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Note

Characterizations of competition multigraphs

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ABSTRACT

The notion of a competition multigraph was introduced by C. A. Anderson, K. F. Jones, J. R. Lundgren, and T. A. McKee [C. A. Anderson, K. F. Jones, J. R. Lundgren, and T. A. McKee: Competition multigraphs and the multicompetition number, *Ars Combinatoria* 29B (1990) 185–192] as a generalization of the competition graphs of digraphs.

In this note, we give a characterization of competition multigraphs of arbitrary digraphs and a characterization of competition multigraphs of loopless digraphs. Moreover, we characterize multigraphs whose multicompetition numbers are at most m, where m is a given nonnegative integer and give characterizations of competition multihypergraphs.

1. Introduction

Cohen [2] introduced the notion of a competition graph in connection with a problem in ecology in 1968 (see also [3]). Let $D = (V, A)$ be a digraph that corresponds to a food web. A vertex in the digraph D stands for a species in the food web, and an arc $(x, v) \in A$ in D means that the species x preys on the species v. For vertices $x, v \in V$, we call v a *prey* of x if there is an arc $(x, v) \in A$. If two species x and y have a common prey v, they will compete for the prey v. Cohen defined a graph which represents the competition among the species in the food web. The *competition graph* $C(D)$ of a digraph $D = (V, A)$ is an undirected graph $G = (V, E)$ which has the same vertex set V and has an edge between two distinct vertices $x, y \in V$ if there exists a vertex $v \in V$ such that $(x, v), (y, v) \in A$. We say that a graph G is a *competition graph* if there exists a digraph D such that $C(D) = G$. This notion is applicable not only in ecology but also in channel assignments, coding, and modeling of complex economic and energy systems (see [6]).

Dutton and Brigham [4] gave a characterization of competition graphs, and also characterized the competition graphs of acyclic digraphs. Roberts and Steif [8] characterized the competition graphs of loopless digraphs. Opsut [5] showed that the problem of determining whether a graph is the competition graph of an acyclic digraph or not is an NP-complete problem.

Competition graphs are closely related to edge clique covers and the edge clique cover numbers of graphs. A *clique* of a graph G is a subset of the vertex set of G such that its induced subgraph of G is a complete graph. For a clique S of a graph G and an edge xy of G, we say *xy is covered by S* if both x and y are contained in S. An *edge clique cover* of a graph G is a family of cliques of G such that each edge of G is covered by some clique in the family. The minimum size of an edge clique cover of G is called the *edge clique cover number* of the graph G, and is denoted by $\theta_e(G)$.

Anderson, Jones, Lundgren, and McKee [1] generalized the notion of competition graphs to competition multigraphs. A *multigraph* $M = (V, E, \mu)$ consists of a graph (V, E) and a *multiplicity* $\mu : E \to \mathbb{N}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$ denotes the set of positive integers.

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Definition. The *competition multigraph* $CM(D)$ of a digraph $D = (V, A)$ is a multigraph $M = (V, E, \mu)$ such that (V, E) is the competition graph of D, and the multiplicity $\mu(xy)$ of an edge $xy \in E$ is the number of common preys of x and y in D. We say that a multigraph M is a *competition multigraph* if there exists a digraph D such that $CM(D) = M$.

Anderson et al. [1] also gave a characterization of the competition multigraphs of acyclic digraphs by using edge clique partitions of multigraphs. For a multigraph $M = (V, E, \mu)$, we call a clique of the graph (V, E) a *clique* of M.

Definition. Let $M = (V, E, \mu)$ be a multigraph. A multifamily $\mathcal{F} = \{S_1, \dots, S_r\}$ of cliques of M is called an *edge clique partition* of the multigraph M if each edge xy of M is covered by exactly $\mu(xy)$ cliques S_i in the multifamily $\mathcal{F}$.

The minimum size of an edge clique partition of a multigraph M is called the *edge clique partition number* of the multigraph M, and is denoted by $\theta^*(M)$.

Note that the edge clique partition number of a multigraph $M = (V, E, \mu)$ and the edge clique cover number of the underlying graph $G = (V, E)$ of M have the relation $\theta_e(G) \leq \theta^*(M)$ since an edge clique partition of M is an edge clique cover of G.

A characterization of the competition multigraph of an acyclic digraph given by Anderson et al. is the following.

Theorem 1.1 *([1, Theorem 1]). Let $M = (V, E, \mu)$ be a multigraph. Then, M is the competition multigraph of an acyclic digraph if and only if there exist an ordering $v_1, \dots, v_n$ of the vertices of M and an edge clique partition $\{S_1, \dots, S_n\}$ of M such that $v_i \in S_j \Rightarrow i < j$.*

Now let us recall the definition of the multicompetition number of a multigraph. Note that the notion of the (ordinary) competition number of a graph was introduced by Roberts [7].

Definition. The *multicompetition number* of a multigraph $M = (V, E, \mu)$ is the smallest nonnegative integer k such that $(V \cup I_k, E, \mu)$ is the multicompetition graph of some acyclic digraph D, where I_k denotes a set of k isolated vertices and $V \cap I_k = \emptyset$. The multicompetition number of a multigraph M is denoted by $k^*(M)$.

This note is organized as follows: In Section 2, we give a characterization of competition multigraphs of arbitrary digraphs and a characterization of competition multigraphs of loopless digraphs. In Section 3, we characterize multigraphs whose multicompetition numbers are at most m, where m is a given nonnegative integer, which is a generalization of Theorem 1.1. In Section 4, we introduce the notion of a competition multihypergraph, which is a generalization of both a competition multigraph and a competition hypergraph introduced by Sonntag and Teichert [9], and we give characterizations of competition multihypergraphs.

2. Competition multigraphs

Theorem 2.1. *Let M be a multigraph with n vertices. Then, M is the competition multigraph of a digraph if and only if $\theta^*(M) \leq n$.*

Proof. Let $M = (V, E, \mu)$ be a multigraph with $V = \{v_1, \dots, v_n\}$.

Suppose that M is a competition multigraph. Then there exists a digraph $D = (V, A)$ such that $CM(D) = M$. Put $S_j := \{v_i \in V \mid (v_i, v_j) \in A\} (j = 1, \dots, n)$. For any x and y in S_j, xy is an edge in M since v_j is a common prey of x and y in D, and thus S_j is a clique of M. If $xy \in E$ is an edge with multiplicity $\mu(xy) = p$, then there exist exactly p common preys $v_{i_1}, \dots, v_{i_p} \in V$ of x and y in D. Then exactly p cliques $S_{i_1}, \dots, S_{i_p}$ contain both x and y. Hence the family $\{S_1, \dots, S_n\}$ is an edge clique partition of M, and thus we conclude $\theta^*(M) \leq n$.

Next, suppose that $\theta^*(M) \leq n$. Then there exists an edge clique partition $\mathcal{F} = \{S_1, \dots, S_r\}$ of M with $r \leq n$. If $xy \in E$ is an edge with multiplicity p in M, then there exist exactly p cliques $S_{i_1}, \dots, S_{i_p} \in \mathcal{F}$ such that each clique contains both x and y. Now we define a digraph D as follows; $V(D) = V$, and $A(D) = \cup_{j=1}^{r}\{(v_i, v_j) \mid v_i \in S_j\}$. Then the competition graph of this digraph D is the graph (V, E), and x and y have exactly p common preys $v_{i_1}, \dots, v_{i_p} \in V$ in the digraph D. Thus the multiplicity of the edge xy in $CM(D)$ is equal to p. Hence we have $CM(D) = M$, and thus we conclude M is a competition multigraph. □

In ordinary situations, it is natural to assume that there are no species that prey on themselves in a food web. This assumption corresponds to the condition that a digraph D is *loopless*, i.e., D does not have an arc with the form (v, v).

Let V be a finite set, and D_i be a subset of V and $v_i \in V$ for each $i = 1, \dots, r$. Then, $(v_1, \dots, v_r)$ is called a *system of distinct representatives* for $\{D_1, \dots, D_r\}$ if $v_1, \dots, v_r$ are distinct and $v_i \in D_i$ for $i = 1, \dots, r$.

Theorem 2.2. *Let M be a multigraph. Then the following statements are equivalent.*

(a) *M is the competition multigraph of a loopless digraph.*

(b) *There exist an ordering $v_1, \dots, v_n$ of the vertices of M and an edge clique partition $\{S_1, \dots, S_r\}$ of M such that $r \leq n$ and $v_j \notin S_j (j = 1, \dots, r)$.*

(c) *There exists an edge clique partition $\{S_1, \dots, S_r\}$ of M such that $r \leq n$ and $\{D_1, \dots, D_r\}$ has a system of distinct representatives, where $D_j := V(M) - S_j$ $(j = 1, \dots, r)$.*

Proof. Let $M = (V, E, \mu)$ be a multigraph with $V = \{v_1, \ldots, v_n\}$.

(a)⇒(b): Let $D = (V, A)$ be a loopless digraph such that $CM(D) = M$. Put $S_j := \{v_i \in V \mid (v_i, v_j) \in A\}$ $(j = 1, \ldots, n)$. Then $\{S_1, \ldots, S_n\}$ is an edge clique partition of M. Since D is loopless, we have $v_j \notin S_j$ $(j = 1, \ldots, n)$.

(b)⇒(c): Let $v_1, \ldots, v_n$ be an ordering of the vertices of M and $\{S_1, \ldots, S_r\}$ be an edge clique partition of M such that $r \le n$ and $v_j \notin S_j$ $(j = 1, \ldots, r)$. Then $(v_1, \ldots, v_r)$ is a system of distinct representatives for $\{D_1, \ldots, D_r\}$.

(c)⇒(a): Let $\{S_1, \ldots, S_r\}$ be an edge clique partition of M such that $r \le n$ and that $\{D_1, \ldots, D_r\}$ has a system of distinct representatives $(v_1, \ldots, v_r)$. Then we have $v_j \notin S_j$ $(j = 1, \ldots, r)$. We define a digraph D as follows; $V(D) = V$, and $A(D) = \cup_{j=1}^{r}\{(v_i, v_j) \mid v_i \in S_j\}$. Then we have $CM(D) = M$, and that D has no loops since $v_j \notin S_j$. □

3. Multigraphs with a bounded multicompetition number

In this section, we give a characterization of multigraphs whose multicompetition numbers are at most m.

Theorem 3.1. *Let M be a multigraph with n vertices, and m be a nonnegative integer. Then, $k^*(M) \le m$ if and only if there exist an ordering $v_1, \ldots, v_n$ of the vertices of M and an edge clique partition $\{S_1, \ldots, S_{n+m}\}$ of M such that $v_i \in S_j \Rightarrow i < j$.*

Proof. Suppose that $k^*(M) \le m$. Let $M' := M \cup I_m = M \cup \{z_1, \ldots, z_m\}$, where $z_1, \ldots, z_m$ are extra isolated vertices. Then the multigraph M' is the competition multigraph of some acyclic digraph D. By Theorem 1.1, there exist an ordering $v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+m}$ of the vertices of M' and an edge clique partition $\{S_1, \ldots, S_{n+m}\}$ of M' such that $v_i \in S_j \Rightarrow i < j$. Here we may assume that $v_{n+1}, \ldots, v_{n+m}$ are isolated vertices in M' since M' has at least m isolated vertices and isolated vertices have no necessity to be contained in the cliques S_j. Hence $M \cong M' - \{v_{n+1}, \ldots, v_{n+m}\}$, and $\{S_1, \ldots, S_{n+m}\}$ is an edge clique partition of $M' - \{v_{n+1}, \ldots, v_{n+m}\}$. Therefore the ordering $v_1, \ldots, v_n$ given above and the edge clique partition $\{S_1, \ldots, S_{n+m}\}$ satisfy the condition $v_i \in S_j \Rightarrow i < j$.

Conversely, suppose that there exist an ordering $v_1, \ldots, v_n$ of the vertices of M and an edge clique partition $\{S_1, \ldots, S_{n+m}\}$ of M such that $v_i \in S_j \Rightarrow i < j$. Put $M' := M \cup I_m = M \cup \{v_{n+1}, \ldots, v_{n+m}\}$, where $v_{n+1}, \ldots, v_{n+m}$ are new isolated vertices. Then, the edge clique partition $\{S_1, \ldots, S_{n+m}\}$ of M is also that of M'. Now the ordering $v_1, \ldots, v_n, v_{n+1}, \ldots, v_{n+m}$ of the vertices of M' and the edge clique partition $\{S_1, \ldots, S_{n+m}\}$ of M' satisfy the condition $v_i \in S_j \Rightarrow i < j$. Thus, by Theorem 1.1, $M' = M \cup I_m$ is the competition multigraph of an acyclic digraph. Hence we have $k^*(M) \le m$. □

Remark 3.2. By definition, a multigraph M is the competition multigraph of an acyclic digraph if and only if the multicompetition number is 0. So Theorem 1.1 follows as a corollary of Theorem 3.1 in the case $m = 0$.

Corollary 3.3. *Let $M = (V, E, \mu)$ be a multigraph with n vertices. Then the following statements are equivalent.*

(a) *The multicompetition number of M is at most m.*

(b) *There exist an ordering $v_1, \ldots, v_n$ of the vertices of M and an edge clique partition $\{S_1, \ldots, S_{n+m}\}$ of M such that $v_i \in S_j \Rightarrow i < j$.*

(c) *There exist an ordering $v_1, \ldots, v_n$ of the vertices of M and an edge clique partition $\{S_1, \ldots, S_{n+m-1}\}$ of M such that $v_i \in S_j \Rightarrow i \le j$.*

(d) *There exist an ordering $v_1, \ldots, v_n$ of the vertices of M and an edge clique partition $\{S_1, \ldots, S_{n+m-2}\}$ of M such that $v_i \in S_j \Rightarrow i \le j + 1$.*

Proof. Let $M = (V, E, \mu)$ be a multigraph with n vertices. Theorem 3.1 includes the equivalence of (a) and (b).

Suppose that (b) holds. Let $v_1, \ldots, v_n$ be an ordering of the vertices of M and $\{S_1, \ldots, S_{n+m}\}$ be an edge clique partition of M such that $v_i \in S_j \Rightarrow i < j$. Then we have $S_1 = \emptyset$. Putting $S'_j := S_{j+1}$ for $j = 1, \ldots, n + m - 1$, we have an edge clique partition $\{S'_1, \ldots, S'_{n+m-1}\}$ of M with $v_i \in S_j \Rightarrow i \le j$, and thus (c) holds.

Suppose that (c) holds. Let $v_1, \ldots, v_n$ be an ordering of the vertices of M and $\{S_1, \ldots, S_{n+m-1}\}$ be an edge clique partition of M such that $v_i \in S_j \Rightarrow i \le j$. Then we have $S_1 \subseteq \{v_1\}$. So S_1 is not covering any edge of M. Putting $S'_j := S_{j+1}$ for $j = 1, \ldots, n + m - 2$, we have an edge clique partition $\{S'_1, \ldots, S'_{n+m-2}\}$ of M with $v_i \in S_j \Rightarrow i \le j + 1$, and thus (d) holds.

Suppose that (d) holds. Let $v_1, \ldots, v_n$ be an ordering of the vertices of M and $\{S_1, \ldots, S_{n+m-2}\}$ be an edge clique partition of M such that $v_i \in S_j \Rightarrow i \le j + 1$. Putting $S'_1 := \emptyset$ $S'_2 := \emptyset$, and $S'_j := S_{j-2}$ for $j = 3, \ldots, n + m$, we have an edge clique partition $\{S'_1, \ldots, S'_{n+m}\}$ of M with $v_i \in S_j \Rightarrow i < j$, and thus (b) holds. □

Corollary 3.4. *Let $M = (V, E, \mu)$ be a multigraph with n vertices. Then, $k^*(M) \le 1$ if and only if there exist an ordering $v_1, \ldots, v_n$ of the vertices of M and an edge clique partition $\{S_1, \ldots, S_n\}$ of M such that $v_i \in S_j \Rightarrow i \le j$.*

Proof. It follows from the equivalence of (a) and (c) in Corollary 3.3 with $m = 1$. □

Corollary 3.5. *Let $M = (V, E, \mu)$ be a multigraph with n vertices. Then, $k^*(M) \le 2$ if and only if there exist an ordering $v_1, \ldots, v_n$ of the vertices of M and an edge clique partition $\{S_1, \ldots, S_n\}$ of M such that $v_i \in S_j \Rightarrow i \le j + 1$.*

Proof. It follows from the equivalence of (a) and (d) in Corollary 3.3 with $m = 2$. □

Remark 3.6. Note that, if $m \geq 3$, then it seems to be impossible in general to characterize multigraphs with n vertices and the multicompetition number at most m by using edge clique partitions of size n, like Corollaries 3.4 and 3.5.

At the end of this section, we give a characterization of nontrivial triangle-free connected multigraphs with the multicompetition number 1. A multigraph M is called *triangle-free* if M does not contain a clique of size 3. A *trivial* multigraph is the multigraph which consists of one vertex and no edges. The multicompetition number of a nontrivial triangle-free connected multigraph M is given explicitly as follows.

Theorem 3.7 (*[1, Corollary 1]*). *Let $M = (V, E, \mu)$ be a nontrivial multigraph. If M is triangle-free and connected, then*

$$k^*(M) = \sum_{e \in E} \mu(e) - |V| + 2.$$

As a corollary of the above theorem, we have the following result.

Corollary 3.8. *Let $M = (V, E, \mu)$ be a nontrivial triangle-free connected multigraph. Then, $k^*(M) = 1$ if and only if the underlying graph (V, E) of M is a tree and $\mu(e) = 1$ for all $e \in E$.*

Proof. Let $M = (V, E, \mu)$ be a nontrivial triangle-free connected multigraph. Note that a connected graph (V, E) is a tree if and only if $|V| - 1 = |E|$ holds. If (V, E) is a tree and $\mu(e) = 1$ for all $e \in E$, then we have $k^*(M) = 1$ by Theorem 3.7.

Suppose that $k^*(M) = 1$. Then we have $\sum_{e \in E} \mu(e) - |V| + 2 = 1$ by Theorem 3.7. This equation implies $|V| - 1 = \sum_{e \in E} \mu(e) \geq |E|$. Since M is connected, we have $|E| \geq |V| - 1$. Thus we have $|E| = |V| - 1$ and $\sum_{e \in E} \mu(e) = |E|$. Hence (V, E) is a tree and $\mu(e) = 1$ for all $e \in E$. □

4. Competition multihypergraphs

Sonntag and Teichert [9] introduced *competition hypergraphs* and characterized them. The *competition hypergraph* of a digraph $D = (V, A)$ is a hypergraph $(V, \mathcal{E})$ which has the same vertex set as D and $e \subseteq V$ is a hyperedge if and only if there exists a vertex $v \in V$ such that $|e| \geq 2$ and $e = \{u \in V \mid (u, v) \in A\}$.

In this section, we generalize competition multigraphs and competition hypergraphs to "competition multihypergraphs". A *multihypergraph* $(V, \mathcal{E}, \mu)$ consists of a hypergraph $(V, \mathcal{E})$ and a *multiplicity* $\mu : \mathcal{E} \to \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers.

Definition. The *competition multihypergraph* of a digraph $D = (V, A)$ is a multihypergraph $\mathcal{M} = (V, \mathcal{E}, \mu)$ such that $(V, \mathcal{E})$ is the competition hypergraph of D, and the multiplicity $\mu(e)$ of a hyperedge $e \in \mathcal{E}$ is the number of vertices v such that $e = \{u \in V \mid (u, v) \in A\}$.

Theorem 4.1. *Let $\mathcal{M} = (V, \mathcal{E}, \mu)$ be a multihypergraph. Then, $\mathcal{M}$ is the competition multihypergraph of a digraph if and only if $\sum_{e \in \mathcal{E}} \mu(e) \leq |V|$.*

Proof. Let $\mathcal{M} = (V, \mathcal{E}, \mu)$ be a multihypergraph with $V = \{v_1, \ldots, v_n\}$.

Suppose that $\mathcal{M}$ is the competition multihypergraph of a digraph. Then there exists a digraph $D = (V, A)$ such that its competition multihypergraph is $\mathcal{M}$. Put

$$e'_j := \{v_i \in V \mid (v_i, v_j) \in A\} \quad (j = 1, \ldots, n).$$

Then we have a multifamily $\mathcal{E}' := \{e'_1, \ldots, e'_n\}$. Then the hyperedge set $\mathcal{E}$ of $\mathcal{M}$ is given by $\mathcal{E} = \{e'_j \in \mathcal{E}' \mid |e_j| \geq 2\} =: \{e_1, \ldots, e_t\}$ and the multiplicity $\mu(e_j)$ of $e_j \in \mathcal{E}$ is given by $\mu(e_j) = |\{e' \in \mathcal{E}' \mid e_j = e'\}|$. Thus we conclude $\sum_{e \in \mathcal{E}} \mu(e) \leq |\mathcal{E}'| = n$.

Next, suppose that $\sum_{e \in \mathcal{E}} \mu(e) \leq n$. Let $\mathcal{E} = \{e_1, \ldots, e_t\}$. We define a digraph D as follows;

$$V(D) := V, \qquad A(D) := \bigcup_{j=1}^{t} \bigcup_{k=\sum_{l=1}^{j-1} \mu(e_l)+1}^{\sum_{l=1}^{j} \mu(e_l)} \{(v_i, v_k) \mid v_i \in e_j\}.$$

Then we can check that $\mathcal{M}$ is the competition multihypergraph of this digraph D. □

We can show the following characterizations similarly (we omit proofs).

Theorem 4.2. *Let $\mathcal{M} = (V, \mathcal{E}, \mu)$ be a multihypergraph. Then, $\mathcal{M}$ is the competition multihypergraph of a loopless digraph if and only if there exist an ordering $v_1, \ldots, v_n$ of the vertices of $\mathcal{M}$ and an ordering $e_1, \ldots, e_t$ of the hyperedges of $\mathcal{M}$ such that $\sum_{j=1}^{t} \mu(e_j) \leq n$ and $v_j \notin e_j (j = 1, \ldots, t)$.*

Theorem 4.3. *Let $\mathcal{M} = (V, \mathcal{E}, \mu)$ be a multihypergraph. Then, $\mathcal{M}$ is the competition multihypergraph of an acyclic digraph if and only if there exist an ordering $v_1, \ldots, v_n$ of the vertices of $\mathcal{M}$ and an ordering $e_1, \ldots, e_t$ of the hyperedges of $\mathcal{M}$ such that $v_i \in e_j$ implies $i < \sum_{l=1}^{j} \mu(e_l)$.*

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