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ScienceDirect
J. Math. Anal. Appl. 338 (2008) 223–243
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

Generalized elliptic integrals and the Legendre $\mathcal{M}$-function

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Received 15 January 2007
Available online 18 May 2007
Submitted by B.C. Berndt

Abstract

We study monotonicity and convexity properties of functions arising in the theory of elliptic integrals, and in particular in the case of a Schwarz–Christoffel conformal mapping from a half-plane to a trapezoid. We obtain sharp monotonicity and convexity results for combinations of these functions, as well as functional inequalities and a linearization property.

Keywords: Hypergeometric function; Elliptic integrals; Modular function; Legendre $\mathcal{M}$-function

1. Introduction

In this paper we continue the study of the modular function $\varphi_K^{a,b,c}$ and the generalized modulus $\mu_{a,b,c}$ started in [12], as well as the generalized elliptic integrals $\mathcal{K}_{a,b,c}$ and $\mathcal{E}_{a,b,c}$ (for the notation, see (1.3), (1.5), (2.1) and (2.2) below). In general, the more freedom the parameter values a, b and c are allowed, the more complex and hard-to-handle these functions will be. As in [12] we are here particularly interested in the case $b = c - a$. Geometrically this case corresponds to the Schwarz–Christoffel problem from the unit disk onto a trapezoid, i.e. a quadrilateral with two parallel sides (see [12, Theorem 2.3]). In the case $c = 1$, (and $b = 1 - a$) these functions coincide with the special cases φ_K^a, μ_a, $\mathcal{K}_a$, and $\mathcal{E}_a$ which were studied extensively in e.g. [3], and relate to the case of a parallelogram.

Given complex numbers a, b, and c with $c \neq 0, -1, -2, \ldots$, the *Gaussian hypergeometric function* is the analytic continuation to the slit plane $\mathbb{C} \setminus [1, \infty)$ of the series

$$F(a, b; c; z) = {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a, n)(b, n)}{(c, n)} \frac{z^n}{n!}, \qquad |z| < 1. \tag{1.1}$$

Here $(a, 0) = 1$ for $a \neq 0$, and (a, n) is the *shifted factorial function* or the *Appell symbol*

$$(a, n) = a(a + 1)(a + 2) \cdots (a + n - 1)$$

for $n \in \mathbb{N} \setminus \{0\}$, where $\mathbb{N} = \{0, 1, 2, \ldots\}$. As usual, we let $\mathbb{C}$, $\mathbb{R}$ and $\mathbb{Z}$ denote respectively, the sets of complex numbers, real numbers, and integers.

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0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.05.020

A *generalized modular equation of order* (*or degree*) $p > 0$ is

$$\frac{F(a,b;c;1-s^2)}{F(a,b;c;s^2)} = p\frac{F(a,b;c;1-r^2)}{F(a,b;c;r^2)}, \quad 0 < r < 1. \tag{1.2}$$

Sometimes we just call this an (a,b,c)-modular equation of order p and we usually assume that $a,b,c > 0$ with $a+b \geqslant c$, in which case this equation uniquely defines s as a function of r, see [12, Lemma 4.5].

Many particular cases of (1.2) have been studied in the literature on both analytic number theory and geometric function theory, [3,4,9,10]. Rational modular equations were studied most recently by R.S. Maier in [15]. The classical case $(a,b,c) = (\frac{1}{2}, \frac{1}{2}, 1)$ was studied already by Jacobi and many others in the nineteenth century. In 1995 B. Berndt, S. Bhargava, and F. Garvan published an important paper [9] in which they studied the case $(a,b,c) = (a, 1-a, 1)$ and p an integer. For several rational values of a such as $a = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$ and integers p (e.g. $p = 2, 3, 5, 7, 11, \ldots$) they were able to give proofs for numerous algebraic identities stated by Ramanujan in his unpublished notebooks. These identities involve r and s from (1.2).

To abbreviate (1.2), we use the decreasing homeomorphism $\mu_{a,b,c} : (0,1) \to (0,\infty)$, defined by

$$\mu(r) = \mu_{a,b,c}(r) = \frac{B(a,b)}{2}\frac{F(a,b;c;r'^2)}{F(a,b;c;r^2)}, \quad r \in (0,1), \tag{1.3}$$

for $a,b,c > 0$, $a+b \geqslant c$, where B is the beta function, and r' is the *complementary argument* $r' = \sqrt{1-r^2}$. We call $\mu_{a,b,c}$ the *generalized modulus*, cf. [14, (2.2)]. Now (1.2) can be rewritten as

$$\mu_{a,b,c}(s) = p\mu_{a,b,c}(r), \quad 0 < r < 1. \tag{1.4}$$

With $p = 1/K$, $K > 0$, the solution of (1.2) is then given by

$$s = \varphi_K^{a,b,c}(r) = \mu_{a,b,c}^{-1}\left(\mu_{a,b,c}(r)/K\right). \tag{1.5}$$

We call the function $\varphi_K^{a,b,c}$ defined by (1.5) *the* (a,b,c)*-modular function with degree* $p = 1/K$ [9], [3, (1.5)]. In the case $a < c$ we also use the notation

$$\mu_{a,c} = \mu_{a,c-a,c}, \qquad \varphi_K^{a,c} = \varphi_K^{a,c-a,c}.$$

This article is organized as follows. In Section 2 we introduce the necessary notation and the functions studied, as well as known results used in the sequel. In Section 3 we obtain various generalizations of monotonicity results for certain combinations of the generalized elliptic integrals. The most important results here are Theorems 3.6 and 3.12, where in particular the latter one concerning the Legendre $\mathcal{M}$-function leads to many of the results in Section 4. In Section 4 we present a number of interesting results, which include the monotonicity properties for functions symmetric with respect to r and $s = \varphi_K^{a,c}(r)$ (Lemma 4.1), the functional inequalities for $\mu_{a,c}$ and $\varphi_K^{a,c}(r)$, and a linearization result, Theorem 4.7. Finally, in Section 5 the dependence on the parameter c for the functions $\mu_{a,c}$ and $\varphi_K^{a,c}(r)$ is studied. The main results in this section are Corollary 5.5 and Theorems 5.7 and 5.8. In the final section some open problems are presented.

2. Preliminaries and definitions

For $0 < a < \min\{c, 1\}$ and $0 < b < c \leqslant a+b$, define the *generalized complete elliptic integrals of the first and second kinds* (cf. [3, (1.9), (1.10), (1.3), and (1.5)]) on $[0,1]$ by

$$\mathcal{K} = \mathcal{K}_{a,b,c} = \mathcal{K}_{a,b,c}(r) = \frac{B(a,b)}{2}F\big(a,b;c;r^2\big), \tag{2.1}$$

$$\mathcal{E} = \mathcal{E}_{a,b,c} = \mathcal{E}_{a,b,c}(r) = \frac{B(a,b)}{2}F\big(a-1,b;c;r^2\big), \tag{2.2}$$

$$\mathcal{K}' = \mathcal{K}'_{a,b,c} = \mathcal{K}_{a,b,c}(r') \quad \text{and} \quad \mathcal{E}' = \mathcal{E}'_{a,b,c} = \mathcal{E}_{a,b,c}(r') \tag{2.3}$$

for $r \in (0,1)$, $r' = \sqrt{1-r^2}$. The end values are defined by limits as r tends to 0^+ and 1^-, respectively. In particular, we denote $\mathcal{K}_{a,c} = \mathcal{K}_{a,c-a,c}$ and $\mathcal{E}_{a,c} = \mathcal{E}_{a,c-a,c}$. Thus, by (2.9) below,

$$\mathcal{K}_{a,b,c}(0) = \mathcal{E}_{a,b,c}(0) = \frac{B(a,b)}{2}$$

and

$$\mathcal{E}_{a,b,c}(1) = \frac{1}{2}\frac{B(a,b)B(c,c+1-a-b)}{B(c+1-a,c-b)}, \qquad \mathcal{K}_{a,b,c}(1) = \infty.$$

Note that the restrictions on a, b and c ensure that the function $\mathcal{K}_{a,b,c}$ is increasing and unbounded whereas $\mathcal{E}_{a,b,c}$ is decreasing and bounded, as in the classical case $a = b = \frac{1}{2}$, $c = 1$.

Let Γ denote Euler's *gamma function* and let Ψ be its logarithmic derivative (also called the *digamma function*), $\Psi(z) = \Gamma'(z)/\Gamma(z)$. By [2, p. 198] the function Ψ and its derivative have the series expansions

$$\Psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty}\frac{z}{n(n+z)}, \qquad \Psi'(z) = \sum_{n=0}^{\infty}\frac{1}{(n+z)^2}, \tag{2.4}$$

where $\gamma = -\Psi(1) = \lim_{n\to\infty}(\sum_{k=1}^{n} 1/k - \log n) = 0.57721\ldots$ is the *Euler–Mascheroni constant*. From (2.4) it is seen that Ψ is strictly increasing on $(0,\infty)$ and that Ψ' is strictly decreasing there, so that Ψ is concave. Moreover, $\Psi(z+1) = \Psi(z) + 1/z$ and $\Psi(\frac{1}{2}) = -\gamma - 2\log 2$, see [1, Chapter 6].

For all $z \in \mathbb{C} \setminus \{0,-1,-2,\ldots\}$ and for all $n \in \mathbb{N}$ we have

$$\Gamma(z+n) = (z,n)\Gamma(z), \tag{2.5}$$

a fact which follows by induction [18, 12.12]. This enables us to extend the Appell symbol for all complex values of a and $a+t$, except for non-positive integer values, by

$$(a,t) = \frac{\Gamma(a+t)}{\Gamma(a)}. \tag{2.6}$$

Furthermore, the gamma function satisfies the reflection formula [18, 12.14]

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \tag{2.7}$$

for all $z \notin \mathbb{Z}$. In particular, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

The *beta function* is defined for $\operatorname{Re} x > 0$, $\operatorname{Re} y > 0$ by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{2.8}$$

As in this article we are mostly interested in cases where the hypergeometric parameters satisfy $0 < a < c < 1$ and $b = c - a$, we will shorten $B := B(a, c-a)$ if no risk for confusion is apparent.

We will make use of the standard notation for contiguous hypergeometric functions (cf. [17])

$$F = F(a,b;c;z), \qquad F(a+) = F(a+1,b;c;z), \qquad F(a-) = F(a-1,b;c;z),$$

etc. We also let

$$v = v(z) = F, \qquad u = u(z) = F(a-), \qquad v_1 = v_1(z) = v(1-z), \quad \text{and} \quad u_1 = u_1(z) = u(1-z).$$

The behavior of the hypergeometric function near $z = 1$ in the three cases $\operatorname{Re}(a+b-c) < 0$, $a+b=c$, and $\operatorname{Re}(a+b-c) > 0$, respectively, is given by

$$\begin{cases} F(a,b;c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \\ B(a,b)F(a,b;a+b;z) + \log(1-z) = R(a,b) + \mathrm{O}\big((1-z)\log(1-z)\big), \\ F(a,b;c;z) = (1-z)^{c-a-b}F(c-a,c-b;c;z), \end{cases} \tag{2.9}$$

where $R(a,b) = -\Psi(a) - \Psi(b) - 2\gamma$. The above asymptotic formula for the *zero-balanced* case $a+b=c$ is due to Ramanujan (see [8]). This formula is implied by [1, 15.3.10]. Note that $R(\frac{1}{2},\frac{1}{2}) = \log 16$.

For complex a, b, c, and z, with $|z| < 1$, we now let

$$\mathcal{M}(z) = \mathcal{M}(a,b,c,z) = z(1-z)\left(v_1(z)\frac{dv}{dz} - v(z)\frac{dv_1}{dz}\right). \tag{2.10}$$

Using the Gauss contiguous relations [17, p. 61], it is easy to see that

$$\mathcal{M} = (c-a)(uv_1 + u_1 v) + (2(a-c)+b)vv_1 = (c-a)(uv_1 + u_1 v - vv_1) + (a+b-c)vv_1 \tag{2.11}$$

and that

$$(B/2)^2 \mathcal{M}\big(r^2\big) = (a+b-c)\mathcal{K}\mathcal{K}' + (c-a)[\mathcal{K}\mathcal{E}' + \mathcal{K}'\mathcal{E} - \mathcal{K}\mathcal{K}']. \tag{2.12}$$

It follows from [3, Corollary 3.13(5)] that

$$\mathcal{M}(a, 1-a, 1, r) = \frac{1-a}{\Gamma(a)\Gamma(2-a)} = \frac{\sin(\pi a)}{\pi} \tag{2.13}$$

for $0 < a < 1$ and $0 \leqslant r < 1$. In particular, we get the classical Legendre relation [7,11]

$$\mathcal{M}(1/2, 1/2, 1, r) = \frac{1}{\pi}. \tag{2.14}$$

The function $\mathcal{M}$ will be referred to as *the Legendre $\mathcal{M}$-function*, and it has a central role for the generalizations considered in this article. It has the following useful symmetry and convexity properties, some of which were established already in [12, 3.17] (properties (1)–(3)).

2.15. Theorem. *For positive constants a, b, c the restriction to* $(0, 1)$ *of the continuous function $\mathcal{M}$ has the following properties.*

(1) $\mathcal{M}(x) = \mathcal{M}(1-x) > 0$ *for all* $x \in (0, 1)$.
(2) *If* $a + b \leqslant c$*, then* $\mathcal{M}(x)$ *is bounded and extends continuously to* $[0, 1]$. *In particular, if* $a + b = c = 1$*, then* $\mathcal{M}(x)$ *equals the constant* $\sin(\pi a)/\pi$.
(3) *If* $a + b > c$*, then* $\mathcal{M}$ *is unbounded on* $(0, 1)$ *with* $\mathcal{M}(0^+) = \mathcal{M}(1^-) = \infty$.
(4) *If* $(a + b - 1)(c - b) > 0$, $a + b \geqslant c \geqslant a$ *and* $ab/(a + b + 1) < c$*, then* $\mathcal{M}(a, b, c, r)$ *is strictly convex, decreasing in* $(0, 1/2]$ *and increasing in* $[1/2, 1)$.
(5) *If* $(a + b - 1)(c - b) < 0$, $a + b \leqslant c$*, and* $ab/(a + b + 1) < c$*, then* $\mathcal{M}(a, b, c, r)$ *is strictly concave, increasing in* $(0, 1/2]$ *and decreasing in* $[1/2, 1)$.
(6) *If* $a + b \geqslant c$ *then* $\mathcal{M}(r) > ab/c$ *for all* $r \in (0, 1)$.

Proof. Parts (1)–(3) are proved in the above mentioned article.

For (4) and (5) note that by (2.11) the function $\mathcal{M}$ can be written as

$$\mathcal{M}(a, b, c, r) = (c-a)(uv_1 + u_1 v - vv_1) + (a+b-c)vv_1.$$

In both cases (4) and (5) the constant $(c - a)$ is positive, so concavity/convexity of $(c - a)(uv_1 + u_1 v - vv_1)$ follows from the assumptions by [13, 2.1]. The functions v and v_1, are both log-convex by [5, 1.4], which follows from the parameter assumption $ab/(a + b + 1) < c$. Then, so is the product vv_1 (by e.g. [4, 1.38(5)]), and thus it is convex. Then the convexity/concavity of $(a + b - c)vv_1$ in the asserted cases also follows.

For (6), we see that

$$\begin{aligned}\mathcal{M}(r) &= (a+b-c)vv_1 + (c-a)\big[vv_1(a-) + v_1 v(a-) - vv_1\big] \\ &= (a+b-c)vv_1 + \big[(c-a)(c-b)/c\big]\big[(1-r)vv_1(c+) + r v_1 v(c+)\big] \\ &> (a+b-c) + \big[(c-a)(c-b)/c\big] = ab/c. \qquad \square\end{aligned}$$

Next we record some elementary but useful results for deriving monotonicity properties and obtaining inequalities. The first one is the so-called *l'Hôpital's monotone rule*, see [4, 1.25] and [6].

2.16. Lemma. *Let* $-\infty < a < b < \infty$*, and let* $f, g : [a, b] \to \mathbb{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b). *Let* $g'(x) \neq 0$ *on* (a, b). *Then, if* $f'(x)/g'(x)$ *is increasing* (*decreasing*) *on* (a, b)*, so are*

$$\big[f(x) - f(a)\big]\big/\big[g(x) - g(a)\big] \quad \textit{and} \quad \big[f(x) - f(b)\big]\big/\big[g(x) - g(b)\big].$$

If $f'(x)/g'(x)$ *is strictly monotone, then the monotonicity in the conclusion is also strict.*

The second result follows from direct differentiation, and concerns the monotonicity of certain rational functions.

2.17. Proposition. *Assume that $f, g : I \to \mathbb{R}$ are differentiable on an interval $I \subset \mathbb{R}$, and that $a, b, c, d \in \mathbb{R}$. Then*

$$\operatorname{sign}\left((ad-bc)\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right)\right) = \operatorname{sign}\frac{d}{dx}\left(\frac{af(x)+bg(x)}{cf(x)+dg(x)}\right).$$

Finally, we record some of the most useful differentiation formulae for the functions defined in (1.3), (1.5), (2.1), (2.2) and (2.10) (cf. [12]);

$$\frac{d\mathcal{K}}{dr} = \frac{2}{rr'^2}\big((c-a)\mathcal{E} + \big(br^2+a-c\big)\mathcal{K}\big), \tag{2.18}$$

$$\frac{d\mathcal{E}}{dr} = \frac{2(a-1)}{r}(\mathcal{K}-\mathcal{E}), \tag{2.19}$$

$$\frac{d}{dr}(\mathcal{K}-\mathcal{E}) = \frac{2}{rr'^2}\big(\big((c-a)+(1-a)r'^2\big)\mathcal{E} + \big((a+b)r^2 - c + r'^2\big)\mathcal{K}\big), \tag{2.20}$$

$$\frac{d}{dr}\big(\mathcal{E}-r'^2\mathcal{K}\big) = \frac{2}{r}\big((1-c)\mathcal{E} + \big(c-1-(b-1)r^2\big)\mathcal{K}\big), \tag{2.21}$$

$$\frac{d}{dr}\mu(r) = -\frac{B(a,b)\mathcal{M}(r^2)}{rr'^2 v(r^2)^2} = -\frac{B(a,b)^3\mathcal{M}(r^2)}{4rr'^2\mathcal{K}^2}, \tag{2.22}$$

$$\frac{\mathcal{M}(s^2)}{\mathcal{M}(r^2)}\frac{ds}{dr} = \frac{1}{K}\frac{ss'^2 v(s^2)^2}{rr'^2 v(r^2)^2} = \frac{1}{K}\frac{ss'^2\mathcal{K}(s)^2}{rr'^2\mathcal{K}(r)^2}, \quad s = \varphi_K(r), \tag{2.23}$$

$$\begin{aligned}\frac{d\mathcal{M}}{dr} = \frac{1}{r(1-r)}\big((c-a)\big[\big(1-c+(a+b-1)r\big)u(r)v_1(r) + \big(-a-b+c+(a+b-1)r\big)u_1(r)v(r)\big] \\ + (1-2r)\big[(c-a)(a+2b-1)-b^2\big]v(r)v_1(r)\big).\end{aligned} \tag{2.24}$$

Note that for the case $(a,b,c) = (1/2, 1/2, 1)$ the above formulas reduce to the classical ones [4,11].

3. Monotonicity and bounds

In studying monotonicity and convexity of modular functions, a useful method is to combine rational functions consisting of generalized elliptic integrals whose monotonicity properties are known in different ways. In the following lemmas we collect some useful properties of such functions, proved in [12, 4.21, 4.13, 4.24].

3.1. Lemma. *For $0 < a, b < \min\{c, 1\}$ and $c \leqslant a+b$, denote $\mathcal{K} = \mathcal{K}_{a,b,c}$ and $\mathcal{E} = \mathcal{E}_{a,b,c}$. Then the function*

(1) *$f_1(r) = (\mathcal{K}-\mathcal{E})/(r^2\mathcal{K})$ is strictly increasing from $(0,1)$ onto $(b/c, 1)$. In particular, we have the sharp inequality,*
$$\frac{b}{c} < \frac{\mathcal{K}-\mathcal{E}}{r^2\mathcal{K}} < 1$$
for all $r \in (0,1)$.

(2) *$f_2(r) = (\mathcal{E}-r'^2\mathcal{K})/r^2$ has positive Maclaurin coefficients and maps $(0,1)$ onto $(B(a,b)(c-b)/(2c), d)$, where*
$$d = \frac{B(a,b)B(c, c+1-a-b)}{2B(c+1-a, c-b)}.$$

(3) *$f_5(r) = (r')^{-2}\mathcal{E}$ has positive Maclaurin coefficients and maps $[0,1)$ onto $[B(a,b)/2, \infty)$.*

(4) *$f_6(r) = r'^2\mathcal{K}$ has negative Maclaurin coefficients, except for the constant term, and maps $[0,1)$ onto $(0, B(a,b)/2]$.*

(5) *$f_7(r) = \mathcal{K}$ has positive Maclaurin coefficients and is log-convex from $[0,1)$ onto $[B(a,b)/2, \infty)$. In fact, $(d/dr)(\log\mathcal{K})$ also has positive Maclaurin coefficients.*

(6) *$f_8(r) = (\mathcal{E}-r'^2\mathcal{K})/(r^2\mathcal{K})$ is strictly decreasing from $(0,1)$ onto $(0, 1-(b/c))$.*

(7) *$f_9(r) = (\mathcal{K}-\mathcal{E})/(\mathcal{E}-r'^2\mathcal{K})$ is strictly increasing from $(0,1)$ onto $(b/(c-b), \infty)$.*

3.2. Lemma.

(1) *For* $0 < a < c$ *and* $b = c - a$, *the function* $h(r) = r^2\mathcal{K}_{a,c}(r)/\log(1/r')$ *is strictly decreasing* (*respectively, increasing*) *from* $(0, 1)$ *onto* $(1, B(a, b))$ *if* $a, b \in (0, 1)$ (*respectively, onto* $(B(a, b), 1)$, *if* $a, b \in (1, \infty)$).
(2) *For* $0 < a, b < c$ *and* $2ab < c \leqslant a + b < c + 1/2$, *the function* $f(r) = r'\mathcal{K}(r)$ *is strictly decreasing from* $[0, 1)$ *onto* $(0, B(a, b)/2]$.

We start with some further monotonicity results for the generalized elliptic integrals, proved in [3] for the case $c = 1$, $b = 1 - a$. Note that part (1) extends [12, 4.38], as the condition $c \leqslant a + (1/2)$ is not needed.

3.3. Theorem. *For* $c \in (0, 1]$, $a \in (0, c)$ *and* $b = c - a$, *we have that the function*

(1) $f_1(r) = r\mathcal{K}_{a,c}(r)/\operatorname{arth}(r)$ *is strictly decreasing from* $(0, 1)$ *onto* $(1, B/2)$.
(2) $f_2(r) = ((B/2)^2 - (r'\mathcal{K}_{a,c}(r))^2)/(\mathcal{E}_{a,c}(r) - r'^2\mathcal{K}_{a,c}(r))$ *is strictly increasing from* $(0, 1)$ *onto* $(B(c - 2ac + 2a^2)/(2a), B^2(c - a)/2)$.
(3) $f_3(r) = r'^2(\mathcal{K}_{a,c}(r) - \mathcal{E}_{a,c}(r))/(r^2\mathcal{E}_{a,c}(r))$ *is strictly decreasing from* $(0, 1)$ *to* $(0, (c - a)/c)$.

Proof. (1) Clearly $f_1(0^+) = B/2$. By l'Hôpital's rule, Lemma 2.16, (2.5), (2.8) and the transformation formula and evaluation at 1 for hypergeometric functions given in (2.9), we see that

$$\begin{aligned} f_1(1^-) &= (B/2)\lim_{r\to 1^-} 2(a/c)(c-a)r'^2 F\big(a+1, c-a+1; c+1; r^2\big) \\ &= B(a/c)(c-a)\lim_{r\to 1^-} F\big(a, c-a; c+1; r^2\big) = B\frac{a}{c}(c-a)\frac{\Gamma(c+1)}{\Gamma(c-a+1)\Gamma(a+1)} \\ &= B\frac{\Gamma(c)}{\Gamma(c-a)\Gamma(a)} = B\frac{1}{B} = 1. \end{aligned}$$

Next, let $F_1(r) = rF(a, c - a; c; r^2)$ and $F_2(r) = \operatorname{arth}(r)$. By differentiation we get

$$\begin{aligned} \frac{F_1'(r)}{F_2'(r)} &= r'^2 F\big(a, c-a; c, r^2\big) + 2(a/c)(c-a)r^2 F\big(a, c-a; c+1; r^2\big) \\ &= \sum_{n=0}^{\infty}\frac{(a,n)(c-a,n)}{(c,n)}\frac{r^{2n}}{n!} - \sum_{n=0}^{\infty}\frac{(a,n)(c-a,n)}{(c,n)}\frac{r^{2(n+1)}}{n!} + 2(a/c)(c-a)\sum_{n=0}^{\infty}\frac{(a,n)(c-a,n)}{(c+1,n)}\frac{r^{2(n+1)}}{n!} \\ &= 1 - \sum_{n=1}^{\infty}\frac{(a,n-1)(c-a,n-1)}{(c,n)n!}\cdot\big[n\big(1-2a(c-a)\big) - (1-a)(1-c+a)\big]r^{2n}, \end{aligned}$$

which is strictly decreasing on $(0, 1)$, since

$$\begin{aligned} n\big(1-2a(c-a)\big) - (1-a)(1-c+a) &\geqslant 1 - 2a(c-a) - (1-a)(1-c+a) \\ &> c - 3a(c-a) \geqslant c - \frac{3}{4}c^2 \geqslant \frac{1}{4}c^2 > 0. \end{aligned}$$

Then, by l'Hôpital's rule the function f is also decreasing.

(2) Let $F(r) = (B/2)^2 - (r'\mathcal{K}_{a,c}(r))^2$ and $G(r) = \mathcal{E}_{a,c}(r) - r'^2\mathcal{K}_{a,c}(r)$. Then, using the differentiation formulas (2.18) and (2.20), we see that

$$\frac{F'(r)}{G'(r)} = \mathcal{K}_{a,c}\left(\frac{r^2\mathcal{K}_{a,c} - 2(c-a)(\mathcal{E}_{a,c} - r'^2\mathcal{K}_{a,c})}{ar^2\mathcal{K}_{a,c} + (1-c)(\mathcal{E}_{a,c} - r'^2\mathcal{K}_{a,c})}\right).$$

By Lemma 2.16 we need to show that this ratio is strictly increasing. However, since $\mathcal{K}_{a,c}$ is strictly increasing, and also $r \mapsto r^2\mathcal{K}_{a,c}/(\mathcal{E}_{a,c} - r'^2\mathcal{K}_{a,c})$ is, by Lemma 3.1(6), the result follows from Proposition 2.17 and the fact that $(1 - c) + 2a(c - a) > 0$. Also, by Lemma 3.1(6)

$$\lim_{r\to 0}\frac{\mathcal{E}_{a,c} - r'^2\mathcal{K}_{a,c}}{r^2\mathcal{K}_{a,c}} = \frac{a}{c},$$

and so we see that

$$\lim_{r\to 0^+}\frac{F(r)}{G(r)}=\lim_{r\to 0^+}\mathcal{K}_{a,c}\left(\frac{1-2(c-a)\frac{\mathcal{E}_{a,c}-r'^2\mathcal{K}_{a,c}}{r^2\mathcal{K}_{a,c}}}{a+(1-c)\frac{\mathcal{E}_{a,c}-r'^2\mathcal{K}_{a,c}}{r^2\mathcal{K}_{a,c}}}\right)=\frac{B}{2}\frac{c-2ac+2a^2}{a}.$$

Furthermore, using the value of $\mathcal{E}_{a,c}(1)$ and the fact that $\lim_{r\to 1^-} r'\mathcal{K}_{a,c}=0$, we see that $\lim_{r\to 1^-} F(r)/G(r)=(c-a)B^2/2$.

(3) Follows directly from the fact that $f_3(r)=1-g(r)/\mathcal{E}_{a,c}(r)$, where g is the function f_2 in Lemma 3.1(2). □

The following result extends part of [3, 5.4].

3.4. Lemma. *Let* $0<a<c\leqslant 1$, $B=B(a,b)$ *with* $b=c-a$, *and* $\mathcal{K}=\mathcal{K}_{a,c}$, $\mathcal{E}=\mathcal{E}_{a,c}$. *Then the function*

(1) $f_1(r)=r'^p\mathcal{K}(r)$ *is decreasing if and only if* $p\geqslant 2\frac{a}{c}(c-a)$, *in which case* $r'^p\mathcal{K}(r)$ *is decreasing from* $(0,1)$ *onto* $(0,B/2)$. *In particular,* $\sqrt{r'}\mathcal{K}(r)$ *is decreasing on* $[0,1)$.
(2) $f_2(r)=r'^p\mathcal{E}(r)$ *is increasing if and only if* $p\leqslant -\frac{2}{c}(1-a)(c-a)$, *in which case it is increasing from* $(0,1)$ *onto* $(B/2,\infty)$. *In particular,* $\mathcal{E}(r)/r'^2$ *is increasing on* $[0,1)$.

Proof. (1) Differentiating we get that

$$r(r')^{2-p}f_1'(r)=-pr^2\mathcal{K}(r)+2(c-a)\big(\mathcal{E}(r)-r'^2\mathcal{K}(r)\big).$$

This is non-positive if and only if

$$p\geqslant 2(c-a)\sup_r\frac{\mathcal{E}(r)-r'^2\mathcal{K}(r)}{r^2\mathcal{K}(r)}=2\frac{a}{c}(c-a),$$

by Lemma 3.1(6). Finally, since $\max\{2\frac{a}{c}(c-a)\mid 0<a<c\leqslant 1\}=1/2$, the function $\sqrt{r'}\,\mathcal{K}(r)$ will be decreasing for all appropriate values of a and c. The limiting value at $r=0$ is obvious, and the one at $r=1$ follows from l'Hôpital's rule and Lemma 3.1(2).

(2) Differentiating yields

$$rf_2'(r)=-p(r')^{p-2}r^2\mathcal{E}(r)+2(a-1)r'^p\big(\mathcal{K}(r)-\mathcal{E}(r)\big),$$

which is non-negative if and only if

$$-p\geqslant 2(1-a)\sup_r\frac{r'^2(\mathcal{K}(r)-\mathcal{E}(r))}{r^2\mathcal{E}(r)}=\frac{2}{c}(1-a)(c-a),$$

where the value of the supremum follows from Theorem 3.3(3). Since $\sup\{\frac{2}{c}(1-a)(c-a)\mid 0<a<c\leqslant 1\}=2$, the function $\mathcal{E}(r)/r'^2$ will be increasing for all appropriate values of a and c. The limiting values are obvious. □

3.5. Lemma. *For* $0<a,b<\min\{c,1\}$ *and* $a+b\geqslant c$, $r\in(0,1)$, *we have that the function*

(1) $f_1(r)=(r')^{2(a+b-c)}\mathcal{K}_{a,b,c}(r)$ *has positive Maclaurin coefficients and is log-convex on* $(0,1)$ *with range* $(B(a,b)/2,B(c,a+b-c)/2)$.
(2) $f_2(r)=(r')^{2(a+b-c-1)}\mathcal{E}_{a,b,c}(r)$ *has positive Maclaurin coefficients and is log-convex on* $(0,1)$ *with range* $(B(a,b)/2,\infty)$.

Proof. (1) From (2.9), we have that $f_1(r)=(B(a,b)/2)F(c-a,c-b;c;r^2)$, so that $(c-a)(c-b)<c(2c-a-b)$ if and only if $ab<c^2+c$, which is true. Hence the assertion follows from [5, Theorem 3.2(1)].

(2) From (2.9), we have that $f_2(r)=(B(a,b)/2)F(c+1-a,c-b;c;r^2)$, so that $(c+1-a)(c-b)<c(2c+2-a-b)$, if and only if $(a-1)b<c^2+c$, which is true. Hence the assertion follows from [5, Theorem 3.2(1)]. □

We next derive some monotonicity results for functions combined with the $\mu_{a,c}$-function.

3.6. Theorem. *Let* $0 < a < c \leqslant 1$. *Then the function*

(1) $f_1(r) = \mu_{a,c}(r) + \log r$ *is strictly decreasing from* $(0, 1]$ *onto* $[0, R(a, c-a)/2)$, *where* $R(a, c-a)$ *is as in* (2.9).
(2) $f_2(r) = \frac{r'^2 \log r'}{r^2 \log r} \mu_{a,c}(r)$ *is strictly increasing from* $(0, 1]$ *onto* $(1/2, B^2/2]$.
(3) $f_3(r) = \frac{r' \operatorname{arth}(r)}{r \operatorname{arth}(r')} \mu_{a,c}(r)$ *is strictly increasing from* $(0, 1)$ *onto* $(1, (B/2)^2]$.
(4) $f_4(r) = r' \mu_{a,c}(r)/\log(1/r)$ *is strictly increasing from* $(0, 1)$ *onto* $(1, \infty)$. *Thus the function* $\tilde{f}_4(r) = \mu_{a,c}(r)/\log(1/r)$ *is also strictly increasing from* $(0, 1)$ *onto* $(1, \infty)$.
(5) $f_5(r) = \mu_{a,c}(r) \operatorname{arth}(r)$ *is strictly increasing from* $(0, 1)$ *onto* $(0, (B/2)^2)$.
(6) $f_6(r) = \mu_{a,c}(r) \log(r/r')$ *is increasing from* $[1/\sqrt{2}, 1)$ *onto* $[0, (B/2)^2)$.

Proof. (1) Clearly $f_1(1) = 0$, and by [4, 1.52(2)] it follows that $f_1(0^+) = R(a, c-a)/2$. From [12, (4.19)] we find that

$$f_1'(r) = \frac{1}{r} - \frac{B(a, c-a)^3 \mathcal{M}(r^2)}{4rr'^2 \mathcal{K}_{a,c}^2(r)} = \frac{1}{r}\left(1 - \frac{(B(a, c-a)/2)^2 \; B(a, c-a)\mathcal{M}(r^2)}{(r'\mathcal{K}_{a,c}(r))^2}\right).$$

It now suffices to show that this derivative is negative, which is true if, denoting $B = B(a, c-a)$, we have

$$\frac{(B/2)^2 B \mathcal{M}(r^2)}{(r'\mathcal{K}_{a,c}(r))^2} > 1 \tag{3.7}$$

for $r \in (0, 1)$. From Lemma 3.4(1) it follows that $g(r) = r'\mathcal{K}_{a,c}(r)$ is strictly decreasing from $[0, 1)$ onto $(0, B/2]$. By Theorem 2.15 we see that $\mathcal{M}(r^2)$ gets its smallest value for $\mathcal{M}(0^+) = \mathcal{M}(1^-) = 1/B$. Then we see that

$$\frac{(B/2)^2 B \mathcal{M}(r^2)}{(r'\mathcal{K}_{a,c}(r))^2} \geqslant B\mathcal{M}\big(r^2\big)/r' > B\mathcal{M}\big(0^+\big)/r' = \frac{1}{r'} > 1.$$

The claim follows.

(2) The function f_2 can be rewritten as

$$f_2(r) = \frac{B}{2} \cdot \frac{r'^2 F(a, c-a; c; r'^2)}{\log(1/r^2)} \cdot \frac{\log(1/r'^2)}{r^2 F(a, c-a; c; r^2)}.$$

By Lemma 3.2(1) the second fraction is strictly increasing onto $(2/B, 2]$, and the third onto $(1/2, B/2]$, so the claim follows.

(3) The function f_3 can be rewritten as

$$f_3(r) = \frac{B}{2} \cdot \frac{r' F(a, c-a; c; r'^2)}{\operatorname{arth}(r')} \cdot \frac{\operatorname{arth}(r)}{r F(a, c-a; c; r^2)}.$$

Then, in the same way as in part (2), the claim follows from Theorem 3.3(1).

(4) Clearly

$$f_4(r) = \frac{r' F(a, c-a; c; r'^2)}{\log(1/r) F(a, c-a; c; r^2)} = \frac{r'^2 F(a, c-a; c; r'^2)}{\log(1/r)} \cdot \frac{1}{r' F(a, c-a; c; r^2)}.$$

By Lemma 3.2(1) and part (2) it is then the product of two increasing functions. The limiting values also follow immediately. As $r \mapsto r'$ is decreasing, $\lim_{r\to 0^+} r' = 1$ and $\lim_{r \to 1^-} r' = 0$, the statements for $\tilde{f}_4$ also follow.

(5) We see that

$$f_5(r) = \frac{B}{2} \frac{\operatorname{arth}(r)}{r\mathcal{K}(r)} r\mathcal{K}'(r).$$

Then it is a product of two increasing functions, by Theorem 3.3(1) and Lemma 3.2(2). Hence, f is increasing itself. The limiting values are obvious.

(6) The value $f_6(1/\sqrt{2}) = 0$ is obvious, while the limit as $r \to 1$ follows from (4) and the symmetry property $\mu_{a,c}(r)\mu_{a,c}(r') = (B/2)^2$. Next, $f_6(r) = (1/2) f_2(r) g(r)$, where

$$g(r) = \left(\frac{r}{r'}\right)^2 \frac{\log r}{\log r'} \log\left(\frac{r}{r'}\right)^2.$$

Hence, by (2) it suffices to prove that $g(r)$ is increasing on $[1/\sqrt{2}, 1)$. Put $t = (r/r')^2$, so that

$$g(r) = \frac{t \log t}{\log(t+1)} \log \frac{t+1}{t}$$

for $t \in [1, \infty)$. Clearly $\log t/\log(t+1)$ is increasing on $[1/\sqrt{2}, 1)$. Let $h(t) = t \log((t+1)/t)$. Then $h'(t) = \log((t+1)/t) - 1/(t+1)$ and $h''(t) = -1/(t(t+1)^2) < 0$, so that $h'(t)$ is decreasing. Since $\lim_{t\to\infty} h'(t) = 0$, we get $h'(t) > 0$ on $[1, \infty)$ and thus $h(t)$ is increasing on $[1, \infty)$. □

For a quotient of hypergeometric functions with different parameters we obtain the following results.

3.8. Theorem. *Let a, b, c, a', b', c' be positive constants, satisfying the conditions $a' \geqslant a$, $b' \geqslant b$, and $c' \leqslant c$, with at least one inequality being strict, and let* $\max\{a', b'\} < c'$. *Then the function $f(r) := F(a', b'; c'; r)/F(a, b; c; r)$ is strictly increasing on* $[0, 1)$ *onto* $[1, L)$, *where*

$$L = \frac{B(c', c'-a'-b')B(c-a, c-b)}{B(c, c-a-b)B(c'-a', c'-b')}$$

in case $a' + b' < c'$, and $L = \infty$ in case $a' + b' \geqslant c'$.

Proof. First, $f(0) = 1$ is obvious. Next, let T_n denote the nth coefficient-quotient, that is $T_n = a_n/b_n$, where a_n and b_n are the nth Maclaurin coefficients of $F(a', b'; c'; r)$ and $F(a, b; c; r)$, respectively. Then

$$T_n = \frac{(a', n)(b', n)(c, n)}{(a, n)(b, n)(c', n)},$$

so that $T_{n+1}/T_n = (a'+n)(b'+n)(c+n)/[(a+n)(b+n)(c'+n)] > 1$. Hence the assertion on monotonicity follows from [12, Theorem 4.3].

Now assume that $a' + b' < c'$. Then $a + b \leqslant a' + b' < c' \leqslant c$, so by (2.9) the assertion for L follows.

Next, let $a' + b' > c'$, and $a + b > c$. Then $(a' + b' - c') - (a + b - c) = p > 0$. Hence, by (2.9), we get

$$f(r) = (1-r)^{-p} \frac{F(c'-a', c'-b'; c'; r)}{F(c-a, c-b; c; r)},$$

so that $f(1^-) = \infty$.

Next, if $a' + b' > c'$ and $a + b = c$, then $a' + b' - c' > a + b - c = 0$, so by (2.9) it follows that $L = \infty$.

Finally, let $a + b < c$, but $a' + b' \geqslant c'$. Again, from (2.9) it follows that $L = \infty$. □

3.9. Corollary. *With notation for contiguous hypergeometric functions as in* [17, p. 50], *let a, b, c be positive constants, and let $f = F(a+)/F$, $g = F(b+)/F$ and $h = F/F(c+)$. Then f, g and h are all increasing on* $[0, 1)$, *with $f(0) = g(0) = h(0) = 1$. Furthermore,*

(1) *$f(1^-) = (c-a-1)/(c-a-b-1)$ if $a+b+1 < c$ and $= \infty$ otherwise.*
(2) *$g(1^-) = (c-b-1)/(c-a-b-1)$ if $a+b+1 < c$ and $= \infty$ otherwise.*
(3) *$h(1^-) = (c-a)(c-b)/[c(c-a-b)]$ if $a+b < c$ and $= \infty$ otherwise.*

The particular case $0 < a < c < 1$, $b = c - a$ requires that we have some knowledge about the Legendre $\mathcal{M}$-function, a phenomenon which does not show in the case $c = 1$, as then $\mathcal{M}(a, c-a, c, r) = \mathcal{M}(a, 1-a, 1, r)$ is constant by (2.13). In the following theorems we derive some more useful properties of the $\mathcal{M}$-function.

3.10. Theorem. *Denote $f(r) = (r(1-r))^{a+b-c}\mathcal{M}(a, b, c, r)$. Then the following hold for positive a, b, c with $a \leqslant c$, $b \leqslant c$ and $r \in (0, 1)$.*

(1) *If $a + b > c$, then the function $f(r)$ is bounded.*
(2) *If $a = c$ or $b = c$, then the function $f(r)$ is the constant b or a, respectively.*

(3) *If* $a+b+1=2c$, *then* $f(r)=d$, *a constant, that is*, $\mathcal{M}(r)=d(r(1-r))^{1-c}$, *where* $d=\frac{\Gamma(c)^2}{\Gamma(a)\Gamma(b)}$. *In particular,* $\mathcal{M}(r)$ *is constant if and only if* $c=1$.

Proof. (1) By [12, 3.17(7)] we know that the limit of f at $r=0$ is $(a+b-c)B(c,a+b-c)/B(a,b)$. By symmetry of f, it is also the limit at $r=1$. Therefore f is bounded if $a+b>c$.

(2) Assume that $c=a$. By (2.11) we see that

$$f(r)=b\big(r(1-r)\big)^b v(a,b,a,r)v_1(a,b,a,r).$$

By [1, 15.1.8] we have that $F(a,b;a,r)=F(b,a;a;r)=(1-r)^{-b}$. Thus

$$f(r)=b\big(r(1-r)\big)^b(1-r)^{-b}\,r^{-b}=b,$$

which proves the statement. Since the parameters a and b are interchangeable in hypergeometric functions, the proof is the same in the case $c=b$.

(3) Let $N(r)=\mathcal{M}(r)/(r(1-r))$. Then, by (2.10)

$$N(r)=v_1(r)v'(r)-v(r)v_1'(r),$$

and

$$N'(r)=v_1(r)v''(r)-v(r)v_1''(r).$$

As v satisfies the hypergeometric differential equation (see [17, (3), p. 54]), we have

$$r(1-r)v''(r)+c(1-2r)v'(r)-abv(r)=0.$$

Now, $v_1'(r)=-v'(1-r)$ and $v_1''(r)=v''(1-r)$. Hence,

$$r(1-r)v_1''(r)+c(1-2r)v_1'(r)-abv_1(r)=0,$$

and thus

$$r(1-r)N'(r)+c(1-2r)N(r)=0.$$

Hence

$$\frac{d}{dr}\big[\big(r(1-r)\big)^cN(r)\big]=0=\frac{d}{dr}\big[\big(r(1-r)\big)^{c-1}\mathcal{M}(r)\big],$$

so that $\mathcal{M}(r)=d(r(1-r))^{1-c}$, where d is a constant.

We now show that $d=\Gamma(c)^2/(\Gamma(a)\Gamma(b))$. Taking the limit as $r\to 0^+$, we have $d=f(0^+)$.

Case (i). $c=1$, so that $b=1-a$. Then by (2.9)

$$f(r)=r(1-r)v_1(r)v'(r)+a(1-a)(1-r)v(r)v(1-a,a;2;1-r),$$

so that

$$d=a(1-a)v(1-a,a;2;1)=a(1-a)\frac{\Gamma(2)\Gamma(1)}{\Gamma(1+a)\Gamma(2-a)}=\frac{1}{\Gamma(a)\Gamma(1-a)},$$

as required. Note that in this case $d=\sin(\pi a)/\pi$.

Case (ii). $0<c<1$. In this case we have $0<a+b<c<1$. Then

$$\begin{aligned}f(r)&=\big(r(1-r)\big)^c\big(v_1(r)v'(r)+(ab/c)v(r)v(a+1,b+1;c+1;1-r)\big)\\&=\big(r(1-r)\big)^c\big[v_1(r)v'(r)+(ab/c)r^{-c}v(r)v(c-a,c-b;c+1;1-r)\big],\end{aligned}$$

so that

$$f\big(0^+\big)=0+(ab/c)v(c-a,c-b:c+1;1)=(ab/c)\frac{\Gamma(c+1)\Gamma(c)}{\Gamma(a+1)\Gamma(b+1)}=\frac{\Gamma(c)^2}{\Gamma(a)\Gamma(b)}.$$

Case (iii). $c > 1$. This is similar to case (ii). □

The following corollary is a direct consequence of Theorem 3.10(3) and the formulas (2.22) and (2.23).

3.11. Corollary. *Let $\mu = \mu_{a,b,c}$ and let $s = \varphi_K^{a,b,c}(r)$. If a, b, c are positive with $a \leqslant c$ and $b \leqslant c$, $r \in (0,1)$ and $a + b + 1 = 2c$, then we have the following generalized derivative formulas.*

(1) $$\frac{d\mu}{dr} = -\frac{D}{r^{2c-1} r'^{2c} \mathcal{K}(r)^2},$$

where $D = \frac{(\Gamma(a)\Gamma(b)\Gamma(c))^2}{4\Gamma(a+b)^3}$.

(2) $$\frac{ds}{dr} = \frac{1}{K}\left(\frac{s}{r}\right)^{2c-1}\left(\frac{s'}{r'}\right)^{2c}\left(\frac{\mathcal{K}(s)}{\mathcal{K}(r)}\right)^2.$$

3.12. Theorem. *Let* $0 < a < c \leqslant 1$, $b = c - a$ *and* $\mathcal{M}(r) = \mathcal{M}(a, c - a, c, r)$. *Then*

(1) *The inequality*
$$\mathcal{M}(r^2) - 2r^2\mathcal{M}'(r^2) \geqslant (c - a)a > 0$$
holds for all $r \in [0, 1]$. *In particular the function* $f(r) = r/\mathcal{M}(r^2) - a(c - a)r$ *is increasing from* $[0, 1]$ *onto* $[0, B - a(c - a)]$.

(2) *The function* $g(r) = f(r')$ *is decreasing from* $[0, 1]$ *onto* $[0, B - a(c - a)]$.

Proof. (1) First, if $c = 1$, then $\mathcal{M}(r^2)$ is a positive constant, hence the assertion is trivial. We then assume that $0 < c < 1$. In this case, $(a + b - 1)(c - b) = (c - 1)a < 0$, so by Theorem 2.15 $\mathcal{M}'(r^2) > 0$ for $r \in (0, 1/\sqrt{2})$ and < 0 for $r \in (1/\sqrt{2}, 1)$. Let
$$F_1 = F(c; r'^2)F(c+; r^2) \quad \text{and} \quad F_2 = F(c; r^2)F(c+; r'^2),$$
where the parameter triple of F is $(a, c - a; c)$. Then we see that both F_1 and F_2 are non-negative, and in fact $\geqslant 1$. As in [13] (11) and (27), we see that
$$\mathcal{M}(r^2) = (c - a)\frac{a}{c}(r^2F_1 + r'^2F_2) \quad \text{and} \quad \mathcal{M}'(r^2) = (c - a)\frac{a}{c}(1 - c)(F_1 - F_2).$$
Now $\mathcal{M}'(r^2)$ is negative in $(1/\sqrt{2}, 1)$, so from the equation above we see that in this interval $F_1 - F_2$ is also negative. Then
$$\begin{aligned}\mathcal{M}(r^2)^2\frac{d}{dr}\frac{r}{\mathcal{M}(r^2)} &= \mathcal{M}(r^2) - 2r^2\mathcal{M}'(r^2) = (c - a)\frac{a}{c}\big(r^2F_1 + r'^2F_2 - 2(1 - c)r^2(F_1 - F_2)\big)\\ &\geqslant (c - a)\frac{a}{c}(r^2F_1 + r'^2F_2) \geqslant (c - a)\frac{a}{c}(r^2 + r'^2) = \frac{(c - a)a}{c}.\end{aligned}$$
In the case $r \in (0, 1/\sqrt{2})$ and $c \geqslant 1/2$ both $F_1 - F_2$ and $(2c - 1)$ are non-negative. Then we see that
$$\begin{aligned}\mathcal{M}(r^2) - 2r^2\mathcal{M}'(r^2) &= (c - a)\frac{a}{c}\big(r^2F_1 + r'^2F_2 - 2(1 - c)r^2(F_1 - F_2)\big)\\ &= (c - a)\frac{a}{c}\big(F_2 + (2c - 1)(F_1 - F_2)r^2\big) \geqslant (c - a)\frac{a}{c}F_2 \geqslant (c - a)\frac{a}{c}.\end{aligned}$$
For $c \leqslant 1/2$ the expression $(2c - 1)$ is non-positive. Thus, using (2.9), and the inequality $rF(a, c - a; c; r'^2) \leqslant 1$ which follows from Lemma 3.2(2), we get
$$\begin{aligned}(c - a)\frac{a}{c}\big(r^2(2c - 1)F_1 + \big(1 - r^2(2c - 1)\big)F_2\big) &\geqslant (c - a)\frac{a}{c}\left((2c - 1)\frac{c}{(c - a)a}\frac{1}{B}r + 1 - r^2(2c - 1)\right)\\ &\geqslant (2c - 1)\frac{1}{B}r + r^2\frac{a(c - a)(1 - 2c)}{c} + \frac{(c - a)a}{c}.\end{aligned}$$

Finally, using the inequality $1/B(a,b) \leqslant 2ab/(a+b)$ (see [4, 1.50]) together with the fact that $r(1-r) \leqslant 1/4$, we obtain

$$\mathcal{M}(r^2)^2 \frac{d}{dr}\frac{r}{\mathcal{M}(r^2)} \geqslant \frac{(c-a)a}{c}\big(1-2(1-2c)r+(1-2c)r^2\big)$$
$$= \frac{(c-a)a}{c}\big(2c+(1-2c)-2(1-2c)r+(1-2c)r^2\big)$$
$$= \frac{(c-a)a}{c}\big(2c+(1-2c)(1-r)^2\big) \geqslant \frac{(c-a)a}{c}2c = 2a(c-a).$$

This proves the statement.

Part (2) follows directly from the equality $\mathcal{M}(x) = \mathcal{M}(1-x)$ by interchanging x with x' in part (1). □

4. Functional inequalities and linearization

In this section we generalize the functional inequalities for the modular function $\varphi_K^a(r)$ proved in [3] to hold also for the generalized modular function $\varphi_K^{a,b,c}(r)$ in the case $b = c-a$. We start by a generalization of the results in [3, 6.2].

4.1. Lemma. *Let $a < c \leqslant 1$, $K \in (1,\infty)$, $r \in (0,1)$, and let $s = \varphi_K^{a,c}(r)$ and $t = \varphi_{1/K}^{a,c}(r)$. Then the function*

(1) $f_1(r) = s/r$ *is decreasing from* $(0,1)$ *onto* $(1,\infty)$,
(2) $f_2(r) = s'/r'$ *is decreasing from* $(0,1)$ *onto* $(0,1)$,
(3) $f_3(r) = \mathcal{K}(s)/\mathcal{K}(r)$ *is increasing from* $(0,1)$ *onto* $(1,K)$,
(4) $f_4(r) = \mathcal{K}'(s)/\mathcal{K}'(r)$ *is increasing from* $(0,1)$ *onto* $(1/K,1)$,
(5) $f_5(r) = s'\mathcal{K}_{a,c}(s)^2/(r'\mathcal{K}_{a,c}(r)^2)$ *is decreasing from* $(0,1)$ *onto* $(0,1)$,
(6) $f_6(r) = s\mathcal{K}'_{a,c}(s)^2/(r\mathcal{K}'_{a,c}(r)^2)$ *is decreasing from* $(0,1)$ *onto* $(1,\infty)$,
(7) $g_1(r) = t/r$ *is increasing from* $(0,1)$ *onto* $(0,1)$,
(8) $g_2(r) = t'/r'$ *is increasing from* $(0,1)$ *onto* $(1,\infty)$,
(9) $g_3(r) = \mathcal{K}(t)/\mathcal{K}(r)$ *is decreasing from* $(0,1)$ *onto* $(1/K,1)$,
(10) $g_4(r) = \mathcal{K}'(t)/\mathcal{K}'(r)$ *is decreasing from* $(0,1)$ *onto* $(1,K)$,
(11) $g_5(r) = t'\mathcal{K}_{a,c}(t)^2/(r'\mathcal{K}_{a,c}(r)^2)$ *is increasing from* $(0,1)$ *onto* $(1,\infty)$,
(12) $g_6(r) = t\mathcal{K}'_{a,c}(t)^2/(r\mathcal{K}'_{a,c}(r)^2)$ *is increasing from* $(0,1)$ *onto* $(0,1)$.

Proof. (1) Differentiating we see that

$$f_1'(r) = \frac{ss'^2\mathcal{K}(s)\mathcal{K}'(s)\mathcal{M}(r^2)}{rr'^2\mathcal{K}(r)\mathcal{K}'(r)\mathcal{M}(s^2)} \cdot r - s \leqslant 0$$

if and only if

$$\frac{s'^2\mathcal{K}(s)\mathcal{K}'(s)}{\mathcal{M}(s^2)} \leqslant \frac{r'^2\mathcal{K}(r)\mathcal{K}'(r)}{\mathcal{M}(r^2)}. \tag{4.2}$$

As $\mu_{a,c}(s) = \mu_{a,c}(r)/K$, we see that $s > r$ for all $r \in (0,1)$, and thus (4.2) holds if $x \mapsto x'^2\mathcal{K}(x)\mathcal{K}'(x)/\mathcal{M}(x^2)$ is decreasing, which is true by Theorem 3.12(2) and Lemma 3.2(2). The limiting value at 1 is clear. For the limiting value at 0 we see that since $s/r = (s/r^{1/K})(1/r^{1-1/K})$, we get

$$\log(s/r) = (1-1/K)\log(1/r) + \big(\log s - (1/K)\log r\big)$$
$$= (1-1/K)\log(1/r) + \big(\big(\mu(s)+\log s\big) - (1/K)(\mu r + \log r)\big),$$

which by Theorem 3.6(1) tends to ∞.

(3) Differentiating we have that

$$\mathcal{K}_{a,c}(r)^2 f_1'(r) = 2(c-a)\left[\mathcal{K}_{a,c}(r)\frac{\mathcal{E}_{a,c}(s)-s'^2\mathcal{K}_{a,c}(s)}{ss'^2}\frac{ds}{dr} - \mathcal{K}_{a,c}(s)\frac{\mathcal{E}_{a,c}(r)-r'^2\mathcal{K}_{a,c}(r)}{rr'^2}\right]$$
$$= \frac{2(c-a)\mathcal{K}_{a,c}(s)\mathcal{M}(r^2)}{rr'^2\mathcal{K}'_{a,c}(r)}\left[\frac{\mathcal{K}'_{a,c}(s)(\mathcal{E}_{a,c}(s)-s'^2\mathcal{K}_{a,c}(s))}{\mathcal{M}(s^2)} - \frac{\mathcal{K}'_{a,c}(r)(\mathcal{E}_{a,c}(r)-r'^2\mathcal{K}_{a,c}(r))}{\mathcal{M}(r^2)}\right].$$

Then, since by Theorem 4.4 $r < r^{1/K} < s$, it suffices to show that

$$\frac{\mathcal{K}'_{a,c}(x)(\mathcal{E}_{a,c}(x)-x'^2\mathcal{K}_{a,c}(x))}{\mathcal{M}(x^2)} = \frac{x^2\mathcal{K}'_{a,c}(x)}{\mathcal{M}(x^2)}\cdot\frac{\mathcal{E}_{a,c}(x)-x'^2\mathcal{K}_{a,c}(x)}{x^2}.$$

is increasing. But this follows from Theorem 3.12(1) together with parts (2) and (9) of Theorem 3.2(2). The limiting values are clear.

(5) Differentiating, we see that $f_3'(r)$ is negative if and only if the function

$$F(x) = \frac{x^2\mathcal{K}_{a,c}(x)\mathcal{K}'_{a,c}(x)}{\mathcal{M}(x^2)}\left(1-4(c-a)\frac{\mathcal{E}_{a,c}(x)-x'^2\mathcal{K}_{a,c}(x)}{x^2\mathcal{K}_{a,c}(x)}\right)$$

is increasing. But this follows from Lemma 3.1(6) together with 3.12(3). The limiting values follow from the limiting values in parts (2) and (3), as

$$\lim_{r\to 0}\frac{s'\mathcal{K}(s)^2}{r'\mathcal{K}(r)^2} = \lim_{r\to 0}\frac{s'}{r'}\cdot\lim_{r\to 0}\frac{\mathcal{K}(s)^2}{\mathcal{K}(r)^2} = 1\cdot 1 = 1$$

and

$$\lim_{r\to 1}\frac{s'\mathcal{K}(s)^2}{r'\mathcal{K}(r)^2} = \lim_{r\to 1}\frac{s'}{r'}\cdot\lim_{r\to 1}\frac{\mathcal{K}(s)^2}{\mathcal{K}(r)^2} = 0\cdot K^2 = 0.$$

As $f_2(r) = 1/f_1(s')$, $f_4(r) = 1/f_3(s')$ and $f_6(r) = 1/f_5(s')$, parts (2), (4) and (6) follow. The parts (7)–(12) follow from (1)–(6), as $g_i(r) = 1/f_i(t)$ for $i = 1,2,3,4,5,6$. □

We continue by proving some functional inequalities for the function $\mu_{a,c}$.

4.3. Theorem. *Let $0 < a < c \leqslant 1$. Then, denoting $f(r) = \mu_{a,c}(r) = \mu(r)$, the function $g_1(r) = (1-r)f'(r)$ is increasing, and the function $g_2(r) = rf'(r)$ is decreasing. In particular, the inequalities*

$$\mu_{a,c}\big(1-\sqrt{(1-u)(1-t)}\big) \leqslant \frac{\mu_{a,c}(u)+\mu_{a,c}(t)}{2} \leqslant \mu_{a,c}(\sqrt{ut})$$

hold for all $u, t \in (0,1)$ with equality if and only if $u = t$.

Proof. We first see that for the function $g_1(r)$

$$-g_1(r) = \frac{B^3}{4}\frac{\mathcal{M}(r^2)}{r}\frac{1}{(1+r)\mathcal{K}(r)^2}.$$

Clearly this is decreasing by Theorem 3.12(1), so that $g_1(r)$ is increasing. Also

$$-g_2(r) = \frac{B^3}{4}\frac{\mathcal{M}(r^2)}{r'}\frac{1}{r'\mathcal{K}(r)^2},$$

which is increasing by Theorems 3.12(2) and 3.4(1), so that $g_2(r)$ is decreasing. These monotone properties imply that the function $f(1-e^{-t})$ is convex on $(0,\infty)$ and that the function $f(e^{-t})$ is concave on $(0,\infty)$, and so the asserted inequalities follow. □

4.4. Theorem. *For each* $0 < a < c \leqslant 1$ *and* $K > 1$, *the function* $f(r) = \varphi_K^{a,c}(r)/r^{1/K}$ *is strictly decreasing from* $(0,1]$ *onto* $[1, e^{(1-(1/K))R(a,c-a)/2})$. *In particular,*

$$r^{1/K} < \varphi_K^{a,c}(r) < e^{(1-(1/K))R(a,c-a)/2} r^{1/K}.$$

Also, the function $g(r) = \varphi_{1/K}^{a,c}(r)/r^K$ *is strictly increasing from* $(0,1]$ *onto* $(e^{(1-K)R(a,c-a)/2}, 1]$. *In particular*

$$r^K > \varphi_{1/K}^{a,c}(r) > e^{(1-K)R(a,c-a)/2} r^K.$$

Proof. If $s = \varphi_K^{a,c}(r)$, then $\mu_{a,c}(s) = \mu_{a,c}(r)/K$, and $s > r$, for all $r \in (0,1)$ and $K > 1$. Differentiating we get

$$\frac{f'(r)}{f(r)} = \frac{1}{Kr}\left(\left(\frac{s'\mathcal{K}(s)}{r'\mathcal{K}(r)}\right)^2 \frac{\mathcal{M}(r^2)}{\mathcal{M}(s^2)} - 1\right).$$

This derivative is negative if and only if $(s'^2\mathcal{K}(s)^2)/\mathcal{M}(s^2) \leqslant (r'^2\mathcal{K}(r)^2)/\mathcal{M}(r^2)$, that is, if the function $x \mapsto (x'^2\mathcal{K}(x)^2)/\mathcal{M}(x^2)$ is decreasing. This, however, follows from Theorems 3.12(1) and 3.4(1), as

$$\frac{x'^2\mathcal{K}(x)^2}{\mathcal{M}(x^2)} = \frac{x'}{\mathcal{M}(x^2)}\big(\sqrt{x'}\mathcal{K}(x)\big)^2.$$

Then f is indeed strictly decreasing. By Theorem 3.6(1)

$$\log\big(s/r^{1/K}\big) = \big[\mu(s) + \log(s)\big] - (1/K)\big[\mu(r) + \log(r)\big]$$

tends to $(1-(1/K))R(a, c-a)/2$, as $r \to 0$. The proof for the function g follows the same pattern. □

4.5. Remark. We observe that in Theorem 4.4 for $a = 1/2$, and $c = 1$, the coefficient in the upper bound reduces to the classical constant $4^{1-(1/K)}$ [14].

4.6. Theorem. *Let* $0 < a < c \leqslant 1$ *and* $K \in (1, \infty)$. *Then the function*

(1) *the function* $f_1(r) = \log(\varphi_K(r'))$ *is decreasing and concave on* $(0,1)$. *In particular*

$$\varphi_K(u')\varphi_K(t') \leqslant \varphi_K\left(\sqrt{1 - \left(\frac{u+t}{2}\right)^2}\right)^2,$$

and

$$\varphi_K(u)\varphi_K(t) \leqslant \varphi_K\left(\sqrt{1 - \sqrt{(1-u^2)(1-t^2)}}\right)^2$$

for all $u, t \in (0,1)$, *with equality if and only if* $u = t$.

(2) *The function* $f_2(r) = \log(\varphi_K(r'^2))$ *is decreasing and concave on* $(0,1)$. *In particular*

$$\varphi_K\big(u'^2\big)\varphi_K\big(t'^2\big) \leqslant \varphi_K\left(1 - \left(\frac{u+t}{2}\right)^2\right)^2,$$

and

$$\varphi_K(u)\varphi_K(t) \leqslant \varphi_K\Big(1 - \sqrt{(1-u^2)(1-t^2)}\Big)^2$$

for all $u, t \in (0,1)$, *with equality if and only if* $u = t$.

(3) *The function* $f_3(r) = \log(\varphi_K(1 - e^{-r}))$ *is increasing and concave on* $(0, \infty)$. *In particular*

$$\varphi_K(1-u)\varphi_K(1-t) \leqslant \varphi_K(1 - \sqrt{ut})^2,$$

and

$$\varphi_K(u)\varphi_K(t) \leqslant \varphi_K\Big(1 - \sqrt{(1-u)(1-t)}\Big)^2$$

for all $u, t \in (0,1)$, *with equality if and only if* $u = t$.

Proof. (1) Denote $t=\varphi_{1/K}(r)$. Then we see that $\varphi_K(r')=\sqrt{1-\varphi_{1/K}(r)^2}=\sqrt{1-t^2}=t'$. Now

$$\frac{d(t')}{dr}=\frac{tt'^2\mathcal{K}(t)\mathcal{K}'(t)\mathcal{M}(r^2)}{rr'^2\mathcal{K}(r)\mathcal{K}'(r)\mathcal{M}(t^2)}\cdot\left(-\frac{t}{t'}\right)=-\frac{t't^2\mathcal{K}(t)\mathcal{K}'(t)\mathcal{M}(r^2)}{rr'^2\mathcal{K}(r)\mathcal{K}'(r)\mathcal{M}(t^2)}=-\frac{1}{K}\frac{t't^2\mathcal{K}'(t)^2\mathcal{M}(r^2)}{rr'^2\mathcal{K}'(r)^2\mathcal{M}(t^2)}.$$

Thus

$$\frac{df_1}{dr}=\frac{1}{t'}\frac{d(t')}{dr}=-\frac{1}{K}\left(\frac{t\mathcal{K}'(t)^2}{r\mathcal{K}'(r)^2}\right)\left(\frac{t}{\mathcal{M}(t^2)}\right)\left(\frac{\mathcal{M}(r^2)}{r'^2}\right),$$

where each of the bracketed functions is positive and increasing, by Theorems 4.1(12) and 3.12. Thus df_1/dr is negative and decreasing, and f_1 is decreasing and concave. Then the convexity inequality $f((x+y)/2)\geqslant(f(x)+f(y))/2$ directly yields the first inequality. The rewritten inequality follows from change of variables.

(2) Again, let $t=\varphi_{1/K}(r)$, and $u(r)=\sqrt{2r^2-r^4}$. The function $u(r)$ is easily shown to be increasing in $(0,1)$. Now

$$\begin{aligned}\frac{df_2}{dr}&=\frac{t'(u)t(u)^2\mathcal{K}(t(u))\mathcal{K}'(t(u))\mathcal{M}(u^2)}{u'u^2\mathcal{K}(u)\mathcal{K}'(u)\mathcal{M}(t(u)^2)}\cdot(-2r)\cdot\frac{1}{t'(u)}\\&=-\frac{1}{K}\left(\frac{t(u)\mathcal{K}'(t(u))^2}{u\mathcal{K}'(u)^2}\right)\left(\frac{t(u)}{\mathcal{M}(t(u)^2)}\right)\left(\frac{\mathcal{M}(u^2)}{u'}\right)\left(\frac{2r}{\sqrt{2r^2-r^4}}\right).\end{aligned}$$

Also here all the bracketed functions are positive and increasing, and thus df_2/dr is negative and decreasing, and f_1 is decreasing and concave. The rest of the statement is proved as in (1).

(3) With $x=1-e^{-r}$ and $s=\varphi_K(x)$ we have

$$f_3'(r)=\left(\frac{1-x}{Kx}\right)\left(\frac{s\mathcal{K}(s)^2}{x\mathcal{K}(x)^2}\right)\left(\frac{\mathcal{M}(x^2)}{x'}\right)\left(\frac{s}{\mathcal{M}(s^2)}\right),$$

which is decreasing by Theorem 4.1(6). The rest of the statement is proved as in the previous cases. □

4.7. Theorem. *Let $p:(0,1)\to(-\infty,\infty)$ and $q:(-\infty,\infty)\to(0,1)$ be given by $p(x)=2\log(x/x')$ and $q(x)=p^{-1}(x)=\sqrt{e^x/(e^x+1)}$, respectively, and for $a\in(0,1)$, $c\in(a,1]$, $K\in(1,\infty)$, let $g,h:(-\infty,\infty)\to(-\infty,\infty)$ be defined by $g(x)=p(\varphi_K^{a,c}(q(x)))$ and $h(x)=p(\varphi_{1/K}^{a,c}(q(x)))$. Then*

$$g(x)\geqslant\begin{cases}Kx, & \text{if } x\geqslant 0,\\ \frac{x}{K}, & \text{if } x<0\end{cases}\quad\text{and}\quad h(x)\leqslant\begin{cases}\frac{x}{K}, & \text{if } x\geqslant 0,\\ Kx, & \text{if } x<0.\end{cases}$$

Proof. First, if $x>0$, then

$$\begin{aligned}g(x)\geqslant Kx\quad&\Leftrightarrow\quad\varphi_K^{a,c}\big(q(x)\big)\geqslant q(Kx)\quad\Leftrightarrow\quad\mu_{a,c}^{-1}\left(\frac{1}{K}\mu_{a,c}\left(\sqrt{\frac{e^x}{e^x+1}}\right)\right)\geqslant\sqrt{\frac{e^{Kx}}{e^{Kx}+1}}\\&\Leftrightarrow\quad\mu_{a,c}\left(\sqrt{\frac{e^x}{e^x+1}}\right)\leqslant K\mu_{a,c}\left(\sqrt{\frac{e^{Kx}}{e^{Kx}+1}}\right).\end{aligned}$$

This will be true if $f(K)=K\mu_{a,c}(\sqrt{e^{Kx}/(e^{Kx}+1)})$ is increasing on $[1,\infty)$.

Now, setting $r=\sqrt{e^{Kx}/(e^{Kx}+1)}$, we have $r^2=e^{Kx}/(e^{Kx}+1)$, and $r'^2=1/(e^{Kx}+1)$. Then $f(K)=(2/x)f_6(r)$, where f_6 is as in Theorem 3.6(6), and thus increasing, as $r(K)$ is increasing as a function of K.

Let still $x>0$. Then

$$\begin{aligned}g(-x)\geqslant -x/K\quad&\Leftrightarrow\quad\varphi_K^{a,c}\left(\sqrt{\frac{e^{-x}}{e^{-x}+1}}\right)\geqslant\sqrt{\frac{e^{-x/K}}{e^{-x/K}+1}}\quad\Leftrightarrow\quad\mu_{a,c}^{-1}\left(\frac{1}{K}\mu_{a,c}\left(\sqrt{\frac{1}{e^x+1}}\right)\right)\geqslant\sqrt{\frac{1}{e^{x/K}+1}}\\&\Leftrightarrow\quad\mu_{a,c}\left(\frac{1}{\sqrt{e^x+1}}\right)\leqslant K\mu_{a,c}\left(\frac{1}{\sqrt{e^{x/K}+1}}\right).\end{aligned}$$

This is true if $F(K)=K\mu_{a,c}(1/\sqrt{e^{x/K}+1})$ is increasing on $[1,\infty)$. Let $t=1/\sqrt{e^{x/K}+1}$. Then $t\in(0,1/\sqrt{2})$ and $t^2=1/(e^{x/K}+1)$, $t'^2=e^{x/K}/(e^{x/K}+1)$, $x=2K\log(t'/t)$. Now $f(K)=(B^2/8)(x/f_6(t'))$, where f_6 is as in Theorem 3.6(6), and thus increasing, as $t'(K)$ is decreasing as a function of K. Finally, the proof of $h(x)$ is similar. □

5. Dependence on *c*

In this section we study how the functions $\mu_{a,c}$, $\mu_{a,c}^{-1}$ and $\varphi_K^{a,c}$ depend on the parameter c. corresponding results for the case $c = 1$ can be found in the articles [3,16].

5.1. Notation. For $0 < a < c$ and $t > 0$ we denote

$$P(a,c,t) = \Psi(c-a+t) - \Psi(c+t),$$
$$A = A_t = A(a,c,t) = \frac{(c-a,t)}{(c,t)} = \frac{\Gamma(c-a+t)\Gamma(c)}{\Gamma(c+t)\Gamma(c-a)},$$
$$\tilde{A} = \tilde{A}_t = \tilde{A}(a,c,t) = (a,t)A_t,$$

and

$$B = B_t = B(a,c,t) = P(a,c,t) - P(a,c,0).$$

5.2. Lemma. *Let f, g, and h be real valued functions defined on $[0,\infty)$ such that f is strictly increasing, f' is strictly decreasing, $0 < g(x) < h(x)$, and $g'(x) \geqslant h'(x) > 0$ for all $x \in [0,\infty)$. Let $F(x) = f(g(x)) - f(h(x))$. Then*

(1) *F is strictly increasing on $[0,\infty)$.*

In particular, with notation as in 5.1, the function B is strictly increasing in t, so that $B(a,c,t) \geqslant 0$ with equality if and only if $t = 0$.

(2) $\partial A/\partial c = AB$.

Proof. (1) By the assumptions,

$$F'(x) = f'\big(g(x)\big)g'(x) - f'\big(h(x)\big)h'(x) > f'\big(g(x)\big)g'(x) - f'\big(g(x)\big)g'(x) = 0.$$

We now take $f = \Psi$, $g(x) = c-a+x$, and $h(x) = c+x$. Then by the above, $F(x) = \Psi(c-a+x) - \Psi(c+x)$ is strictly increasing on $[0,\infty)$ so that $F(x) - F(0) \geqslant 0$ with equality if and only if $x = 0$. By the definition of B, this means that $B(a,c,t) \geqslant 0$ with equality if and only if $t = 0$.

(2) By logarithmic differentiation we get

$$\frac{\partial A/\partial c}{A} = \frac{\Gamma'(c-a+t)}{\Gamma(c-a+t)} - \frac{\Gamma'(c+t)}{\Gamma(c+t)} - \left(\frac{\Gamma'(c-a)}{\Gamma(c-a)} - \frac{\Gamma'(c)}{\Gamma(c)}\right)$$
$$= \Psi(c-a+t) - \Psi(c+t) - \big(\Psi(c-a) - \Psi(c)\big) = P(a,c,t) - P(a,c,0) = B(a,c,t). \qquad \square$$

5.3. Theorem. *For $a > 0$ and $x, y \in (0,1)$, the function f defined on (a,∞) by*

$$f(c) = B(a,c-a)\frac{F(a,c-a;c;x)}{F(a,c-a;c;y)}$$

is strictly decreasing from (a,∞) onto $(0,\infty)$.

Proof. First, since

$$1 \leqslant F(a,c-a;c;x) \leqslant 1 + \frac{c-a}{c}F(a,1;1;x),$$

it follows that $F(a;c-a;c;x) \to 1$ as $c \to a^+$. Hence

$$f(a+) = \lim_{c\to a+} B(a,c-a) = \lim_{c\to a+} \Gamma(c-a) = \infty.$$

Next, we note that for $n \geqslant 1$, $c \mapsto (c-a,n)/(c,n)$ is increasing by [4, 1.58(32)] with limit 1 as $c \to \infty$. Hence, using [4, 1.20(1)], we get

$$F(a,c-a;c;r) \leqslant F(a,1;1;r) = (1-r)^{-a}. \tag{5.4}$$

Now let $F(c,r)=F(a,c-a;c;r)$, $h(r)=(1-r)^{-a}$ and

$$F_n(c,r)=\sum_{k=0}^{n}\frac{(a,k)(c-a,k)}{(c,k)}\frac{r^k}{k!}\quad\text{and}\quad h_n(r)=\sum_{k=0}^{n}(a,k)\frac{r^k}{k!}.$$

Then let $r\in(0,1)$ and $\varepsilon>0$. Now let m_0 be such that $h(r)-h_m(r)<\varepsilon$ for all $m>m_0$. Then there exists a c_0 such that when $c>c_0$ and all $0\leqslant m\leqslant m_0$ we have $(c-a,m)/(c,m)>1-\varepsilon$. Thus, for $c>c_0$ and $p=m_0$ we have

$$\begin{aligned}F(c,r)>F_p(c,r)&=\sum_{n=0}^{p}\frac{(a,n)(c-a,n)}{(c,n)}\frac{r^n}{n!}>(1-\varepsilon)\sum_{n=0}^{p}(a,n)\frac{r^n}{n!}\\&=(1-\varepsilon)h_p(r)>(1-\varepsilon)\big(h(r)-\varepsilon\big).\end{aligned}$$

From this and (5.4) we see that $F(a,c-a;c;r)\to(1-r)^{-a}$ as $c\to\infty$. Applying for $z=x,y$ as $c\to\infty$, we see that $F(a,c-a;c,x)/F(a,c-a;c;y)\to((1-x)/(1-y))^{-a}$, which is finite. As $c\to\infty$, by Stirling's formula [18, 12.33],

$$\frac{B(a,c-a)}{\Gamma(a)}=\frac{\Gamma(c-a)}{\Gamma(c)}\sim\left(\left(1-\frac{a}{c}\right)^{c-(1/2)}\right)\left(\frac{e}{c-a}\right)^{a}\to\big(e^{-a}\big)(0)=0.$$

Hence $f(c)\to0$, as $c\to\infty$.

Logarithmic differentiation together with the Notation 5.1 and Lemma 5.2(2) yield

$$\begin{aligned}\frac{f'(c)}{f(c)}&=\frac{1}{f(c)}\frac{\partial}{\partial c}\left(B(a,c-a)\frac{F(a,c-a;c;x)}{F(a,c-a;c;y)}\right)\\&=\frac{1}{f(c)}\left(\left(\frac{\partial}{\partial c}\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\right)\frac{F(a,c-a;c;x)}{F(a,c-a;c;y)}+\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\frac{\partial}{\partial c}\frac{F(a,c-a;c;x)}{F(a,c-a;c;y)}\right)\\&=\frac{1}{f(c)}\left(\frac{\Gamma'(c-a)\Gamma(a)\Gamma(c)-\Gamma'(c)\Gamma(a)\Gamma(c-a)}{\Gamma(c)^2}\frac{F(a,c-a;c;x)}{F(a,c-a;c;y)}\right.\\&\quad+\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\frac{1}{F(a,c-a;c;y)^2}\left(\left(\frac{\partial}{\partial c}F(a,c-a;c;x)\right)F(a,c-a;c;y)\right.\\&\quad\left.\left.-\left(\frac{\partial}{\partial c}F(a,c-a;c;y)\right)F(a,c-a;c;x)\right)\right)\\&=\frac{\Gamma'(c-a)\Gamma(a)\Gamma(c)-\Gamma'(c)\Gamma(a)\Gamma(c-a)}{\Gamma(c)^2}\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\\&\quad+\frac{1}{F(a,c-a;c;x)}\sum_{n=0}^{\infty}\left(\frac{\partial}{\partial c}\tilde A_n\right)\frac{x^n}{n!}-\frac{1}{F(a,c-a;c;y)}\sum_{n=0}^{\infty}\left(\frac{\partial}{\partial c}\tilde A_n\right)\frac{y^n}{n!}\\&=\Psi(c-a)-\Psi(c)+\frac{1}{F(a,c-a;c;x)}\sum_{n=0}^{\infty}\tilde A_nB_n\frac{x^n}{n!}-\frac{1}{F(a,c-a;c;y)}\sum_{n=0}^{\infty}\tilde A_nB_n\frac{y^n}{n!}.\end{aligned}$$

It follows that

$$\begin{aligned}h(c)&=\frac{1}{B(a,c-a)}F(a,c-a;c;y)^2f'(c)\\&=F(a,c-a;c;y)F(a,c-a;c;x)\big(\Psi(c-a)-\Psi(c)\big)+F(a,c-a;c;y)\sum_{n=0}^{\infty}\frac{\tilde A_nB_n}{n!}x^n\\&\quad-F(a,c-a;c;x)\sum_{n=0}^{\infty}\frac{\tilde A_nB_n}{n!}y^n\\&=\big(\Psi(c-a)-\Psi(c)\big)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\tilde A_n\tilde A_m}{n!m!}y^nx^m+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\tilde A_n\tilde A_mB_m}{n!m!}x^my^n-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\tilde A_n\tilde A_mB_m}{n!m!}y^mx^n\end{aligned}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\tilde{A}_n \tilde{A}_m}{n!m!} G_{m,n}(a,c,r)(xy)^m,$$

where

$$G_{m,n}(a,c,r) = \big(\Psi(c-a) - \Psi(c)\big)y^{n-m} + B_m y^{n-m} - B_m x^{n-m}$$
$$= y^{n-m}\big(\Psi(c-a+m) - \Psi(c+m)\big) - B_m x^{n-m}.$$

Since Ψ is strictly increasing, we have $\Psi(c-a+m) - \Psi(c+m) < 0$ and by Lemma 5.2(1), $B_m \geqslant 0$. Hence $G_{m,n}(a,c,r) < 0$. It follows that $h(c) < 0$ and as $B(a, c-a) = \Gamma(a)\Gamma(c-a)/\Gamma(c) > 0$ for $0 < a < c$, we get that $f'(c) < 0$ for $c \in (a,\infty)$. □

5.5. Corollary. *For $a > 0$ and $r \in (0,1)$ the function $\tilde{f}(c)$ defined on (a,∞) by $\tilde{f}(c) = \mu_{a,c}(r)$ is strictly decreasing from (a,∞) onto $(0,\infty)$ with $\tilde{f}(1) = \mu_a(r)$ if $a < 1$.*

5.6. Lemma. *Let $z = f(x,y) = f_x(y) = f_y(x)$ be continuously differentiable for x and y in some real intervals. Suppose that $(\partial f/\partial x)(\partial f/\partial y) > 0$. Let $y = f_x^{-1}(z) = g(x,z)$. Then*

$$\frac{\partial y}{\partial x} = \frac{\partial g}{\partial x} < 0.$$

Proof. By implicit differentiation partial to x, we get

$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

Hence

$$\frac{\partial g}{\partial x} = -\frac{\partial f/\partial x}{\partial f/\partial y} < 0. \qquad \square$$

5.7. Theorem. *Let $a, x > 0$ be fixed. Then the function*

$$g : c \mapsto \mu_{a,c}^{-1}(x)$$

is strictly decreasing from (a,∞) onto $(0,1)$ with $g(1) = \mu_a^{-1}(x)$ if $a < 1$.

Proof. Denote $r = \mu_{a,c}^{-1}(x) = h(c,x)$. Then $x = \mu_{a,c}(r) = f(c,r)$. Now $\partial f/\partial r < 0$ and by Corollary 5.5 $\partial f/\partial c < 0$, so that $\partial g/\partial c < 0$ and the monotonicity of g follows from Lemma 5.6.

Since $\mu_{a,1} = \mu_a$, we get

$$x = \mu_a\big(\mu_a^{-1}(x)\big) = \mu_{a,1}\big(\mu_a^{-1}(x)\big)$$

so that

$$g(1) = \mu_{a,1}^{-1}(x) = \mu_a^{-1}(x).$$

We claim that $\lim_{c\to\infty} h(c,x) = 0$. Assume on the contrary that $\lim_{c\to\infty} h(c,x) = r_0 > 0$. Then $h(c,x) > r_0$ for all $c \in (a,\infty)$. Hence

$$x = \mu_{a,c}\big(h(c,x)\big) < \mu_{a,c}(r_0).$$

Letting $c \to \infty$, Corollary 5.5 implies that $x \leqslant 0$, which is a contradiction.

It remains to show that $h(a^+, x) = 1$. Suppose that $h(a^+, x) = r_0 \in (0,1)$. Then $h(c,x) < r_0$ for all $c \in (a,\infty)$. Hence

$$x = \mu_{a,c}\big(h(c,x)\big) > \mu_{a,c}(r_0).$$

Letting $c \to a^+$, we get, by Theorem 5.5, that $x = \infty$ which is a contradiction. Thus $h(a^+, x) = 1$. □

5.8. Theorem. *Let $a, r \in (0,1)$ and $K \in (1,\infty)$ be fixed. Then the function*

$$c \mapsto \varphi_K^{a,c}(r)$$

is strictly decreasing from $(a,1]$ onto $[\varphi_K^a(r), 1)$ and the function

$$c \mapsto \varphi_{1/K}^{a,c}(r)$$

is strictly increasing from $(a,1]$ onto $(0, \varphi_{1/K}^a(r)]$.

Proof. It is obvious (see [12, Remark 4.12]) that we have $\varphi_K^{a,c} = \tilde{\mu}_{a,c}^{-1}(\tilde{\mu}_{a,c}(r)/K)$, where

$$\tilde{\mu}_{a,c}(r) = \frac{F(a, c-a; c, r'^2)}{F(a, c-a; c; r^2)}.$$

Denote $s = \varphi_K^{a,c}(r)$ and

$$Q(a,c,r) = F\big(a, c-a; c; r^2\big).$$

By definition,

$$\frac{Q(a,c,s')}{Q(a,c,s)} = \frac{1}{K}\frac{Q(a,c,r')}{Q(a,c,r)}. \tag{5.9}$$

We apply logarithmic differentiation with respect to c to (5.9) and get

$$\frac{1}{Q(a,c,s')}\left(\frac{\partial Q(a,c,s')}{\partial c} - \frac{\partial Q(a,c,s')}{\partial s'}\frac{s}{s'}\frac{\partial s}{\partial c}\right) - \frac{1}{Q(a,c,s)}\left(\frac{\partial Q(a,c,s)}{\partial c} + \frac{\partial Q(a,c,s)}{\partial s}\frac{\partial s}{\partial c}\right)$$
$$= \frac{1}{Q(a,c,r')}\frac{\partial Q(a,c,r')}{\partial c} - \frac{1}{Q(a,c,r)}\frac{\partial Q(a,c,r)}{\partial c},$$

which is equivalent to

$$\left(\frac{\partial Q(a,c,s')}{\partial s'}\frac{s}{s'}\frac{1}{Q(a,c,s')} + \frac{1}{Q(a,c,s)}\frac{\partial Q(a,c,s)}{\partial s}\right)\frac{\partial s}{\partial c}$$
$$= \big(Q_1(a,c,s') - Q_1(a,c,r')\big) + \big(Q_1(a,c,r) - Q_1(a,c,s)\big), \tag{5.10}$$

where

$$Q_1(a,c,x) = \frac{1}{Q(a,c,x)}\frac{\partial Q(a,c,x)}{\partial c}.$$

Then for $0 < a < c \leqslant 1$ we get that

$$\frac{\partial Q(a,c,x)}{\partial x} = \frac{2a(c-a)}{c} x F\big(a+1, c-a+1; c+1; x^2\big) > 0$$

for all $x \in (0,1)$. Hence the coefficient of $\partial s/\partial c$ in (5.10) is positive. We turn our attention to the right-hand side of (5.10). By Lemma 5.2(2) we have that

$$Q_1(a,c,x) = \frac{\sum_{n=0}^{\infty} \frac{\tilde{A}_n B_n}{n!} r^{2n}}{\sum_{n=0}^{\infty} \frac{\tilde{A}_n}{n!} r^{2n}} = \frac{\sum_{n=0}^{\infty} \alpha_n r^{2n}}{\sum_{n=0}^{\infty} \beta_n r^{2n}},$$

where $\alpha_n = \tilde{A}_n B_n/n!$ and $\beta_n = \tilde{A}_n/n!$. Then

$$\frac{\alpha_n}{\beta_n} = B_n = \Psi(c) - \Psi(c-a) - \big(\Psi(c+n) - \Psi(c-a+n)\big),$$

where, by (2.4),

$$-\big(\Psi(c+n)-\Psi(c-a+n)\big)=\frac{1}{c+n}-\sum_{k=1}^{\infty}\frac{c+n}{k(k+c+n)}-\frac{1}{c-a+n}+\sum_{k=1}^{\infty}\frac{c-a+n}{k(k+c-a+n)}$$
$$=-\frac{a}{(c+n)(c-a+n)}+\sum_{k=1}^{\infty}\frac{-a}{(k+c+n)(k+c-a+n)}$$
$$=-a\sum_{k=0}^{\infty}\frac{1}{(k+c+n)(k+c-a+n)},$$

which is clearly increasing in n. Hence α_n/β_n is increasing in n and [12, Theorem 4.4] implies that $Q_1(a,c,x)$ is strictly increasing in x. Since $K>1$, it is immediate that $s>r$ and $r'>s'$ and it follows that the right-hand side of (5.10) is negative. Hence $\partial s/\partial c<0$, which proves the first monotonicity claim. On the other hand, if $s=\varphi_{1/K}^{a,c}(r)$, then $s<r$ and $r'<s'$ and the right-hand side of (5.10) together with $\partial s/\partial c$ are positive and the second monotonicity claim follows.

It remains to consider the ranges of the functions. The values at $c=1$ follow from the fact that for all $k>0$,

$$\tilde{\varphi}_k^{a,1}(r)=\varphi_k^a(r). \tag{5.11}$$

To show that (5.11) holds, we write

$$\mu_a\big(\tilde{\mu}_{a,1}^{-1}(t)\big)=\frac{\pi}{2\sin(\pi a)}\tilde{\mu}_{a,1}\big(\tilde{\mu}_{a,1}^{-1}(t)\big)=\frac{\pi}{2\sin(\pi a)}t$$

and put $t=\tilde{\mu}_{a,1}(r)/k$ to get

$$\mu_a\big(\tilde{\mu}_{a,1}^{-1}\big(\tilde{\mu}_{a,1}(r)/k\big)\big)=\frac{\pi}{2\sin(\pi a)}\frac{\tilde{\mu}_{a,1}(r)}{k}=\mu_a(r)/k$$

which implies (5.11).

To conclude the proof we need to show that as $c\to a+$, $\tilde{\varphi}_K^{a,c}(r)\nearrow 1$ and $\tilde{\varphi}_{1/K}^{a,c}(r)\searrow 0$. We prove the first fact and note that the proof of the second one is similar. Let $L=\tilde{\varphi}_K^{a,a+}(r)$. Assume that $L<1$. By the monotonicity in c, it follows that $\tilde{\varphi}_K^{a,c}(r)<L$ for all $c\in(a,1]$. Hence $\tilde{\mu}_{a,c}(L)<\tilde{\mu}_{a,c}(r)/K$, so that

$$\frac{\tilde{\mu}_{a,c}(L)}{\tilde{\mu}_{a,c}(r)}<1/K.$$

Letting $c\to a+$, we get $1/K\geqslant 1$, which is a contradiction, since $K>1$. Hence $L=1$. □

5.12. Theorem. *For $a,r\in(0,1)$, let f and g be functions defined on (a,∞) by*

(1) $f(c)=\mathcal{K}_{a,c}-(B/2)$,
(2) $g(c)=(B/2)-\mathcal{E}_{a,c}$, *where $B=B(a,c-a)$. Then, both f and g are strictly decreasing, with $f(a+)=\log(1/r')$, $f(\infty)=0=g(\infty)$,*

$$g(a+)=\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{r^{2n}}{a+n-1}\right)-\log(1/r').$$

Proof. The assertion $f(\infty)=0=g(\infty)$ follows immediately from Stirling's formula, as in the proof of Theorem 5.3 (cf. [4, 1.49]). Next, the coefficient of r^{2n} in the Maclaurin series of $f(c)$ is

$$f_n(c)=\Gamma(a+n)\Gamma(c-a+n)/\big(2(n!)\Gamma(c+n)\big),$$

so that $f_n'(c)/f_n(c)=\Psi(c-a+n)-\Psi(c+n)<0$, since Ψ is strictly increasing. Similarly, it can be shown that an analogous assertion holds for $g(c)$, thus proving the monotonicity of these functions. Finally,

$$f(a+)=\sum_{n=1}^{\infty}\frac{r^{2n}}{2n}=\log(1/r'),$$

and

$$g(a+) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(1-a)r^{2n}}{n(a+n-1)} = -\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{n} - \frac{r^{2n}}{a+n-1}\right) = \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{r^{2n}}{a+n-1}\right) - \log(1/r'). \qquad \square$$

Finally, we make some conjectures regarding the behavior of the Legendre $\mathcal{M}$-function combined with other functions. Such problems seem to be quite difficult, and apart from the functions in Theorem 3.12 and immediate consequences, we are not aware of any results in this direction. In particular, solving any one of the following problems immediately yields several interesting functional inequalities generalizing those stated in [3, 1.14, 1.15].

5.13. Conjecture. *Based on experimental evidence, we make the following conjectures.*

(1) *Let* $0 < a < c < 1$. *Then the function* $f(r) = \sqrt{r}/\mathcal{M}(r^2)$ *is strictly increasing from* $(0,1)$ *onto* $(0,B)$, *and* $g(r) = \sqrt{r'}/\mathcal{M}(r^2)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,B)$.

(2) *Let* $0 < a < c < 1$, $K > 1$, *and* $s = \varphi_K^{a,c}(r)$. *Then the function*
  (i) $f_1(r) = (s\mathcal{M}(r^2))/(r\mathcal{M}(s^2))$ *is decreasing from* $(0,1)$ *onto* $(1,\infty)$.
  (ii) $f_2(r) = (s'\mathcal{M}(r^2))/(r'\mathcal{M}(s^2))$ *is decreasing from* $(0,1)$ *onto* $(0,1)$.
  (iii) $f_3(r) = (\mathcal{K}(r)\mathcal{M}(r^2))/(\mathcal{K}(s)\mathcal{M}(s^2))$ *is decreasing from* $(0,1)$ *onto* $(1/K,1)$.
  (iv) $f_4(r) = (\mathcal{K}'(r)\mathcal{M}(r^2))/(\mathcal{K}'(s)\mathcal{M}(s^2))$ *is decreasing from* $(0,1)$ *onto* $(1,K)$.

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