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Remarks on the Euler Equation*

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INTRODUCTION

Let Ω be a bounded domain of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outward normal n. The motion of an incompressible perfect fluid is described by the Euler equation

$$\partial u_i/\partial t + \sum_{j=1}^{N} u_j(\partial u_i/\partial x_j) = f_i + \partial\tilde{\omega}/\partial x_i, \qquad 1 \leqslant i \leqslant N,$$

$$\text{on} \quad \Omega \times (0, T), \tag{1}$$

$$\operatorname{div} u = 0 \qquad \text{on} \quad \Omega \times (0, T), \tag{2}$$

$$u \cdot n = 0 \qquad \text{on} \quad \partial\Omega \times (0, T), \tag{3}$$

$$u\,|_{t=0} = u_0 \qquad \text{on} \quad \Omega, \tag{4}$$

where $f(x, t)$ and $u_0(x)$ are given, while the velocity $u(x, t)$ and the pressure $\tilde{\omega}(x, t)$ are to be determined.

The Euler equation has been considered by several authors including L. Lichtenstein (1925–30), J. Leray (1932–37), M. Wolibner (1938). T. Kato proved the existence of a global solution for $N = 2$ [3] and of a local solution for $\Omega = \mathbb{R}^3$ [4]. Recently, D. Ebin and J. Marsden [2] have proved the existence of a local solution in the general case. Their proof relies heavily on techniques of Riemannian geometry on infinite dimensional manifolds. Our purpose is to present

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a more "classical" proof of their result by reducing (1)–(4) to an ordinary differential equation on a closed set of a Banach space; actually, we get a slightly more general result valid for L^p data instead of L^2 data.

The main theorem is the following

THEOREM 1. *Let $1 < p < +\infty$, and let $s > (N/p) + 1$ be an integer. Suppose $u_0 \in W^{s,p}(\Omega; \mathbb{R}^N)$ with* $\operatorname{div} u_0 = 0$ *on Ω and $u_0 \cdot n = 0$ on $\partial\Omega$. Suppose $f \in C([0, T]; C^{s+1+\alpha}(\Omega; \mathbb{R}^N))$ with $0 < \alpha < 1$*[1]. *Then there exists $0 < T_0 \leqslant T$ and a unique function*

$$u \in C([0, T_0]; W^{s,p}(\Omega; \mathbb{R}^N))$$

satisfying (1)–(4).

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1. Notations and Preliminaries

Let $W^{s,p}$ be the Sobolev space of real-valued functions in L^p such that all their derivatives up to order s are in L^p. In the following we assume that $s > (N/p) + 1$ so that by the Sobolev embedding theorem $W^{s,p}(\Omega) \subset C^{1+\alpha}(\bar{\Omega})$ with $\alpha = s - 1 - N/p$. The norm in $W^{s,p}$ is denoted by $\| \ \|_{s,p}$. Let

$$\mathscr{D}^{s,p} = \{\eta \in W^{s,p}(\Omega; \mathbb{R}^N);$$
$$\eta \text{ is bijective from } \bar{\Omega} \text{ onto } \bar{\Omega} \quad \text{and} \quad \eta^{-1} \in W^{s,p}(\Omega; \mathbb{R}^N)\}.$$

Note that $\eta \in \mathscr{D}^{s,p}$ if and only if $\eta \in W^{s,p}(\Omega; \mathbb{R}^N)$ and η is a C^1 diffeomorphism with $\eta(\partial\Omega) \subset \partial\Omega$.

Let

$$\mathscr{D}^{s,p}_\mu = \{\eta \in \mathscr{D}^{s,p}; \mid \operatorname{Jac} \eta \mid = 1 \text{ on } \Omega\},$$

where $\operatorname{Jac} \eta$ denotes the Jacobian matrix of η and $\mid \operatorname{Jac} \eta \mid$ its determinant. Note that $\eta \in \mathscr{D}^{s,p}_\mu$ if and only if $\eta \in W^{s,p}(\Omega; \mathbb{R}^N)$, $\mid \operatorname{Jac} \eta \mid = 1$ on Ω and $\eta(\partial\Omega) \subset \partial\Omega$.

Let

$$T_e\mathscr{D}^{s,p} = \{u \in W^{s,p}(\Omega; \mathbb{R}^N); u \cdot n = 0 \text{ on } \partial\Omega\}$$

and

$$T_e\mathscr{D}^{s,p}_\mu = \{u \in T_e\mathscr{D}^{s,p}; \operatorname{div} u = 0 \text{ in } \Omega\}.$$

[1] In fact, it is sufficient to assume $f \in C([0, T]; W^{s+1,p}(\Omega; \mathbb{R}^N))$.

Recall that if $V(x, t) \in C^1(\bar{\Omega} \times [0, T])$ is such that V is tangent to the boundary, i.e., $V(x, t) \cdot n(x) = 0$ on $\partial\Omega \times [0, T]$ and if $\eta(x, t)$ is the flow generated by V, i.e. the solution of

$$(d\eta/dt)(x, t) = V(\eta(x, t), t),$$

then

$$(d/dt) \,|\, \text{Jac}\, \eta(x, t)|_{t=\tau} = (\text{div}\, V)(\eta(x, \tau), \tau) \,|\, \text{Jac}\, \eta(x, \tau)|. \tag{5}$$

So that in particular if div $V = 0$ on $\Omega \times [0, T]$, then

$$|\, \text{Jac}\, \eta(x, t)| = |\, \text{Jac}\, \eta(x, 0)| \qquad \text{on} \quad \Omega \times [0, T].$$

The following lemmas are well-known (see, e.g., [5]).

LEMMA 1 (Neumann problem). *Given an* $f \in W^{k,p}(\Omega)$ ($k \geqslant 0$ *an integer*) *and a* $g \in W^{k+1-1/p,p}(\partial\Omega)$ *such that*

$$\int_\Omega f\, dx = \int_{\partial\Omega} g\, d\sigma,$$

there exists a $u \in W^{k+2,p}(\Omega)$ *satisfying*

$$\Delta u = f \qquad \text{on} \quad \Omega,$$

$$\frac{\partial u}{\partial n} = g \qquad \text{on} \quad \partial\Omega.$$

In addition,

$$\|\, \text{grad}\, u \,\|_{k+1,p} \leqslant C(\|\, f \,\|_{k,p} + \|\, g \,\|_{k+1-1/p,p}).$$

LEMMA 2. *Given an* $f \in W^{k,p}(\Omega; \mathbb{R}^N)$, *there exists a unique* $g \in T_e\mathcal{D}_\mu^{k,p}$ *and a* $\bar{\omega} \in W^{k+1,p}(\Omega)$ *such that*

$$f = g + \text{grad}\, \bar{\omega}.$$

We set $g = P(f)$. *P is called the projection on divergence free vector fields; it is a bounded operator in* $W^{k,p}(\Omega; \mathbb{R}^N)$. *P is related to the solution of the Neumann problem in the following way: let* $\bar{\omega} \in W^{k+1,p}(\Omega)$ *be a solution of*

$$\begin{cases} \Delta \bar{\omega} = \text{div}\, f & \text{on} \quad \Omega, \\ \dfrac{\partial \bar{\omega}}{\partial n} = f \cdot n & \text{on} \quad \partial\Omega. \end{cases}$$

Then

$$g = Pf = f - \text{grad}\, \bar{\omega}.$$

2. Reduction of the Euler Equation to an Ordinary Differential Equation

Following an idea of V. Arnold [1], we shall work as in [2] with Lagrange variables. So, we use the configuration η of the fluid (i.e. the flow generated by u) as unknown. As we shall see, this leads us to the study of a second-order "ordinary" differential equation.

Assuming (1)–(4) has a solution u, let η be the flow of u:

$$(d\eta/dt)(x, t) = u(\eta(x, t), t), \qquad \eta(x, 0) = x. \tag{6}$$

Let us rewrite the equation (1)–(4) in terms of η. Equation (4) becomes

$$(d\eta/dt)(x, 0) = u_0(x). \tag{4'}$$

Equation (3) corresponds to the fact that, for each t, $\eta(\cdot, t)$ is a diffeomorphism from $\bar{\Omega}$ onto itself and Eq. (2) is equivalent to

$$|\operatorname{Jac} \eta(x, t)| = 1 \qquad \text{on} \quad \Omega \times [0, T]. \tag{2'}$$

In order to write down (1) in terms of η, we eliminate the pressure $\tilde{\omega}$ by applying P to (1). Using (2) we get

$$(\partial u/\partial t) + P\left(\sum_j u_j(\partial u/\partial x_j)\right) = Pf.$$

On the other hand, by differentiating (6) with respect to t, we obtain

$$\begin{aligned}(\partial^2\eta/\partial t^2)(x, t) &= \sum_i (\partial u/\partial x_i)(\eta(x, t), t)(\partial\eta_i/\partial t)(x, t) + (\partial u/\partial t)(\eta(x, t), t)\\ &= \sum_i u_i(\eta(x, t), t)(\partial u/\partial x_i)(\eta(x, t), t) + (\partial u/\partial t)(\eta(x, t), t).\end{aligned}$$

Therefore,

$$(\partial^2\eta/\partial t^2)(x, t) = \left[(I - P)\sum_i u_i(\partial u/\partial x_i)\right](\eta(x, t), t) + (Pf)(\eta(x, t), t). \tag{7}$$

If we keep in mind that

$$u = (\partial\eta/\partial t)(\eta^{-1}, t),$$

we can consider (7) as an equation involving only η.

A crucial observation is that (7) should not be regarded as a partial differential equation in η but rather as an ordinary differential equation in η (this fact is outlined in [2, p. 147]).

We first write (7) as a system

$$\begin{cases} \dfrac{d\eta}{dt} = v \\ \dfrac{dv}{dt} = \left[(I - P)\sum_i (v \circ \eta^{-1})_i \dfrac{\partial}{\partial x_i} (v \circ \eta^{-1})\right] (\eta, t) + (Pf)(\eta, t) \end{cases}$$

or

$$(d/dt)(\eta, v) = A(t; \eta, v), \tag{8}$$

where

$$A(t; \eta, v) = (v, B(v \circ \eta^{-1}) \circ \eta + (Pf)(\eta, t)) \tag{9}$$

and

$$Bv = (I - P)\left(\sum_i v_i \frac{\partial v}{\partial x_i}\right). \tag{10}$$

We shall work in the space $X = W^{s,p}(\Omega; \mathbb{R}^N) \times W^{s,p}(\Omega; \mathbb{R}^N)$. Clearly, A is not everywhere defined on X and not even on an open subset because of the additional requirement $\eta \in \mathscr{D}_\mu^{s,p}$. Thus we cannot apply standard existence theorems for ordinary differential equations, but shall use the following theorem which is a particular case of a result of R. Martin [6].

THEOREM 2. *Let F be a closed subset of a Banach space X, and let $A(t, z)$: $[0, T) \times F \to X$ be locally Lipschitz in z and continuous in t. Suppose that for each $(t, z) \in [0, T] \times F$ the following holds*

$$\lim_{h \downarrow 0} \frac{1}{h} \operatorname{dist}(z + hA(t, z), F) = 0.^2 \tag{11}$$

Then for every $z_0 \in F$ the equation

$$dz/dt = A(t, z), \qquad z(0) = z_0,$$

admits a local solution $z \in C^1([0, T_0]; F)$.

We shall apply Theorem 2 with $F = \{(\eta, v) \in X;\ \eta \in \mathscr{D}_\mu^{s,p}$ and $v \circ \eta^{-1} \in T_e\mathscr{D}_\mu^{s,p}\}$ which is clearly closed in X.

The main steps in proving Theorem 1 are the following:

(a) Prove that $A(t; \eta, v)$ is locally Lipschitz in (η, v) from F into X (see Section 3).

[2] Where $\operatorname{dist}(\cdot, F)$ denotes the distance to F.

One has to be rather careful because the mapping $\eta \mapsto \eta^{-1}$ is *not* locally Lipschitz from $\mathscr{D}_\mu^{s,p}$ into itself (it is only continuous); similarly, the mapping $[\psi, \eta] \mapsto \psi \circ \eta$ is *not* locally Lipschitz from $\mathscr{D}_\mu^{s,p} \times \mathscr{D}_\mu^{s,p}$ into $\mathscr{D}_\mu^{s,p}$.

(b) Prove that $A(t; \eta, v)$ is tangent to F in the sense of (11) (see Section 4).

Remark. In case $f = 0$, Eq. (7) represents the equation of geodesics on the manifold $\mathscr{D}_\mu^{s,2}$ for an appropriate weak Riemannian metric. Since the metric is weak (i.e. the topology induced by this metric is weaker than the topology of $\mathscr{D}_\mu^{s,2}$), the existence of a Riemannian connection and of geodesics does not follow at once, but is proved in [2].

3. A is Locally Lipschitz

First of all, we observe the following.

LEMMA 3. *Let f be as in Theorem 1. The mapping $(t, \eta) \mapsto (Pf)(\eta, t)$ is continuous in t and locally Lipschitz in η.*

Proof. As $t \to t_0$, $f(\cdot, t) \to f(\cdot, t_0)$ *in* $C^s(\bar{\Omega}; \mathbb{R}^N)$, and therefore $Pf(\cdot, t) \to Pf(\cdot, t_0)$ in $W^{s,p}(\Omega; \mathbb{R}^N)$. We conclude by Lemma A.4 that $Pf(\eta, t) \to Pf(\eta, t_0)$ in $W^{s,p}(\Omega; \mathbb{R}^N)$.

For a fixed t, $f(\cdot, t) \in C^{s+1+\alpha}(\bar{\Omega})$ and so $Pf(\cdot, t) \in C^{s+1+\alpha}(\bar{\Omega})$. Thus, by Lemma A.3, $\eta \mapsto (Pf)(\eta, t)$ is locally Lipschitz from $\mathscr{D}_\mu^{s,p}$ into $W^{s,p}(\Omega; \mathbb{R}^N)$. ∎

Remark. It is actually sufficient to assume that $f \in W^{s+1,p}(\Omega, \mathbb{R}^N)$ and use the remark following Lemma A.5 instead of Lemma A.3.

We shall now prove

THEOREM 3. *The mapping $(\eta, v) \mapsto B(v \circ \eta^{-1}) \circ \eta$ (B is defined in (10)) is locally Lipschitz from F into $W^{s,p}(\Omega; \mathbb{R}^N)$.*

The proof of Theorem 3 relies on an appropriate factorization of B. Note that if $u \in T_e\mathscr{D}_\mu^{s,p}$, we have by Lemma 2, $Bu = \operatorname{grad} \bar{\omega}$ where $\bar{\omega}$ is a solution of

$$\Delta\bar{\omega} = \operatorname{div}\left(\sum_i u_i \frac{\partial u}{\partial x_i}\right) \quad \text{on} \quad \Omega,$$

$$\frac{\partial\bar{\omega}}{\partial n} = \left(\sum_i u_i \frac{\partial u}{\partial x_i}\right) \cdot n \quad \text{on} \quad \partial\Omega.$$

But

$$\operatorname{div}\Big(\sum_i u_i \frac{\partial u}{\partial x_i}\Big) = \sum_{i,j} \frac{\partial}{\partial x_j}\Big(u_i \frac{\partial u_j}{\partial x_i}\Big) = \sum_{i,j} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}$$

(since $\operatorname{div} u = 0$) and

$$\Big(\sum_i u_i \frac{\partial u}{\partial x_i}\Big)\cdot n = \sum_{i,j} u_i \frac{\partial u_j}{\partial x_i} n_j = \beta(\cdot\,; u, u)$$

where $\beta(x; u, u)$ denotes the second fundamental form of $\partial\Omega$. More precisely, let $\delta(x)$ be a smooth function on $\mathbb{R}^N$ such that

$$\Omega = \{x \in \mathbb{R}^N; \delta(x) > 0\},$$
$$\partial\Omega = \{x \in \mathbb{R}^N; \delta(x) = 0\},$$

and $\operatorname{grad}\delta = -n$ on $\partial\Omega$. For $u \in T_e\mathscr{D}_\mu^{s,p}$, we have $u.\operatorname{grad}\delta = 0$ on $\partial\Omega$ and by differentiation we obtain

$$u.\operatorname{grad}[u.\operatorname{grad}\delta] = 0 \qquad \text{on} \quad \partial\Omega,$$

i.e.,

$$\sum_{i,j} u_i \frac{\partial}{\partial x_i}\Big(u_j \frac{\partial \delta_i}{\partial x_j}\Big) = 0 \qquad \text{on} \quad \partial\Omega.$$

Therefore on $\partial\Omega$ we have

$$\sum_{i,j} u_i \frac{\partial u_j}{\partial x_i} n_j = \sum_{i,j} \frac{\partial^2 \delta}{\partial x_i\,\partial x_j} u_i u_j = \beta(\cdot\,; u, u). \tag{12}$$

Note that β is a quadratic form in u depending smoothly on $x \in \partial\Omega$. We consider first the mapping Q defined by

$$Q(\eta, v) = \Big(\eta, \sum_{i,j}\Big(\frac{\partial u_j}{\partial x_j}\frac{\partial u_j}{\partial x_i}\Big)\circ\eta, \beta(\eta; v, v)\Big),$$

where $u = v \circ \bar{\eta}^1$, which maps F into Z, where

$$Z = \Big\{(\eta, f, g) \in \mathscr{D}_\mu^{s,p} \times W^{s-1,p}(\Omega) \times W^{s-1/p,p}(\partial\Omega); \int_\Omega f\,dx = \int_{\partial\Omega} g\circ\eta^{-1}\,d\sigma\Big\}.$$

Next, let $S(\eta, f, g)$ be defined from Z into $W^{s,p}(\Omega; \mathbb{R}^N)$ by

$$S(\eta, f, g) = (\operatorname{grad}\pi)\circ\eta,$$

where π is a solution of

$$\Delta\pi = f \circ \eta^{-1} \quad \text{on} \quad \Omega,$$
$$\frac{\partial \pi}{\partial n} = g \circ \eta^{-1} \quad \text{on} \quad \partial\Omega.$$

Therefore we obtain

$$B(v \circ \eta^{-1}) \circ \eta = (S \circ Q)(\eta, v),$$

and it is sufficient to prove the following propositions:

PROPOSITION 1. *The mapping* $(\eta, v) \mapsto Q(\eta, v)$ *is locally Lipschitz from* F *into* Z.

PROPOSITION 2. *The mapping* $(\eta, f, g) \mapsto S(\eta, f, g)$ *is locally Lipschitz from* Z *into* $W^{s,p}(\Omega; \mathbb{R}^N)$.

The following lemma will be very useful.

LEMMA 4. *Let* $f \in W^{s,p}(\Omega)$ *and* $\eta \in \mathscr{D}_\mu^{s,p}$. *Then*

$$\|(\operatorname{grad}(f \circ \eta^{-1})) \circ \eta - \operatorname{grad} f\|_{s-1,p} \leqslant C_\eta \|\eta - e\|_{s,p} \|f\|_{s,p},$$

where e *denotes the identity of* Ω *and* C_η *a constant depending only on* $\|\eta\|_{s,p}$.

Proof of Lemma 4. We have

$$\operatorname{grad}(f \circ \eta^{-1}) = {}^t(\operatorname{Jac} \eta^{-1}) \cdot (\operatorname{grad} f)(\eta^{-1})$$

and

$$(\operatorname{grad}(f \circ \eta^{-1})) \circ \eta = {}^t(\operatorname{Jac} \eta^{-1})(\eta) \operatorname{grad} f = (\operatorname{Jac} \eta)^{-1} \cdot \operatorname{grad} f.$$

We deduce from Lemma A.1 that

$$\begin{aligned}\|(\operatorname{grad}(f \circ \eta^{-1})) \circ \eta - \operatorname{grad} f\|_{s-1,p} &\leqslant C \|(\operatorname{Jac} \eta)^{-1} - I\|_{s-1,p} \|\operatorname{grad} f\|_{s-1,p} \\ &\leqslant C \|(\operatorname{Jac} \eta)^{-1} \circ (I - \operatorname{Jac} \eta)\|_{s-1,p} \|f\|_{s,p}.\end{aligned}$$

Remark. Lemma 4 holds true for any first-order differential operator and in a particular grad can be replaced by div or by curl.

Proof of Proposition 1. From Lemma 4, it follows easily that $(\eta, f) \mapsto (\operatorname{grad}(f \circ \eta^{-1})) \circ \eta$ is locally Lipschitz from $\mathscr{D}_\mu^{s,p} \times W^{s,p}(\Omega)$

into $W^{s-1,p}(\Omega; \mathbb{R}^N)$. Indeed, by Lemma A.4 (applied with $\alpha = s - 1$ and $q = p^*$), we have

$$\|(\operatorname{grad}(f \circ \eta_1^{-1})) \circ \eta_1 - (\operatorname{grad}(f \circ \eta_2^{-1})) \circ \eta_2 \|_{s-1,p}$$
$$\leqslant C \|(\operatorname{grad}(f \circ \eta_1^{-1})) \circ \eta_1 \circ \eta_2^{-1} - \operatorname{grad}(f \circ \eta_2^{-1})\|_{s-1,p}(\| \eta_2 \|_{s,p}^{s-1} + 1)$$
$$\leqslant C(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} \| f \|_{s,p}$$

where $C(\eta_1, \eta_2)$ is locally bounded. Hence, by Lemma A.1, the mapping

$$(\eta, v) \mapsto \sum_{i,j} \frac{\partial(v_i \circ \eta^{-1})}{\partial x_j}(\eta) \frac{\partial(v_j \circ \eta^{-1})}{\partial x_i}(\eta)$$

is locally Lipschitz.

It remains to check that $(\eta, v) \mapsto \beta(\eta; v, v)$ is locally Lipschitz from F into $W^{s-1/p,p}(\partial\Omega)$. This is clear (by Lemma A.5) since $\beta(x; v, v)$ is smooth in x and quadratic in v. ∎

In the proof of Proposition 2, we shall use the following:

LEMMA 5. *There is a positive constant* α *such that*

$$\alpha \| w \|_{s,p} \leqslant \| \operatorname{div} w \|_{s-1,p} + \| \operatorname{curl} w \|_{s-1,p} + \| w \cdot n \|_{s-1/p,p} + \| w \|_{s-1,p}$$

for all $w \in W^{s,p}(\Omega; \mathbb{R}^N)$, *where* curl u *denotes the matrix with coefficients* $\varphi_{ij} = (\partial w_i/\partial x_j) - (\partial w_j/\partial x_i)$.

Proof of Lemma 5. We have

$$(\partial^2 w_i/\partial x_i\, \partial x_j) - \partial^2 w_j/\partial x_i^2 = \partial\varphi_{ij}/\partial x_i ,$$

and thus for all $1 \leqslant j \leqslant N$,

$$\frac{\partial}{\partial x_j}(\operatorname{div} w) - \Delta w_j = \sum_i \frac{\partial \varphi_{ij}}{\partial x_i}. \tag{13}$$

Let $\nu = (\nu_j) \in C^\infty(\bar{\Omega}; \mathbb{R}^N)$ be such that $\nu = n$ on $\partial\Omega$ and let $U = \sum_j \nu_j w_j$. So that

$$\Delta U = \sum_j \nu_j \frac{\partial}{\partial x_j}(\operatorname{div} w) - \sum_{i,j} \nu_j \frac{\partial \varphi_{ij}}{\partial x_i} + 2 \sum_{i,j} \frac{\partial \nu_j}{\partial x_i}\frac{\partial w_j}{\partial x_i} + \sum_j (\Delta \nu_j)\, w_j .$$

Therefore, by a regularity theorem for the Dirichlet problem ((see [5]), we have

$$\| U \|_{s,p} \leqslant C(\| \Delta U \|_{s-2,p} + \| U_{|\partial\Omega} \|_{s-1/p,p})$$
$$\leqslant C'(\| \operatorname{div} w \|_{s-1,p} + \| \operatorname{curl} w \|_{s-1,p} + \| w \|_{s-1,p} + \| w \cdot n \|_{s-1/p,p}).$$

Finally, for all $1 \leqslant i \leqslant N$,

$$V_i = \sum_j \nu_j \frac{\partial w_i}{\partial x_j} = \sum_j \frac{\partial}{\partial x_i} (\nu_j w_j) - \sum_j \frac{\partial \nu_j}{\partial x_i} w_j + \sum_j \nu_j \varphi_{ij}$$
$$= \frac{\partial U}{\partial x_i} - \sum_j \frac{\partial \nu_j}{\partial x_i} w_j + \sum \nu_j \varphi_{ij} .$$

Hence, $\partial w_i/\partial \eta = V_{i|\partial\Omega} \in W^{s-1-1/p,p}(\partial\Omega)$ and we have the estimate

$$\left\| \frac{\partial w_i}{\partial n} \right\|_{s-1-1/p,p} \leqslant C(\| U \|_{s,p} + \| w \|_{s-1,p} + \| \operatorname{curl} w \|_{s-1,p}).$$

On the other hand, by (13), $\Delta w_i \in W^{s-2,p}(\Omega)$. Moreover,

$$\| \operatorname{grad} w_i \|_{s-1,p} \leqslant C \left(\| \Delta w_i \|_{s-2,p} + \left\| \frac{\partial w_i}{\partial n} \right\|_{s-1-1/p,p} \right)$$

so that by (13) and the previous estimate we get

$$\| w \|_{s,p} \leqslant C(\| \operatorname{div} w \|_{s-1,p} + \| \operatorname{curl} w \|_{s-1,p} + \| w \|_{s-1,p} + \| w \cdot n \|_{s-1/p,p}). \blacksquare$$

Remark. For any norm $||| \cdot |||$ on $W^{s-1,p}$ which is weaker than $\| \ \|_{s-1,p}$, there is a constant $\alpha > 0$ such that

$$\alpha \| w \|_{s,p} \leqslant \| \operatorname{div} w \|_{s-1,p} + \| \operatorname{curl} w \|_{s-1,p} + \| w \cdot n \|_{s-1/p,p} + ||| w |||,$$

since the injection $W^{s,p} \subset W^{s-1,p}$ is compact.

Proof of Proposition 2. We have to estimate

$$X = \|(\operatorname{grad} \pi_1) \circ \eta_1 - (\operatorname{grad} \pi_2) \circ \eta_2 \|_{s,p}$$

where

$$\Delta \pi_i = f_i \circ \eta_i^{-1} \quad \text{on } \Omega, \qquad (\partial \pi_i/\partial n) = g_i \circ \eta_i^{-1} \quad \text{on } \partial\Omega, \quad i = 1, 2.$$

By Lemma A.4 we know that

$$X \leqslant C(\eta_2) \|(\operatorname{grad} \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \operatorname{grad} \pi_2 \|_{s,p}.$$

We shall use the Remark following Lemma 5 to estimate

$$\|(\operatorname{grad} \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \operatorname{grad} \pi_2 \|_{s,p} .$$

Let

$$X_1 = \| \operatorname{div}[(\operatorname{grad} \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \operatorname{grad} \pi_2]\|_{s-1,p}$$

$$X_2 = \| \operatorname{curl}[(\operatorname{grad} \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \operatorname{grad} \pi_2]\|_{s-1,p}$$

$$X_3 = \|[(\operatorname{grad} \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \operatorname{grad} \pi_2] \cdot n \|_{s-1/p,p}$$

$$X_4 = |||(\operatorname{grad} \pi_1) \circ \eta_1 \circ \eta_2^{-1} - \operatorname{grad} \pi_2 |||,$$

where we choose

$$||| u ||| = \sup \left\{ \int_\Omega u \cdot \zeta \, dx ;\ \zeta \in C^s(\bar{\Omega}; \mathbb{R}^N),\ \zeta = 0 \text{ on } \partial\Omega \text{ and } \| \zeta \|_{C^s} \leqslant 1 \right\}.$$

We have

$$\operatorname{div} \operatorname{grad} \pi_2 = \Delta \pi_2 = f_2 \circ \eta_2^{-1}$$

and

$$\operatorname{div}[(\operatorname{grad} \pi_1) \circ \eta_1 \circ \eta_2^{-1}] = [\operatorname{div}(\operatorname{grad} \pi_1)] \circ \eta_1 \circ \eta_2^{-1} + R$$

where, by the Remark following Lemma 4 (used with $f = (\operatorname{grad} \pi_1) \circ \eta$ and $\eta = \eta_1 \circ \eta_2^{-1}$), we have

$$\begin{aligned} \| R \|_{s-1,p} &\leqslant C(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} \| \operatorname{grad} \pi_1 \|_{s,p} \\ &\leqslant C'(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \circ \eta_1^{-1} \|_{s-1,p} + \| g_1 \circ \eta_1^{-1} \|_{s-1/p,p}) \\ &\leqslant C''(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \|_{s-1,p} + \| g_1 \|_{s-1/p,p}). \end{aligned}$$

Hence

$$\begin{aligned} X_1 \leqslant\ & C''(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \|_{s-1,p} + \| g_1 \|_{s-1/p,p}) \\ & + \| f_1 \circ \eta_2^{-1} - f_2 \circ \eta_2^{-1} \|_{s-1,p} \end{aligned}$$

and thus

$$X_1 \leqslant C'''(\eta_1, \eta_2)[\| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \|_{s-1,p} + \| g_1 \|_{s-1/p,p}) + \| f_1 - f_2 \|_{s-1,p}].$$

Similarly, since curl grad $= 0$, we get

$$X_2 \leqslant C(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \|_{s-1,p} + \| g_1 \|_{s-1/p,p}).$$

Next letting $\eta = \eta_1 \circ \eta_2^{-1}$ we have

$$X_3 \leqslant \|(\text{grad}\, \pi_1) \circ \eta \cdot (n - n \circ \eta)\|_{s-1/p,p} + \left\| \frac{\partial \pi_1}{\partial n}(\eta) - \frac{\partial \pi_2}{\partial n} \right\|_{s-1/p,p}$$
$$\leqslant C(\eta_1, \eta_2)[\| \eta_1 - \eta_2 \|_{s,p} (\| f_1 \|_{s-1,p} + \| g_1 \|_{s-1/p,p}) + \| g_1 - g_2 \|_{s-1/p,p}].$$

Finally we estimate X_4; let $\zeta \in C^s(\bar{\Omega}; \mathbb{R}^N)$ be such that $\zeta = 0$ on $\partial\Omega$. Let

$$K(\zeta) = \int_\Omega [(\text{grad}\, \pi_1) \circ \eta - \text{grad}\, \pi_2] \cdot \zeta \, dx$$
$$= \int_\Omega [(\text{grad}\, \pi_1) \cdot (\zeta \circ \eta^{-1}) - \text{grad}\, \pi_2 \cdot \zeta] \, dx.$$

Let ω and ω_η be solutions of the equations

$$\begin{cases} \Delta\omega = \operatorname{div} \zeta & \text{on } \Omega \\ \dfrac{\partial \omega}{\partial n} = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} \Delta\omega_\eta = \operatorname{div}(\zeta \circ \eta^{-1}) & \text{on } \Omega \\ \dfrac{\partial \omega_\eta}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

We can always assume that

$$\| \omega \|_{C^s} \leqslant C \| \zeta \|_{C^s},$$
$$\| \omega_\eta - \omega \|_{s,p} \leqslant C \| \zeta \circ \eta^{-1} - \zeta \|_{s-1,p} \leqslant C(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} \| \zeta \|_{C^s}$$

by Lemma A.3. Thus

$$\| \omega_\eta \circ \eta - \omega \|_{s-1,p} \leqslant \| \omega_\eta \circ \eta - \omega \circ \eta \|_{s-1,p} + \| \omega \circ \eta - \omega \|_{s-1,p}$$
$$\leqslant C_\eta (\| \omega_\eta - \omega \|_{s-1,p} + \| \omega \|_{C^s} \| \eta - e \|_{s,p})$$

by Lemma A.3 and A.4. Hence

$$\| \omega_\eta \circ \eta - \omega \|_{s-1,p} \leqslant C(\eta_1, \eta_2) \| \eta_1 - \eta_2 \|_{s,p} \| \zeta \|_{C^s}.$$

But

$$K(\zeta) = \int_\Omega [\pi_1 \cdot \Delta\omega_\eta - \pi_2 \cdot \Delta\omega] \, dx$$
$$= \int_\Omega (\Delta\pi_1 \cdot \omega_\eta - \Delta\pi_2 \cdot \omega) \, dx - \int_{\partial\Omega} (g_1 \circ \eta_1^{-1} \cdot \omega_\eta - g_2 \circ \eta_2^{-1} \cdot \omega) \, d\sigma$$
$$= \int_\Omega [(f_1 \circ \eta_1^{-1}) \cdot \omega_\eta - (f_2 \circ \eta_2^{-1}) \cdot \omega] \, dx$$
$$- \int_{\partial\Omega} [(g_1 \circ \eta_1^{-1}) \cdot \omega_\eta - (g_2 \circ \eta_2^{-1}) \cdot \omega] \, d\sigma.$$

The first term can be estimated by

$$\|f_1 - f_2\|_{L^\infty(\Omega)} \|\omega\|_{L^1(\Omega)} + \|f_1\|_{L^1(\Omega)} \|\omega_\eta \circ \eta - \omega\|_{L^\infty(\Omega)},$$

while the second term can be estimated by

$$\|g_1 - g_2\|_{L^\infty(\partial\Omega)} \|\omega_\eta\|_{L^1(\partial\Omega)} + \|g_2\|_{L^\infty(\partial\Omega)} \|\omega_\eta - \omega\|_{L^1(\partial\Omega)}$$

$$+ \|g_2\|_{Lip} \|\eta_1^{-1} - \eta_2^{-1}\|_{L^1(\partial\Omega)} \|\omega_\eta\|_{L^\infty(\partial\Omega)}.$$

So finally

$$K(\zeta) \leqslant C(\eta_1, \eta_2) \|\zeta\|_{C^s} [\|f_1 - f_2\|_{L^\infty(\Omega)} + \|\eta_1 - \eta_2\|_{s,p} \|f_1\|_{L^1(\Omega)}$$

$$+ \|g_1 - g_2\|_{L^\infty(\partial\Omega)} + \|\eta_1 - \eta_2\|_{s,p} (\|g_2\|_{L^\infty(\partial\Omega)} + \|g_2\|_{Lip})],$$

and

$$X_4 = \sup_\zeta \frac{K(\zeta)}{\|\zeta\|_{C^s}}. \quad \blacksquare$$

4. A IS "TANGENT" TO THE CLOSED SET F

Let u and γ be given so that $u \in W^{s,p}(\Omega; \mathbb{R}^N)$ with $\operatorname{div} u = 0$ on Ω and $u \cdot n = 0$ on $\partial\Omega$ and $\gamma \in W^{s,p}(\Omega; \mathbb{R}^N)$ satisfying

$$\operatorname{div}\left(\gamma - \sum_i u_i \frac{\partial u}{\partial x_i}\right) = 0 \quad \text{on } \Omega, \qquad \left(\gamma - \sum_i u_i \frac{\partial u}{\partial x_i}\right) \cdot n = 0 \quad \text{on } \partial\Omega.$$

In order to prove that A is tangent to F, we shall exhibit a curve $\eta \in C^2(I; \mathscr{D}_\mu^{s,p})$ ($I = [0, t_0]$, t_0 small enough) such that $\eta_0 = e$, $\dot\eta_0 = u$, $\ddot\eta_0 = \gamma$. This curve will be a "good approximation" in $\mathscr{D}_\mu^{s,p}$ of $e + tu + (t^2/2)\gamma$.

THEOREM 4. *Let $u \in \mathscr{D}_\mu^{s,p}$ and $\gamma \in W^{s,p}(\Omega; \mathbb{R}^N)$ with $s > (N/p) + 1$ such that*

$$\operatorname{div}\left(\gamma - \sum_i u_i \frac{\partial u}{\partial x_i}\right) = 0 \quad \textit{on } \Omega, \qquad \left(\gamma - \sum_i u_i \frac{\partial u}{\partial x_i}\right) \cdot n = 0 \quad \textit{on } \partial\Omega.$$

Then there exists a curve η_t *satisfying* $\eta \in C^2(I; \mathscr{D}_\mu^{s,p})$

$$\eta_0 = e, \tag{14}$$

$$\dot{\eta}_0 = u, \tag{15}$$

$$\ddot{\eta}_0 = \gamma. \tag{16}$$

Remark. Conversely, if η is a curve satisfying (14), then $u = \dot{\eta}_0 \in T_e\mathscr{D}_\mu^{s,p}$ and $\gamma = \ddot{\eta}_0 \in W^{s,p}(\Omega; \mathbb{R}^N)$ verify

$$\operatorname{div}\left(\gamma - \sum_i u_i \frac{\partial u}{\partial x_i}\right) = 0 \quad \text{on } \Omega \quad \text{and} \quad \left(\gamma - \sum_i u_i \frac{\partial u}{\partial x_i}\right) \cdot n = 0 \quad \text{on } \partial\Omega.$$

The proofs of Theorem 4 and its Remark are based on the following lemma.

LEMMA 6. *Let* $\mathcal{A}$ *and* $\mathcal{B}$ *be Banach spaces, and let* φ *be a* C^2 *mapping defined on a neighborhood of* 0 *in* $\mathcal{A}$ *with values into* $\mathcal{B}$, *such that* $\varphi(0) = 0$ *and* $D_0\varphi$ *is a split surjection (i.e.* $D_0\varphi$ *is onto* $\mathcal{B}$ *and* $\ker D_0\varphi$ *has a topological complement in* $\mathcal{A}$*).*

Given U, V *in* $\mathcal{A}$, *there exists a curve* $\zeta \in C^2(I; \mathcal{A})$ *such that*

$$\varphi(\zeta_t) = 0 \quad \textit{for } t \in I, \qquad \zeta_0 = 0, \tag{17}$$

$$\dot{\zeta}_0 = U, \tag{18}$$

$$\ddot{\zeta}_0 = V, \tag{19}$$

if and only if U *and* V *satisfy*

$$D_0\varphi \cdot U = 0, \tag{20}$$

$$D_0\varphi \cdot V + D_0{}^2\varphi(U, U) = 0. \tag{21}$$

Proof of Lemma 6. It is easy to check that $U = \dot{\zeta}_0$ and $V = \ddot{\zeta}_0$ satisfy necessarily (20) and (21) by differentiating (17). The converse relies on the implicit function theorem. Let $\mathscr{C} = \ker D_0\varphi$, and let P be a continuous projection from $\mathcal{A}$ onto $\mathscr{C}$. Define $\psi: \mathcal{A} \to \mathcal{B} \times \mathscr{C}$ by $\psi(u) = (\varphi(u), Pu)$, so that $D_0\psi = D_0\varphi \times P$ is an isomorphism from $\mathcal{A}$ onto $\mathcal{B} \times \mathscr{C}$. Therefore, by the implicit function theorem, ψ is a C^2 isomorphism from a neighborhood of 0 in $\mathcal{A}$ onto a neighborhood of 0 in $\mathcal{B} \times \mathscr{C}$. For t small enough, consider

$$\zeta_t = \psi^{-1}(0, tU + (t^2/2)\, PV).$$

Therefore, $\varphi(\zeta_t) = 0$ and $P\zeta_t = tU + (t^2/2)\,PV$. Consequently, $D_0\varphi \cdot \dot{\zeta}_0 = 0$ and $P\dot{\zeta}_0 = U$, which implies $\dot{\zeta}_0 = U$. Also,

$$D_0\varphi \cdot \ddot{\zeta}_0 + D_0^2\varphi(U, U) = 0$$

and $P\ddot{\zeta}_0 = PV$. Hence, $D_0\varphi(\ddot{\zeta}_0 - V) = 0$ and $P(\ddot{\zeta}_0 - V) = 0$, which implies $\ddot{\zeta}_0 = V$. ∎

Proof of Theorem 4. Let $\mathcal{A} = W^{s,p}(\Omega; \mathbb{R}^N)$ and let

$$\mathcal{B} = \left\{(f, g) \in W^{s-1,p}(\Omega) \times W^{s-1/p,p}(\partial\Omega); \int_\Omega f\,dx = \int_{\partial\Omega} g\,d\sigma\right\}.$$

We consider the mapping φ defined on $\mathcal{A}$ by $\varphi(u) = (\varphi_1(u), \varphi_2(u))$ where

$$\varphi_1(u) = |\operatorname{Jac}(e + u)| - \frac{1}{\operatorname{Vol}\Omega}\int_\Omega |\operatorname{Jac}(e + u)|\,dx - \frac{1}{\operatorname{Vol}\Omega}\int_{\partial\Omega} \delta \circ (e + u)\,d\sigma,$$

$$\varphi_2(u) = -\delta \circ (e + u)|_{\partial\Omega}$$

(recall that δ is smooth and $\partial\Omega = \{x; \delta(x) = 0\}$). Observe that φ takes its values in $\mathcal{B}$ and that φ is C^∞ since $|\operatorname{Jac}|$ is a polynomial in the first derivatives (we suppose $s > (N/p) + 1$; cf. Lemma A.1) and since δ is C^∞. For u small enough, $\varphi(u) = 0$ implies that $(e + u) \in \mathcal{D}_\mu^{s,p}$. Indeed, $\eta = (e + u)$ is a C^1 diffeomorphism and $\eta(\partial\Omega) \subset \partial\Omega$. Therefore, $\eta \in \mathcal{D}^{s,p}$ and since $|\operatorname{Jac}\eta| = C$ is constant on Ω, we have $\operatorname{Vol}\Omega = \operatorname{Vol}\eta(\Omega) = \int_\Omega |\operatorname{Jac}\eta|\,dx = C\operatorname{Vol}\Omega$; so that $C = 1$ and $\eta \in \mathcal{D}_\mu^{s,p}$. For $v \in \mathcal{A}$, we have the expansion

$$|\operatorname{Jac}(e + tv)| = 1 + t\operatorname{div} v + \frac{t^2}{2}\left(|\operatorname{div} v|^2 - \sum_{i,j=1}^{N} \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\right) + \cdots$$

since for any matrix $M = (m_{ij})$ we know that

$$|I + \epsilon M| = 1 + \epsilon\operatorname{tr} M + \frac{\epsilon^2}{2}\left(|\operatorname{tr} M|^2 - \sum_{i,j=1}^{N} m_{ij}m_{ji}\right) + \cdots$$

Hence,

$$D_0\varphi_1 \cdot v = \operatorname{div} v - \frac{1}{\operatorname{Vol}\Omega}\int_\Omega \operatorname{div} v\,dx + \frac{1}{\operatorname{Vol}\Omega}\int_{\partial\Omega} v \cdot n\,d\sigma = \operatorname{div} v;$$

and $D_0\varphi_2 \cdot v = v \cdot n$. Consequently, $D_0\varphi \cdot v = (\operatorname{div} v, v \cdot n)$ is a split surjection onto $\mathscr{B}$. Also

$$D_0{}^2\varphi_1(v, v) = \lim_{\epsilon\to 0} \frac{\varphi_1(\epsilon v) + \varphi_1(-\epsilon v)}{\epsilon^2} = |\operatorname{div} v|^2 - \sum_{i,j=1}^{N} \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}$$

$$- \frac{1}{\operatorname{Vol} \Omega}\int_\Omega \left(|\operatorname{div} v|^2 - \sum_{i,j=1}^{N} \frac{\partial v_i}{\partial x_j}\frac{\partial v_j}{\partial x_i}\right) dx$$

$$- \frac{1}{\operatorname{Vol} \Omega}\int_{\partial\Omega} \sum_{i,j=1}^{N} \frac{\partial^2 \delta}{\partial x_i\, \partial x_j} v_i v_j \, d\sigma,$$

and

$$D_0{}^2\varphi_2(v, v) = -\sum_{i,j=1}^{N} \frac{\partial^2 \delta}{\partial x_i\, \partial x_j} v_i v_j = -\beta(\,\cdot\,; v, v).$$

We apply now Lemma 6 with $U = u$ and $V = \gamma$. Conditions (20) and (21) are satisfied since

$$D_0\varphi \cdot u = (\operatorname{div} u, u \cdot n) = 0,$$

and by (12),

$$D_0\varphi_1 \cdot \gamma + D_0{}^2\varphi_1(u, u) = \operatorname{div} \gamma - \sum_{i,j=1}^{N} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i} + \frac{1}{\operatorname{Vol} \Omega}\int_\Omega \sum_{i,j=1}^{N} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\, dx$$

$$- \frac{1}{\operatorname{Vol} \Omega}\int_{\partial\Omega} \sum_{i,j=1}^{N} \frac{\partial^2 \delta}{\partial x_i\, \partial x_j} u_i u_j \, d\sigma = 0,$$

$$D_0\varphi_2 \cdot \gamma + D_0{}^2\varphi_2(u, u) = \gamma \cdot n - \sum_{i,j=1}^{N} \frac{\partial^2 \delta}{\partial x_i\, \partial x_j} u_i u_j = 0. \quad \blacksquare$$

THEOREM 5. *A is "tangent" to F in the following sense:*

$$\lim_{h\to 0} \frac{\operatorname{dist}((\eta, v) + hA(t; \eta, v), F)}{h} = 0 \qquad \textit{for all} \quad (\eta, v) \in F, \tag{22}$$

where $\operatorname{dist}(\cdot, F)$ *denotes the distance to the closed set F in the space* $X = W^{s,p}(\Omega; \mathbb{R}^N) \times W^{s,p}(\Omega; \mathbb{R}^N)$.

Proof of Theorem 5. We recall that

$$A(t; \eta, u) = (u, B(u \circ \eta^{-1}) \circ \eta + P(f_t) \circ \eta)$$

(where f_t is the given field of external forces),

$$F = \{(\eta, u) \in X;\ \eta \in \mathscr{D}_\mu^{s,p} \text{ and } u \circ \eta^{-1} \in T_e\mathscr{D}_\mu^{s,p}\}.$$

We start by proving (22) for the case $\eta = e$. We observe then that $u \in T_e\mathscr{D}_\mu^{s,p}$ and $\gamma = B(u) + P(f_t)$ meets the requirements of Theorem 4, i.e.,

$$\operatorname{div}\left(\gamma - \sum_i u_i \frac{\partial u}{\partial x_i}\right) = 0 \quad \text{on } \Omega \qquad \text{and} \qquad \left(\gamma - \sum_i u_i \frac{\partial u}{\partial x_i}\right) \cdot n = 0 \quad \text{on } \partial\Omega$$

since $\gamma - \sum_i u_i(\partial u/\partial x_i) = P(f - \sum_i u_i\,(\partial u/\partial x_i))$ by the definition of B. From Theorem 4 we know that there exists a curve $\eta \in C^2(I; \mathscr{D}_\mu^{s,p})$ with initial data (e, u, γ). Since $(\eta_h, \dot\eta_h) \in F$, we have

$$(1/h)\operatorname{dist}[(e, u) + hA(t; e, u), F] \leqslant (1/h)\operatorname{dist}[(e, u) + hA(t; e, u), (\eta_h, \dot\eta_h)].$$

By construction of η, the right-hand side tends to 0 as $h \to 0$, which proves Theorem 5 at $\eta = e$. For the general case, we just have to notice that

$$A(t; \eta, u) = A(t; e, u \circ \eta^{-1}) \circ \eta,$$

that $\eta(F) = F$ for $\eta \in \mathscr{D}_\mu^{s,p}$, and that the map $v \mapsto v \circ \eta$ is continuous (cf. Lemma A.4). Therefore, we can apply the result at e, completing the proof of Theorem 5. ∎

APPENDIX: Product and Composition of Functions in Sobolev Spaces

1. Product of Two Functions

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary.

Lemma A.1. *Let* $\alpha \geqslant 1$ *be an integer, and let* $1 \leqslant p \leqslant +\infty$, $1 \leqslant q \leqslant +\infty$.

If $u \in W^{\alpha,p}(\Omega)$ *and* $v \in W^{\alpha,q}(\Omega)$, *then* $u, v \in W^{\alpha,r}(\Omega)$, *where* r *is defined by*

$$1/r = (1/p) + (1/q) - \alpha/N \qquad \textit{when} \quad \max\{p, q\} < N/\alpha, \tag{1}$$

$$r \textit{ arbitrary} < \min\{p, q\} \qquad \textit{when} \quad \max\{p, q\} = N/\alpha \tag{2}$$

$$(r = 1 \textit{ if } p = q = N = \alpha = 1),$$

$$r = \min\{p, q\} \qquad \textit{when} \quad \max\{p, q\} > N/\alpha. \tag{3}$$

In addition, $\| u \cdot v \|_{W^{\alpha,r}} \leqslant C \| u \|_{W^{\alpha,p}} \| v \|_{W^{\alpha,q}}$, *where* C *depends only on* α, p, q, r, *and* Ω.

Proof. By induction on α, the proof is easy for $\alpha = 1$. In order to show that $u \cdot v \in W^{\alpha,r}(\Omega)$, we have to prove that $u \cdot v \in L^r(\Omega)$ (which is straightforward) and that $Du \cdot v + u \cdot Dv \in W^{\alpha-1,r}(\Omega)$. By symmetry , it is sufficient to check that $Du \cdot v \in W^{\alpha-1,r}(\Omega)$. But $Du \in W^{\alpha-1,p}(\Omega)$ and $v \in W^{\alpha,q}(\Omega) \subset W^{\alpha-1,q^*}(\Omega)$, where q^* is determined by

$$\frac{1}{q^*} = \begin{cases} \dfrac{1}{q} - \dfrac{1}{N} & \text{when } \; q < N, \\ \text{arbitrarily small with} & \\ \dfrac{1}{q^*} < \dfrac{1}{q} & \text{when } \; q = N, \\ 0 & \text{when } \; q > N. \end{cases}$$

We have now to distinguish three cases:

Case 1. $\max\{p, q\} < N/\alpha$ and thus $\max\{p, q^*\} < N/(\alpha - 1)$. By the induction assumption, we know that $Du \cdot v \in W^{\alpha-1,s}(\Omega)$ where $1/s = (1/p) + (1/q^*) - (\alpha - 1)/N = (1/p) + (1/q) - \alpha/N$.

Case 2. $\max\{p, q\} = N/\alpha$. *Either* $p \leqslant q = N/\alpha$, so that $q^* = N/(\alpha - 1)$. Thus, $\max\{p, q^*\} = N/(\alpha - 1)$ and by the induction assumption we know that $Du \cdot v \in W^{\alpha-1,s}(\Omega)$ for any $s < \min\{p, q^*\} = p = \min\{p, q\}$. *Or* $q < p = N/\alpha$, so that $\max\{p, q^*\} < N/(\alpha - 1)$ and by the induction assumption $Du \cdot v \in W^{\alpha-1,s}(\Omega)$ with

$$1/s = (1/p) + (1/q^*) - (\alpha - 1)/N = (1/p) + (1/q) - \alpha/N = 1/q.$$

Hence $Du \cdot v \in W^{\alpha-1,s}(\Omega)$ with $s = \min\{p, q\}$.

Case 3. $\max\{p, q\} > N/\alpha$. *Either* $q > N/\alpha$ so that

$$\max\{p, q^*\} > N/(\alpha - 1)$$

and by the induction assumption $Du \cdot v \in W^{\alpha-1,s}(\Omega)$ with $s = \min\{p, q^*\} \geqslant \min\{p, q\}$. *Or* $p > N/\alpha$ and $q \leqslant N/\alpha$; by the induction assumption $Du \cdot v \in W^{\alpha-1,s}(\Omega)$, for s as follows: when

$$\max\{p, q^*\} < N/(\alpha - 1)$$

we have $1/s = (1/p) + (1/q^*) - (\alpha - 1)/N$ and $1/s < 1/q$. Therefore, $Du \cdot v \in W^{\alpha-1,s}(\Omega)$ with $s = \min\{p, q\}$. When

$$\max\{p, q^*\} \geqslant N/(\alpha - 1),$$

we have $Du \cdot v \in W^{\alpha-1,s}(\Omega)$ for any $s < \min\{p, q^*\}$ and in particular we can choose $s = \min\{p, q\}$. ∎

2. Composition of Two Mappings

Let $\Omega' \subset \mathbb{R}^M$ be a bounded domain with smooth boundary.

LEMMA A.2. *Let $\alpha \geqslant 1$ be an integer, and let $1 \leqslant p \leqslant +\infty$ with $\alpha > N/p$. Let $F \in C^\alpha(\bar{\Omega}')$, and let $G \in W^{\alpha,p}(\Omega; \mathbb{R}^M)$ such that $G(\Omega) \subset \bar{\Omega}'$. Then $F \circ G \in W^{\alpha,p}(\Omega)$ and*

$$\| F \circ G \|_{W^{\alpha,p}} \leqslant C \| F \|_{C^\alpha} (\| G \|^\alpha_{W^{\alpha,p}} + 1),$$

where C depends only on α, p, Ω, and Ω'.

Proof. By induction on α, the proof is easy for $\alpha = 1$. In order to show that $F \circ G \in W^{\alpha,p}(\Omega)$, we have to check that $F \circ G \in L^p(\Omega)$ (which is obvious) and that $(DF \circ G) \cdot DG \in W^{\alpha-1,p}(\Omega)$.

Since $\alpha - 1 > N/p^*$, we know by the induction assumption that $DF \circ G \in W^{\alpha-1,p^*}(\Omega)$ with

$$\| DF \circ G \|_{W^{\alpha-1,p^*}} \leqslant C \| F \|_{C^\alpha} (\| G \|^{\alpha-1}_{W^{\alpha-1,p^*}} + 1).$$

But $DG \in W^{\alpha-1,p}(\Omega)$ and from Lemma A.1 (Case 3) we get $(DF \circ G) \cdot DG \in W^{\alpha-1,p}(\Omega)$ with the corresponding estimate. ∎

Remark. A slightly sharper version of Lemma A.2 can be found in [7].

LEMMA A.3. *Let $\alpha \geqslant 1$ be an integer and let $1 \leqslant p \leqslant +\infty$ with $\alpha > N/p$. Let $F \in C^{\alpha+1}(\bar{\Omega}')$, and let $G \in W^{\alpha,p}(\Omega; \mathbb{R}^M)$ and*

$$H \in W^{\alpha,p}(\Omega; \mathbb{R}^M)$$

such that $G(\Omega) \subset \bar{\Omega}'$, $H(\Omega) \subset \bar{\Omega}'$. Then

$$\begin{aligned} &\| F \circ G - F \circ H \|_{W^{\alpha,p}} \\ &\qquad \leqslant C \| F \|_{C^{\alpha+1}} \| G - H \|_{W^{\alpha,p}} (\| G \|^\alpha_{W^{\alpha,p}} + \| H \|^\alpha_{W^{\alpha,p}} + 1), \end{aligned}$$

where C depends only on α, p, Ω and Ω'.

Proof. By induction on α, the proof is easy for $\alpha = 1$. In order to show that (4) holds, we have to check that

$$\|F \circ G - F \circ H\|_{L^p} \leqslant C \|F\|_{C^1} \|G - H\|_{W^{1,p}}$$

(which is obvious) and that

$$\|(DF \circ G) \cdot DG - (DF \circ H) \cdot DH\|_{W^{\alpha-1,}}$$

can be bounded by the right-hand side in (4). But

$$\begin{aligned}(DF \circ G) \cdot DG &- (DF \circ H) \cdot DH \\ &= (DF \circ G - DF \circ H) \cdot DG + (DF \circ H) \cdot (DG - DH).\end{aligned}$$

The first term in the right-hand side is bounded in $W^{\alpha-1,p}(\Omega)$ by

$$C \|F\|_{C^{\alpha+1}} \|G - H\|_{W^{\alpha-1,p^*}} (\|G\|^{\alpha-1}_{W^{\alpha-1,p}} + \|H\|^{\alpha-1}_{W^{\alpha-1,p^*}} + 1) \|G\|_{W^{\alpha,p}}$$

(using the induction assumption and Lemma A.1 with $q = p^*$), while the second term in the right-hand side is bounded in $W^{\alpha-1,p}$ by

$$C \|G - H\|_{W^{\alpha,p}} \|F\|_{C^\alpha} (\|H\|^{\alpha-1}_{W^{\alpha-1,p^*}} + 1)$$

(using Lemmas A.1 and A.2). ■

The following result differs essentially from Lemma A.2 by the fact that we assume only that $F \in W^{\alpha,p}$ (instead of C^α), but G is here a diffeomorphism.

LEMMA A.4. *Let $\alpha \geqslant 2$ be an integer, and let $1 \leqslant p \leqslant q \leqslant +\infty$ such that $\alpha > (N/q) + 1$. Let $F \in W^{\alpha,p}(\Omega)$, and let $G \in \mathscr{D}^{\alpha,q}(\Omega)$* (i.e. *$G \in W^{\alpha,q}(\Omega; \mathbb{R}^N)$ and G is a C^1 diffeomorphism from $\bar{\Omega}$ onto $\bar{\Omega}$). Then $F \circ G \in W^{\alpha,p}(\Omega)$ and*

$$\|F \circ G\|_{W^{\alpha,p}} \leqslant C \|F\|_{W^{\alpha,p}} \frac{1}{\inf |\operatorname{Jac} G|^{1/p}} (\|G\|^{\alpha}_{W^{\alpha,q}} + 1),$$

where C depends only on α, p, q and Ω.

Proof. By induction on α, we consider first the case where $\alpha = 2$. It is clear that $F \circ G \in L^p(\Omega)$ and

$$\|F \circ G\|_{L^p} \leqslant \frac{1}{\inf |\operatorname{Jac} G|^{1/p}} \|F\|_{L^p}.$$

Also, $D(F \circ G) = (DF \circ G) \cdot DG$ belongs to $W^{1,p}(\Omega)$ by Lemma A.1 since $DG \in W^{1,q}(\Omega)$ $(q > N)$ and $DF \circ G \in W^{1,p}(\Omega)$ with

$$\| DF \circ G \|_{W^{1,p}} \leqslant \frac{1}{\inf | \operatorname{Jac} G |^{1/p}} (\| DF \|_{L^p} + \| D^2F \|_{L^p} \| DG \|_{L^\infty}).$$

In the general case, we have to check that $F \circ G \in L^p(\Omega)$ and that $(DF \circ G) \cdot DG \in W^{\alpha-1,p}(\Omega)$. By the induction assumption, we know that $DF \circ G \in W^{\alpha-1,p}(\Omega)$ (since $\alpha - 1 > (N/q^*) + 1$) and

$$\| DF \circ G \|_{W^{\alpha-1,p}} \leqslant C \| F \|_{W^{\alpha-1,p}} \frac{1}{\inf | \operatorname{Jac} G |^{1/p}} (\| G \|^{\alpha-1}_{W^{\alpha-1,q^*}} + 1).$$

From Lemma A.1, we conclude that $(DF \circ G) \cdot DG$ belongs to $W^{\alpha-1,p}(\Omega)$ with the corresponding estimate. ∎

LEMMA A.5. *Let $\alpha \geqslant 2$ be an integer, and let $1 \leqslant p \leqslant q \leqslant +\infty$ be such that $p < +\infty$ and $\alpha > (N/q) + 1$. Let $F \in W^{\alpha,p}(\Omega)$; then the mapping $G \mapsto F \circ G$ is continuous from $\mathscr{D}^{\alpha,q}(\Omega)$ into $W^{\alpha,p}(\Omega)$.*

Proof. Given $\delta > 0$, there exists $\tilde{F} \in C^{\alpha+1}(\bar{\Omega})$ such that

$$\| F - \tilde{F} \|_{W^{\alpha,p}} < \delta.$$

We have

$$F \circ G - F \circ H = (F \circ G - \tilde{F} \circ G) + (\tilde{F} \circ G - \tilde{F} \circ H) + (\tilde{F} \circ H - F \circ H).$$

The first and third terms in the right-hand side can be bounded in $W^{\alpha,p}(\Omega)$ (using Lemma A.4) by

$$C\delta \frac{1}{\inf | \operatorname{Jac} G |^{1/p}} (\| G \|^{\alpha}_{W^{\alpha,q}} + 1) + C\delta \frac{1}{\inf | \operatorname{Jac} H |^{1/p}} (\| H \|^{\alpha}_{W^{\alpha,q}} + 1),$$

while the second term can be bounded in $W^{\alpha,q}(\Omega)$ (and *a fortiori* in $W^{\alpha,p}(\Omega)$), using Lemma A.3, by

$$C \| \tilde{F} \|_{C^{\alpha+1}} \| G - H \|_{W^{\alpha,q}} (\| G \|^{\alpha}_{W^{\alpha,q}} + \| H \|^{\alpha}_{W^{\alpha,q}} + 1). \quad ∎$$

Remark. More generally, one can show, under the assumptions of Lemma A.5, that if $F \in W^{\alpha+\beta,p}(\Omega)$, then the mapping $G \mapsto F \circ G$ is of class C^β from $\mathscr{D}^{\alpha,q}(\Omega)$ into $W^{\alpha,p}(\Omega)$[$\mathscr{D}^{\alpha,q}(\Omega)$ is provided with an appropriate manifold structure].

3. Integration of Vector Fields

Let $F(x, t)\colon \Omega \times [0, T] \to \mathbb{R}^N$ be a vector field tangent to $\partial\Omega$ on $\partial\Omega$ (i.e. $F(x, t) \cdot n(x) = 0$ for $x \in \partial\Omega$ and $t \in [0, T]$).

Lemma A.6. *Assume* $F \in C([0, T]; W^{\alpha,p}(\Omega; \mathbb{R}^N))$ *with*

$$\alpha > (N/p) + 1 \qquad \text{and} \qquad 1 \leqslant p < +\infty.$$

Then the differential equation

$$(du/dt)(x, t) = F(u(x, t), t)$$
$$u(x, 0) = x$$

has a solution $u \in C^1([0, T]; \mathscr{D}^{\alpha,p}(\Omega))$.

Remark. Lemma A.6 is not used in our paper, but it answers a question raised by Ebin and Marsden [2] who proved the same result for the case where $p = 2$ and $\alpha > (N/2) + 2$.

Proof. When $\alpha = 2$ (so that $p > N$), we have

$$F \in C([0, T]; C^{1,\lambda}(\bar{\Omega}; \mathbb{R}^N)),$$

where $\lambda = 1 - N/p$. In this case, it is well-known that there exists a solution $u \in C^1([0, T]; C^{1,\lambda}(\bar{\Omega}; \mathbb{R}^N))$ and in addition $(d/dt)\, Du = DF(u, t) \cdot Du$. On the other hand, $x \mapsto u(x, t)$ is a diffeomorphism for all $t \in [0, T]$ since

$$(d/dt) \mid \operatorname{Jac} u(x, t)|_{t=\tau} = \operatorname{div} F(u(x, \tau), \tau) \mid \operatorname{Jac} u(x, \tau)| \geqslant -C \mid \operatorname{Jac} u(x, \tau)|$$

and thus $| \operatorname{Jac} u(x, t)| \geqslant e^{-Ct}$. Hence, $DF(u(x, t), t) \in W^{1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ for all $t \in [0, T]$; more precisely, the mapping $t \mapsto DF(u(x, t), t)$ is continuous from $[0, T]$ into $W^{1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ (as in the proof of Lemma A.5). For a fixed $u \in C^1(\bar{\Omega}, \bar{\Omega})$, the operator $v \mapsto DF(u, t) \cdot v$ is bounded from $W^{1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ into itself (by Lemma A.1). Therefore, the linear differential equation $dv/dt = DF(u, t) \cdot v$ (considered in the Banach space $W^{1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$) has a solution

$$v \in C^1([0, T]; W^{1,p}(\Omega, \mathbb{R}^N \times \mathbb{R}^N)).$$

Consequently, $Du \in C^1([0, T]; W^{1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N))$ and

$$u \in C^1([0, T]; W^{2,p}(\Omega; \mathbb{R}^N)).$$

In the general case, the proof is by induction on α. Since

$$F \in C([0, T]; W^{\alpha-1,p^*}(\Omega; \mathbb{R}^N)),$$

we know from the induction assumption that $u \in C^1([0, T]; \mathscr{D}^{\alpha-1,q}(\Omega))$, where $q = p^*$ for $p \leqslant N$ and q is any finite number for $p > N$.

Lemma A.4 shows that $DF(u, t) \in W^{\alpha-1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$ for all $t \in [0, T]$; more precisely, it follows from Lemma A.5 that the mapping $t \mapsto DF(u(x, t), t)$ is continuous from $[0, T]$ into $W^{\alpha-1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N)$. Therefore, the linear differential equation

$$dv/dt = DF(u, t) \cdot v$$

has a solution $v \in C^1([0, T]; W^{\alpha-1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N))$. Consequently, $Du \in C^1([0, T]; W^{\alpha-1,p}(\Omega; \mathbb{R}^N \times \mathbb{R}^N))$ and $u \in C^1([0, T]; W^{\alpha,p}(\Omega; \mathbb{R}^N))$. ∎

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