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A numerical algorithm for nonlinear multi-point boundary value problems

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ABSTRACT

In this paper, an algorithm is presented for solving second-order nonlinear multi-point boundary value problems (BVPs). The method is based on an iterative technique and the reproducing kernel method (RKM). Two numerical examples are provided to show the reliability and efficiency of the present method.

1. Introduction

In this paper, we consider the following second-order multi-point boundary value problem:

$$\begin{cases} u''(x) + b(x)u'(x) + c(x)u(x) = f(x, u), \quad 0 \le x \le 1, \\ u(0) = \sum_{i=1}^{m_0} \alpha_i u(\xi_i) + \lambda_1, \, u(1) = \sum_{i=1}^{m_1} \beta_i u(\eta_i) + \lambda_2, \end{cases} \tag{1.1}$$

where $b(x), c(x) \in C[0, 1]$, $0 < \xi_i, \eta_i < 1$, and $f(x, u)$ is a nonlinear function of u.

Multi-point boundary value problems (BVPs) arise in a variety of applied mathematics and physics. For instance, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point BVP, as in [1]; also, many problems in the theory of elastic stability can be handled by the method of multi-point problems [2]. The existence and multiplicity of solutions of multi-point boundary value problems have been studied by many authors; see [3–6] and the references therein. For two-point BVPs, there are many solution methods such as orthonormalization, invariant imbedding algorithms, finite difference, collocation methods, etc. [7–9]. However, there seems to be little discussion about numerical solutions of multi-point boundary value problems. The shooting method is used to solve multi-point boundary value problems in [10,11]. However, the shooting method is a trial-and-error method, and it is often sensitive to the initial guess. This makes computation by the conventional shooting method expensive and ineffective. Geng [12] proposed a method for a class of second-order three-point BVPs by converting the original problem into an equivalent integro–differential equation. Lin and Lin [13] introduced an algorithm for solving a class of multi-point BVPs by constructing a reproducing kernel satisfying the multi-point boundary conditions. However, the method introduced in [13,14] for obtaining a reproducing kernel satisfying multi-point boundary conditions is very complicated, and the form of the reproducing kernel obtained is also very complicated. Hence, the computational cost of this method is very high. Tatari and Mehghan [15] introduced the Adomian decomposition method (ADM) for multi-point BVPs. Yao [16] proposed a successive iteration method for three-point BVPs. Li and Wu [17] developed a method for solving linear multi-point BVPs. Motivated by the interesting paper [17], we shall present an effective method for solving nonlinear multi-point BVPs.

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The rest of the paper is organized as follows. In Section 2, the algorithm for solving nonlinear multi-point BVP (1.1) is introduced. Numerical examples are presented in Section 3. Section 4 ends this paper with a brief conclusion.

2. The algorithm for solving multi-point BVP (1.1)

Reproducing kernel theory has important applications in numerical analysis, differential equations, probability, and statistics, amongst other fields [18–28]. Recently, using the reproducing kernel method (RKM), various two-point BVPs have been discussed [21–28].

In this section, based on the idea of iteration and the RKM for linear two-point BVPs, we shall introduce an effective algorithm for multi-point BVP (1.1).

2.1. Algorithm

The steps of the algorithm are as follows.

Step A: Choose a reasonable initial approximation $u_0(x)$ which satisfies the boundary condition of (1.1) for the function $u(x)$ in $f(x, u)$ and approximate (1.1) in the following fashion:

$$\begin{cases} u_k''(x) + b(x)u_k'(x) + c(x)u_k(x) = f(x, u_{k-1}), \quad 0 \le x \le 1, \ k = 1, 2, \dots \\ u_k(0) = \sum_{i=1}^{m_0} \alpha_i u_k(\xi_i) + \lambda_1, \, u_k(1) = \sum_{i=1}^{m_1} \beta_i u_k(\eta_i) + \lambda_2. \end{cases} \tag{2.1}$$

Step B: Construct auxiliary two-point boundary conditions for (2.1),

$$u_k(0) = \gamma_0, \qquad u_k(1) = \gamma_1,$$

where γ_0 and γ_1 are constants to be determined.

Step C: Solve the following two-point BVP by means of the RKM presented in [22]:

$$\begin{cases} u_k''(x) + b(x)u_k'(x) + c(x)u_k(x) = f(x, u_{k-1}) \triangleq h(x), \quad 0 \le x \le 1, \\ u_k(0) = \gamma_0, \qquad u_k(1) = \gamma_1. \end{cases} \tag{2.2}$$

The detailed process is as follows.

Introduce a new unknown function

$$v(x) = u_k(x) - \phi(x),$$

where $\phi(x)$ satisfies $\phi(0) = \gamma_0, \phi(1) = \gamma_1$, and $\phi(x) = \gamma_0 + (\gamma_1 - \gamma_0)x$.

Problem (2.2) with inhomogeneous boundary conditions can be equivalently reduced to the problem of finding a function $v(x)$ satisfying

$$\begin{cases} v''(x) + b(x)v'(x) + c(x)v(x) = g(x, \gamma_0, \gamma_1), \quad 0 \le x \le 1, \\ v(0) = 0, \qquad v(1) = 0, \end{cases} \tag{2.3}$$

where

$$g(x, \gamma_0, \gamma_1) = h(x) - b(x)\phi'(x) - c(x)\phi(x) = h(x) + \gamma_0[b(x) - c(x) + xc(x)] - \gamma_1[b(x) + xc(x)].$$

By using the RKM presented in [22], the solution and its n-term approximation can be obtained respectively (see Section 2.2 for details):

$$v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k, \gamma_0, \gamma_1)\overline{\psi}_i(x), \qquad v_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} g(x_k, \gamma_0, \gamma_1)\overline{\psi}_i(x),$$

where β_{ik}, x_k, and $\overline{\psi}_i(x)$ are all given.

Then the solution to (2.1) and its n-term approximation are obtained immediately:

$$u_k(x) = \phi(x) + \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k, \gamma_0, \gamma_1)\overline{\psi}_i(x), \qquad u_{k,n}(x) = \phi(x) + \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} g(x_k, \gamma_0, \gamma_1)\overline{\psi}_i(x).$$

Step D: Incorporating the multi-point boundary conditions of (2.1) into $u_{k,n}(x)$, it follows that

$$u_{k,n}(0) = \sum_{i=1}^{m_0} \alpha_i u_{k,n}(\xi_i) + \lambda_1, \qquad u_{k,n}(1) = \sum_{i=1}^{m_1} \beta_i u_{k,n}(\eta_i) + \lambda_2. \tag{2.4}$$

Clearly, (2.4) is a system of two linear equations in two unknowns γ_0 and γ_1, and the constants γ_0 and γ_1 can be determined easily.

Step E: Substituting the obtained γ_0 and γ_1 in $u_{k,n}(x)$, the kth n-term approximate solution $u_{k,n}(x)$ of multi-point BVP (1.1) is obtained.

2.2. Reproducing kernel method for solving (2.3)

In order to solve (2.3) using the RKM presented in [18,22], first, we construct a reproducing kernel space $W_2^3[0, 1]$ in which every function satisfies the homogenous boundary conditions of (2.2).

The reproducing kernel Hilbert space $W_2^3[0, 1]$ is defined as $W_2^3[0, 1] = \{u(x) \mid u''(x)$ is an absolutely continuous real-valued function, $u'''(x) \in L^2[0, 1], u(0) = 0, u(1) = 0\}$. The inner product and norm in $W_2^3[0, 1]$ are given, respectively, by

$$(u(y), v(y))_{W_2^3} = u(0)v(0) + u'(0)v'(0) + u(1)v(1) + \int_0^1 u'''v'''dy$$

and

$$\|u\|_{W_2^3} = \sqrt{(u, u)_{W_2^3}}, \quad u, v \in W_2^3[0, 1].$$

By Cui and Lin [18] and Cui and Geng [22], it is easy to obtain its reproducing kernel (RK),

$$k(x, y) = \begin{cases} k_1(x, y), & y \le x, \\ k_1(y, x), & y > x, \end{cases} \tag{2.5}$$

where $k_1(x, y) = -\frac{1}{120}(x-1)y\left(yx^4 - 4yx^3 + 6yx^2 + \left(y^4 - 5y^3 - 120y + 120\right)x + y^4\right)$.

In (2.3), letting $Lv(x) = v''(x) + b(x)v'(x) + c(x)v(x)$, it is clear that $L : W_2^3[0, 1] \to W_2^1[0, 1]$ is a bounded linear operator. Put $\varphi_i(x) = \overline{k}(x_i, x)$ and $\psi_i(x) = L^*\varphi_i(x)$, where $\overline{k}(x_i, x)$ is the RK of $W_2^1[0, 1]$, L^* is the adjoint operator of L. The orthonormal system $\{\overline{\psi}_i(x)\}_{i=1}^{\infty}$ of $W_2^3[0, 1]$ can be derived from the Gram–Schmidt orthogonalization process of $\{\psi_i(x)\}_{i=1}^{\infty}$,

$$\overline{\psi}_i(x) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x), \quad (\beta_{ii} > 0, i = 1, 2, \ldots). \tag{2.6}$$

By the RKM presented in [18,22], we have the following theorem.

Theorem 2.1. *For (2.3), if $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$, then $\{\psi_i(x)\}_{i=1}^{\infty}$ is the complete system of $W_2^3[0, 1]$ and $\psi_i(x) = L_s k_\alpha(x, s)|_{s=x_i}$.*

Theorem 2.2. *If $\{x_i\}_{i=1}^{\infty}$ is dense on $[0, 1]$ and the solution of (2.3) is unique, then the solution of (2.3) is*

$$v(x) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik} g(x_k, \gamma_0, \gamma_1)\overline{\psi}_i(x). \tag{2.7}$$

The approximate solution $v_n(x)$ can be obtained by taking finitely many terms in the series representation of $v(x)$ and

$$v_n(x) = \sum_{i=1}^{n}\sum_{k=1}^{i} \beta_{ik} g(x_k, \gamma_0, \gamma_1)\overline{\psi}_i(x).$$

3. Numerical examples

In this section, two numerical examples are studied to demonstrate the accuracy of the present method. All computations are performed by using Mathematica 5.1.

Example 3.1. Consider the following three-point second-order nonlinear ordinary differential equation [15,16]:

$$u'' + \frac{3}{8}u + \frac{2}{1089}[u']^2 + 1 = 0,$$

with the boundary conditions

$$u(0) = 0, \qquad u(1) - u(1/3) = 0.$$

Using the present method, it is easy to obtain the approximate solution of this problem. Taking $x_i = \frac{i-1}{n-1}, i = 1, 2, \ldots, n$, $n = 21$, initial approximation $u_0(x) = 0$, and performing the iteration twice, the numerical results obtained are compared with those from other methods in Table 1.

Example 3.2. Consider the following singular multi-point boundary value problem:

$$\begin{cases} x(1-x)u''(x) + 6u'(x) + 2u(x) + u^2(x) = f(x), \quad 0 \le x \le 1, \\ u(0) + u\left(\frac{2}{3}\right) = \sinh\frac{2}{3}, u(1) + \frac{1}{2}u\left(\frac{4}{5}\right) = \frac{\sinh\frac{4}{5}}{2} + \sinh 1, \end{cases} \tag{3.1}$$

where $f(x) = 6\cosh x + \sinh x\left(2 + x - x^2 + \sinh x\right)$. The exact solution is given by $u(x) = \sinh x$.

Table 1
Numerical results for Example 3.1.

Nodes	Present method	Adomian decomposition method [15]	Successive iteration method [16]
0.1	0.0656	0.0656	0.0656
0.2	0.1209	0.1209	0.1211
0.3	0.1658	0.1658	0.1661
0.4	0.2001	0.2001	0.2004
0.5	0.2236	0.2236	0.2240
0.6	0.2363	0.2363	0.2367
0.7	0.2382	0.2382	0.2385
0.8	0.2291	0.2291	0.2295
0.9	0.2092	0.2091	0.2095

Table 2
Numerical results for Example 3.2.

Nodes	Exact solution	Relative error (Present method)	Relative error (method in [12])
0.08	0.080085	7.9E−05	3.0E−08
0.24	0.242311	1.9E−05	2.6E−07
0.40	0.410752	7.0E−06	3.1E−07
0.48	0.498646	3.6E−06	4.0E−07
0.64	0.684594	6.6E−07	3.0E−07
0.72	0.783840	1.7E−06	5.2E−09
0.80	0.888106	2.2E−06	1.8E−06
0.88	0.998058	2.2E−06	2.7E−06
0.96	1.114400	2.8E−06	2.2E−05

Using the present method, it is easy to obtain the approximate solution of this problem. Taking $x_i = \frac{i-1}{n-1}, i = 1, 2, \ldots, n, n = 21$, initial approximation $u_0(x) = \frac{1}{54}((-45\sinh(\frac{2}{3})+30(\sinh(\frac{4}{5})+2\sinh(1)))x-10(\sinh(\frac{4}{5})+2\sinh(1))+42\sinh(\frac{2}{3}))$, and performing iteration the five times, the numerical results obtained are compared with those from the method in [12] in Table 2. In [12], it is required to converted the equation into an equivalent integro–differential equation. The present method can avoid this step and reduce the cost of computational work.

The detailed process is as follows.

Step A: Choose an initial approximation $u_0(x)$ satisfying the multi-point boundary conditions of (3.1) in the form of $a + bx$ for the function $u(x)$ in $f(x) - u^2(x)$. Clearly,

$$u_0(x) = \frac{1}{54}\left(\left(-45\sinh\left(\frac{2}{3}\right) + 30\left(\sinh\left(\frac{4}{5}\right) + 2\sinh(1)\right)\right)x - 10\left(\sinh\left(\frac{4}{5}\right) + 2\sinh(1)\right) + 42\sinh\left(\frac{2}{3}\right)\right).$$

Approximate (3.1) in the following fashion:

$$\begin{cases} x(1-x)u_k''(x) + 6u_k'(x) + 2u_k(x) = f(x) - u_{k-1}^2(x) \triangleq F(x), \quad 0 \le x \le 1,\ k = 1, 2, 3, 4, 5, \\ u_k(0) + u_k\left(\frac{2}{3}\right) = \sinh\frac{2}{3}, u_k(1) + \frac{1}{2}u_k\left(\frac{4}{5}\right) = \frac{\sinh\frac{4}{5}}{2} + \sinh 1. \end{cases} \tag{3.2}$$

In comparison with the homotopy perturbation method (HPM) and the ADM, the technique for dealing with nonlinearity can avoid the computation of so-called Adomian polynomials and only requires the continuity of the nonlinear term $f(x, u)$.

Step B: Construct auxiliary two-point boundary conditions for (3.2)

$$u_k(0) = \gamma_0, \qquad u_k(1) = \gamma_1,$$

where γ_0 and γ_1 are constants to be determined.

Step C: Solve the following two-point BVP:

$$\begin{cases} x(1-x)u_k''(x) + 6u_k'(x) + 2u_k(x) = F(x), \quad 0 \le x \le 1, \\ u_k(0) = \gamma_0, \qquad u_k(1) = \gamma_1. \end{cases} \tag{3.3}$$

Introduce a new unknown function,

$$v(x) = u_k(x) - \phi(x),$$

where $\phi(x) = \gamma_0 + (\gamma_1 - \gamma_0)x$.

Then problem (3.3) with inhomogeneous boundary conditions is equivalently reduced to the problem of finding a function $v(x)$ satisfying

$$\begin{cases} x(1-x)v''(x) + 6v'(x) + 2v(x) = g(x, \gamma_0, \gamma_1), \quad 0 \le x \le 1, \\ v(0) = 0, \qquad v(1) = 0, \end{cases} \tag{3.4}$$

where

$$g(x,\gamma_0,\gamma_1)=F(x)-6\phi'(x)-2\phi(x)=F(x)+2(2+x)\gamma_0-2(3+x)\gamma_1.$$

By using the RKM presented in Section 2.2, the n-term approximation of (3.4) can be obtained:

$$v_n(x)=\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}g(x_k,\gamma_0,\gamma_1)\overline{\psi}_i(x),$$

where β_{ik}, x_k, and $\overline{\psi}_i(x)$ are all given.

Then the n-term approximate solution of (3.2) is obtained immediately:

$$u_{k,n}(x)=\phi(x)+\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}g(x_k,\gamma_0,\gamma_1)\overline{\psi}_i(x)=\gamma_0+(\gamma_1-\gamma_0)x+\sum_{i=1}^{n}\sum_{k=1}^{i}\beta_{ik}g(x_k,\gamma_0,\gamma_1)\overline{\psi}_i(x).$$

Step D: Incorporating the multi-point boundary conditions of (3.2) into $u_{k,n}(x)$, it follows that

$$u_{k,n}(0)+u_{k,n}\left(\frac{2}{3}\right)=\sinh\frac{2}{3},\qquad u_{k,n}(1)+\frac{1}{2}u_{k,n}\left(\frac{4}{5}\right)=\frac{\sinh\frac{4}{5}}{2}+\sinh 1. \tag{3.5}$$

Clearly, (3.5) is a system of two linear equations in the unknowns γ_0 and γ_1, and γ_0 and γ_1 can be determined easily.

Step E: Substituting the obtained γ_0 and γ_1 in $u_{k,n}(x)$, the kth n-term approximate solution $u_{k,n}(x)$ of multi-point BVP (3.1) is then obtained.

4. Conclusion

In this paper, a technique for dealing with nonlinearity and a method for handling nonlocal boundary conditions are combined to solve nonlinear second-order multi-point BVPs. The main advantages of the present technique for dealing with nonlinearity over the HPM and the ADM are that it can avoid the computation of the so-called Adomian polynomials and it has fewer requirements for nonlinear terms. Also, the present method can be extended to BVPs with nonlinear boundary conditions.

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