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On N-differential graded algebras

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Abstract

We introduce the concept of N-differential graded algebras (N-dga), and study the moduli space of deformations of the differential of an N-dga. We prove that it is controlled by what we call the (M, N)-Maurer–Cartan equation.

MSC: 16E45; 13N99; 13N15; 13D10

0. Introduction

The goal of this paper is to take the first step towards finding a generalization of Homological Mirror Symmetry (HMS) [11,12] to the context of N-homological algebra [5,10]. In [7] Fukaya introduced HMS as the equivalence of the deformation functor of the differential of a differential graded algebra associated with the holomorphic structure, with the deformation functor of an A_∞-algebra associated with the symplectic structure of a Calabi–Yau variety. This idea motivated us to define deformation functors of the differential of an N-differential graded algebra. An N-dga is a graded associative algebra A, provided with an operator $d : A \to A$ of degree 1 such that $d(ab) = d(a)b + (-1)^{\bar{a}} ad(b)$ and $d^N = 0$. A nilpotent differential graded algebra (Nil-dga) will be an N-dga for some integer $N \geq 2$. Theorem 10 endows the category of Nil-differential graded algebras with a symmetric monoidal structure. We remark that such a monoidal structure cannot be constructed in a natural way for a fixed N (except for $N = 2$), not even using the q-deformed Leibniz rule, see [13].

In Section 2 we consider deformations of a 2-dga into an N-dga. By deforming 2-dgas one is able to construct a plethora of examples of N-dgas. Roughly speaking Theorem 16 tell us that a derivation of a 2-dga $d_A + e$ is an N-differential iff

$$(d_{\mathrm{End}}(e) + e^2)^{\frac{N-1}{2}}(d_A + e) = 0 \quad \text{for } N \text{ odd},$$
$$(d_{\mathrm{End}}(e) + e^2)^{\frac{N}{2}} = 0 \quad \text{for } N \text{ even}.$$

In Section 3 we introduce a general formalism for discrete quantum mechanics. We introduce these models since they turn out, in a totally unexpected way, to be relevant in the problem of deforming an M-differential into an N-differential with $N \geq M$. Section 4 contains our main result, Theorem 19 which provides an explicit identity called the

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(M, N)-Maurer–Cartan equation that controls deformations of an M-complex into an N-complex. The construction of the (M, N)-Maurer–Cartan equation is based on an explicit description of coefficients c_k such that

$$(d_A + e)^N = \sum_{k=0}^{N-1} c_k d_A^k,$$

where c_k depends on d_A and e. In Section 5 we define a functional $cs_{2,2N}$ whose critical points are naturally determined by the $(2, 2N)$-Maurer–Cartan equation.

In conclusion in this paper we introduce the moduli space of deformations of the differential of an N-dga and prove that it is controlled by a generalized Maurer–Cartan equation. We point out that our methods and ideas can be applied in a wide variety of contexts. Examples of N-dga's coming from differential geometry are developed in [1]. A q-analogue, for q a primitive N-th root of unity, of our main result Theorem 19 is provided in [2]. In [3] we state an N-generalized Deligne's principle and use the constructions of this paper to study A_∞-algebras of depth N.

1. N-differential graded algebras and modules

Throughout this paper we shall work with the abelian category of **k**-modules over a commutative ring **k** with unit [6]. We will denote by $A^\bullet \mathbb{Z}$-graded **k**-modules $\oplus_{i\in\mathbb{Z}} A^i$. We let $\bar{a} \in \mathbb{Z}$ denote the degree of the element $a \in A^{\bar{a}}$. The following definition is taken from [9].

Definition 1. Let $N \geq 1$ an integer. An N-*complex* is a pair $(A^\bullet, d)$, where $A^\bullet$ is a $\mathbb{Z}$-graded object and $d : A^\bullet \to A^\bullet$ is a morphism of degree 1 such that $d^N = 0$.

Clearly an N-complex is a P-complex for all $P \geq N$. If **k** is a field, then an N-complex $(A^\bullet, d)$ is referred to as an N-*differential graded vector space* (N-dgvect). An N-complex $(A^\bullet, d)$ such that $d^{N-1} \neq 0$ is said to be a *proper N-complex*. Let $(A^\bullet, d_A)$ be an M-complex and $(B^\bullet, d_B)$ be an N-complex, a *morphism* $f : (A^\bullet, d_A) \to (B^\bullet, d_B)$ is a morphism $f : A^\bullet \to B^\bullet$ of **k**-modules such that $d_B f = f d_A$.

Lemma 2. *Let $(A^\bullet, d_A)$ be a proper M-complex, $(B^\bullet, d_B)$ be a proper N-complex and $f : (A^\bullet, d_A) \to (B^\bullet, d_B)$ be a morphism, then* (1) *If* $\mathrm{Ker}(f) = 0$, *then $M \leq N$;* (2) *If* $\mathrm{Im}(f) = B^\bullet$, *then $M \geq N$ and* (3) *If* $\mathrm{Ker}(f) = 0$ *and* $\mathrm{Im}(f) = B^\bullet$, *then $M = N$.*

Proof. (1) Assume that $N < M$ and let $a \in A^\bullet$ then $f(d_A^N(a)) = d_B^N(f(a)) = 0$. This implies that $d_A^N(a) \in \mathrm{Ker}(f) = 0$, and therefore $d_A^N(a) = 0$ which is in contradiction with the fact that $(A^\bullet, d_A)$ is a proper M-complex. The proof of (2) is analogous to (1), (3) follows from (1) and (2). □

Example 3. Consider $V = \mathbb{C}\langle e_1, e_2, e_3\rangle$ the complex vector space generated by e_1, e_2, e_3. We endow V with a $\mathbb{Z}$-graduation declaring $\bar{e_1} = 0$, $\bar{e_2} = 1$ and $\bar{e_3} = 2$. Define the linear map $d : V \to V$ on generators by

$$d(e_1) = e_2, \qquad d(e_2) = e_3, \quad \text{and} \quad d(e_3) = 0.$$

(V, d) is a proper 3-complex.

Definition 4. Let $(A^\bullet, d)$ be an N-complex, we say that an element $a \in A^i$ is **p**-*closed* if $d^p(a) = 0$ and is **p**-*exact* if there exists an element $b \in A^{i-N+p}$ such that $d^{N-p}(b) = a$, for $1 \leq p < N$ fixed. The *cohomology groups* are the **k**-modules

$${}_pH^i(A) = \frac{\mathrm{Ker}\{d^p : A^i \to A^{i+p}\}}{\mathrm{Im}\{d^{N-p} : A^{i-N+p} \to A^i\}},$$

where $i \in \mathbb{Z}$, $p = 1, 2, \ldots, N-1$. We set ${}_kH^*(A) = 0$ for $k \geq N$.

Notice that a 2-complex $A^\bullet$ is just a complex in the usual sense and in this case p is necessarily equal to 1 and ${}_1H^i(A)$ agrees with $H^i(A)$ for all $i \in \mathbb{Z}$.

Definition 5. (a) Let $N \geq 1$ be an integer. An N-*differential graded algebra* or N-dga over **k**, is a triple $(A^\bullet, m, d)$ where $m : A^k \otimes A^l \to A^{k+l}$ and $d : A^k \to A^{k+1}$ are **k**-modules homomorphisms satisfying

(1) The pair $(A^\bullet, m)$ is a graded associative algebra.
(2) For all $a, b \in A^\bullet$, d satisfies the graded Leibniz rule $d(ab) = d(a)b + (-1)^{\bar{a}} a d(b)$.
(3) $d^N = 0$, i.e., $(A^\bullet, d)$ is an N-complex.

(b) A *nilpotent differential graded algebra* (Nil-dga) is an N-dga for some integer $N \geq 2$.

A 1-dga is a graded associative algebra. A 2-dga is a differential graded algebra.

Lemma 6. *Let $(A^\bullet, m, d)$ be an N-dga, then if a is p-closed and b is q-closed then ab is $(p + q - 1)$-closed.*

Proof. The lemma follows from the identity

$$d^n(ab) = \sum_{i=0}^{n} \begin{Bmatrix} n \\ i \end{Bmatrix}_{\bar{a}} d^i(a) d^{n-i}(b),$$

where $\begin{Bmatrix} n \\ 0 \end{Bmatrix}_{\bar{a}} = (-1)^{\bar{a}}$, and for $j \geq 1$, $\begin{Bmatrix} n+1 \\ j \end{Bmatrix}_{\bar{a}} = \begin{Bmatrix} n \\ j-1 \end{Bmatrix}_{\bar{a}} + (-1)^{\bar{a}+j} \begin{Bmatrix} n \\ j \end{Bmatrix}_{\bar{a}}$.

When $n = p+q-1$, since $d^i(a) = 0$ for $i \geq p$, we only consider the case $i < p$, then $n-i = p+q-1-i > q-1$ and $d^{n-i}(b) = 0$, because $d^j(b) = 0$ for $j \geq q$. Thus either $d^i(a) = 0$ or $d^{n-i}(b) = 0$ for all i, and we have that ab is $(p + q - 1)$-closed. □

Definition 7. Let $(A^\bullet, m_A, d_A)$ be an M-dga and $(B^\bullet, m_B, d_B)$ be an N-dga. A *morphism* $f : A^\bullet \to B^\bullet$ is a linear map such that $f m_A = m_B(f \otimes Id) + m_B(Id \otimes f)$ and $d_B f = f d_A$.

A morphism $f : A^\bullet \to B^\bullet$ such that $f(A^i) \subset B^{i+k}$ is said to be a morphism of degree k. A pair of morphisms $f, g : A^\bullet \to B^\bullet$ of N-dga are homotopic, if there exist $h : A^\bullet \to B^\bullet$ of degree $N - 1$ such that

$$f - g = \sum_{i=0}^{N-1} d_B^{N-1-i} h d_A^i.$$

We remark that if two morphisms $f, g : A^\bullet \to B^\bullet$ of Nil-dga are homotopic then they induce the same maps in cohomology.

Let $(A^\bullet, m_A, d_A)$ and $(B^\bullet, m_B, d_B)$ be an M-dga and an N-dga, respectively. Defining $d_{A\otimes B} = d_A \otimes Id + Id \otimes d_B$, the identity

$$d^n_{A\otimes B}(a \otimes b) = \sum_{k=0}^{n} (-1)^{\bar{a}(n-k)} d_A^k(a) \otimes d_B^{n-k}(b) \quad \text{implies,}$$

Proposition 8. *The triple $(A^\bullet \otimes B^\bullet, m_{A\otimes B}, d_{A\otimes B})$ is an $(M + N - 1)$-dga, where $m_{A\otimes B} = m_A \otimes m_B$.*

Example 9. Let (V, d) be the 3-complex of in Example 3. On the space $V \otimes V^*$ consider the base given by $E_{ij} = e_i \otimes e_j^*$, $i, j = 1, 2, 3$, and define

$$D(E_{ij}) = E_{(i+1)j} + (-1)^{i+j} E_{i(j-1)},$$

by Proposition 8 and since $D^4(E_{13}) \neq 0$, then $(V \otimes V^*, D)$ is a proper 5-dga.

Theorem 10. *The category Nil-dgvect is a symmetric monoidal category. Nil-dga is the category of monoids in Nil-dgvect. Nil-dga inherits a symmetric monoidal structure from Nil-dgvetc.*

Let $V^\bullet$ be an N-dga. By Proposition 8, $(V^\bullet)^{\otimes 2}$ is a $(2N - 1)$-dga, $(V^\bullet)^{\otimes 3}$ is a $(3N - 2)$-dga and in general $(V^\bullet)^{\otimes k}$ is a $[k(N - 1) + 1]$-dga.

Definition 11. Let $(A^\bullet, m_A, d_A)$ be an N-dga and $M^\bullet$ a graded **k**-module. Let $K \geq 2$ be an integer. A *K-differential graded module* (K-dgm) over $(A^\bullet, m_A, d_A)$, is a triple $(M^\bullet, m_M, d_M)$ with $m_M : A^k \otimes M^l \to M^{k+l}$ and $d_M : M^k \to M^{k+1}$, **k**-module morphisms satisfying the following properties

(1) For all $a, b \in A^\bullet$ and $m \in M^\bullet$, $m_M(a, m_M(b, m)) = m_M(m_A(a, b), m)$. If no confusion arises, we denote $m_M(a, m)$ by am.
(2) For all $a \in A^\bullet$ and $m \in M^\bullet$, $d_M(am) = d_A(a)m + (-1)^{\bar{a}} a d_M(m)$.
(3) The pair $(M^\bullet, d_M)$ is a K-complex, $d_M^K = 0$.

Let $(M^\bullet, m_M, d_M)$ be a K-dgm and $(N^\bullet, m_N, d_N)$ be an L-dgm both over an N-dga $(A^\bullet, m_A, d_A)$. A morphism $f : M^\bullet \to N^\bullet$ of degree k is a linear map such that $f(m_M(a, b)) = (-1)^{\bar{a}\bar{f}} m_N(a, f(b))$ and $d_M(f(b)) = f(d_N(b))$, for all $a \in A^\bullet$ and $b \in M^\bullet$. Now let $(M^\bullet, m_M, d_M)$ be a K-dgm over an M-dga $(A^\bullet, m_A, d_A)$ and $(N^\bullet, m_N, d_N)$ an L-dgm over an N-dga $(B^\bullet, m_B, d_B)$. The triple $(M \otimes N, m_{N\otimes M}, d_{M\otimes N})$ turns out to be a $(K + L - 1)$-dgm over $(A \otimes B, m_{A\otimes B}, d_{A\otimes B})$, where $m_{M\otimes N}$ and $d_{M\otimes N}$ are defined as before.

Definition 12. The space of endomorphisms of degree k of $M^\bullet$ is $\mathrm{End}^k(M) = \prod_{i\in\mathbb{Z}} \mathrm{Hom}(M^i, M^{i+k})$, that is, $\mathrm{End}^k(M)$ consists of maps $f : M^\bullet \to M^\bullet$ of degree k which are linear in regard to the action of $A^\bullet$ but which does not necessarily satisfy the relation $d_M f = (-1)^{\bar{f}} f d_M$.

There are operators $\circ_M : \mathrm{End}(M) \otimes M^\bullet \to M^\bullet$ and $\circ_E : \mathrm{End}(M) \otimes \mathrm{End}(M) \to \mathrm{End}(M)$. Similarly to Proposition 8, Proposition 13 below provides the natural algebraic structure on $\mathrm{End}(M)$.

Proposition 13. *Define* $d_{\mathrm{End}}(f) := d_M(f) - (-1)^{\bar{f}} f(d_M)$, *for* $f \in \mathrm{End}(M)$. *The triple* $(\mathrm{End}(M), \circ_E, d_{\mathrm{End}})$ *is a* $(2N-1)$*-dga, and* $(M^\bullet, \circ_M, d_M)$ *is an* N*-dgm over* $(\mathrm{End}(M), \circ_E, d_{\mathrm{End}})$.

Proof. Associativity of $\circ_E$ follows from the associativity of morphism composition. The Leibniz rule for d_{End} is a consequence of the Leibniz rule for d_M. From the definition of d_{End} we obtain the identity

$$d^n_{\mathrm{End}}(f) = \sum_{k=0}^{n} (-1)^{\bar{f}(n-k)} d_M^k \circ f \circ d_M^{n-k}$$

which can be proved by induction and holds for all $n \geq 1$. Let $n = 2N - 1$ if $k < N$ then $N - 1 < n - k$ and thus $d_M^{n-k} = 0$. Similarly if $n - k < N$ then $d_M^k = 0$. □

2. Deformation theory of 2-dgas into N-dgas

Let $\mathbf{k}$ be a field and consider the category *Artin* of finite dimensional local $\mathbf{k}$-algebras. If $\mathcal{R} \in Ob\ (Artin)$ with maximal ideal $\mathcal{R}_+$ then $\mathbf{k} \cong \mathcal{R}/\mathcal{R}_+$ ($\mathcal{R} = \mathbf{k}[[t]]$ and $\mathcal{R}_+ = t\mathbf{k}[[t]]$ are examples to keep in mind). Since $\mathbf{k} \cong \mathcal{R}/\mathcal{R}_+$ then $\mathcal{R} \cong \mathbf{k} \oplus \mathcal{R}_+$ as vector spaces. We study deformation theory using the formalism which considers deformations as functors from Artin algebras to Sets for later convenience.

Definition 14. Let $A^\bullet$ be an M-dga, an N*-deformation* of $A^\bullet$ over $\mathcal{R}$ is an N-dga $A^\bullet_{\mathcal{R}}$ over $\mathcal{R}$, with $N \geq M$, such that $A^\bullet_{\mathcal{R}}/\mathcal{R}_+ A^\bullet_{\mathcal{R}}$ is isomorphic to $A^\bullet$ as an N-dga. Two N-deformations $A^\bullet_{\mathcal{R}}$ and $B^\bullet_{\mathcal{R}}$ are said to be *isomorphic* if there exist an isomorphism $\Phi : A^\bullet_{\mathcal{R}} \to B^\bullet_{\mathcal{R}}$ of N-dgas such that the induced isomorphism $\bar{\Phi} : A^\bullet_{\mathcal{R}}/\mathcal{R}_+ A^\bullet_{\mathcal{R}} \to B^\bullet_{\mathcal{R}}/\mathcal{R}_+ B^\bullet_{\mathcal{R}}$ satisfies $i_B \bar{\Phi} = i_A$, where i_A and i_B are the isomorphism $i_A : A^\bullet_{\mathcal{R}}/\mathcal{R}_+ A^\bullet_{\mathcal{R}} \to A^\bullet$ and $i_B : B^\bullet_{\mathcal{R}}/\mathcal{R}_+ B^\bullet_{\mathcal{R}} \to A^\bullet$.

The core of Definition 14 is to require that $d_{A_{\mathcal{R}}}$ reduces to d_A, and $m_{A_{\mathcal{R}}}$ reduces to m_A under the natural projection $\pi : A^\bullet_{\mathcal{R}} \to A^\bullet_{\mathcal{R}}/\mathcal{R}_+ A^\bullet_{\mathcal{R}} \cong A^\bullet$. Assume that $A^\bullet_{\mathcal{R}} = A^\bullet \otimes \mathcal{R}$ as graded algebras. We have the following decomposition

$$A^\bullet_{\mathcal{R}} = A^\bullet \otimes \mathcal{R} = A^\bullet \otimes (\mathbf{k} \oplus \mathcal{R}_+) = (A^\bullet \otimes \mathbf{k}) \oplus (A^\bullet \otimes \mathcal{R}_+) = A^\bullet \oplus (A^\bullet \otimes \mathcal{R}_+).$$

Thus, since $d_{A_{\mathcal{R}}}$ reduces to d_A under the projection π, we must have

$$d_{A_{\mathcal{R}}} = d_A + e$$

where $e \in \mathrm{Der}(A^\bullet \otimes \mathcal{R}_+)$ has degree 1. Moreover, the fact that $d^N_{A_{\mathcal{R}}} = 0$ implies that e is required to satisfy an identity which we call the (M, N)-Maurer–Cartan equation. The next proposition is well known and considers the classical case, that is, the $(2, 2)$-Maurer–Cartan equation.

Proposition 15. *Let $A^\bullet$ be a 2-dga and $A^\bullet_{\mathcal{R}} = A^\bullet \otimes \mathcal{R}$ be a 2-deformation over $\mathcal{R}$, $d_{A_{\mathcal{R}}} = d_A + e$ where $e \in \mathrm{Der}(A^\bullet \otimes \mathcal{R}_+)$, then e satisfies the $(2,2)$-Maurer–Cartan equation given by*

$$d_{\mathrm{End}}(e) + e^2 = 0.$$

Proof. We have

$$\begin{aligned} d^2_{A_{\mathcal{R}}}(a) &= (d_A + e)(d_A + e)(a) \\ &= d_A^2(a) + d_A(e(a)) + e(d_A(a)) + e^2(a) \\ &= d_{\mathrm{End}}(e)(a) + e^2(a), \quad \text{for all } a \in A^\bullet. \quad \square \end{aligned}$$

Suppose that $N = 2k + n$, $n \in \{0, 1\}$ and $k \in \mathbb{N}$, then

$$d^N_{A_{\mathcal{R}}} = d^{2k+n}_{A_{\mathcal{R}}} = (d^2_{A_{\mathcal{R}}})^k d^n_{A_{\mathcal{R}}} = (d_{\mathrm{End}}(e) + e^2)^k d^n_{A_{\mathcal{R}}}, \quad \text{thus}$$

Theorem 16. *Let $A^\bullet$ be a 2-dga. $A^\bullet_{\mathcal{R}} = A^\bullet \otimes \mathcal{R}$ is an N-deformation over $\mathcal{R}$ with $d_{A_{\mathcal{R}}} = d_A + e$ where $e \in \mathrm{Der}(A^\bullet \otimes \mathcal{R}_+)$ of degree 1, iff e satisfies*

$$\begin{aligned} &(d_{\mathrm{End}}(e) + e^2)^{\frac{N-1}{2}}(d_A + e) = 0 \quad \text{for } N \text{ odd}, \\ &(d_{\mathrm{End}}(e) + e^2)^{\frac{N}{2}} = 0 \quad \text{for } N \text{ even}. \end{aligned}$$

Theorem 16 can be easily extended to study deformations of the differential of a 2-dgm $M^\bullet$ over a 2-dga $A^\bullet$ as follows.

Theorem 17. *Let $M^\bullet$ be a 2-dgm over a 2-dga $A^\bullet$. Then $M^\bullet_{\mathcal{R}} = M^\bullet \otimes \mathcal{R}$ is an N-deformation over $\mathcal{R}$ with $d_{A_{\mathcal{R}}} = d_A + e$ where $e \in \mathrm{End}(M^\bullet \otimes \mathcal{R}_+)$ has degree 1, iff e satisfies*

$$\begin{aligned} &(d_{\mathrm{End}}(e) + e^2)^{\frac{N-1}{2}}(d_M + e) = 0 \quad \text{for } N \text{ odd}, \\ &(d_{\mathrm{End}}(e) + e^2)^{\frac{N}{2}} = 0 \quad \text{for } N \text{ even}. \end{aligned}$$

Let M be a 3-dimensional smooth manifold. The space $(\Omega^\bullet(M), d)$ of differential forms on M is a differential graded algebra with d the de Rham differential. Let $\pi : E \to M$ be a vector bundle, the space $(\Omega^\bullet(M, E), d_E)$ of E-valued forms is a differential graded module over $(\Omega^\bullet(M), d)$, where d_E is the differential induced by d. Let $A \in \Omega^1(M)$ and consider the endomorphism e_A induced by A, defined by $e_A(\omega) = A \wedge \omega$ for all $\omega \in \Omega^\bullet(M, E)$. The pair $(\Omega^\bullet(M, E), d_E + e_A)$ is a 4-dgm for any A. Moreover, according to Theorem 17 $(\Omega^\bullet(M, E), d + e_A)$ is a 3-dgm if and only if for all ω

$$d_{\mathrm{End}}(e_A)(d + e_A)\omega = 0.$$

Since $d_{\mathrm{End}}(e_A)(d + e_A)$ is an operator of degree 3, the identity $d_{\mathrm{End}}(e_A)(d + e_A)\omega = 0$ holds for any k-form ω, $k \geq 1$. Thus $(\Omega^\bullet(M, E), d + e_A)$ is a 3-dgm if and only if for any 0-form ω

$$d_{\mathrm{End}}(e_A)(d + e_A)\omega = d(A) \wedge (d_E(\omega) + A \wedge \omega) = 0.$$

Similarly, it is easy to deduce from Theorem 17 that if M is an n-dimensional smooth manifold and $n < m$, then $(\Omega^\bullet(M, E), d + e_A)$ is a m-complex. Let now M be a $2n$-dimensional smooth manifold. Using local coordinates the 2-form $d_{\mathrm{End}}(e_A)$ can be written as $F_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j$ where $F_{ij} = \partial_i A_j - \partial_j A_i$. Furthermore,

$$(F_{ij}\mathrm{d}x^i \wedge \mathrm{d}x^j)^n = \left(\sum_{\alpha \in P(2n)} \prod_{i=1}^{n} \mathrm{sign}(\alpha) F_{a_i, b_i} \right) \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^{2n},$$

where $P(2n)$ is the set of ordered pairings of $[2n] = \{1, \ldots, 2n\}$. Recall that a ordered pairing $\alpha \in P(2n)$ is a sequence $\{(a_i, b_i)\}_{i=1}^n$ such that $[2n] = \bigsqcup_{i=1}^n \{a_i, b_i\}$ and $a_i < b_i$. By Theorem 17, $(\Omega^\bullet(M, E), d + e_A)$ is a $2n$-complex if and only if the 2-form $F_{ij} \mathrm{d}x^i \wedge \mathrm{d}x^j$ satisfies

$$\sum_{\alpha \in P(2n)} \mathrm{sign}(\alpha) \prod_{i=1}^n F_{a_i, b_i} = 0.$$

Let M be a complex manifold and consider the differential graded algebra $(\Omega(M), \wedge, \bar{\partial})$, where $\bar{\partial}$ is the Dolbault differential. Let $\pi : E \to M$ be a complex vector bundle, we consider $\Omega(M, E)$ the forms with values in E. Recall [8] that a holomorphic structure on E is given by a left differential graded module structure $(\Omega(M, E), \wedge_E, \bar{\partial}_E)$ over the 2-dga $(\Omega(M), \wedge, \bar{\partial})$. Suppose that on $(\Omega(M, E), \wedge_E, \bar{\partial}_E)$ there is a left N-differential graded module structure over the 2-dga $(\Omega(M), \wedge, \bar{\partial})$, then in this case we say that E carries an *N-holomorphic structure*.

3. Discrete quantum theory

Generally speaking the following data constitute the basic set up for a (non-relativistic) *quantum mechanical system*: A finite dimensional Riemannian manifold M which is thought as the configuration space of the quantum system; A Lagrangian function $L : TM \to \mathbb{R}$ which assigns weights to points in phase space.

Associated to this data is the Hilbert space $\mathcal{H}$ of quantum states which is usually taken to be $L^2(M)$, the space of square integrable functions on M. The dynamics of the quantum system is determined by operators $U_t : \mathcal{H} \to \mathcal{H}$, where $t \in \mathbb{R}$ represents time. The kernel ω_t of U_t is such that

$$(U_t f)(y) = \int_M \omega_t(y, x) f(x) \mathrm{d}x.$$

The key insight of Feynman is that $\omega_t(y, x)$ admits an integral representation

$$\omega_t(y, x) = \int \mathrm{e}^{\mathrm{i} \int_0^t L(\gamma, \dot{\gamma}) \mathrm{d}t} D(\gamma).$$

The integral above runs over all paths $\gamma : [0, t] \to M$ such that $\gamma(0) = x$ and $\gamma(t) = y$. Making rigorous sense of this integral is the main obstacle in turning quantum mechanics a fully rigorous mathematical theory. Recall that a directed graph Γ is given by: (i) A set V_Γ called the set of vertices, (ii) A set E_Γ called the set of edges and (iii) A map $(s, t) : E_\Gamma \to V_\Gamma \times V_\Gamma$. Following the pattern above, one may define a *discrete quantum mechanical system* as being given by the following data

(1) A directed graph Γ (finite or infinite) which plays the role of configuration space.
(2) A map $L : E_\Gamma \to \mathbb{R}$ called the Lagrangian map of the system.

The associated Hilbert space is $\mathcal{H} = \mathbb{C}^{V_\Gamma}$. The operators $U_n : \mathcal{H} \to \mathcal{H}$, where $n \in \mathbb{Z}$ represents discretized time are given by

$$(U_n f)(y) = \sum_{x \in V_\Gamma} \omega_n(y, x) f(x),$$

where the discretized kernel $\omega_n(y, x)$ admits the following representation

$$\omega_n(y, x) = \sum_{\gamma \in P_n(\Gamma, x, y)} \prod_{e \in \gamma} \mathrm{e}^{\mathrm{i} L(e)}.$$

Here $P_n(\Gamma, x, y)$ denotes the set of length n paths in Γ from x to y, i.e., sequences $(e_1, \ldots, e_n)$ of edges in Γ such that $s(e_1) = x$, $t(e_i) = s(e_{i+1})$, $i = 1, \ldots, n-1$ and $t(e_n) = y$.

In Section 4 we show that the generalized Maurer–Cartan equation controlling deformations of N-dgas is determined by the kernel of a discrete quantum mechanical system L which we proceed to introduce. Let us first explain our notation and conventions which generalize those introduced in [4].

For $s = (s_1, \ldots, s_n) \in \mathbb{N}^n$ we set $l(s) = n$, the length of the vector s, and $|s| = \sum_i s_i$. For $1 \le i < n$, $s_{>i}$ denotes the vector given by $s_{>i} = (s_{i+1}, \ldots, s_n)$, for $1 < i \le n$, $s_{<i}$ stands for $s_{<i} = (s_1, \ldots, s_{i-1})$, we also set $s_{>n} = s_{<1} = \emptyset$. $\mathbb{N}^{(\infty)}$ denotes the set $\bigsqcup_{n=0}^{(\infty)} \mathbb{N}^n$, where by convention $\mathbb{N}^{(0)} = \{\emptyset\}$.

We define maps $\delta_i, \eta_i : \mathbb{N}^n \to \{0, 1\}$, for $1 \le i \le n$, as follows

$$\delta_i(s) = \begin{cases} 1 & \text{if } s_i = 0, \\ 0 & \text{otherwise.} \end{cases} \qquad \eta_i(s) = \begin{cases} 1 & \text{if } s_i \ge 1, \\ 0 & \text{otherwise.} \end{cases}$$

For an M-dga $A^\bullet$ and $e \in \mathrm{End}(A^\bullet)$ and $s \in \mathbb{N}^n$ we define $e^{(s)} = e^{(s_1)} \cdots e^{(s_n)}$, where $e^{(l)} = d^l_{\mathrm{End}}(e)$ if $l \ge 1$, $e^{(0)} = e$ and $e^{\emptyset} = 1$. In the case that $e_a \in \mathrm{End}(A^\bullet)$ is given by

$$e_a(\phi) = a\phi, \quad \text{for } a \in A^1 \text{ fixed and all } \phi \in A^\bullet,$$

then $e_a^{(l)} = d^l_{\mathrm{End}}(e_a)$ reduces to $e_a^{(l)} = e_{d^l(a)}$, thus

$$e_a^{(s)} = e_a^{(s_1)} \cdots e_a^{(s_n)} = e_{d^{s_1}(a)} \cdots e_{d^{s_n}(a)}$$

where $[k]$ denotes the set $\{1, 2, \ldots, k\}$. For $N \in \mathbb{N}$ we define $E_N = \{s \in \mathbb{N}^{(\infty)} : |s| + l(s) \le i\}$ and for $s \in E_N$ we define $N(s) \in \mathbb{Z}$ by $N(s) = N - |s| - l(s)$.

We introduce the discrete quantum mechanical system L by

(1) $V_L = \mathbb{N}^{(\infty)}$.
(2) There is a unique directed edge in L from vertex s to t if and only if $t \in \{(0, s), s, (s + e_i)\}$ where $e_i = (0, \ldots, \underbrace{1}_{i\text{-th}}, \ldots, 0) \in \mathbb{N}^{l(s)}$, in this case we set source$(e) = s$ and target$(e) = t$.
(3) Edges in L are weighted according to the following table.

Source(e)	Target(e)	Weight(e)
s	$(0, s)$	1
s	s	$(-1)^{\lvert s\rvert+l(s)}$
s	$(s + e_i)$	$(-1)^{\lvert s_{<i}\rvert+i-1}$

The set $P_N(\emptyset, s)$ consists of all paths $\gamma = (e_1, \ldots, e_N)$, such that source$(e_1) = \emptyset$, target$(e_N) = s$ and source$(e_{l+1}) =$ target(e_l). For $\gamma \in P_N(\emptyset, s)$ we define the weight $\omega(\gamma)$ of γ as

$$\omega(\gamma) = \prod_{i=1}^{N} \omega(e_i).$$

4. The (M, N)-Maurer–Cartan equation

Lemma 18. *Let $A^\bullet$ be an M-dga and $\mathcal{R} \in Ob$ (Artin). We define $d_{A_\mathcal{R}} = d_A + e$ where $e \in \mathrm{Der}(A^\bullet \otimes \mathcal{R}_+)$ has degree 1, then*

$$(d_{A_\mathcal{R}})^N = \sum_{s \in E_N} c(s, N) e^{(s)} d_A^{N(s)},$$

where the coefficient $c(s, N+1)$ is equal to

$$\delta_1(s) c(s_{>1}, N) + (-1)^{|s|+l(s)} c(s, N) + \sum_{i=1}^{l(s)} \eta_i(s) (-1)^{|s_{<i}|+i-1} c(s - e_i, N), \tag{1}$$

and $c(\emptyset, 1) = c(0, 1) = 1$.

Proof. We use an induction on N. For $N = 1$, since $E_1 = \{s = \emptyset, s = 0\}$

$$\begin{aligned} d_{A_\mathcal{R}} &= \sum_{s \in E_1} c(s, 1) e^{(s)} d_A^{1(s)} = c(\emptyset, 1) e^{(\emptyset)} d_A^{1-|\emptyset|-l(\emptyset)} + c(0, 1) e^{(0)} d_A^{1-|0|-l(0)} \\ &= c(\emptyset, 1) d_A + c(0, 1) e. \end{aligned}$$

Suppose our formula holds for N and let us check it for $N+1$

$$\begin{aligned}(d_{A_{\mathcal{R}}})^{N+1} &= (d_A+e)(d_{A_{\mathcal{R}}})^N\\ &= (d_A+e)\left(\sum_{s\in E_N} c(s,N)e^{(s)}d_A^{N(s)}\right)\\ &= d_A\left(\sum_{s\in E_N} c(s,N)e^{(s)}d_A^{N(s)}\right) + e\left(\sum_{s\in E_N} c(s,N)e^{(s)}d_A^{N(s)}\right)\\ &= \sum_{s\in E_N} c(s,N)d_A(e^{(s)}d_A^{N(s)}) + \sum_{s\in E_N} c(s,N)ee^{(s)}d_A^{N(s)}. \qquad (2)\end{aligned}$$

Consider the second term of the right hand side of (2)

$$\begin{aligned}\sum_{s\in E_N} c(s,N)ee^{(s)}d_A^{N(s)} &= \sum_{s\in E_N} c(s,N)e^{(0)}e^{(s)}d_A^{N(s)}\\ &= \sum_{\substack{t\in E_{N+1}\\ t_1=0}} c(t_{>1},N)e^{(t)}d_A^{N-|t_{>1}|-l(t_{>1})} \qquad (3)\\ &= \sum_{s\in E_{N+1}} \delta_1(s)c(s_{>1},N)e^{(s)}d_A^{N(s)+1}. \qquad (4)\end{aligned}$$

In (3) we put $t=(0,s)$ thus $|t|=|s|$ and $l(t)=l(s)+1$ and (4) is obtained by rewriting and changing t to s.

Now consider the first term of the right hand side of (2)

$$\begin{aligned}\sum_{s\in E_N} c(s,N)d_A(e^{(s)}d_A^{N(s)}) &= \sum_{\substack{s\in E_N\\ 1\le i\le l(s)}} (-1)^{|s_{<i}|+i-1}c(s,N)e^{(s+e_i)}d_A^{N(s)}\\ &\quad + \sum_{s\in E_N}(-1)^{|s|+l(s)}c(s,N)e^{(s)}d_A^{N(s)+1} \qquad (5)\\ &= \sum_{t\in E_{N+1}}\sum_{\substack{i=1\\ t_i\ge 1}}^{l(t)}(-1)^{|t<i|+i-1}c(t-e_i,N)e^{(t)}d_A^{N-|t-e_i|-l(t)}\\ &\quad + \sum_{s\in E_N}(-1)^{|s|+l(s)}c(s,N)e^{(s)}d_A^{N(s)+1} \qquad (6)\\ &= \sum_{s\in E_{N+1}}\sum_{i=1}^{l(s)}\eta_i(s)(-1)^{|s<i|+i-1}c(s-e_i,N)e^{(s)}d_A^{N(s)+1}\\ &\quad + \sum_{s\in E_N}(-1)^{|s|+l(s)}c(s,N)e^{(s)}d_A^{N(s)+1}. \qquad (7)\end{aligned}$$

Putting $t=s+e_i$ in the first term of (5) we obtain (6) and rewriting and changing t to s we obtain (7). Finally collecting similar terms in (4) and (7), and using the recurrence formula we get

$$(d_A+e)^{N+1} = \sum_{s\in E_{N+1}} c(s,N+1)e^{(s)}d_A^{N(s)+1},$$

thus the proof is completed. □

The following result generalizes Theorem 16. It provides an explicit formula for the coefficients of the generalized Maurer–Cartan equation introduced below.

Theorem 19. *We have,*

$$(d_{A_{\mathcal{R}}})^N = \sum_{k=0}^{N-1} c_k d_A^k,$$

where

$$c_k = \sum_{\substack{s \in E_N \\ N(s)=k \\ s_i < M}} c(s,N)e^{(s)} \quad \textit{and} \quad c(s,N) = \sum_{\gamma \in P_N(\emptyset,s)} \omega(\gamma).$$

Proof. One checks that the coefficients $c(s,N) = \sum_{\gamma \in P_N(\emptyset,s)} \omega(\gamma)$ satisfy the recurrence formula of Lemma 18. For this one checks that $P_{N+1}(\emptyset, s)$ is naturally partitioned in three blocks. The first block contains paths that are the composition of a path $\gamma : \emptyset \to s$ in $P_N(\emptyset, s_{>1})$ with an edge $s_{>1} \to (0, s_{>1})$ and corresponds with the first term in (1). The second block consists of paths that are the composition of a path $\gamma : \emptyset \to s$ in $P_N(\emptyset, s)$ with an edge $s \to s$ and corresponds with the second term in (1), finally the last block consists of paths that are the composition of a path $\gamma : \emptyset \to s - e_i$ in $P_N(\emptyset, s - e_i)$ with an edge $s - e_i \to s$ and corresponds with the last term of (1). □

Let $A^\bullet$ be an M-dga and $A^\bullet_{\mathcal{R}}$ an N-deformation over $\mathcal{R}$ with $A^\bullet_{\mathcal{R}} = A^\bullet \otimes \mathcal{R}$. For $a \in A^1 \otimes \mathcal{R}_+$ we define $e_a : A^\bullet_{\mathcal{R}} \to A^\bullet_{\mathcal{R}}$ by

$$e_a(b) = ab - (-1)^{\bar{b}} ba.$$

We are assuming that the product is not graded commutative. It is easy to see that e_b is a derivation of degree 1 on $A^\bullet \otimes \mathcal{R}_+$. Then $d_{A_{\mathcal{R}}} = d_A + e_a$ is an N-deformation of d_A iff e_a satisfies the equation

$$\sum_{\substack{s \in E_N \\ s_i < M}} c(s,N) e_a^{(s)} d_A^{N-|s|-l(s)} = 0. \tag{8}$$

Eq. (8) will be called the (M, N)-*Maurer–Cartan equation*. We closed this section by formally introducing the (M, N)-Maurer–Cartan functor $MC_M^N(A)$ which controls deformations of the differential d_A of an N-dga $A^\bullet$.

Definition 20. For $N \geq M$, $a \in A^1 \otimes \mathcal{R}_+$ is said to be an (M, N)-*Maurer–Cartan element* of $A^\bullet \otimes \mathcal{R}$ if e_a satisfies the (M, N)-Maurer–Cartan equation (8). We say that a is homotopic to a', if e_a is homotopic to $e_{a'}$ as morphisms of N-dgas.

Definition 21. We define the (M, N)-*Maurer–Cartan* functor $MC_M^N(A) : \textit{Artin} \to \textit{Set}$ for each M-dga $A^\bullet$ over $\mathbf{k}$. Functor $MC_M^N(A)$ is given by

(1) Let $\mathcal{R}$ be an object of *Artin*. $MC_M^N(A)(\mathcal{R})$ is the set of homotopy classes of all (M, N)-Maurer–Cartan elements of $A^\bullet \otimes \mathcal{R}$.
(2) If $\varphi : \mathcal{R} \to \mathcal{R}'$ is a morphism of the category *Artin* and a is an (M, N)-Maurer–Cartan element of $A^\bullet \otimes \mathcal{R}$, then $(1 \otimes \varphi)(a)$ is an (M, N)-Maurer–Cartan elements of $A^\bullet \otimes \mathcal{R}'$. Thus we obtain a map $\varphi_* : MC_M^N(A)(\mathcal{R}) \to MC_M^N(A)(\mathcal{R}')$.

The deformation theory of K-dgms over an M-dga can be defined similarly.

5. Chern–Simons actions

Let $(A^\bullet, m_A, d_A)$ be a 2-dga over $\mathbf{k}$ and let $(M^\bullet, m_M, d_M)$ be a 2-dgm over $(A^\bullet, m_A, d_A)$, consider its $2K$-Maurer–Cartan equation, that is the equation that arises when we deform the 2-dgm $(M^\bullet, m_M, d_M)$ into a $2K$-dgm, $MC_{2K}(a) = (d_{\mathrm{End}}(a) + a^2)^K = 0$, where $a \in \mathrm{End}(M^\bullet)$ has degree 1. Let us assume that there exists a linear functional $\int : \mathrm{End}(M^\bullet) \to \mathbf{k}$ of degree $2K+1$, (i.e., $\int b = 0$ if $\bar{b} \neq 2K+1$) satisfying the following conditions:

(1) $\int$ is non degenerate, that is, $\int ab = 0$ for all a, then $b = 0$.
(2) $\int d(a) = 0$ for all a, where $d = d_{\mathrm{End}(M^\bullet)}$.
(3) $\int$ is cyclic, that is $\int a_1 a_2 \cdots a_n = (-1)^{\bar{a}_1(\bar{a}_2 \cdots \bar{a}_n)} \int a_2 \cdots a_n a_1$.

We define the *Chern–Simons* functional $cs_{2,2K} : \mathrm{End}(M^\bullet) \to \mathbf{k}$ by

$$cs_{2,2K}(a) = 2K \int \pi(\#^{-1}(a(d_{\mathrm{End}}(a) + a^2)^K)),$$

where

(1) $\mathbf{k} < a, d(a) >$ denotes the free $\mathbf{k}$-algebra generated by symbols a and $d(a)$.
(2) $\# : \mathbf{k} < a, d(a) > \longrightarrow \mathbf{k} < a, d(a) >$ is the linear map defined by

$$\#(a^{i_1} d(a)^{j_1} \cdots a^{i_k} d(a)^{j_k}) = (i_1 + \cdots + i_k + j_1 + \cdots + j_k) a^{i_1} d(a)^{j_1} \cdots a^{i_k} d(a)^{j_k}.$$

(3) $\pi : \mathbf{k} < a, d(a) > \longrightarrow \mathrm{End}(M^\bullet)$ is the canonical projection.

For $K = 1$ we have that $cs_{2,2}(a)$ is equal to

$$2 \int \pi(\#^{-1}(a(d(a) + a^2))) = 2 \int \pi(\#^{-1}(ad(a) + a^3)) = \int ad(a) + \frac{2}{3} a^3,$$

which is the Chern–Simons functional. In general we have the following result.

Theorem 22. *Let $K \geq 1$ be an integer. The Chern–Simons functional $cs_{2,2K}$ is a Lagrangian for the $2K$-Maurer–Cartan equation, i.e., $a \in \mathrm{End}^1(M^\bullet)$ is a critical point of $cs_{2,2K}$ if and only if $(d(a) + a^2)^K = 0$.*

Proof. We check that $\frac{\partial}{\partial \varepsilon} cs_{2,2K+2}(a + b\varepsilon)|_{\varepsilon=0} = (2K + 2) \int bMC_{2K+2}(a)$.

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon} cs_{2,2K+2}(a + b\varepsilon)|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon}(2K + 2)\pi \int (\#^{-1}((a + b\varepsilon) MC_{2K+2}(a + b\varepsilon)))|_{\varepsilon=0} \\
&= (2K + 2) \int \pi \left(\#^{-1} \left(\frac{\partial}{\partial \varepsilon}(a + b\varepsilon) MC_{2K}(a + b\varepsilon) MC_2(a + b\varepsilon)\right)\right)\Big|_{\varepsilon=0} \\
&= (2K + 2) \int \pi \left(\#^{-1} \left(\frac{\partial}{\partial \varepsilon}(a + b\varepsilon) MC_{2K}(a + b\varepsilon)\right)\right)\Big|_{\varepsilon=0} MC_2(a) \\
&\quad + (2K + 2) \int \pi \left(\#^{-1} \left(aMC_{2K}(a) \frac{\partial}{\partial \varepsilon} MC_2(a + b\varepsilon)\right)\right)\Big|_{\varepsilon=0}.
\end{aligned} \tag{9}$$

For degree reasons, the second term of (9) vanishes, the inductive hypothesis yields

$$\begin{aligned}
\frac{\partial}{\partial \varepsilon} cs_{2,2K+2}(a + b\varepsilon)|_{\varepsilon=0} &= (2K + 2) \int bMC_{2K}(a) MC_2(a) \\
&= (2K + 2) \int bMC_{2K+2}(a). \quad \square
\end{aligned}$$

For $K = 2, 3$ the Chern–Simons functional $cs_{2,2K}(a)$ is given by

$$cs_{2,4}(a) = \int \frac{4}{3} a(d(a))^2 + 2a^3 d(a) + \frac{4}{5} a^5.$$

$$cs_{2,6}(a) = \int \frac{3}{2} a(d(a))^3 + \frac{12}{5} a^3 (d(a))^2 + \frac{6}{5} ad(a) a^2 d(a) + 3a^5 d(a) + \frac{6}{7} a^7.$$

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