Transversal Greedoids

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In a natural way, a new class of set system is defined in terms of transversals of a family of sets. Examination of the structure of these systems reveals that they are in fact examples of strong greedoids (see [2]), namely greedoids which are greedy algorithm compatible, and we call them transversal greedoids. In this paper, such set systems are shown to be characterized by a simple exchange property.

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1. Basic Definitions and Results

A set system is a pair \((E, \mathcal{A})\), where \(E\) is a finite non-empty set and \(\mathcal{A}\) is a collection of subsets of \(E\). A transversal (or set of distinct representatives) of a family \(\mathcal{A} = (A_1, \ldots, A_n)\) of subsets of \(E\) is a set \(X\) with \(|X| = n\) the elements of which can be labelled \(\{x_1, \ldots, x_n\}\) with \(x_i \in A_i\) for \(i = 1, \ldots, n\). A partial transversal of \(\mathcal{A}\) is a transversal of some subfamily of \(\mathcal{A}\). A matroid is a set system \((E, \mathcal{A})\) which satisfies the following axioms:

(M1) \(\emptyset \in \mathcal{A}\);

(M2) if \(X \subseteq Y \in \mathcal{A}\), then \(X \in \mathcal{A}\);

(M3) if \(X, Y \in \mathcal{A}\) with \(|X| = |Y| + 1\), then there exists \(x \in X \setminus Y\) with \(Y \cup \{x\} \in \mathcal{A}\).

It is well known (see [3], [5] or [7]) that the collection of partial transversals of a family \(\mathcal{A}\) forms a matroid, known as a transversal matroid. We now use a related idea to define a new class of set system. Let \(E\) be any set and let \(\mathcal{A} = (A_1, \ldots, A_n)\) be a family of subsets of \(E\). Let \(\mathcal{A}(\mathcal{A})\) be the collection of all transversals of subfamilies of \(\mathcal{A}\) of the form \((A_1, \ldots, A_r)\) for some \(r \leq n\) (including the empty family). We call \(\mathcal{A}(\mathcal{A})\) the ordered transversal structure of \(\mathcal{A}\). It is readily seen from the trivial example of a family of disjoint sets that the ordered transversal structure of a family \(\mathcal{A}\) is not, in general, a matroid.

Korte and Lovasz [4] define a greedoid to be a set system \((E, \mathcal{A})\) satisfying the matroid axioms (M1) and (M3) and the following weaker version of (M2):

(G2) if \(x \in \mathcal{A}\) and \(X \neq \emptyset\), then there exists \(x \in X\) such that \(X \setminus \{x\} \in \mathcal{A}\).

In [2], strong greedoids are defined as greedoids which satisfy the following stronger version of (M3), known as the strong exchange property:

(M3)' if \(X, Y \in \mathcal{A}\) with \(|X| = |Y| + 1\), then there exists \(x \in X \setminus Y\) with \(Y \cup \{x\} \in \mathcal{A}\) and \(X \setminus \{x\} \in \mathcal{A}\).

The next theorem shows that the ordered transversal structures are examples of strong greedoids which, as we showed in [2], are precisely the structures which are fully compatible with the greedy algorithm. Indeed, it was the search for a large class of compatible set systems which led us to the ordered transversal structures.

Theorem 1.1. Let \(E\) be any set and let \(\mathcal{A} = (A_1, \ldots, A_n)\) be a family of subsets of \(E\). Then \((E, \mathcal{A}(\mathcal{A}))\) is a strong greedoid.

Proof. By induction, \(\emptyset \in \mathcal{A}(\mathcal{A})\) and so (M1) is satisfied.

Next, let \(X \in \mathcal{A}(\mathcal{A})\), with \(X \neq \emptyset\). Then we can write \(X = \{x_1, \ldots, x_r\}\) for some \(r \leq n\).
with \( 1 \leq r \leq n \), where \( x_i \in A_i \) for each \( 1 \leq i \leq r \). But then \( \{x_1, \ldots, x_r\} \) is a transversal of \( (A_1, \ldots, A_{r-1}) \), so \( X \{x_i\} \in \mathcal{F}(\mathcal{A}) \) and (G2) is satisfied.

It only remains to show that the strong exchange property \((\text{M3})\) is satisfied. To this end, let \( X, Y \in \mathcal{F}(\mathcal{A}) \) with \( |X| = |Y| + 1 = r \), say, and for each \( j, 1 \leq j \leq n \), let \( I_j = \{1, \ldots, j\} \). We now follow the same lines as a proof in [3] for the result on transversal matroids, and note that \( X \in \mathcal{F}(\mathcal{A}) \) is equivalent to the existence of a matching between \( X \) and \( I_r \) in the usual bipartite graph \( G = (E, \Delta, I_r) \) associated with the family \( \mathcal{A} \). So there exist matchings \( \Delta_X, \Delta_Y \) \((\subseteq \Delta)\) between \( X \) and \( I_r \) and \( Y \) and \( I_{r-1} \) respectively. We shall first see that \( x_i \in A_i \) for \( 1 \leq i \leq r \). Then the family \( X \) is a transversal of \( (A_1, \ldots, A_r) \), and so \( X \cup \{x_i\} \in \mathcal{F}(\mathcal{A}) \). Hence \( X \cup \{x_i\} \) is a transversal of \( (A_1, \ldots, A_r) \) and \( X \cup \{x_i\} \in \mathcal{F}(\mathcal{A}) \). Now we can also write the alternating path in the following way:

\[
\begin{align*}
&x_i = \Delta_X(i - 1) \quad i_0 = \Delta_X(x_i) \quad x_{i-1} = \Delta_X(i_{i-1}) \\
&(\in I_r) \quad (\in X \cap Y) \quad (\in I_{r-1}) \quad (\in X \cap Y) \\
&\vdots \\
&x_i = \Delta_X(i - 1) \quad i_0 = \Delta_X(x_i) \quad x_{i-1} = \Delta_X(i_{i-1}) \\
&(\in X \cap Y) \quad (\in I_{r-1}) \quad (\in X \cup Y).
\end{align*}
\]

But then \( x_i \in A_i \) for \( 1 \leq i \leq r \). Then the family \( \mathcal{A} = (A_1, \ldots, A_n) \) is called the family associated with \((E, \mathcal{A})\).

Since all ordered transversal structures are greedoids we are now able to call them \textit{transversal greedoids}. As in the case of transversal matroids [1, 7], we call a family \( \mathcal{A} \) a \textit{presentation} of the transversal greedoid \((E, \mathcal{A})\) if \( \mathcal{E} = \mathcal{F}(\mathcal{A}) \). It is easy to find examples of different presentations of a transversal greedoid, and so, in general, reconstruction of the original family is not possible. However, given a transversal greedoid \((E, \mathcal{A})\), it is possible to construct the unique maximal presentation, giving a result comparable to that for transversal matroids in [1]. Given any set system \((E, \mathcal{A})\) of rank \( n \), let \( A_1 = \{x \in E : \{x\} \in \mathcal{E}\} \) and, for \( 2 \leq r \leq n \), let

\[
A_r = \{x \in E : \{y_1, \ldots, y_{r-1}, x\} \in \mathcal{E} \quad \text{for each distinct} \quad y_1, \ldots, y_{r-1} \in A_{r-1} \text{ with } x \notin \{y_1, \ldots, y_{r-1}\}\}.
\]

Then the family \( \mathcal{A} = (A_1, \ldots, A_n) \) is called the family associated with \((E, \mathcal{E})\).
THEOREM 2.2. Let \((E, \mathfrak{E})\) be a transversal greedoid and let \(\mathcal{A}\) be the family associated with \((E, \mathfrak{E})\). Then \(\mathcal{A}\) is a presentation of \(\mathfrak{E}\). In addition, if \(\mathcal{A} = (A_1, \ldots, A_n)\), and \(\mathcal{B} = (B_1, \ldots, B_n)\) is another presentation of \(\mathfrak{E}\), then \(B_i \subseteq A_i\) for each \(i\). Hence the family associated with \((E, \mathfrak{E})\) is the unique maximal presentation of \((E, \mathfrak{E})\).

PROOF. Let \(\mathfrak{E} = \mathfrak{E}(\mathcal{B})\), where \(\mathcal{B} = (B_1, \ldots, B_n)\), and let \(\mathcal{A} = (A_1, \ldots, A_n)\) be the family associated with \((E, \mathfrak{E})\). We aim to show that \(\mathfrak{E} = \mathfrak{E}(\mathcal{A})\) and that \(B_i \subseteq A_i\) for each \(i\).

As commented above, it is clear that \(\mathfrak{E}(\mathcal{A}) \subseteq \mathfrak{E}\). To show the opposite inclusion, we first show that \(B_i \subseteq A_i\) for each \(i\). Let \(y_1 \in A_1, \ldots, y_{i-1} \in A_{i-1}\) and \(x \in B_i\), with \(y_1, \ldots, y_{i-1}, x\) distinct. To show that \(x \in A_i\), we must show that \(\{y_1, \ldots, y_{i-1}, x\} \in \mathfrak{E}\).

Clearly, \(\{y_1, \ldots, y_{i-1}\} \in \mathfrak{E}(\mathcal{A}) \subseteq \mathfrak{E} = \mathfrak{E}(\mathcal{B})\) and so \(\{y_1, \ldots, y_{i-1}\}\) is a transversal of \((B_1, \ldots, B_{i-1})\). Hence \(\{y_1, \ldots, y_{i-1}, x\}\) is clearly a transversal of \((B_1, \ldots, B_i)\) and is in \(\mathfrak{E}\). Therefore \(x \in A_i\) and \(B_i \subseteq A_i\), as claimed.

It also follows that \(\mathfrak{E} \subseteq \mathfrak{E}(\mathcal{A})\) and hence that \(\mathfrak{E} = \mathfrak{E}(\mathcal{A})\), as required.

COROLLARY. A set system \((E, \mathfrak{E})\) is a transversal greedoid if \(\mathfrak{E} = \mathfrak{E}(\mathcal{A})\), where \(\mathcal{A}\) is the family associated with \((E, \mathfrak{E})\).

2. CHARACTERIZATION OF THE TRANSVERSAL GREEDOIDS

In the last corollary we saw one characterization of the transversal greedoids; namely, that a set system is a transversal greedoid iff it is the transversal greedoid of its associated family. In the remainder of this paper we look for more tangible characterizations.

Transversal matroids have the property of base orderability [7]; namely, that if \(B_1\) and \(B_2\) are bases of the matroid, then there exists a bijection \(\pi : B_1 \to B_2\) such that, for each \(x \in B_1\), \((B_1 \setminus \{x\}) \cup \{\pi(x)\}\) and \((B_2 \setminus \{\pi(x)\}) \cup \{x\}\) are also bases of the matroid. It is easy to see that transversal greedoids have the same property (and, indeed, that they have a corresponding property for any pair of equal sized independent sets). However, it is well known that this property does not characterize those matroids which are transversal, and so it is not surprising that neither does it characterize those strong greedoids which are transversal.

Mason [6] characterizes transversal matroids in terms of their rank functions using the notion of a fully dependent set (one which is a finite union of circuits of the matroid). In the case of transversal greedoids, however, the union of circuits may be independent, as we see from the next example.

EXAMPLE. Let \(E = \{a, b, c, d, e\}\) and let \(\mathcal{A} = (A_1, A_2, A_3, A_4, A_5)\) be the family of subsets of \(E\) given by \(A_1 = \{a, b, c, e\}\), \(A_2 = \{d, e\}\), \(A_3 = \{a\}\), \(A_4 = \{b\}\) and \(A_5 = \{c\}\). If \(\mathfrak{E} = \mathfrak{E}(\mathcal{A})\) then it is clear that both \(\{a, b\}\) and \(\{c, d, e\}\) are circuits of \((E, \mathfrak{E})\); yet \(\{a, b\} \cup \{c, d, e\} = \{a, b, c, d, e\} \in \mathfrak{E}\).

There is, however, a neat characterization of the transversal greedoids in terms of a further exchange property, as we now see.

THEOREM 2.1. For a set system \((E, \mathfrak{E})\) the following two properties are equivalent:
1. \((E, \mathfrak{E})\) is a transversal greedoid.
2. \((E, \mathfrak{E})\) is a strong greedoid with the following additional property:
   (M3) \(^1\) if \(X \in \mathfrak{E}\) and \(Z \neq \emptyset\), then there exists \(x \in X\) with \(X \setminus \{x\} \in \mathfrak{E}\) and such that \(Y \cup \{x\} \in \mathfrak{E}\) for all \(Y \in \mathfrak{E}\) with \(|Y| = |X| - 1\).
Prooef. (1) $\Rightarrow$ (2). Let $(E, \mathcal{F})$ be a transversal greedoid and let $\mathcal{A} = (A_1, \ldots, A_n)$ be its associated family. Then, by Theorem 1.1, $(E, \mathcal{F})$ is a strong greedoid and it remains only to deduce the additional property (M3)' of $(E, \mathcal{F})$. So let $X \in \mathcal{F}$ with $X = \{x_1, \ldots, x_j\}$ and $x_i \in A_i$ for each $i$. Then the $x_i$ required by (M3)' can clearly be taken as $x_i$. For, first, $X \setminus \{x_i\} = \{x_1, \ldots, x_{j-1}\}$ is a transversal of $(A_1, \ldots, A_{j-1})$ and is hence in $\mathcal{F}$. Second, if $Y = \{y_1, \ldots, y_{r-1}\} \in \mathcal{F}$ with $y_i \in A_i$ for each $i$, let $1 \leq i \leq r$, then either $x_i \in Y$ and $Y \setminus \{x_i\} = Y \setminus \mathcal{F}$ or $x_i \not\in Y$ and $Y \cup \{x_i\} = Y \cup \mathcal{F}$. Hence (M3)' is established and (2) follows. 

(2) $\Rightarrow$ (1). Conversely, suppose that $(E, \mathcal{F})$ is a strong greedoid with the additional property (M3)' and let $\mathcal{A} = (A_1, \ldots, A_n)$ be the family associated with $(E, \mathcal{F})$. We shall now show that $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{F}$ as commented earlier, $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{F}$ and we shall assume that $\mathcal{F}(\mathcal{A}) \subseteq \mathcal{F}$ and let $X$ be minimal in $\mathcal{F}(\mathcal{A})$: we shall now deduce a contradiction.

Clearly, $|X| > 1$, since all singleton sets in $\mathcal{F}$ are also in $\mathcal{F}(\mathcal{A})$. Using (M3)' we can find an $x \in X$ with $X \setminus \{x\} \in \mathcal{F}$ such that if $Y \in \mathcal{F}$ with $|Y| = |X| - 1$ then $Y \cup \{x\} \in \mathcal{F}$. By the minimality of $X$, $X \setminus \{x\} \in \mathcal{F}(\mathcal{A})$, and so we can write $X \setminus \{x\}$ as $\{x_1, \ldots, x_{j-1}\}$, where $x_i \in A_i$ for $1 \leq i \leq r$. Now, to show that $x \in A_i$, let $y_1 \in A_1, \ldots, y_{r-1} \in A_{r-1}$ be such that $y_1, \ldots, y_{r-1}$ are distinct, so that $\{y_1, \ldots, y_{r-1}\} \in \mathcal{F}(\mathcal{A})$. But then, by (M3)', $\{y_1, \ldots, y_{r-1}, x\} \in \mathcal{F}(\mathcal{A})$ and so, by the construction of the associated family $\mathcal{A}$, $x \in A_i$. But then $X = \{x_1, \ldots, x_{r-1}, x\} \in \mathcal{F}(\mathcal{A})$, which is a required contradiction, and (1) follows. This completes the proof.

It is worth noting that the properties (M3)' and (M3)" cannot be combined to form the single property:

(*) if $X, Y \in \mathcal{F}$ with $|X| = |Y| + 1$, then there exists $x \in X \setminus Y$ with $X \setminus \{x\} \in \mathcal{F}$, $Y \cup \{x\} \in \mathcal{F}$ and such that $Y \cup \{x\} \in \mathcal{F}$ whenever $|Y| = |X| - 1$. This is because the $x$ in (M3)' and the $x$ in (M3)" cannot necessarily be chosen simultaneously.

Our last theorem characterizes those transversal greedoids which are also matroids.

**Theorem 2.2.** A transversal greedoid $(E, \mathcal{F})$, with associated family $\mathcal{A} = (A_1, \ldots, A_n)$ is a matroid iff $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$.

**Proof.** Suppose first that the transversal greedoid $(E, \mathcal{F})$ is a matroid and let $\mathcal{A} = (A_1, \ldots, A_n)$ be the family associated with $(E, \mathcal{F})$. Suppose that for some $i$ ($1 \leq i \leq n - 1$), $A_i \supseteq A_{i+1}$ and that $i$ is the lowest such. Let $x \in A_{i+1} \setminus A_i$ so that, in particular, $x \not\in A_i$ and there exists $y_1 \in A_1, \ldots, y_{r-1} \in A_{r-1}$ with $y_1, \ldots, y_{r-1}$ distinct and $\{y_1, \ldots, y_{r-1}, x\} \not\in \mathcal{F}$. Now $\{y_1, \ldots, y_{r-1}\} \in \mathcal{F}$ and so it can be extended to a set $\{y_1, \ldots, y_{r-1}, y\} \in \mathcal{F}$. It follows that $\{y_1, \ldots, y_{r-1}, y\}$ is a transversal of $(A_1, \ldots, A_n)$ and that $y_1, \ldots, y_{r-1}$, $y$ and $x$ are distinct. Since $x \in A_{i+1}$, it follows from the construction of $\mathcal{A}$ that $\{y_1, \ldots, y_{r-1}, x\} \not\in \mathcal{F}$ which, together with the fact that $\{y_1, \ldots, y_{r-1}, x\} \not\in \mathcal{F}$, contradicts matroid axiom (M2). Hence $A_1 \supseteq A_{i+1}$ for $1 \leq i \leq n - 1$.

Conversely, let $(E, \mathcal{F})$ be a transversal greedoid the associated family $\mathcal{A} = (A_1, \ldots, A_n)$ of which has the property that $A_1 \supseteq A_2 \supseteq \cdots \supseteq A_n$. Now, by Theorem 1.1, $(E, \mathcal{F})$ satisfies matroid axioms (M1) and (M3). It therefore remains to establish that $(E, \mathcal{F})$ satisfies (M2).

Let $X \in \mathcal{F}$ be non-empty, so that $X$ can be written $X = \{x_1, \ldots, x_j\}$, where $x_i \in A_i$ for $1 \leq i \leq r$. Let $X' \subseteq X$ be non-empty. Then $X' = \{x_{i_1}, \ldots, x_{i_k}\}$, with $1 \leq i_1 < i_2 < \cdots < i_k \leq r$ for some $q < r$. It follows that $i_j > i_j$ for each $j$ and so $x_{i_1} \in A_{i_1} \subseteq A_{i_2}, \ldots, x_{i_k} \in A_{i_k} \subseteq A_{i_{k+1}}$. Hence $X' = \{x_{i_1}, \ldots, x_{i_k}\}$ is a transversal of $(A_1, \ldots, A_n)$ and so $X' \in \mathcal{F}$.

Thus (M2) holds and $(E, \mathcal{F})$ is a matroid.
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