 2 \); the graphs obtained from \( P_n(n \geq 3) \) by adding exactly two leaves at each vertex of degree \( 2 \) of \( P_n \), the graphs consisting of vertices \( a_0, a_1, \ldots, a_n, b_0, b_1, \ldots, b_n \) with edges \( a_ia_{i+1}, b_ib_{i+1} \) for \( i = 0, \ldots, n - 1 \) and \( a_ib_i \) for \( i = 1, \ldots, n - 1 \); the graphs obtained from a star by adjoining to each end vertex the path \( P_3 \) or by adjoining to each end vertex \( P_4 \). She conjectures that all trees are odd graceful and proved the conjecture for all trees with order up to 10.

2.5. Graceful-like labelings

As a means of attacking graph decomposition problems, Rosa [Ga67] invented another analogue of graceful labelings by permitting the vertices of a graph with \( q \) edges to assume labels from the set \( \{0, 1, \ldots, q + 1\} \), while the edge labels induced by the absolute value of the difference of the vertex labels are \( \{1, 2, \ldots, q - 1, q\} \) or \( \{1, 2, \ldots, q - 1, q + 1\} \). He calls these nearly graceful labelings, or \( \rho \)-labelings. Frucht [21] has shown that the following graphs have nearly graceful labelings with edge labels from \( \{1, 2, \ldots, q - 1, q + 1\} : P_m \cup P_n; S_m \cup S_n; S_m \cup P_n; G \cup K_2 \) where \( G \) is graceful; \( C_3 \cup K_2 \cup S_m \) where \( m \) is even or \( m \equiv 3 \pmod{14} \).

In the same paper, Rosa defined a triangular snake as a connected graph all of whose blocks are triangles and whose block-cutpoint-graph is a path. He further conjectured that triangular snakes with \( t \equiv 0 \) or \( 1 \pmod{4} \) blocks are graceful and those with \( t \equiv 2 \) or \( 3 \pmod{4} \) blocks are nearly graceful (a parity condition ensures that the graphs in the latter case cannot be graceful). Moulton [62] proved Rosa's conjecture while introducing the slightly stronger concept of almost graceful by permitting the vertex labels to come from \( \{0, 1, 2, \ldots, q - 1, q + 1\} \) while the edge labels are \( \{1, 2, \ldots, q - 1, q\} \), or \( \{1, 2, \ldots, q - 1, q + 1\} \).

For graphs with the property that \( p = q + 1 \) (i.e. graphs that are trees or the disjoint union of a tree and unicyclic graphs), Frucht [22] has introduced a stronger version of almost graceful graphs by permitting as vertex labels \( \{1, 2, \ldots, q - 1, q + 1\} \) and as edge labels \( \{1, 2, \ldots, q\} \). He calls such a labeling pseudograceful. Frucht proved that...
\[ P_n (n \geq 3), \text{combs (i.e. graphs obtained by joining a single pendent edge to each vertex of a path), sparklers (i.e., graphs obtained by joining an end vertex of a path to the center of a star), } C_3 \cup P_n (n \neq 3), \text{ and } C_4 \cup P_n (n \neq 1) \text{ are pseudograceful while } K_{1,n} (n \geq 3) \text{ is not.} \]

McTavish [61] has investigated labelings where the vertex and edge labels are from \( \{0, \ldots, q, q + 1\} \). She calls these \( \tilde{p}\)-labelings. Graphs that have \( \tilde{p}\)-labelings include cycles and the disjoint union of \( P_n \) or \( S_n \) with any graceful graph.

Frucht [23] has made an observation about graceful labelings that yields nearly graceful analogs of \( \alpha\)-labelings and weakly \( \alpha\)-labelings in a natural way. Suppose \( G(V,E) \) is a graceful graph with the vertex labeling \( f \). For each edge \( xy \) in \( E \), let \( \lbrack f(x), f(y) \rbrack \) (where \( f(x) \leq f(y) \)) denote the interval of real numbers \( r \) with \( f(x) < r < f(y) \). Then the intersection \( \bigcap \lbrack f(x), f(y) \rbrack \) over all edges \( xy \in E \) is a unit interval, a single point, or empty. Indeed, if \( f \) is an \( \alpha\)-labeling of \( G \) then the intersection is a unit interval; if \( f \) is a weakly \( \alpha\)-labeling, but not an \( \alpha\)-labeling, then the intersection is a point; and if \( f \) is graceful but not a weakly \( \alpha\)-labeling, then the intersection is empty. For nearly graceful labelings, the intersection also gives three distinct classes. To illustrate the three cases consider the following three nearly graceful labelings of \( P_5 \cup P_3 \):

(i) \[ \begin{array}{cccccc}
\, & \, & \, & \, & \, & \, \\
0 & 7 & 4 & 5 & 3 & 1 \\
\, & \, & \, & \, & \, & \, \\
\end{array} \]

(ii) \[ \begin{array}{cccccc}
\, & \, & \, & \, & \, & \, \\
0 & 7 & 4 & 3 & 5 & 1 \\
\, & \, & \, & \, & \, & \, \\
\end{array} \]

(iii) \[ \begin{array}{cccccc}
\, & \, & \, & \, & \, & \, \\
0 & 7 & 3 & 1 & 6 & 2 \\
\, & \, & \, & \, & \, & \, \\
\end{array} \]

In (i), the intersection is \( \lbrack 4, 5 \rbrack \), in (ii) it is \( \{4\} \), and in (iii) it is empty.

### 2.6. Cordial labelings

Cahit [Ga20] has introduced a variation of both graceful and harmonious labelings. Let \( f \) be a function from the vertices of \( G \) to \( \{0, 1\} \) and for each edge \( xy \) assign the label \( \lvert f(x) - f(y) \rvert \). Call \( f \) a cordial labeling of \( G \) if the number of vertices labeled 0 and the number of vertices labeled \( 1 \) differ by at most 1, and the number of edges labeled 0 and the number of edges labeled \( 1 \) differ at most by 1. Cahit [Ga20] proved the following: every tree is cordial; \( K_n \) is cordial if and only if \( n < 3 \); \( K_{m,n} \) is cordial for all \( m \) and \( n \); the friendship graph \( F_n \) is cordial if and only if \( n \not\equiv 2 \pmod{4} \); all fans are cordial; the wheel \( W_n \) is cordial if and only if \( n \not\equiv 3 \pmod{4} \); and an Eulerian graph is not cordial if its size is congruent to \( 2 \pmod{4} \). A \( k\)-angular cactus is a connected graph all of whose blocks are cycles with \( k \) vertices. In [15] Cahit proved that a \( k\)-angular cactus with \( t \) cycles is cordial if and only if \( kt \not\equiv 2 \pmod{4} \). This was improved by Kirchherr [42] who showed that any cactus whose blocks are cycles is cordial if and only if the size of the graph is not congruent to \( 2 \pmod{4} \). Kirchherr [43] also gave a characterization of cordial graphs in terms of their adjacency matrices and conjectures that determining the set of cordial graphs is NP-complete. Ho et al. [36] proved:
$P_n \times P_n$ is cordial for all $n \geq 2$; $P_n \times C_{4m}$ is cordial for all $m$ and all odd $n$; the composition $G[H]$ is cordial if $G$ is cordial and $H$ is cordial and has odd order and even size; for $n \geq 4$ the composition $C_n[K_2]$ is cordial if and only if $n \neq 2 \pmod{4}$; the cartesian product of two cordial graphs of even sizes is cordial. The same authors [35] showed that a unicyclic graph is cordial unless it is $C_{4k+2}$ and that the generalized Petersen graph $P(n,k)$ is cordial if and only if $n \neq 2 \pmod{4}$, $(P(n,k)$, where $n \geq 5$ and $1 \leq k \leq n$, has vertex set \{a_0, a_1, \ldots, a_{n-1}, b_0, b_1, \ldots, b_{n-1}\} and edge set \{a_ia_{i+1} | i = 0, 1, \ldots, n-1\} \cup \{a_ib_i | i = 0, 1, \ldots, n-1\} \cup \{b_ib_{i+k} | i = 0, 1, \ldots, n-1\}$ where all subscripts are taken modulo $n$.) Bensen and Lee [10] investigated the regular windmill graphs $K_n$ and determined precisely which ones are cordial for $n < 14$.

Hovey [37] has introduced a simultaneous generalization of harmonious and cordial labelings. For any Abelian group $A$ and graph $G(V,E)$ he defines $G$ to be $A$-cordial if there is a labeling of $V$ with elements of $A$ so that for all $a$ and $b$ in $A$ when the edge $ab$ is labeled with $f(a) + f(b)$, the number of vertices labeled with $a$ and the number of vertices labeled $b$ differ by at most one and the number of edges labeled with $a$ and the number labeled with $b$ differ by at most one. In the case where $A = \mathbb{Z}_k$, the labeling is called $k$-cordial. With this definition we have: $G(V,E)$ is harmonious if and only if $G$ is $|E|$-cordial; $G$ is cordial if and only if $G$ is 2-cordial.

Hovey has obtained the following: caterpillars are $k$-cordial for all $k$; all trees are $k$-cordial for $k = 3, 4$ and 5; odd cycles with pendent edges attached are $k$-cordial for all $k$; cycles are $k$-cordial for all odd $k$; for $k$ even, $C_{2mk+j}$ is $k$-cordial when $0 \leq j \leq \frac{1}{2}k + 2$ and when $k < j < 2k; C_{(2m+1)k}$ is not $k$-cordial; $K_{2m}$ is 3-cordial, and, for $k$ even, $K_{2m}$ is 3-cordial if and only if $m = 1$.

Hovey advances the following conjectures: all trees are $k$-cordial for all $k$; and all connected graphs are 3-cordial.

### 2.7. $k$-equitable labelings

Rosa [64] has proposed the idea of distributing the vertex and edge labels among \{0, 1, \ldots, k - 1\} as evenly as possible to obtain a generalization of graceful labelings as follows. For any graph $G(V,E)$ and any positive integer $k$, assign vertex labels from \{0, 1, \ldots, k - 1\} so that when the edge labels induced by the absolute value of the difference of the vertex labels, the number of vertices labeled with $i$ and the number of edges labeled with $j$ differ by at most one. Rosa [64] has called a graph with such an assignment of labels $k$-equitable. Obviously, $G(V,E)$ is graceful if and only if it is $|E| + 1$-equitable and $G(V,E)$ is cordial if and only if it is 2-equitable. Rosa has conjectured that every tree is $k$-equitable for all $k$.

Cahit [15] has shown the following: $C_n$ is 3-equitable if and only if $n \neq 3 \pmod{6}$; a triangular snake with $n$ blocks is 3-equitable if and only if $n$ is even; the friendship graph $F_n$ is 3-equitable if and only if $n$ is even; $W_n$ is 3-equitable if and only if $n \neq 3 \pmod{6}$; an Eulerian graph with $q \equiv 3 \pmod{6}$ edges is not 3-equitable, and all caterpillars are 3-equitable [16]. He conjectures [15] that a triangular cactus with $n$ blocks is 3-equitable if and only if $n$ is even.

Bloom has used the term $k$-equitable to describe another kind of labeling (see [72]). He calls a graph $k$-equitable if the edge labels induced by the absolute value of the
difference of the vertex labels have the property that every edge label induced occurs exactly $k$ times. A graph of order $n$ is called minimally $k$-equitable if the vertex labels are $1, 2, \ldots, n$ and it is $k$-equitable. Both Bloom and Wojciechowski [72] proved that $C_n$ is minimally $k$-equitable if and only if $k$ is a proper divisor of $n$.

2.8. Update on graceful labelings

In this section, we update the earlier survey on graceful labelings by the author [25]. Huang and Skiena [40] have shown the following graphs are graceful: $C_m \times P_n$ for all $n$ when $m$ is even and for all $n$ with $3 \leq n \leq 12$ when $m$ is odd; $P^2_n$ (the square of $P_n$) for $n \leq 32$; and windmills $K^*_n$ for all $4 \leq m \leq 22$. Cahit [16] has proved that all trees of diameter 4 are graceful.

For $3 \leq p \leq n - r$, let $C_n(p, r)$ denote the $n$-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the $r$ chords $v_1v_p, v_1v_{p+1}, \ldots, v_1v_{p+r-1}$ have been added. The survey [25] included the cases $(p, r) = (3, n - 3), (p, 1), (p, 2)$ and $(p, 3)$. Since then, Ma [60] has shown that $C_n(p, n - p)$ is graceful when $p \equiv 0, 3 \pmod{4}$. He also proved that if one adds to the graph $C_n(3, n - 3)$ any number $k_i$ of paths of length 2 from the vertex $v_1$ to the vertex $v_i$ for $i = 2, \ldots, n$, the resulting graph is graceful. Zhi-Zeng [74] has shown that apart from four exceptional cases a graph consisting of three independent paths joining two vertices is graceful. This generalizes the result that a cycle plus a chord is graceful ([Ga24, Ga42]).

In 1985 Frucht and Salinas [Ga27] conjectured that $C_s \cup P_n$ is graceful if and only if $s + n \geq 7$ and proved the conjecture for the case that $s = 4$. Frucht [22] has now done the case that $s = 3$ and the case that $s = 2n + 1$.

Liu [58] has shown that the $n$-cycle with consecutive vertices $v_1, v_2, \ldots, v_n$ to which the chords $v_1v_k$ and $v_1v_{k+2}$ ($2 \leq k \leq n - 3$) are adjoined is graceful.

Yuan and Zhu [73] proved that $K_{m,n} + K_2$ is graceful. Aravamudhan and Murugan [5] have shown that the complete tripartite graph $K_{1,m,n}$ is graceful while Gnanajothi [28, pp. 25–31] has shown that $K_{2,m,n}, K_{1,1,m,n}$, and quadrilateral snakes (a connected graph all of whose blocks are quadrilaterals and whose block-cutpoint-graph is a path) are graceful.

Gnanajothi [28, p. 50] has shown that the graph that consists of $n$ copies of $C_6$ that have exactly $P_4$ in common is graceful if and only if $n$ is even. For a fixed $n$, let $v_{i1}, v_{i2}, v_{i3}$ and $v_{i4}$ ($1 \leq i \leq n$) be consecutive vertices of a 4-cycle. Gnanajothi [28, p. 35] also proves that the graph obtained by joining each $v_{i1}$ to $v_{i+1,3}$ is graceful for all $n$.

Gnanajothi [28, p. 51] calls a graph $G$ bigraceful if both $G$ and its line graph are graceful. She shows the following are bigraceful: $P_m$; $P_m \times P_n$; $C_n$ if and only if $n \equiv 0, 3 \pmod{4}$; $S_n$; $K_n$ if and only if $n \equiv 3 \pmod{4}$; $B_n$ if and only if $n \equiv 3 \pmod{4}$. She also shows that $K_{m,n}$ is not bigraceful when $n \equiv 3 \pmod{4}$. (Gangopadhyay and Rao Hebbare [27] used the term bigraceful to mean a bipartite graceful graph.)

Chen et al. [17] define a firecracker as a graph obtained from the concatenation of stars by linking one leaf from each. They also define a banana tree as a graph obtained by connecting a vertex $v$ to one leaf of each of any number of stars ($v$ is not in any of the stars). They proved that firecrackers are graceful and conjecture that banana trees are graceful.
3. Variations of harmonious labelings

3.1. Sequential and strongly c-harmonious labelings

Chang et al. [Ga21] and Grace [Ga33, Ga34] have investigated subclasses of harmonious graphs. Chang et al. define an injective labeling $f$ of a graph $G$ with $q$ vertices as strongly $c$-harmonious if the vertex labels are from $\{0, 1, \ldots, q - 1\}$ and the edge labels induced by $f(x) + f(y)$ for each edge $xy$ are $c, \ldots, c + q - 1$. Grace called such a labeling sequential. In the case of a tree, Chang et al. modify the definition to permit exactly one vertex label to be assigned to two vertices while Grace allows the vertex labels to range from 0 to $q$ with no vertex label used twice. By taking the edge labels of a sequentially labeled graph modulo $q$ we obviously obtain a harmoniously labeled graph. It is not known if there is a graph that can be harmoniously labeled but not sequentially labeled. Grace [Ga34] proved that caterpillars, caterpillars with a pendant edge, cycles $C_{2n+1}$ with zero or more pendant edges, trees with $x$-labelings, wheels $K_m \cup K_n$, and $P_n$, are sequential. Liu and Zhang [56] finished off the crowns $C_{2n} \cup K_1$ (the case $C_{2n+1} \cup K_1$ was a special case of Grace’s results). Both Grace [Ga33] and Reid (see [Ga29]) have found sequential labelings for the book $B_{2n}$. Jungreis and Reid [Ga41] have shown the following graphs are sequential: $P_m \times P_n (m, n) \neq (2,2)$; $C_{4m} \times P_n (m, n) \neq (1,2)$; $C_{4m+2} \times P_{2n}$; $C_{2m+1} \times P_n$; and $C_4 \times C_{2n} (n > 1)$. The graphs $C_{4m+2} \times C_{2n+1}$ and $C_{2m+1} \times C_{2n+1}$ fail to satisfy a necessary condition given by Graham and Sloane [Ga36]. The remaining cases of $C_m \times P_n$ and $C_m \times C_n$ are open. Gallian et al. [Ga30] proved that all graphs $C_n \times P_2$ with a vertex or edge deleted are sequential.

Gnanajothi [28, pp. 68–78] has shown the following graphs are sequential: $K_{1,m,n}$, the disjoint union of $m$ copies of $C_n$, if and only if $m$ and $n$ are odd; books with triangular pages or pentagonal pages, and books of the form $B_{4n+1}$, thereby answering a question and proving a conjecture of Gallian and Jungreis [Ga29].

Yuan and Zhu [73] have shown that $mC_n$ and $C^{(m)}_n$, the one-point union of $m$ cycles of length $n$, are sequential when $m$ and $n$ are odd.

If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ denote the consecutive vertices of two disjoint paths with $n$ vertices, the M"obius ladder $M_n$ is the graph obtained by joining $a_i$ to $b_i$ for $i = 1, \ldots, n, a_i$ to $b_i$, and $b_1$ to $a_n$. Although Graham and Sloane [Ga36] proved that $M_3$ is not harmonious, Gallian [Ga28] established that all other M"obius ladders are sequential.

Among the strongly 1-harmonious (also called strongly harmonious) graphs found by Chang et al. are fans $f_n$ with $n \geq 2$ [Ga21], wheels $W_n$ with $n \not\equiv 2$ (mod 3) [Ga21], $K_{m,n} + K_1$ [Ga21], French windmills $K^0_t$ [Ga38], and the friendship graphs $K^0_t$ if and only if $t \equiv 0$ or 1 (mod 4) [Ga38].

Acharya and Hegde [3] have generalized sequential labelings as follows. Let $G$ be a graph with $q$ edges and let $k$ and $d$ be positive integers. A labeling $f$ of $G$ is said to be $(k,d)$-arithmetic if the vertex labels are distinct nonnegative integers and the edge labels induced by $f(x) + f(y)$ for each edge $xy$ are $k, k + d, k + 2d, \ldots, k + (q - 1)d$. They obtained a number of necessary conditions for various kinds of graphs to have a $(k,d)$-arithmetic labeling.
3.2. Elegant labelings

An elegant labeling $f$ of a graph $G$ with $q$ edges is an injective function from the vertices of $G$ to the set $\{0, 1, \ldots, q\}$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod{q + 1}$ the resulting edge labels are distinct and nonzero. This notion was introduced by Chang et al. [Ga21] in 1981. Note that in contrast to the definition of a harmonious labeling, it is not necessary to make an exception for trees. While the cycle $C_n$ is harmonious if and only if $n$ is odd, Chang et al. [Ga21] proved that $C_n$ is elegant when $n \equiv 0$ or $3 \pmod{4}$ and not elegant when $n \equiv 1 \pmod{4}$. Chang et al. further showed that all fans are elegant and the paths $P_n$ are elegant for $n \not\equiv 0 \pmod{4}$. Cahit [14] then showed that $P_4$ is the only path that is not elegant.

One can extend the notion of harmoniousness to arbitrary finite Abelian groups as follows. Let $G$ be a graph with $q$ edges and $H$ a finite Abelian group (under addition) of order $q$. Define $G$ to be $H$-harmonious if there is an injection $f$ from the vertices of $G$ to $H$ such that when each edge $xy$ is assigned the label $f(x) + f(y)$ the resulting edge labels are distinct. When $G$ is a tree, one label may be used on exactly two vertices. Beals et al. [9] have shown that if $H$ is a finite Abelian group of order $n > 1$ then $C_n$ is $H$-harmonious if and only if $H$ has a noncyclic or trivial Sylow 2-subgroup and $H$ is not an elementary 2-group. Thus, for example, $C_{12}$ is not $Z_{12}$-harmonious but is $(Z_2 \times Z_2 \times Z_3)$-harmonious. Analogously, the notion of an elegant graph can be extended to arbitrary finite Abelian groups. Let $G$ be a graph with $q$ edges and $H$ a finite Abelian group (under addition) with $q + 1$ elements. We say $G$ is $H$-elegant if there is an injection $f$ from the vertices of $G$ to $H$ such that when each edge $xy$ is assigned the label $f(x) + f(y) \pmod{H}$ the resulting set of edge labels is $H^*$, the nonidentity elements of $H$. Beals et al. [9] proved that if $H$ is a finite Abelian group of order $n$ with $n \neq 1$ and $n \neq 3$ then $C_{n-1}$ is $H$-elegant using only the nonidentity elements of $H$ as vertex labels if and only if $H$ has either a noncyclic or trivial Sylow 2-subgroup. This result completed a partial characterization of elegant cycles given by Chang et al. [Ga21] by showing that $C_n$ is elegant when $n \equiv 2 \pmod{4}$.

3.3. Felicitous labelings

Another generalization of harmonious labelings are felicitous labelings. An injective function $f$ from the vertices of a graph $G$ with $q$ edges to the set $\{0, 1, \ldots, q\}$ is called felicitous if the edge labels induced by $f(x) + f(y) \pmod{q}$ for each edge $xy$ are distinct. This definition first appeared in a paper by Lee et al. in [Ga55] and is attributed to E. Choo. Among the graphs known to be felicitous are: $C_n$ except when $n \equiv 2 \pmod{4}$ [Ga55]; $K_{m,n}$ when $m,n > 1$ [Ga55]; $P_2 \cup C_{2n+1}$ [Ga55]; $P_3 \cup C_{2n+1}$ [Ga55]; $S_m \cup C_{2n+1}$ [Ga55]; the friendship graph $K^{t\ell}$ for $t$ odd [Ga55]; $P_n \cup C_2$ [Ga70]; and the one-point union of an odd cycle and a caterpillar [Ga70]. Shee [Ga68] conjectured that $P_m \cup C_n$ is felicitous when $n > 2$ and $m > 3$. Lee et al. [Ga55] ask for which $m$ and $n$ is the one-point union of $n$ copies of $C_n$ felicitous. They showed that the case where $mn$ is twice an odd integer is not felicitous. In contrast to the situation for felicitous labelings, we remark that $C_{4k}$ and $K_{m,n}$ where $m,n > 1$ are not harmonious and the one-point union of an odd cycle and a caterpillar is not always harmonious.
Chang et al. [Ga21] have given a sequential counterpart to felicitous labelings. They call a graph strongly $c$-elegant if the vertex labels are from $\{0, 1, \ldots, q\}$ and the edge labels induced by addition are $\{c, c + 1, \ldots, c + q - 1\}$. (A strongly $1$-elegant labeling has also been called a consecutive labeling.) Notice that every strongly $c$-elegant graph is felicitous and that strongly $c$-elegant is the same as $(c, 1)$-arithmetic in the case where the vertex labels are from $\{0, 1, \ldots, q\}$. Results on strongly $c$-elegant graphs are meager. Chang et al. [Ga21] have shown: $K_n$ is strongly $1$-elegant if and only if $n = 2, 3, 4$; $C_n$ is strongly $1$-elegant if and only if $n = 3$; $K_{4}^{0}$ is strongly elegant if $t \equiv 0$ or $1 \pmod{4}$; $K_{4}^{t}$ is strongly elegant for $t \geq 1$; and a bipartite graph is strongly $1$-elegant if and only if it is a star. Shee [66] has proved that $K_{m,n}$ is strongly $c$-elegant for a particular value of $c$ and obtained several more specialized results pertaining to graphs formed from complete bipartite graphs.

3.4. Update on harmonious labelings

In this section we present results on harmonious labelings that were not included in an earlier survey [25]. In [Ga36] Graham and Sloane proved that if a harmonious graph has an even number of edges $q$ and the degree of each vertex is divisible by $2^k$ ($k \geq 1$), then $q$ is divisible by $2^{k+1}$. Liu and Zhang [57] have generalized this condition as follows: if a harmonious graph with $q$ edges has degree sequence $d_1, d_2, \ldots, d_p$ then $\text{g.c.d} \,(d_1, d_2, \ldots, d_p, q)$ divides $\frac{1}{2}q(q - 1)$. Liu and Zhang [57] have also obtained the following: $mK_n$, the disjoint union of $m$ copies of $K_n$, is not harmonious for $n$ odd and $m \equiv 2 \pmod{4}$ and is harmonious for $n = 3$ and $m$ odd; helms $H_n$ (i.e. the wheel $W_n$ with a pendant edge at each vertex of the $n$-cycle) are harmonious for $n$ odd; triangular snakes with an odd number of triangles are harmonious while triangular snakes with $n \equiv 2 \pmod{4}$ triangles are not harmonious; the join of any number of copies of $K_2$ is harmonious; the double cone $C_n + K_2$ is not harmonious for $n \equiv 2, 4, 6 \pmod{8}$ and $K_2 + K_2 + \cdots + K_2$ is harmonious.

They conjecture that $mK_3$ is not harmonious when $m \equiv 0 \pmod{4}$. This conjecture is quite likely to be true since Gnanajothi [28, pp. 70] has proved that $mC_n$ is sequential if and only if $m$ and $n$ are odd. Aravamudhan and Murugan [5] have shown that the complete tripartite graph $K_{m,1,n}$ is harmonious. Gnanajothi [28, pp. 80–127] obtained the following: $C_n + K_2$ is harmonious when $n$ is odd and not harmonious when $n \equiv 2, 4, 6 \pmod{8}$; $K_{1,1,m,n}$ is harmonious; $S_n + K_i$ is harmonious; $P_n + K_i$ is harmonious; $S_{m,n}$, the star with $n$ spokes with each spoke $P_m$ is harmonious when $m$ and $n$ are odd; the helm $H_n$ is harmonious (thus proving a conjecture of Liu and Zhang [57]); the web $W_n$ (obtained by joining the pendant vertices of the helm $H_n$ to form a cycle and then adding a single pendant edge to each vertex of this outer cycle) is harmonious when $n$ is odd; the generalized Petersen graph is harmonious in all cases.

Yuan and Zhu [73] define a generalization $P_n(2k)$ of $P_n^2$ as follows: $P_n(2k)$ is the graph obtained from the path $P_n$ by adding edges that join all vertices $x$ and $y$ with $d(x, y) = 2k$. They proved that $P_n(2k)$ is harmonious when $1 \leq k \leq \frac{1}{2}(n - 1)$ and is sequential when $2k - 1 \leq n \leq 4k - 1$. They also proved that $K_{m,n} + K_2$ is harmonious.
Finally, in the harmonious graph table in [25], $S_m + K_1$ was listed as an open problem. However, Chung et al. [Ga21] had shown in 1981 that $K_{m,n} + K_1$, which includes $S_m + K_1$, is sequential.

4. Total labelings

In contrast to the labeling methods discussed thus far in which there is a function from the vertices of a graph to some set of labels, there are numerous methods that involve a function from the vertices and edges to some set of labels.

4.1. $k$-sequential labelings

In 1981 Bange et al. [7] defined a $k$-sequential labeling $f$ of a graph $G(V, E)$ as one for which $f$ is a bijection from $V \cup E$ to $\{k, k + 1, \ldots, |V \cup E| + k - 1\}$ such that for each edge $xy$ in $E$, one has $f(xy) = f(x) - f(y)$. This generalized the notion of simply sequential where $k = 1$ introduced by Slater. Bange et al. showed that cycles are 1-sequential and if $G$ is 1-sequential then the join of $G$ and a point is graceful. In [69], Slater proved: $K_n$ is 1-sequential if and only if $n \leq 3$; for $n \geq 2$, $K_n$ is not $k$-sequential for all $k \geq 2$; and $K_{1,n}$ is $k$-sequential if and only if $k$ divides $n$. Acharya and Hegde [Ga3] proved: $P_n$ is $\frac{1}{2}n$-sequential if $n$ is even; $P_n$ is both $\frac{1}{2}(n - 1)$-sequential and $\frac{1}{2}(n + 1)$-sequential if $n$ is odd; $K_{m,n}$ is $k$-sequential for $k = 1, m, n$; $K_{m,n,1}$ is 1-sequential, and the join of any caterpillar and $K_r$ is 1-sequential. Acharya [2] showed that if $G(E, V)$ is an odd graph with $|E| + |V| \equiv 1 \text{ or } 2 \pmod{4}$ when $k$ is odd or $|E| + |V| \equiv 2 \text{ or } 3 \pmod{4}$ when $k$ is even, then $G$ is not $k$-sequential. Acharya also observed that as a consequence of results of Bermond et al. [Ga9] we have: $mK_d$ is not $k$-sequential for any $k$ when $m$ is odd and $mK_2$ is not $k$-sequential for any odd $k$ when $m \equiv 2$ or 3 (mod 4) or for any even $k$ when $m \equiv 1$ or 2 (mod 4). He further noted that for $m$ and $n$ odd $K_{m,n}$ is not $k$-sequential when $k$ is even, while $K_{m,n}$ is $k$-sequential for all $k$. Acharya [2] points out that the following result of Slater's [69] for $k = 1$ linking $k$-graceful graphs and $k$-sequential graphs holds in general: A graph is $k$-sequential if and only if $G + v$ has a $k$-graceful labeling $f$ with $f(v) = 0$. Slater [69] also proved that a $k$-sequential graph with $p$ vertices and $q > 0$ edges must satisfy $k \leq p - 1$.

4.2. Sequentially additive graphs

In 1983, Bange et al. [8] defined a $k$-sequentially additive labeling $f$ of a graph $G(V, E)$ to be a bijection from $V \cup E$ to $\{k, k + 1, \ldots, |V \cup E| + |E| - 1\}$ such that for each edge $xy$, $f(xy) = f(x) + f(y)$. They proved: $K_n$ is 1-sequentially additive if and only if $n \leq 3$; $C_{3n + 1}$ is not $k$-sequentially additive for $k \equiv 0, 2 \pmod{3}$; $C_{3n + 2}$ is not $k$-sequentially additive for $k \equiv 1, 2 \pmod{3}$; $C_n$ is 1-sequentially additive if and only if $n \equiv 0, 1 \pmod{3}$, and $P_n$ is 1-sequentially additive. They conjecture that all trees are 1-sequentially additive.

Acharya and Hegde [3] have generalized $k$-sequentially additive labelings by defining the image of the bijection to be $\{k, k + d, \ldots, (k + |V \cup E| - 1)d\}$. They call such a labeling additively $(k, d)$-sequential.
4.3. Magic labelings

In 1970 Kotzig and Rosa [45] defined a magic labeling of a graph $G(V,E)$ as a bijection $f$ from $V \cup E$ to $\{1,2,\ldots,|V\cup E|\}$ such that for all edges $xy$, $f(x) + f(y) + f(xy)$ is constant. They proved: $K_{m,n}$ has a magic labeling for all $m$ and $n$; $C_n$ has a magic labeling for all $n \geq 3$; and the disjoint union of $n$ copies of $P_2$ has a magic labeling if and only if $n$ is odd. They further state that $K_m$ has a magic labeling if and only if $m = 2, 3, 4, 5$ or 6 and ask whether all trees have magic labelings.

5. Miscellaneous labelings

5.1. Prime and vertex prime labelings

The notion of a prime labeling originated with Roger Entringer and was introduced in a paper by Tout et al. (see [55]). A graph with vertex set $V$ is said to have a prime labeling if its vertices are labeled with distinct integers $1, 2, \ldots, |V|$ such that for each edge $xy$ the labels assigned to $x$ and $y$ are relatively prime. Around 1980, Entringer conjectured that all trees have a prime labeling. So far, there has been little progress towards proving this conjecture. Among the classes of trees known to have prime labelings are: paths, stars, caterpillars, complete binary trees, spiders (i.e., trees with a unique vertex of degree at least 3 and with all other vertices with degree at most 2) and all trees of order less than 16 (see [24]). Other graphs with prime labelings include all cycles and the disjoint union of $C_{2k}$ and $C_n$ [19]. The complete graph $K_n$ does not have a prime labeling for $n > 4$ and $W_n$ is prime if and only if $n$ is even (see [55]).

Given a finite collection of graphs $G_1, G_2, \ldots, G_n$ and some fixed vertex $v_i$ from each $G_i$, Lee et al. [55] define $Amal\{(G_i, v_i)\}$, the amalgamation of $\{(G_i, v_i)\} i = 1, \ldots, n$, as the graph obtained by taking the union of the $G_i$ and identifying $v_1, v_2, \ldots, v_n$. Lee et al. [55] have shown $Amal\{(G_i, v_i)\}$ has a prime labeling when $G_i$ are paths and when $G_i$ are cycles. They also showed that the amalgamation of any number of copies of $W_n$, $n$ odd, with a common vertex is not prime. They conjecture that for any tree $T$ and $v$ from $T$, the amalgamation of two or more copies of $T$ with $v$ in common is prime. They further conjecture that the amalgamation of two or more copies of $W_n$ that share a common point is prime when $n$ is even ($n \neq 4$).

A dual of prime labelings has been introduced by Deretsky et al. [19]. They say a graph with edge set $E$ has a vertex prime labeling if its edges can be labeled with distinct integers $1, 2, \ldots, |E|$ such that for each vertex of degree at least 2 the greatest common divisor of the labels on its incident edges is 1. Deretsky et al. show the following graphs have vertex prime labelings: forests; all connected graphs; $C_{2k} \cup C_n$; $C_{2m} \cup C_{2n} \cup C_{2k+1}$; $C_{2m} \cup C_{2n} \cup C_{2l} \cup C_{2k}$; and $5C_{2m}$. They further prove that a graph with exactly two components, one of which is not an odd cycle, has a vertex prime labeling and a 2-regular graph with at least two odd cycles does not have a vertex prime labeling. They conjecture that a 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles.
5.2. Edge-graceful labelings

In 1985, Lo [59] introduced the notion of edge-graceful graphs. A graph $G(V, E)$ is said to be edge-graceful if there exists a bijection $f$ from $E$ to $\{1, 2, \ldots, |E|\}$ so that the induced mapping $f^+$ from $V$ to $\{0, 1, \ldots, |V| - 1\}$ given by $f^+(x) = \sum \{f(xy) \mid xy \in E\} \pmod{|V|}$ is a bijection. For a survey on results on edge-graceful graphs we refer the reader to the paper by Lee [46].

5.3. Line graceful labelings

Gnanajothi [28] has defined a concept similar to edge-graceful. She calls a graph with $n$ vertices line graceful if it is possible to label its edges with $0, 1, 2, \ldots, n$ so that when each vertex is assigned the sum modulo $n$ of all the edge labels incident with that vertex the vertex labels are $0, 1, \ldots, n - 1$. A necessary condition for the line gracefulness of a graph is that its order is not congruent to 2 (mod 4). Among line graceful graphs are (see [28, pp. 132-181]): $P_n$ if and only if $n \not\equiv 2 \pmod{4}$; $C_n$ if and only if $n \not\equiv 2 \pmod{4}$; $K_{1,n}$ if and only if $n \not\equiv 1 \pmod{4}$; $P_n \circ K_1$ (combs) if and only if $n$ is even; $(P_n \circ K_1) \circ K_1$ if and only if $n \not\equiv 2 \pmod{4}$; $mC_n$ when $mn$ is odd; $C_n \circ K_1$ (coronas) if and only if $n$ is even; $mC_n$ for all $m$; complete $n$-ary trees when $n$ is even; $K_{1,n} \cup K_{1,n}$ if and only if $n$ is odd; odd cycles with a chord; even cycles with a tail; even cycles with a tail of length 1 and a chord; graphs consisting of two triangles having a common vertex and tails of equal length attached to a vertex other than the common one; the complete $n$-ary tree when $n$ is even; trees for which exactly one vertex has even degree.

She conjectures that all trees with $p \not\equiv 2 \pmod{4}$ vertices are line graceful and proved this for $p \leq 9$.

Gnanajothi [28] has investigated the line gracefulness of several graphs obtained from stars. In particular, the graph obtained from $K_{1,n}$ by subdividing one spoke to form a path of even order (counting the center of the star) is line graceful; the graph obtained from a star by inserting one vertex in a single spoke is line graceful if and only if the star has $p \not\equiv 2 \pmod{4}$ vertices; the graph obtained from $K_{1,n}$ by replacing each spoke with a path of length $m$ (counting the center vertex) is line graceful in the following cases: $n = 2$; $n = 3$ and $m \not\equiv 3 \pmod{4}$; $m$ is even and $mn + 1 \equiv 0 \pmod{4}$.

Gnanajothi studied graphs obtained by joining disjoint graphs $G$ and $H$ with an edge. She proved such graphs are line graceful in the following circumstances: $G = H$; $G = P_n$, $H = P_m$ and $m + n \not\equiv 0 \pmod{4}$; $G = P_n \circ K_1$, $H = P_n \circ K_1$ and $m + n \not\equiv 0 \pmod{4}$.

5.4. Sum graphs

In 1988, Harary [32] introduced the notion of a sum graph. A graph $G(V, E)$ is called a sum graph if there is an injective labeling $f$ from $V$ to a set of positive integers $S$ such that $xy \in E$ if and only if $f(x) + f(y) \in S$. Since the vertex with the highest label in a sum graph cannot be adjacent to any other vertex, every sum graph must contain isolated vertices. For a connected graph $G$, let $s(G)$ denote the minimum number of isolated vertices that must be added to $G$ so that the resulting graph is a sum graph. Ellingham [20] proved the conjecture of Harary [32] that $s(T) = 1$ for every tree.
T \neq K_1. Bergstrand et al. [11] proved that \( s(K_n) = 2n - 3 \). Hartsfield and Smyth [34] showed that \( s(K_{m,n}) = \frac{1}{2}(2m + n - 3) \) when \( n \geq m \). Gould and Rödl [30] investigated bounds on the number of isolated points in a sum graph. A group of six undergraduate students [29] proved that \( s(K_n \text{ - edge}) \leq 2n - 4 \); \( s(W_n) \leq \frac{3}{2}(n + 6) \) when \( n \) is even and \( s(W_n) \leq n + 3 \) when \( n \) is odd (here \( W_n = C_n + K_1 \)). The same group of students also investigated the difference between the largest and smallest labels in a sum graph, which they called the \textit{spurn}. They proved \textit{spurn} of \( K_n \) is \( 4n - 6 \) and the \textit{spurn} of \( C_n \) is at most \( 4n - 10 \).

Alon and Scheinermann [4] generalized sum graphs by replacing the condition \( f(x) + f(y) \in S \) with \( g(f(x), f(y)) \in S \) where \( g \) is an arbitrary symmetric polynomial. They called a graph with this property a \textit{g-graph} and proved that for a given symmetric polynomial \( g \) not all graphs are \textit{g-graphs}. On the other hand, for every symmetric polynomial \( g \) and every graph \( G \) there is some vertex labeling so that \( G \) together with at most \( |E(G)| \) isolated vertices is a \textit{g-graph}.

Boland et al. [13] investigated a modular version of sum graphs. They call a graph \( G(V, E) \) a \textit{mod sum graph} (MSG) if there exists a positive integer \( n \) and an injective labeling from \( V \) to \( \{1, 2, \ldots, n - 1\} \) such that \( xy \in E \) if and only if \( f(x) + f(y) \mod n = f(z) \) for some vertex \( z \). Obviously, all sum graphs are mod sum graphs. However, not all mod sum graphs are sum graphs. Boland et al. [13] have shown the following graphs are MSG: all trees on 3 or more vertices; all cycles on 4 or more vertices; and all \( K_{2, n} \). They also proved that \( K_p \) (\( p \geq 2 \)) is not MSG and conjecture that \( W_p \) is MSG for \( p \geq 4 \).

Grimaldi [31] has investigated labeling the vertices of a graph \( G(V, E) \) with \( n \) vertices with distinct elements of the ring \( \mathbb{Z}_n \) so that \( xy \in E \) whenever \( (x + y)^{-1} \in \mathbb{Z}_n \).

Sum graphs and their generalizations can be efficiently stored in a computer. An array holds the vertex labels, while adjacency can be tested by simple computations followed by a table look-up [4].

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