Difference equations for the co-recursive $r$th associated orthogonal polynomials of the $D_q$-Laguerre–Hahn class

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Abstract

We use some relations between the $r$th associated orthogonal polynomials of the $D_q$-Laguerre–Hahn class to derive the fourth-order $q$-difference equation satisfied by the co-recursive $r$th associated orthogonal polynomials of the $D_q$-Laguerre–Hahn class.

When $r = 1$ and for $q$-semi-classical situations, this $q$-difference equation factorizes as product of two second-order $q$-difference equations. Finally, we study some classical situations, and give some examples relative to the co-recursive associated discrete $q$-Hermite II orthogonal polynomials.

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1. Introduction

Let $\mathcal{U}$ be a regular linear functional on the linear space $\mathcal{P}$ of the polynomials of real variable and $(P_n)_n$ the sequence of monic polynomials orthogonal with respect to $\mathcal{U}$ (see [7] for more details). As any standard orthogonal polynomial family, $(P_n)_n$ satisfies a three-terms recurrence relation

\[ P_{n+1} = (x - \beta_n)P_n - \gamma_n P_{n-1}, \quad n \geq 1, \quad P_0 = 1, \quad P_1 = x - \beta_0, \quad (1) \]

where $\beta_n$ and $\gamma_n$ are complex numbers with $\gamma_n \neq 0 \quad \forall n$. We assume that the linear functionals used in this paper are normalized by: $\langle \mathcal{U}, P_0^2 \rangle = \gamma_0 = 1$.

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• \(\mathcal{U}\) and the corresponding monic orthogonal polynomials family are said to be of the \(\mathcal{D}_q\)-Laguerre–Hahn class if the Stieltjes function \(S(\mathcal{U})\) of \(\mathcal{U}\) satisfies a \(\mathcal{D}_q\)-Riccati \(q\)-difference equation

\[
\phi(qx)\mathcal{D}_qS(x) = G(x,q)S(x)\mathcal{G}_qS(x) + E(x,q)S(x) + F(x,q)\mathcal{G}_qS(x) + H(x,q),
\]

where \(\phi \neq 0, G, E, F, G\) and \(H\) are polynomials in the variable \(x\) and the operators \(\mathcal{D}_q\) and \(\mathcal{G}_q\) are defined by

\[
\mathcal{D}_qP(x) = \frac{P(qx) - P(x)}{x(q - 1)}, \quad \mathcal{G}_qP(x) = P(qx).
\]

The \(q\)-orthogonal polynomials were treated in the thesis of Medem [20] (see also [21]); the peculiar \(q\)-Laguerre–Hahn class was introduced also by Medem [20], and developed in detail in the thesis of Foupouagnigni [10] (see also [13,14] as examples of this class). When \(G = 0\), (2) becomes linear and this correspond to the \(q\)-semi-classical situation. The \(q\)-classical and \(q\)-semi-classical orthogonal polynomials appear, beside the two aforementioned thesis, in Refs. [1,13,14,18,21].

We will from now denote orthogonal polynomials by \(\mathcal{OP}\) and Laguerre–Hahn by \(\mathcal{LH}\).

We define the co-recursive [6] \((P_n^{[\mu]}\) of \((P_n)\) and the \(r\)th associated \((P_n^{(r)}\) of \((P_n)\) as the two families of monic polynomials defined by the following three terms recurrence relations obtained by modifying (1):

\[
P_{n+1}^{[\mu]} = (x - \beta_n)P_n^{[\mu]} - \gamma_n P_{n-1}^{[\mu]}, \quad n \geq 1, \quad P_0^{[\mu]} = 1, \quad P_1^{[\mu]} = x - \beta_0 - \mu,
\]

\[
P_{n+1}^{(r)} = (x - \beta_{n+r})P_n^{(r)} - \gamma_{n+r} P_{n-1}^{(r)}, \quad n \geq 1, \quad P_0^{(r)} = 1, \quad P_1^{(r)} = x - \beta_r.
\]

where \(\mu\) is a complex number.

The co-recursive \(r\)th associated \((P_n^{(r,\mu)}\) of \((P_n)\) is defined as the co-recursive of the \(r\)th associated \((P_n^{(r)}\) of \((P_n)\). This family satisfies the relation

\[
P_{n+1}^{(r,\mu)} = (x - \beta_{n+r})P_n^{(r,\mu)} - \gamma_{n+r} P_{n-1}^{(r,\mu)}, \quad n \geq 1, \quad P_0^{(r,\mu)} = 1, \quad P_1^{(r,\mu)} = x - \beta_r - \mu.
\]

These families, by Favard theorem [7], are orthogonal. We denote by \(\mathcal{U}^{[\mu]}\), \(\mathcal{U}^{(r)}\) and \(\mathcal{U}^{(r,\mu)}\), respectively, the regular normalized functionals associated with these \(\mathcal{OP}\) families.

Obviously, we have the relations

\[
P_n^{[0,\mu]} = P_n^{[\mu]}, \quad P_n^{(r,0)} = P_n^{(r)}.
\]

The families \((P_n^{[\mu]}\)\), \((P_n^{(r)}\) and \((P_n^{(r,\mu)}\) belong, in general, to the LH class if \((P_n)\) belongs to the LH class [9,10,19,23,24]. As a consequence, any polynomial \(P_n^{[\mu]}\), \(P_n^{(r)}\) and \(P_n^{(r,\mu)}\) satisfy a fourth-order differential or difference or \(q\)-difference equation with polynomial coefficients.

The fourth-order differential or difference equation satisfied by \(P_n^{[\mu]}\) was given in [23,24] for classical continuous \(\mathcal{OP}\); and for classical discrete \(\mathcal{OP}\) in [16]. This equation for \(q\)-classical \(\mathcal{OP}\) was given in [12].

Differential, difference and \(q\)-difference equations satisfied by the \(r\)th associated \(\mathcal{OP}\) of the LH class were given in details in [4,10,11,13,14].

On the other hand, the fourth-order differential or difference equation satisfied by the co-recursive associated \(\mathcal{OP}\) was given in [15] for Laguerre and Jacobi \(\mathcal{OP}\); and for Meixner and Charlier \(\mathcal{OP}\) in [16].
In this work, we first prove that the co-recursive $r$th associated $D_q$-LH orthogonal polynomials is a $D_q$-LH orthogonal polynomials; and use relations between the $P_n$, $P_n^{[ν]}$, $P_n^{(r)}$, the co-recursive associated, $P_n^{(r;ν)}$ of $P_n$ and the $q$-difference equations satisfied by the associated OP of the $D_q$-LH class \[ 10,13,11,14 \] to derive the fourth-order $q$-difference equation satisfied by the co-recursive $r$th associated OP of the $D_q$-Laguerre–Hahn class.

This $q$-difference equation is given explicitly for the co-recursive first associated classical OP and also for the co-recursive $r$th associated discrete $q$-Hermite II OP.

The $q$-difference or differential equations obtained in the framework of this paper can be used to:

- Solve connection coefficients and linearization problems;
- To prove that the co-recursive $r$th associated of $q$-classical OP are of $D_q$-Laguerre–Hahn class but neither $q$-classical nor $q$-semi-classical (for $μ ≠ 0$ and $r ≥ 1$).

2. $q$-Difference equations for $P_n^{(r;ν)}$

2.1. The co-recursive $r$th associated orthogonal polynomials

We state and prove the following lemma.

**Lemma 1.** The co-recursive $r$th associated OP of the $D_q$-LH class is of $D_q$-LH class.

**Proof.** Let $U$ be a regular functional of the $D_q$-LH class and let $(P_n)_n$ be the monic family orthogonal with respect to $U$. Let $S$ (resp. $S_r$ and $S_{r;ν}$) be the Stieltjes function of the functional $U$ (resp. $U^{(r)}$ and $U^{(r;ν)}$). It is well known \[ 10,13 \] that when $U$ is of $D_q$-LH class, then $U^{(r)}$ is of $D_q$-LH. The Stieltjes function $S_r$ of $U^{(r)}$ therefore satisfies the Riccati $q$-difference equation:

$$
φ(qx)D_qS_r(x) = G_r(x,q)S_r(x)G_qS_r(x) + E_r(x,q)S_r(x) + F_r(x,q)G_qS_r(x) + H_r(x,q),
$$

where $φ ≠ 0$, $G_r$, $E_r$, $F_r$, $G_r$ and $H_r$ are polynomials. It shall be mentioned that $S = S_0$ satisfies

$$
φ(qx)D_qS(x) = G_0(x,q)S(x)G_qS(x) + E_0(x,q)S(x) + F_0(x,q)G_qS(x) + H_0(x,q),
$$

with the coefficients of the previous equation and those of (2) related by

$$
G_0 = G, \quad E_0 = E, \quad F_0 = F, \quad H_0 = H.
$$

We use the relation linking $S_r$ and $S_{r;ν} = S(U^{(r;ν)})$ \[ 22 \]

$$
S_{r;ν} = \frac{S_r}{1 + νS_r}
$$

and Eq. (5) to get the Riccati $q$-difference equation satisfied by $S_{r;ν}:

$$
φ(qx)D_qS_{r;ν}(x)
= G_{r;ν}(x,q)S_{r;ν}(x)G_qS_{r;ν}(x) + E_{r;ν}(x,q)S_{r;ν}(x) + F_{r;ν}(x,q)G_qS_{r;ν}(x) + H_{r;ν}(x,q),
$$

(6)
with
\[ G_{r,\mu} = G_r - \mu(E + F_r) + \mu^2 H_r, \quad E_{r,\mu} = E_r - \mu H_r, \quad F_{r,\mu} = F_r - \mu H_r, \quad H_{r,\mu} = H_r. \] (7)

\( G^{(r,\mu)} \) is therefore of the \( \mathcal{D}_q \)-LH class. As a consequence, any polynomial \( P_n^{(r,\mu)} \) satisfies a fourth-order linear \( q \)-difference equation with polynomial coefficients. □

2.2. The coupled equations for \( P_n^{(r,\mu)} \)

To derive these fourth-order \( q \)-difference equations, we first recall the following needed lemma giving the coupled equations linking the associated OP of the \( \mathcal{D}_q \)-LH class \( P_n^{(r)} \) and \( P_n^{(r+1)} \).

**Lemma 2** (Foupouagnigni [10], Foupouagnigni et al. [13]). If \( (P_n)_n \) denotes the sequence of monic OP of the \( \mathcal{D}_q \)-LH class, then the \( r \)th associated \( (P_n^{(r)})_n \) and \( (P_n^{(r+1)})_n \) of \( (P_n)_n \) satisfy
\[
D_{r,n}^q[P_n^{(r+1)}] = N_{r+1,n-1}^q[P_n^{(r+1)}],
\]
(8)
\[
D_{r+1,n-1}^q[P_n^{(r+1)}] = N_{r,n}^q[P_n^{(r)}],
\]
(9)
where the \( q \)-difference operators are given by
\[
D_{r,n}^q = a_2(r,n,x)\mathcal{G}_{q}^2 + a_1(r,n,x)\mathcal{G}_q + a_0(r,n,x), \quad N_{r+1,n-1}^q = \tilde{a}_1(r,n,x)\mathcal{G}_q + \tilde{a}_0(r,n,x),
\]
\[
D_{r+1,n-1}^q = b_2(r,n,x)\mathcal{G}_{q}^2 + b_1(r,n,x)\mathcal{G}_q + b_0(r,n,x), \quad N_{r,n}^q = \tilde{b}_1(r,n,x)\mathcal{G}_q + \tilde{b}_0(r,n,x).
\]
The coefficients \( a_j(r,n,x), \tilde{a}_j(r,n,x), b_j(r,n,x) \) and \( \tilde{b}_j(r,n,x) \) are given by
\[
a_2 = K_{3,0}(K_{1,1}K_{7,1} - K_{3,1}K_{8,1}), \quad a_1 = -K_{2,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}),
\]
\[
a_0 = K_{3,1}(K_{2,0}K_{2,1} + K_{4,1}K_{6,0}), \quad \tilde{a}_1 = K_{4,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}), \quad \tilde{a}_0 = -K_{3,1}(K_{2,1}K_{4,0} + K_{4,1}K_{5,0}),
\]
\[
b_2 = K_{3,0}(K_{1,1}K_{7,1} - K_{3,1}K_{8,1}), \quad b_1 = -K_{5,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}),
\]
\[
b_0 = K_{3,1}(K_{5,0}K_{5,1} + K_{4,0}K_{6,1}), \quad \tilde{b}_1 = K_{6,1}(K_{3,0}K_{7,1} + K_{1,0}K_{3,1}), \quad \tilde{b}_0 = -K_{3,1}(K_{5,1}K_{6,0} + K_{6,1}K_{2,0})
\]
(10)
with coefficients \( K_{i,j} \) given by
\[
K_{i,0}(r,n,q;x) = K_i(r,n,q;x) \quad \text{and} \quad K_{i,j}(r,n,q;x) = K_i(r,n,q;q^jx)
\]
and
\[
K_1(r,n,q;x) = \frac{\phi(qx)}{(q-1)x} + E_{n+r+1}(x,q), \quad K_2(r,n,q;x) = \frac{\phi(qx)}{(q-1)x} - F_r(x,q),
\]
\[
K_3(r,n,q;x) = H_{n+r}(x,q), \quad K_4(r,n,q;x) = \begin{cases} \gamma_r \frac{H_{r-1}(x,q)}{\gamma_{r-1}} & \text{if } r \geq 1, \\ \gamma_0 G_0 & \text{if } r = 0, \end{cases}
\]
The proof is obtained using the fact that for any NOOxed that relation (12) is given in [8] but only for the case where coupled relations linking the polynomials families (\( P_n \))

\[
K_5(r, n, q; x) = \frac{\phi(qx)}{(q - 1)x} + E_r(x, q), \quad K_6(r, n, q; x) = -\frac{H_r(x, q)}{\gamma_r},
\]

\[
K_7(r, n, q; x) = \frac{\phi(qx)}{(q - 1)x} - F_{n+r+1}(x, q), \quad K_8(r, n, q; x) = -\gamma_{n+r+1}\frac{H_{n+r+1}(x, q)}{\gamma_{n+r+1}}.
\]

(11)

Secondly, we state and prove the following proposition giving a link between the \( r \)th associated and the co-recursive \( r \)th associated OP.

**Proposition 3.** Given a family of monic OP \((P_n)_n\), its \( r \)th associated \( P_n^{(r)} \) and its co-recursive \( r \)th associated \( P_n^{(r, \mu)} \) satisfy the following relations:

\[
P_n^{(r, \mu)} = P_n^{(r)} - \mu P_{n-1}^{(r+1)},
\]

(12)

\[
P_n^{(r+1, \mu)} = \frac{\mu}{\gamma_{r+1}} P_n^{(r)} + \left( 1 - \frac{\mu(x - \beta_r)}{\gamma_{r+1}} \right) P_{n+1}^{(r+1)},
\]

(13)

where \( \beta_n \) and \( \gamma_n \) are the coefficients of the recurrence relation satisfied by \((P_n)_n\) (see (1)).

**Proof.** The proof is obtained using the fact that for any fixed \( x \), the sequences \((P_n^{(r)}(x))_n\), \((P_{n-1}^{(r+1)}(x))_n\) and \((P_n^{(r, \mu)}(x))_n\) satisfy the same second-order difference equation (see (1)). Since the set of solutions of this second-order difference equations is a two-dimensional vector space containing \((P_n^{(r)}(x))_n\), \((P_{n-1}^{(r+1)}(x))_n\) and \((P_n^{(r, \mu)}(x))_n\), taking care of the fact that the families \((P_n^{(r)}(x))_n\) and \((P_{n-1}^{(r+1)}(x))_n\) are linearly independent, we deduce that there exists two constants \( C_1(x) \), \( C_2(x) \) such that

\[
P_n^{(r, \mu)}(x) = C_1(x)P_n^{(r)}(x) + C_2(x)P_{n-1}^{(r+1)}(x), \quad n \geq 0.
\]

Computations involving Eqs. (1), (3) and (4) gives \( C_1(x) = 1, C_2(x) = -\mu \). It should be mentioned that relation (12) is given in [8] but only for the case where \( r = 0 \).

The same process applied to the families \((P_{n+1}^{(r)}(x))_n\), \((P_{n+1}^{(r, \mu)}(x))_n\) and \((P_n^{(r+1, \mu)}(x))_n\) give, by analogy, relation (13). \( \square \)

### 2.3. \( q \)-Difference equations for \( P_n^{(r, \mu)} \)

In the first step, we eliminate the term \( P_{n-1}^{(r+1)} \) in Eqs. (8) and (9) using Eq. (12) and get the coupled relations linking the polynomials families \((P_n^{(r)}(x))_n\) and \((P_n^{(r, \mu)}(x))_n\), the two relations which used together lead us to the following proposition.

**Proposition 4.** If \((P_n)_n\) denotes the sequence of monic OP of the \( D_q \)-LH class; then the associates \((P_n^{(r)}(x))_n\) and the co-recursive associated \((P_n^{(r, \mu)}(x))_n\) satisfy

\[
D_{r, n}^{\mu; q}[P_n^{(r, \mu)}] = N_{r, n}^{\mu; q}[P_n^{(r)}],
\]

(14)

\[
\bar{D}_{r, n}^{\mu; q}[P_n^{(r)}] = \bar{N}_{r, n}^{\mu; q}[P_n^{(r, \mu)}],
\]

(15)
where the q-difference operators are given by

\[
\begin{align*}
D_{r,n}^{q,\mu} &= c_2(r,n,x)q^2 + c_1(r,n,x)q + c_0(r,n,x), \\
N_{r,n}^{q,\mu} &= \tilde{c}_1(r,n,x)q + \tilde{c}_0(r,n,x), \\
\tilde{D}_{r,n}^{q,\mu} &= d_2(r,n,x)q^2 + d_1(r,n,x)q + d_0(r,n,x), \\
\tilde{N}_{r,n}^{q,\mu} &= \tilde{d}_1(r,n,x)q + \tilde{d}_0(r,n,x),
\end{align*}
\]

with

\[
\begin{align*}
c_j(r,n,x) &= c_j(x), \\
d_j(r,n,x) &= d_j(x), \\
\tilde{c}_j(r,n,x) &= \tilde{c}_j(x), \\
\tilde{d}_j(r,n,x) &= \tilde{d}_j(x)
\end{align*}
\]

and

\[
\begin{align*}
c_2(x) &= -b_2(x)a_2(x)\mu, \\
c_1(x) &= -b_2(x)a_1(x) - a_2(x)\mu b_1(x), \\
c_0(x) &= -b_2(x)a_0(x) - b_0(x)a_2(x)\mu, \\
\tilde{c}_1(x) &= -a_2(x)\mu b_1(x) - b_2(x)a_1(x) + b_2(x)a_1(x)\mu + a_2(x)\mu^2 b_1(x), \\
\tilde{c}_0(x) &= -b_2(x)a_0(x) + a_2(x)\mu^2 b_0(x) + b_2(x)a_0(x)\mu - b_0(x)a_2(x)\mu, \\
d_2(x) &= a_2(x)\mu, \\
d_1(x) &= -a_1(x) + a_1(x)\mu, \\
d_0(x) &= -a_0(x) + a_0(x)\mu, \\
\tilde{d}_1(x) &= -\tilde{a}_1(x), \\
\tilde{d}_0(x) &= -\tilde{a}_0(x).
\end{align*}
\]

The coupled equations linking \(P_n^{(r)}\) and \(P_{n}^{(r,\mu)}\) are essential equations for the derivation of the fourth-order q-difference equations satisfied by the co-recursive associated OP of the \(D_q\)-LH class.

In the first step we apply the operator \(G_q\) to Eq. (14) and use Eq. (15) to eliminate the term \(P_n^{(r)}(q^2x)\) and get

\[
\begin{align*}
(e_3(r,n,x)q^3 + e_2(r,n,x)q^2 + e_1(r,n,x)q + e_0(r,n,x))P_n^{(r,\mu)}
&= (\tilde{e}_3(r,n,x)q + \tilde{e}_0(r,n,x))P_n^{(r)}
\end{align*}
\]

with coefficients \(e_j\) and \(\tilde{e}_j\) given, by \(e_j(r,n,x) = e_j(x)\), \(\tilde{e}_j(r,n,x) = \tilde{e}_j(x)\) and

\[
\begin{align*}
e_3(x) &= c_2(xq)d_2(x), \\
e_2(x) &= c_1(xq)d_2(x), \\
e_1(x) &= c_0(xq)d_2(x) - \tilde{c}_1(xq)d_1(x), \\
e_0(x) &= -\tilde{c}_1(xq)d_0(x),
\end{align*}
\]

\[
\begin{align*}
\tilde{e}_3(x) &= \tilde{c}_1(xq)d_1(x) - \tilde{c}_0(xq)d_2(x), \\
\tilde{e}_0(x) &= \tilde{c}_1(xq)d_0(x).
\end{align*}
\]

In the second step, we apply the operator \(G_q\) to Eq. (18) and use again Eq. (15) to eliminate \(P_n^{(r)}(q^2x)\) to get

\[
\begin{align*}
(f_4(r,n,x)q^3 + f_3(r,n,x)q^2 + f_2(r,n,x)q + f_1(r,n,x))P_n^{(r,\mu)}
&= (\tilde{f}_4(r,n,x)q + \tilde{f}_0(r,n,x))P_n^{(r)}
\end{align*}
\]

with coefficients \(f_j\) and \(\tilde{f}_j\) given by \(f_j(r,n,x) = f_j(x)\); \(\tilde{f}_j(r,n,x) = \tilde{f}_j(x)\) and

\[
\begin{align*}
f_4(x) &= c_2(xq^2)d_2(xq)d_2(x), \\
f_3(x) &= c_2(xq) d_2(x), \\
f_2(x) &= d_2(x)(c_0(xq^2)d_2(xq) - \tilde{c}_1(xq^2)d_1(xq)), \\
f_1(x) &= -\tilde{c}_1(xq^2)d_0(xq)d_2(xq) + \tilde{c}_1(xq^2)d_1(xq)d_1(x) - \tilde{c}_0(xq^2)d_2(xq)d_1(x),
\end{align*}
\]
From elimination of the right-hand side of Eqs. (14), (18) and (20), we see that \( P_{r,n}^{(r,n)} \) satisfies a fourth-order q-difference equation (with polynomials coefficients) easily written in \( 3 \times 3 \) determinant. This equation can be written in terms of the operator \( \mathcal{G}_q \) as

\[
(I_4(r,n)\phi x)\mathcal{G}_q^4 + I_3(r,n)\phi x)\mathcal{G}_q^3 + I_2(r,n)\phi x)\mathcal{G}_q^2 + I_1(r,n)\phi x)\mathcal{G}_q + I_0(r,n)\phi x)\mathcal{G}_q = 0,
\]

(22)

where \( I_k \) are polynomials with fixed degrees.

3. Applications

3.1. Co-recursive \( r \)th associated q-classical orthogonal polynomials

- We suppose that the regular functional \( \mathcal{U} \) is represented by the q-classical weight \( \rho \) (defined on the set \( I \)) satisfying the equation

\[
\mathcal{D}_q(\phi \rho) = \psi \rho,
\]

(23)

where \( \phi \) is a polynomial of degree at most two and \( \psi \) a first-degree polynomial.

Then we deduce that \( \mathcal{U} \) is q-classical (see [1,20,21]) and satisfies the functional equation

\[
\mathcal{D}_q(\phi \mathcal{U}) = \psi \mathcal{U}.
\]

(24)

The coefficients \( \beta_r, \gamma_r, G_r, E_r, F_r, G_r \) and \( H_r \) (see Eq. (5)) in this case are given explicitly in [10,13,20].

The coefficients \( E_{r,\mu}, F_{r,\mu}, G_{r,\mu} \) and \( H_{r,\mu} \) are computed using Eqs. (7) and the expressions of \( G_r, E_r, F_r, G_r \) and \( H_r \) given in [10,13]. In particular, \( G_{r,\mu} \) is given by

\[
G_{r,\mu}(x,q) = -\mu \left( q^r \psi_1 + \frac{q^r - q^{2-r}}{q-1} \phi_2 \right) x + A(r,q),
\]

(25)

where the constant \( A(r,q) \) is space consuming.

- Since \( \psi_1 \neq 0 \), the coefficients \( G_{r,\mu} \) is different from zero for \( r \geq 1 \) and \( \mu \neq 0 \). This result permits us to conclude that the co-recursive \( r \)th associated classical OP is neither classical nor semi-classical.

- Since the coefficients \( I_j(r,n,x) \) are too large, we are going to give, for illustration, coefficients \( a_j, \tilde{a}_j, b_j \) and \( \tilde{b}_j \) for Discrete q-Hermite II case. One can then deduce coefficients \( I_j(r,n,\mu,x) \) since they are written in terms of \( a_j, \tilde{a}_j, b_j \) and \( \tilde{b}_j \).
The polynomials coefficients $a_j$, $\tilde{a}_j$, $b_j$ and $\tilde{b}_j$ of Eqs. (8) and (9) for the discrete $q$-Hermite II case ($\phi(x) = 1$; $\psi(x) = x/(1 - q)$) are given by

$$a_2(x) = (-1 + qx)(qx + 1)q^n q', \quad a_1(x) = -q^n q'(-1 - q + q' q^n q^2 x^2),$$
$$a_0(x) = q(-1 + qx^2 q' - qx^2)q^n q', \quad \tilde{a}_1(x) = -xq(-1 - q + q' q^n q^2 x^2)q^n(q' - 1),$$
$$\tilde{a}_0(x) = xq(q' - 1)(qx^2 q' - q - q')q^n,$$
$$b_2(x) = (-1 + qx)(qx + 1)q^n q', \quad b_1(x) = (-1 - q + q' q^n q^2 x^2)q^n_q(-1 + q' q^2 x^2),$$
$$b_0(x) = (-1 - q + q' q^n q^2 x^2)q^n q(-1 + q' q^2 x^2),$$
$$\tilde{b}_1(x) = xq(-1 - q + q' q^n q^2 x^2)q^n(q')^2, \quad \tilde{b}_0(x) = -xq(-1 + q' q^2 x^2 - q)q^n(q')^2. \quad (26)$$

3.2. Particular cases

Uses of Eqs. (8), (9) and (12)–(15) permit us to recover known results [10,13,12]: The fourth-order $q$-difference equations satisfied by the $r$th associated orthogonal polynomials of the $D_q$-LH class, as well as the fourth-order $q$-difference equations satisfied by the co-recursive orthogonal polynomials of the $D_q$-LH.

3.3. Co-recursive first associated $q$-classical orthogonal polynomials

When we set $r = 1$ in Eq. (22), we get the fourth-order $q$-difference equation satisfied by the co-recursive OP of the $D_q$-LH class. This equation, when the initial family $(P_n)_n$ is $q$-semi-classical, can be factorized as product of two second-order $q$-difference equation. In fact, we eliminate the term $P^{r+1}_{n-1}$ in Eqs. (8) and (9) using relation (13) (instead of (12)); and get the two coupled equations linking $P^{[1,\mu]}_{n-1}$ and $P^{(r)}_n$. These equations for $r = 0$ reach as

$$\tilde{D}^{q;\mu}_{0,n}[P^{[1,\mu]}_{n-1}] = \tilde{N}^{q;\mu}_{0,n}[P_n], \quad (27)$$
$$\tilde{D}^{q;\mu}_{0,n}[P_n] = \tilde{N}^{q;\mu}_{0,n}[P^{[1,\mu]}_{n-1}]. \quad (28)$$

The second-order linear $q$-difference operator, $\tilde{D}^{q;\mu}_{0,n}$, annihilating the second hand of (27) when $(P_n)_n$ is $q$-semi-classical, is obtained thanks to the second-order linear $q$-difference equation satisfied by the $q$-semi-classical OP $(P_n)_n$. The fourth-order difference equation reaches as

$$\tilde{D}^{q;\mu}_{0,n} \tilde{D}^{q;\mu}_{0,n}[P^{[1,\mu]}_{n-1}] = 0.$$
where
\[
\tilde{\mathcal{D}}_{0,n}^{\gamma,\mu} = (\phi(1) + \psi(1)t(1))\mathcal{D}_{2,n-1}^{[1,\mu]} + \tilde{\mathcal{N}}_{0,n}^{\gamma,\mu} = \tilde{g} \mathcal{G}_q + \tilde{h} \mathcal{I}_d
\]
with
\[
\mathcal{D}_{2,n-1}^{[1,\mu]} = MM(1)\phi(2)\gamma_q^2 - MM(2)((1 + q)\phi(1) + \psi t(1) - \delta_{q,n}t^2(1))\gamma_q
\]
\[
+ M(1)M(2)q(\phi + \psi t)\gamma_1 \mathcal{I}_d,
\]
\[
\tilde{g} = (\mu M(1)\phi(2) - \mu M(2)\phi(1) - \mu M(2)\psi t(1) + M(1)M(2)c t(1)\gamma_1)
\]
\[
\times((1 + q)\phi(1) + \psi t(1) - \delta_{q,n}t^2(1))M,
\]
\[
\tilde{h} = -(\mu q M(2)\phi(1) - \mu q M(2)\phi(1) - \mu q M(2)\psi t(1))
\]
\[
+ M(1)M(2)c t(1)\gamma_1 \phi(1) + M(1)M(2)c t(1)\gamma_1 \psi(1) + M(1)M(2)c t(1)\gamma_1 \psi(1))M(1).
\]

where
\[
M_{(j)} \equiv M(q^j x), \quad M_{(0)} \equiv M(x) = (1 - (\mu(x - \beta_0))/\gamma_1),
\]
\[
c = \phi''/2 - \psi', \quad \beta_0 = -\psi_0/\psi_1, \quad \gamma_1 = -\phi(\beta_0)/(\phi_2 + q \psi_1),
\]
\[
\phi(j) = \phi(q^j x), \quad \psi(j) = \psi(q^j x), \quad t(j) = t(q^j x), \quad t(x) = (q - 1)x,
\]
\[
\delta_{q,n} = [n]_q \{\phi'(1) - [n - 1]_q \phi''/2q\}, \quad [n]_q = (1 - q^n)/(1 - q).
\]

The factorized form of the fourth-order \(q\)-difference equation satisfied by the co-recursive first associated \(q\)-classical OP
\[
\mathcal{D}_{2,n-1}^{[1,\mu]} \mathcal{G}_q = 0
\]
is obtained using (29) and the second-order \(q\)-difference equation satisfied by \(P_n\):
\[
(\phi(x) \mathcal{D}_q \mathcal{D}_q + \psi(x) \mathcal{D}_q + \delta_{q,n}) P_n = 0.
\]

In fact, \(\mathcal{D}_{2,n-1}^{[1,\mu]}\) which is a second-order linear \(q\)-difference equation with polynomial coefficients is obtained by applying twice the operator \(\mathcal{D}_q\) to Eq. (29) and using the previous equation to eliminate the term \(P_n(q^2 x)\). Since \(\mathcal{D}_{2,n-1}^{[1,\mu]}\) is space consuming, we decide to give it for the discrete \(q\)-Hermite II case. The operators \(\mathcal{D}_{2,n-1}^{[1,\mu]}\) and \(\mathcal{D}_{2,n-1}^{[1,\mu]}\) of the co-recursive first associated of the discrete \(q\)-Hermite II OP (using Maple V [5] for computations) are given by
\[
\mathcal{D}_{2,n-1}^{[1,\mu]} = -(-q + 1 + \mu q^2 x)(-q + 1 + \mu q x)^2 \mathcal{G}_q
\]
\[
- (-q + 1 + \mu q^2 x)(-q + 1 + \mu q x)(q^2 x^2 q^n - 1 - q) \mathcal{G}_q
\]
\[
+ q(x - 1)(x + 1)(-q + 1 + \mu q^2 x)(-q + 1 + \mu q^2 x),
\]
\[
\mathcal{D}_{2,n-1}^{[1,\mu]} = (q^2 x - 1)(q^2 x + 1)(q x - 1)(q x + 1)(q^4 x^2 q^n - 1 - q - q^2 - q^3 + q^4 x^2) \mathcal{G}_q
\]
\[
- q^2(q x - 1)(q x + 1)(q^4 x^2 q^n - 1 - q - q^2 - q^3 + q^4 x^2)(q^4 x^n q^n)^2 + q^2 x^4 q^n
\]
\[
- q^8 x^2 q^n - q^8 x^2 q^n - q^2 x^2 q^n - q^6 x^2 q^n - q^5 x^2 q^n - q^4 x^2 q^n - q^3 x^2 q^n
\]
\[ + q^4 + 2q^3 + 2q^2 + 2q + 1)(q^3x - 1)(q^2x + 1) \]
\[ (q^6x^2q^n + q^6x^2 - q^3 - q^2 - q - 1)(q^4x^2q^n - 1 - q - q^2 - q^3 + q^4x^2). \]

3.4. Concluding remarks

- If for $q$-classical OP, there are conditions under which associated and co-recursive of classical OP are still classical \([10,14]\), this is not the case for the co-recursive $r$th associated $q$-classical OP (with $\mu \neq 0$, $r \geq 1$).
- Using the results obtained in the framework of this paper, we have deduced the fourth-order differential equation satisfied by the co-recursive $r$th associated LH orthogonal polynomials, as well as the fourth-order difference equation satisfied by the co-recursive $r$th associated OP of the $\Delta$-LH class. This allows us to give the coefficients of the fourth-order differential and difference equation for the co-recursive $r$th associated Bessel, Hermite, Hahn and Krawtchouk OP, results which seem to be new and extend some results given by Letessier.
- The $q$-Difference equations given in this paper can be used to solve linearization problem like in \([3]\) and even in \([2]\) when the expanding family is not orthogonal ($q$-Pochhammer), and also to construct the recurrence relation for the connection coefficients of Fourier coefficients as done in \([17,18]\).

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References


