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Convergence of the Viscosity Method for the Systems of Isentropic Gas Dynamics in Lagrangian Coordinates*

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This paper is a continuation of paper [1]. The intent of this one is to make a first step to solve the systems of isentropic gas dynamics equations in Lagrangian coordinates by the viscosity method.

Consider the Cauchy problem for the systems of isentropic gas dynamics equations in Lagrangian coordinates

$$u_t + p(v)_x = 0, \qquad v_t - u_x = 0, \qquad x \in R,\ t \in R_+. \tag{E}$$

$$(u(x, 0), v(x, 0)) = (u_0(x), v_0(x)), \qquad x \in R, \tag{I}$$

where the pressure $p = p(v) > 0$ is a C^4 function of the specific volume $v > 0$ and u is the velocity of the flow. We assume that $p'(v) < 0$, $p''(v) > 0$. The Jacobian matrix of (E) has two real and distinct eigenvalues $-\lambda(v) = -\sqrt{-p'(v)} < 0 < \sqrt{-p'(v)} = \lambda(v)$.

The main result of this paper is to show that if the flows contain only bounded smooth rarefaction waves in the initial instant, then the solutions of the systems with viscous terms and initial data (I) are uniformly bounded with respect to the small viscosity coefficient $\varepsilon > 0$ in the strip $(-\infty, \infty) \times [0, T)$, where $T > 0$ is any given constant. These solutions uniformly converge to a smooth solution of (E), (I) in $(-\infty, \infty) \times [0, T)$ as $\varepsilon \to 0$.

Consider (E) with viscous terms, namely

$$v_t - u_x = \varepsilon v_{xx}, \qquad u_t + p(v)_x = \varepsilon u_{xx}, \qquad \varepsilon > 0. \tag{E$_\varepsilon$}$$

The Riemann invariants are taken as $r = u + \Phi(v)$, $s = u - \Phi(v)$, where $\Phi(v) = \int_1^v \lambda(s)\, ds$. Then they give a one to one smooth mapping from region

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$\Omega \equiv \{(u, v) \mid u \in R,\ v \in R_+\}$ onto $\Omega_1 \equiv \{2\Phi(0) < r - s < 2\Phi(\infty)\}$. In view of $p_x = -\lambda^2 v_x$ and $\lambda v_{xx} = (\lambda v_x)_x - \lambda_x(\lambda v_x)/\lambda$, it is easy to obtain

$$(r_t + s_t) - \lambda(r_x - s_x) = \varepsilon(r_{xx} + s_{xx}), \tag{1)$_1$}$$

$$(r_t - s_t) - \lambda(r_x + s_x) = \varepsilon(r_{xx} - s_{xx}) - \varepsilon\lambda_x(r_x - s_x)/\lambda \tag{1)$_2$}$$

then

$$r_t - \lambda r_x = \varepsilon r_{xx} - \varepsilon\lambda_x(r_x - s_x)/2\lambda = \varepsilon r_{xx} - \varepsilon\dot{\lambda}(r_x - s_x)^2/4\lambda, \tag{2)$_1$}$$

$$s_t + \lambda s_x = \varepsilon s_{xx} + \varepsilon\lambda_x(r_x - s_x)/2\lambda = \varepsilon s_{xx} + \varepsilon\dot{\lambda}(r_x - s_x)^2/4\lambda, \tag{2)$_2$}$$

where the overdot denotes a differentiator with respect to Φ. Differentiating both sides of $(2)_1$ with respect to x, we have

$$\begin{aligned}\frac{\partial r_x}{\partial t} - \lambda\frac{\partial r_x}{\partial x} &= \varepsilon\frac{\partial^2 r_x}{\partial x^2} + \lambda_x r_x - \frac{\varepsilon}{2}\left(\frac{\lambda_x}{\lambda}\right)_x (r_x - s_x) - \frac{\varepsilon\lambda_x}{2\lambda}(r_{xx} - s_{xx})\\ &= \varepsilon\frac{\partial^2 r_x}{\partial x^2} + \frac{\dot{\lambda}}{2}(r_x - s_x)\, r_x - \frac{\varepsilon}{8}\left(\frac{\dot{\lambda}}{\lambda}\right)_\Phi (r_x - s_x)^3\\ &\quad - \frac{\varepsilon\dot{\lambda}}{2\lambda}(r_x - s_x)(r_{xx} - s_{xx}).\end{aligned}$$

That is

$$\begin{aligned}\frac{\partial r_x}{\partial t} - \lambda\frac{\partial r_x}{\partial x} &= \varepsilon\frac{\partial^2 r_x}{\partial x^2} + \frac{\dot{\lambda}(r_x - s_x)}{2}\\ &\quad\times\left[r_x - \frac{\varepsilon}{4\dot{\lambda}}\left(\frac{\dot{\lambda}}{\lambda}\right)_\Phi (r_x - s_x)^2 - \frac{\varepsilon}{\lambda}(r_{xx} - s_{xx})\right].\end{aligned} \tag{3)$_1$}$$

Similarly, we can prove

$$\begin{aligned}\frac{\partial s_x}{\partial t} + \lambda\frac{\partial s_x}{\partial x} &= \varepsilon\frac{\partial^2 s_x}{\partial x^2} - \frac{\dot{\lambda}(r_x - s_x)}{2}\\ &\quad\times\left[s_x - \frac{\varepsilon}{4\dot{\lambda}}\left(\frac{\dot{\lambda}}{\lambda}\right)_\Phi (r_x - s_x)^2 - \frac{\varepsilon}{\lambda}(r_{xx} - s_{xx})\right].\end{aligned} \tag{3)$_2$}$$

Our main result is the following theorem.

THEOREM 1. *Consider the Cauchy problem* (E), (I), *where* $p = p(v) > 0$ *is a* C^4 *function,* $p'(v) < 0$, $p''(v) > 0$. *Suppose the initial data* $u_0(x)$, $v_0(x) \geq v_{0*} > 0$ *are bounded* C^3 *functions. If*

$$r_0'(x) \geq 0, \qquad s_0'(x) \geq 0, \tag{M}$$

then the solutions of $(\mathrm{E})_\varepsilon$, (I) are uniformly bounded with respect to small $\varepsilon > 0$ in the strip $(-\infty, \infty) \times [0, T)$, where $T > 0$ is any given constant. And those solutions uniformly converge to a smooth function as ε tends to zero. The limit function is a smooth solution of (E), (I), and

$$v_{0*} \leqslant v(x, t) \leqslant v_0^* + LT, \qquad 0 \leqslant t < T, \tag{4}$$

where $(u(x, t), v(x, t))$ is the solution of (E), (I), $v_0^* = \sup_x v_0(x)$, $v_{0*} = \inf_x v_0(x) > 0$.

Proof. Since $v_0(x) \geqslant v_{0*} > 0$ is lower bounded away from zero, then obviously, it is no harm to assume $u_0(x)$ and $v_0(x)$ are constants outside a finite interval $[-X, X]$, where $X > 0$ is any fixed constant (see [1]).

Since $u_0(x)$, $v_0(x) \geqslant v_{0*} > 0$ are bounded smooth functions, by condition (M), we have $0 \leqslant r_0'(x) \leqslant L$, $0 \leqslant s_0'(x) \leqslant L$, and $|r_0''(x)|$, $|s_0''(x)| \leqslant H$ for some constants L, $H > 0$.

Let $u_0(-\infty) = u_{0*}$, $u_0(+\infty) = u_0^*$. For any small constant $\xi > 0$, let

$$D_\xi = \{(u, v) \mid u_{0*} - \xi \leqslant u \leqslant u_0^* + \xi,\ (1 - \xi)\, v_{0*} \leqslant v \leqslant LT + (1 + \xi)\, v_0^*\}.$$

We now prove the following Lemma 2.

LEMMA 2. *For any given $T > 0$, if*

$$\varepsilon c_1(1 - e^{At}) < r_x, \qquad s_x < L + \varepsilon(1 - e^{-At}), \qquad 0 \leqslant t < T, \tag{5}$$

then

$$(u(x, t), v(x, t)) \in D_\xi, \qquad 0 \leqslant t < T, \tag{6}$$

where $(u(x, t), v(x, t))$ are solutions of $(\mathrm{E})_\varepsilon$, (I), $\varepsilon > 0$ is sufficiently small, c_1, A are positive constants that depend only on D_ξ, $p(v)$, T, and L.

Proof. By (5), we have

$$\varepsilon c_1(1 - e^{At}) < u_x < L + \varepsilon(1 - e^{-At}). \tag{7}$$

In view of $v_t = \varepsilon v_{xx} + u_x$, we have

$$v(x, t) = \frac{1}{2\sqrt{\varepsilon \pi t}} \int_{-\infty}^{\infty} v_0(\xi) \exp\left\{-\frac{(x - \xi)^2}{4\varepsilon t}\right\} d\xi$$

$$+ \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{u_x}{\sqrt{\varepsilon(t - \tau)}} \exp\left\{-\frac{(x - \xi)^2}{4\varepsilon(t - \tau)}\right\} d\xi\, d\tau.$$

Thus, by (7),

$$
\begin{aligned}
v(x,t) &= \frac{v_0^*}{2\sqrt{\varepsilon\pi t}} \int_{-\infty}^{\infty} \exp\left\{-\frac{(x-\xi)}{4\varepsilon t}\right\} d\xi \\
&\quad + \frac{1}{\sqrt{\pi}} \int_0^t \int_{-\infty}^{\infty} \frac{L+\varepsilon(1-e^{\tau})}{2\sqrt{\varepsilon(t-\tau)}} \exp\left\{-\frac{(x-\xi)^2}{4\varepsilon(t-\tau)}\right\} d\xi\, d\tau \\
&= v_0^* + \int_0^t (L+\varepsilon(1-e^{-\tau}))\, d\tau \\
&= v_0^* + Lt + \varepsilon(t - e^{-t} + 1) \\
&\leqslant v_0^* + LT + \varepsilon(T+1).
\end{aligned} \tag{8$_1$}
$$

On the other hand, we have

$$
\begin{aligned}
v(x,t) &\geqslant v_{0*} + \int_0^t \varepsilon c_1(1-e^{A\tau})\, d\tau \\
&= v_{0*} + \varepsilon c_1\left(t - \frac{1}{A}e^{At} + \frac{1}{A}\right) \\
&\geqslant v_{0*} - \varepsilon c_1 e^{At}/A.
\end{aligned} \tag{8$_2$}
$$

By (8) if $\varepsilon \leqslant \min\{\xi v_0^*/(T+1),\ \xi A v_{0*} c_1^{-1} e^{-AT}\}$, we have

$$(1-\xi)\, v_{0*} \leqslant v(x,t) \leqslant LT + (1+\xi)\, v_0^*. \tag{9}$$

Since $\varepsilon c_1(1-e^{At}) < u_x(x,t) < L + \varepsilon(1-e^{-t})$, then

$$
\begin{aligned}
u(x,t) &\geqslant u_{0*} + \varepsilon c_1(1-e^{At})(2X + 2t\lambda((1-\xi)\, v_{0*})) \\
&\geqslant u_{0*} - 2\varepsilon c_1 e^{AT}(X + T\lambda((1-\xi)\, v_{0*})).
\end{aligned} \tag{10}
$$

If $\varepsilon < \xi/[2c_1 e^{AT}(X + T\lambda((1-\xi)\, v_{0*}))]$, then by (10), we have

$$u(x,t) > u_{0*} - \xi. \tag{11$_1$}$$

Since $u(x,t)$ is smooth and $\lim_{x\to\infty} u(x,t) = u_0(X) \leqslant u_0^*$, then

$$u(x,t) > u_0^* - \xi. \tag{11$_2$}$$

By (9) and (11), we obtain (6). Q.E.D.

According to Lemma 2, it is easy to see that the major step of the proof is to prove (5). To this end, we let

$$T_1 = \sup\{t \mid \varepsilon c_1(1-e^{At}) < r_x,\ s_x < L + \varepsilon(1-e^{-t}),\ \forall x \in R\},$$

$n_0=\sup_{D_\xi}|\lambda|$, $n_1=\sup_{D_\xi}|\dot{\lambda}|$, $n_2=\sup_{D_\xi}|\ddot{\lambda}|$, $N_0=\sup_{D_\xi}|\dot{\lambda}/\lambda|$, $N_1=\sup_{D_\xi}|(\dot{\lambda}/\lambda)_\Phi|$, $N_2=\sup_{D_\xi}|(\dot{\lambda}/\lambda)_{\Phi\Phi}|$, $b_0=\inf_{D_\xi}|\lambda|$, $b_1=\inf_{D_\xi}|\dot{\lambda}|$, where $c_1>9N_1L^2/16b_1+2c_2/b_0$, $c_2=(H+n_2L^3/4n_1)\exp\{(n_1+3n_1L+2N_1L^2+3N_0L+n_0n_1L)T\}$, $A=\frac{3}{4}Ln_1$. In order to prove (5), it is sufficient to prove $T_1=T$ as $0<\varepsilon<\varepsilon_0$, where ε_0 is a constant which depends only on $p(v)$, D_ξ, L and T.

To do this, we need the following Lemma 3.

LEMMA 3. *If* $\varepsilon c_1(1-e^{At})\leqslant r_x$, $s_x\leqslant L+\varepsilon(1-e^{-t})$, $0\leqslant t<T$, *then*

$$|r_{xx}|,\qquad |s_{xx}|<c_2,\qquad 0\leqslant t<T. \tag{12}$$

We will prove Lemma 3 later.

We now prove $T_1=T$ by contradiction. Suppose $T_1<T$, then it implies that there are only two cases.

Case 1. There exists (x_1,T_1) such that $r_x(x_1,T_1)=L+\varepsilon(1-e^{-T_1})$ (Or $s_x(x_1,T_1)=L+\varepsilon(1-e^{-T_1})$). It implies

$$\left.\frac{\partial r_x}{\partial x}\right|_{(x_1,T_1)}=0,\qquad \left.\frac{\partial^2 r_x}{\partial x^2}\right|_{(x_1,T_1)}\leqslant 0.$$

Thus by (3), we have

$$\left.\frac{\partial r_x}{\partial t}\right|_{(x_1,T_1)}\leqslant\frac{\dot{\lambda}(r_x-s_x)}{2}\left[r_x-\frac{\varepsilon}{4\dot{\lambda}}\left(\frac{\dot{\lambda}}{\lambda}\right)_\Phi(r_x-s_x)^2-\frac{\varepsilon}{\lambda}(r_{xx}-s_{xx})\right]. \tag{13}$$

Since $\dot{\lambda}=p''/2p'<0$, then

$$b_2\equiv\left.\frac{\dot{\lambda}(r_x-s_x)}{2}\right|_{(x_1,T_1)}\leqslant 0.$$

Let $\varepsilon<\frac{1}{2}L(1+c_1e^{AT})^{-1}$, then $|r_x-s_x|<(3/2)L$. Thus

$$\begin{aligned}\left.\frac{\partial r_x}{\partial t}\right|_{(x_1,T_1)}&\leqslant(-b_2)\left(-L-\varepsilon(1-e^{-T_1})+\frac{9}{4}L^2\frac{N_1}{4b_1}\varepsilon+\frac{2c_2}{b_0}\varepsilon\right)\\&\leqslant b_2(L-(9N_1L^2/16b_1+2c_2/b_0)\varepsilon).\end{aligned}$$

Let $\varepsilon\leqslant 16Lb_0b_1/(9N_1b_0L^2+32b_1c_2)$, then

$$\left.\frac{\partial r_x}{\partial t}\right|_{(x_1,T_1)}\leqslant 0. \tag{14}$$

On the other hand, by the definition of T_1,

$$\left.\frac{\partial r_x}{\partial t}\right|_{(x_1, T_1)} = \lim_{t \to T_1 - 0} \frac{r_x(x_1, T_1) - r_x(x_1, t)}{T_1 - t}$$

$$\geqslant \lim_{t \to T_1 - 0} \frac{[L + \varepsilon(1 - e^{-T_1})] - [L + \varepsilon(1 - \varepsilon^{-t})]}{T_1 - t}$$

$$= \varepsilon e^{-T_1} > 0. \tag{15}$$

Equation (15) contradicts (14), thus there is no $T_1 < T$ such that $r_x(x_1, T_1) = L + \varepsilon(1 - e^{-T_1})$. Similarly, there is no $T_1 < T$ such that $s_x(x, T_1) = L + \varepsilon(1 - e^{-T_1})$.

Case 2. There exists (x_1, T_1) such that $r_x(x_1, T_1) = \varepsilon c_1(1 - e^{AT_1})$ (or $s_x(x_1, T_1) = \varepsilon c_1(1 - e^{AT_1})$). It implies

$$\left.\frac{\partial r_x}{\partial x}\right|_{(x_1, T_1)} = 0, \qquad \left.\frac{\partial^2 r_x}{\partial x^2}\right|_{(x_1, T_1)} \geqslant 0.$$

Thus by (3), we have

$$\left.\frac{\partial r_x}{\partial t}\right|_{(x_1, T_1)} \geqslant \frac{\dot{\lambda}(r_x - s_x)}{2} \left[r_x - \frac{\varepsilon}{4\dot{\lambda}} \left(\frac{\dot{\lambda}}{\lambda}\right)_\Phi (r_x - s_x)^2 - \frac{\varepsilon}{\dot{\lambda}} (r_{xx} - s_{xx}) \right]. \tag{16}$$

Since

$$\bar{b}_2 \equiv \left.\frac{\dot{\lambda}(r_x - s_x)}{2}\right|_{(x_1, T_1)} \geqslant 0,$$

then

$$\left.\frac{\partial r_x}{\partial t}\right|_{(x_1, T_1)} \geqslant \bar{b}_2 \left(\varepsilon c_1(1 - e^{AT_1}) - \frac{9N_1 L^2}{16 b_1} \varepsilon - \frac{2c_2}{b_0} \varepsilon \right)$$

$$\geqslant -\bar{b}_2 c_1 \varepsilon e^{AT_1} + \bar{b}_2 \varepsilon \left(c_1 - \frac{9N_1 L^2}{16 b_1} - \frac{2c_2}{b_0} \right). \tag{17}$$

Since $c_1 > (9N_1 L_2/16 b_1) + (2c_2/b_0)$, then

$$\left.\frac{\partial r_x}{\partial t}\right|_{(x_1, T_1)} \geqslant -\bar{b}_2 c_1 \varepsilon e^{AT_1}. \tag{18}$$

On the other hand, by the definition of T_1,

$$\left.\frac{\partial r_x}{\partial t}\right|_{(x_1, T_1)} = \lim_{t \to T_1 - 0} \frac{r_x(x_1, T_1) - r_x(x_1, t)}{T_1 - t}$$

$$\leqslant \lim_{t \to T_1 - 0} \frac{\varepsilon c_1(1 - e^{AT_1}) - \varepsilon c_1(1 - e^{At})}{T_1 - t}$$

$$= -\varepsilon c_1 A e^{AT_1} < -\varepsilon c_1 \bar{b}_2 e^{AT_1}. \tag{19}$$

Equation (19) contradicts (18), thus there is no (x_1, T_1), $T_1 < T$, such that $r_x(x_1, T_1) = \varepsilon c_1(1 - e^{AT_1})$. Similarly, there is no (x_1, T_1), $T_1 < T$, such that $s_x(x_1, T_1) = \varepsilon c_1(1 - e^{AT_1})$.

The contradictions imply $T_1 = T$.

Finally, we only need to prove Lemma 3.

Proof of Lemma 3. Differentiate (3) with respect to x, by straight calculation, we have

$$\begin{aligned}\frac{\partial r_{xx}}{\partial t} - \lambda \frac{\partial r_{xx}}{\partial x} = {} & \varepsilon \frac{\partial^2 r_{xx}}{\partial x} + \frac{\ddot{\lambda}}{4} r_x(r_x - s_x)^2 + \dot{\lambda}(r_x - s_x)\, r_{xx} + \frac{\dot{\lambda}}{2} r_x(r_{xx} - s_{xx}) \\ & - \frac{\varepsilon}{16}\left(\frac{\dot{\lambda}}{\lambda}\right)_{\Phi\Phi} (r_x - s_x)^4 - \frac{5\varepsilon}{8}\left(\frac{\dot{\lambda}}{\lambda}\right)_{\Phi} (r_x - s_x)^2 (r_{xx} - s_{xx}) \\ & - \frac{\varepsilon}{2}\left(\frac{\dot{\lambda}}{\lambda}\right) \frac{\partial}{\partial x} [(r_x - s_x)(r_{xx} - s_{xx})] \\ = {} & \varepsilon \frac{\partial^2 r_x}{\partial x^2} + F(x, t),\end{aligned} \tag{20}$$

where $F(x, t)$ is defined by the equality.

Let $M_0 = \sup_{\pi_T} |r_x - s_x|$, $M_1 = \sup_{\pi_t} \{|r_x|, |s_x|\}$, $M_2 = \sup_{\pi_t} \{|r_{xx}(x, \tau)|, |s_{xx}(x, \tau)|\}$, where $\pi_t = \{(x, \tau) \mid x \in R,\ 0 \leqslant \tau < t\}$. We now prove (12) by contradiction. Let $T_2 = \sup\{t \mid M_2(t) < c_2\}$. Suppose $T_2 < T$, we divide the strip $(-\infty, \infty) \times [0, T_2)$ into small strips $(-\infty, \infty) \times (t_i, t_{i+1}]$, $t_i = i\varepsilon$, $i = 0, 1, ..., T_2\varepsilon^{-1} - 1$, here, it is no harm to assume $T_2\varepsilon^{-1}$ is an integer. For any given point (x_0, t_0), $t_i < t_0 \leqslant t_{i+1}$, we rewrite (20) into

$$\frac{\partial r_{xx}}{\partial t} - \lambda(x_0, t_0) \frac{\partial r_{xx}}{\partial x} = \varepsilon \frac{\partial^2 r_{xx}}{\partial x^2} + F(x, t) + (\lambda(x, t) - \lambda(x_0, t_0)) \frac{\partial r_x}{\partial x}. \tag{21}$$

Define a coordinate transformation $t = t$, $x = y - \lambda(x_0, t_0)(t - t_0)$. Let $\tilde{w}(y, t) = w(x(y, t), t)$, then

$$\frac{\partial \tilde{w}(y, t)}{\partial t} = \frac{\partial w(x, t)}{\partial t} - \lambda(x_0, t_0) \frac{\partial w(x, t)}{\partial x}, \quad \frac{\partial \tilde{w}(y, t)}{\partial y} = \frac{\partial w(x, t)}{\partial x},$$

where w is any smooth function. By (21), we have

$$\frac{\partial \tilde{r}_{yy}}{\partial t} = \varepsilon \frac{\partial^2 \tilde{r}_{yy}}{\partial y^2} + \tilde{F}(y, t) + (\tilde{\lambda}(y, t) - \tilde{\lambda}(y_0, t_0)) \frac{\partial \tilde{r}_{yy}}{\partial y}, \tag{22}$$

where $\tilde{r}(y, t) = r(x(y, t), t)$, $\tilde{F}(y, t) = F(x(y, t), t)$, and $\tilde{\lambda}(y_0, t_0) = \lambda(x_0, t_0)$. Hereafter, we omit the tilde for simplicity in printing.

If we let $r_{yy}(y, t_i)$ be the initial data, then by the Eq. (22), we have

$$r_{yy}(y_0, t_0) = \int_{-\infty}^{\infty} \frac{r_{\xi\xi}(\xi, t_i)}{\sqrt{\varepsilon\pi(t_0 - t_i)}} \exp\left\{\frac{(y_0-\xi)^2}{4\varepsilon(t_0-t_i)}\right\} d\xi$$

$$+ \frac{1}{4\sqrt{\pi}} \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{\ddot{\lambda}(r_\xi - s_\xi)^2 r_\xi}{2\sqrt{\varepsilon(t_0-\tau)}} \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau$$

$$+ \frac{1}{\sqrt{\pi}} \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{\dot{\lambda}(r_\xi - s_\xi)\, r_{\xi\xi}}{\sqrt{\varepsilon(t_0-\tau)}} \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau$$

$$+ \frac{1}{2\sqrt{\pi}} \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{\dot{\lambda} r_\xi (r_{\xi\xi} - s_{\xi\xi})}{2\sqrt{\varepsilon(t_0-\tau)}} \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau$$

$$- \frac{\varepsilon}{16\sqrt{\pi}} \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{(\dot{\lambda}/\lambda)_{\Phi\Phi} (r_\xi - s_\xi)^4}{2\sqrt{\varepsilon(t_0-\tau)}}$$

$$\times \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau$$

$$- \frac{5\varepsilon}{8\sqrt{\pi}} \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{(\dot{\lambda}/\lambda)_{\Phi} (r_\xi - s_\xi)^2 (r_{\xi\xi} - s_{\xi\xi})}{2\sqrt{\varepsilon(t_0-\tau)}}$$

$$\times \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau$$

$$- \frac{\varepsilon}{2\sqrt{\pi}} \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{(\dot{\lambda}/\lambda)(\partial/\partial\xi)\,[(r_\xi - s_\xi)(r_{\xi\xi} - s_{\xi\xi})]}{2\sqrt{\varepsilon(t_0-\tau)}}$$

$$\times \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau$$

$$+ \frac{1}{\sqrt{\pi}} \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{(\lambda(\xi,\tau) - \lambda(y_0, t_0))(\partial \gamma_{\xi\xi}/\partial\xi)}{2\sqrt{\varepsilon(t_0-\tau)}}$$

$$\times \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau.$$

Thus

$$|r_{yy}(y_0, t_0)| \leqslant M_2(t_i) + \frac{n_2 M_1 M_0^2}{4}(t_0 - t_i)$$

$$+ n_1 M_0 \int_{t_i}^{t_0} M_2(\tau)\, d\tau + n_1 M_1 \int_{t_1}^{t_0} M_2(\tau)\, d\tau$$

$$+ \frac{\epsilon}{16} N_2 M_0^4 (t_0 - t_i) + \frac{5\varepsilon}{4} N_1 M_0^2 \int_{t_i}^{t_0} M_2(\tau)\, d\tau + I$$

$$= M_2(t_i) + \left(\frac{n_2 M_1 M_0^2}{4} + \frac{N_2 M_0^4 \varepsilon}{16}\right)(t_0 - t_i)$$

$$+ (n_1 M_0 + n_1 M_1 + \frac{5N_1 M_0^2}{4}\Big) \int_{t_i}^{t_0} M(\tau)\, d\tau + I \tag{23}$$

where

$$I = \frac{\varepsilon}{2\sqrt{\pi}} \left| \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{(\dot{\lambda}/\lambda)(\partial/\partial\xi)\,[(r_\xi - s_\xi)(r_{\xi\xi} - s_{\xi\xi})]}{2\sqrt{\varepsilon(t_0-\tau)}} \times \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau \right|$$
$$+ \frac{1}{\sqrt{\pi}} \left| \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{(\lambda(\xi,\tau) - \lambda(y_0, t_0))(\partial\gamma_{\xi\xi}/\partial\xi)}{2\sqrt{\varepsilon(t_0-\tau)}} \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau \right|.$$

$$I \leqslant \frac{\varepsilon}{2\sqrt{\pi}} \left| \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{(\dot{\lambda}/\lambda)_\xi\,(r_\xi - s_\xi)(r_{\xi\xi} - s_{\xi\xi})}{2\sqrt{\varepsilon(t_0-\tau)}} \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau \right|$$
$$+ \frac{\varepsilon}{2\sqrt{\pi}} \left| \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{2(\dot{\lambda}/\lambda)(r_\xi - s_\xi)(r_{\xi\xi} - s_{\xi\xi})(y_0-\xi)}{(2\sqrt{\varepsilon(t_0-\tau)})^3} \times \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau \right|$$
$$+ \frac{1}{\sqrt{\pi}} \left| \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{\lambda_\xi r_{\xi\xi}}{2\sqrt{\varepsilon(t_0-\tau)}} \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau \right|$$
$$+ \frac{1}{\sqrt{\pi}} \left| \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{2(\lambda(\xi,\tau) - \lambda(y_0,t_0))\, r_{\xi\xi}(y_0-\xi)}{(2\sqrt{\varepsilon(t_0-\tau)})^3} \times \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau \right|$$
$$\leqslant \frac{2\varepsilon}{\sqrt{\pi}} N_0 M_0 \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{M_2(\tau)\,|y_0-\xi|}{(2\sqrt{\varepsilon(t_0-\tau)})^3} \times \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau + n_1 \int_{t_i}^{t_0} M_2(\tau)\, d\tau$$
$$+ \frac{\varepsilon}{4\sqrt{\pi}} \left| \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{(\dot{\lambda}/\lambda)_\Phi\,(r_\xi - s_\xi)^2\,(r_{\xi\xi} - s_{\xi\xi})}{2\sqrt{\varepsilon(t_0-\tau)}} \exp\left\{\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau \right|$$
$$+ \frac{2}{\sqrt{\pi}} \left| \int_{t_i}^{t_0} \int_{-\infty}^{\infty} \frac{(\lambda_t(\eta)(\tau - t_0) + \lambda_\xi(\zeta)(\xi - y_0))\, r_{\xi\xi}(y_0-\xi)}{(2\sqrt{\varepsilon(t_0-\tau)})^3} \times \exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\} d\xi\, d\tau \right|. \tag{24}$$

Since $\lambda_y = (\dot{\lambda}/2)(r_y - s_y)$, $\lambda_t = (\dot{\lambda}/2)[\lambda(r_y + s_y) + \varepsilon(r_{yy} - s_{yy}) - (\varepsilon\dot{\lambda}/\lambda)(r_y - s_y)^2]$.

Let

$$M_3=\sup_{\pi T_2}|\lambda_t|,\qquad z=\frac{y_0-\xi}{2\sqrt{\varepsilon(t_0-\tau)}},$$

then

$$\begin{aligned}
I\leqslant{}&\frac{N_1M_0^2\varepsilon}{2}\int_{t_i}^{t_0}M_2(\tau)\,d\tau\\
&+\frac{2\varepsilon}{\sqrt{\pi}}N_0M_0\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{M_2(\tau)\,|z|}{2\sqrt{\varepsilon(t_0-\tau)}}e^{-z^2}\,dz\,d\tau+n_1\int_{t_i}^{t_0}M_2(\tau)\,d\tau\\
&+\frac{M_3}{2\varepsilon\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{M_2(\tau)\,|y_0-\xi|}{2\sqrt{\varepsilon(t_0-\tau)}}\exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\ )}\right\}d\xi\,d\tau\\
&+\frac{n_1M_0}{\sqrt{\pi}}\int_{t_i}^{t_0}\int_{-\infty}^{\infty}\frac{M_2(\tau)(y_0-\xi)^2}{(2\sqrt{\varepsilon(t-\tau)})^3}\exp\left\{-\frac{(y_0-\xi)^2}{4\varepsilon(t_0-\tau)}\right\}d\xi\,d\tau\\
\leqslant{}&\left(n_1+\frac{N_1M_0^2\varepsilon}{2}\right)\int_{t_i}^{t_0}M_2(\tau)\,d\tau+\frac{2\varepsilon}{\sqrt{\pi}}N_0M_0\frac{1}{2\sqrt{\varepsilon}}2\sqrt{t_0-t_i}M_2(t_0)\\
&+\frac{M_3}{2\varepsilon\pi}\int_{t_i}^{t_0}2\sqrt{\varepsilon(t_0-\tau)}\,M_2(\tau)\,d\tau\int_{-\infty}^{\infty}|z|\,e^{-z^2}\,dz\\
&+\frac{n_1M_0}{\sqrt{\pi}}\int_{t_i}^{t_0}M_2(\tau)\,d\tau\int_{-\infty}^{\infty}z^2e^{-z^2}\,dz\\
\leqslant{}&\left(n_1+\frac{N_1M_0^2\varepsilon}{2}\right)\int_{t_i}^{t_0}M_2(\tau)\,d\tau+\frac{2N_0M_0\varepsilon}{\sqrt{\pi}}M_2(t_0)\\
&+\frac{M_3}{2\varepsilon\sqrt{\pi}}\frac{4\sqrt{\varepsilon}}{3}(\sqrt{t_0-t_i})^3M_2(t_0)\\
&\times\frac{n_1M_0}{\sqrt{\pi}}\frac{\sqrt{\pi}}{2}\int_{t_i}^{t_0}M_2(\tau)\,d\tau\\
\leqslant{}&\left(n_1+\frac{n_1M_0}{2}+\frac{N_1M_0^2\varepsilon}{2}\right)\int_{t_i}^{t_0}M_2(\tau)\,d\tau+\left(\frac{2N_0M_0}{\sqrt{\pi}}+\frac{2M_3}{3\sqrt{\pi}}\right)\varepsilon M_2(t_0).
\end{aligned}\tag{25}$$

Putting (25) into (23), we have

$$\begin{aligned}
|r_{yy}(y_0,t_0)|\leqslant{}&M_2(t_i)+\left(\frac{n_2M_1M_0^2}{4}+\frac{N_2M_0^4\varepsilon}{16}\right)(t_0-t_i)\\
&+\left(n_1M_0+n_1M_1+n_1+\frac{5N_1M_0^2}{4}+\frac{n_1M_0}{2}+\frac{N_1M_0^2\varepsilon}{2}\right)\\
&\times\int_{t_i}^{t_0}M_2(\tau)\,d\tau+\varepsilon\left(\frac{2N_0M_0}{\sqrt{\pi}}+\frac{2M_3}{3\sqrt{\pi}}\right)M_2(t_0).
\end{aligned}\tag{26}$$

Let $a_1 = n_2 M_1 M_0^2/4 + \varepsilon N_0 M_0^4/16$, $a_2 = (3/2) n_1 M_0 + n_1 M_1 + n_1 + 5N_1 M_0^2/4 + N_1 M_0^2 \varepsilon/2$, $a_3 = (6N_0 M_0 + 2M_3)/3\sqrt{\pi}$, then

$$|r_{yy}(y_0, t_0)| \leqslant M_2(t_i) + a_1(t_0 - t_i) + a_2 \int_{t_i}^{t_0} M_2(\tau)\, d\tau + a_3 \varepsilon M_2(t_0). \quad (27)$$

Similarly, we can prove

$$|s_{yy}(y_0, t_0)| \leqslant M_2(t_i) + a_1(t_0 - t_i) + a_2 \int_{t_i}^{t_0} M_2(\tau)\, d\tau + a_3 \varepsilon M_2(t_0). \quad (28)$$

Since (y_0, t_0) is arbitrary, then for any $(x, t)(-\infty, \infty) \times (t_i, t_{i+1})$, we have

$$M_2(t) \leqslant M_2(t_i) + a_1(t - t_i) + a_2 \int_{t_i}^{t} M_2(\tau)\, d\tau + a_3 \varepsilon M_2(t). \quad (29)$$

That is

$$(1 - a_3\varepsilon)\, M_2(t) - a_2 \int_{t_i}^{t} M_2(\tau)\, d\tau \leqslant M_2(t_i) + a_1(t - t_i),$$

or

$$\left(e^{-(a_2/1 - a_3\varepsilon)t} \int_{t_i}^{t} M_2(\tau)\, d\tau\right)_t \leqslant \frac{M_2(t_i) + a_1(t - t_i)}{1 - a_3\varepsilon}\, e^{-(a_2/1 - a_3\varepsilon)t}. \quad (30)$$

Integrate both sides of (30) from t_i to t_{i+1}, by straight calculation, we have

$$M(t_{i+1}) + \frac{a_1}{a_2} \leqslant \frac{1}{1 - a_3\varepsilon}\, e^{(a_2/1 - a_3\varepsilon)(t_{i+1} - t_i)} \left(M(t_i) + \frac{a_1}{a_2}\right). \quad (31)$$

Thus by (31) and the above proof, we have

$$M_2(T_2) + \frac{a_1}{a_2} \leqslant (1 - a_3\varepsilon)^{T_2/\varepsilon + 1}\, e^{(a_2/(1 - a_3\varepsilon))\, T_2} \left(M_2(0) + \frac{a_1}{a_2}\right)$$

Since $\lim_{\varepsilon \to 0} a_1 = \frac{1}{4} n_2 L^3$, $\lim_{\varepsilon \to 0} a_2 = 3n_1 L/2 + n_1 L + n_1 + 5N_1 L^2/4 = 5n_1 L/2 + n_1 + 5N_1 L^2/4$,

$$\lim_{\varepsilon \to 0} a_3 = \frac{6N_0 L + 2n_0 n_1 L}{3\sqrt{\pi}},$$

$$\lim_{\varepsilon \to 0} (1 - a_3\varepsilon)^{-[T_2/\varepsilon] - 1} = \exp\{(6N_0 + 2n_0 n_1)\, LT_2/3\sqrt{\pi}\},$$

then there exists $\varepsilon_0 > 0$ which depends only on $p(v)$, D_ξ, L, and T such that if $0 < \varepsilon < \varepsilon_0$, then

$$\begin{aligned} M_2(T_2) &\leqslant (H + n_2 L^3 (10 n_1 L + 4 n_1 + 5 N_1 L^2)^{-1} \\ &\qquad \times \exp\{(5n_1 L/2 + n_1 + 5N_1 L^2/4)\, T_2 \\ &\qquad + (6N_0 + 2n_0 n_1)\, L T_2/3\sqrt{\pi}\} \\ &< (H + n_2 L^3/4n_1) \exp\{(3n_1 L + n_1 + 2N_1 L^2 + 3N_0 L + n_0 n_1 L)\, T_2\} \\ &\leqslant (H + n_2 L^3/4n_1) \exp(3n_1 L + n_1 + 2N_1 L^2 + 3N_0 L + n_0 n_1 L)\, T\} \\ &= c_2. \end{aligned} \tag{32}$$

Equation (32) contradicts $T_2 < T$. Thus, we obtain (12). Q.E.D.

The remainder of the proof is standard and we omit it. Q.E.D.

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