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Asymptotic behavior of a nonisothermal viscous Cahn–Hilliard equation with inertial term

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Abstract

We consider a differential model describing nonisothermal fast phase separation processes taking place in a three-dimensional bounded domain. This model consists of a viscous Cahn–Hilliard equation characterized by the presence of an inertial term χ_{tt}, χ being the order parameter, which is linearly coupled with an evolution equation for the (relative) temperature ϑ. The latter can be of hyperbolic type if the Cattaneo–Maxwell heat conduction law is assumed. The state variables and the chemical potential are subject to the homogeneous Neumann boundary conditions. We first provide conditions which ensure the well-posedness of the initial and boundary value problem. Then, we prove that the corresponding dynamical system is dissipative and possesses a global attractor. Moreover, assuming that the nonlinear potential is real analytic, we establish that each trajectory converges to a single steady state by using a suitable version of the Łojasiewicz–Simon inequality. We also obtain an estimate of the decay rate to equilibrium.

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1. Introduction

Consider a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$ which contains, for any time $t \geqslant 0$, a two-phase system subject to nonisothermal phase separation. A well-known evolution system which describes this kind of process is (see [8], cf. also [7])

$$\begin{cases} (\vartheta + \chi)_t - \Delta\vartheta = 0, \\ \chi_t - \Delta\big(-\Delta\chi + \phi(\chi) - \vartheta\big) = 0, \end{cases} \tag{1.1}$$

in $\Omega \times (0, \infty)$. Here ϑ denotes the (relative) temperature around a given critical one, χ represents the order parameter (or phase-field) and ϕ is the derivative of a suitable smooth double well potential (e.g., $\phi(r) = r^3 - ar$, $a > 0$). For the sake of simplicity, all the constants have been set equal to one.

In the isothermal case, the following singular perturbation of Cahn–Hilliard equation has been examined in several papers (see [6,14,22,23,55,56] and references therein)

$$\varepsilon\chi_{tt} + \chi_t - \Delta\big(-\Delta\chi + \alpha\chi_t + \phi(\chi)\big) = 0, \tag{1.2}$$

where $\varepsilon > 0$ is a small inertial parameter and $\alpha \geqslant 0$ is a viscosity coefficient. The inertial term $\varepsilon\chi_{tt}$ accounts for fast phase separation processes (see, e.g., [20]) and, according to [21], its presence seems to give a better description of the spinodal decomposition. Regarding the viscous term $\alpha\chi_t$ the reader is referred to [40] for details. The mathematical works quoted above are concerned with the analysis of the infinite-dimensional dissipative dynamical system generated by (1.2) endowed with suitable boundary conditions. We recall that the case $\alpha = 0$ has been analyzed so far in one spatial dimension only, since in two and three dimensions, many issues are still open (see, however, [47,50]).

In this paper we consider Eq. (1.2) in the nonisothermal case, namely,

$$\begin{cases} (\vartheta + \chi)_t + \nabla\cdot\mathbf{q} = 0, \\ \sigma\mathbf{q}_t + \mathbf{q} = -\nabla\vartheta, \\ \varepsilon\chi_{tt} + \chi_t - \Delta\big(-\Delta\chi + \alpha\chi_t + \phi(\chi) - \vartheta\big) = 0, \end{cases} \tag{1.3}$$

where $\sigma \in [0, 1]$. Observe that the standard Fourier law is obtained when $\sigma = 0$. Otherwise, we have the so-called Maxwell–Cattaneo heat conduction law which entails that ϑ propagates at finite speed (see, e.g., [27–29] and their references).

System (1.3) is subject to the initial conditions

$$\vartheta(0) = \vartheta_0, \quad \sigma\mathbf{q}(0) = \sigma\mathbf{q}_0, \quad \chi(0) = \chi_0, \quad \chi_t(0) = \chi_1, \quad \text{in } \Omega, \tag{1.4}$$

and to the no-flux boundary conditions

$$\mathbf{q}\cdot\mathbf{n} = \nabla\chi\cdot\mathbf{n} = \nabla(\Delta\chi)\cdot\mathbf{n} = 0, \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.5}$$

where $\mathbf{n}$ stands for the outward normal derivative and $\cdot$ indicates the usual Euclidean scalar product. Observe that (1.3) reduces to (1.1) when $\varepsilon = \alpha = 0$. Moreover, note that (1.5) are equivalent to assume the first two conditions and $\nabla u \cdot \mathbf{n} = 0$, where $u = -\Delta\chi + \alpha\chi_t + \phi(\chi) - \vartheta$ is the so-called chemical potential.

Here we want to demonstrate first that problem (1.3)–(1.5) is well posed. Thus we can construct a strongly continuous semigroup $S_\sigma(t)$ on an appropriate phase-space. This semigroup possesses a bounded absorbing set which is compact in the phase-space if $\sigma = 0$, otherwise we show the existence of a compact exponentially attracting set which entails the asymptotic compactness of $S_\sigma(t)$. The latter result is based on a recent decomposition of the solution semigroup devised in [41] (see also [23]). Therefore, for any $\sigma \geqslant 0$, we deduce that $S_\sigma(t)$ possesses a (smooth) global attractor. Taking advantage of these results, we can also deduce that any trajectory originating from the phase-space is precompact. Then, we can proceed to analyze the asymptotic behavior of a single trajectory. More precisely, we show that if ϕ is real analytic, then any (weak) solution $(\vartheta(t), \sigma\mathbf{q}(t), \chi(t))$ converges, as t goes to ∞, to a single equilibrium, namely, to a triplet $(\vartheta_\infty, 0, \chi_\infty)$, where ϑ_∞ and χ_∞ satisfy

$$\begin{cases} \vartheta_\infty = |\Omega|^{-1} \displaystyle\int_\Omega (\vartheta_0 - \varepsilon\chi_1), \\ \displaystyle\int_\Omega \chi_\infty = \int_\Omega (\varepsilon\chi_1 + \chi_0), \\ -\Delta\big(-\Delta\chi_\infty + \phi(\chi_\infty)\big) = 0, \quad \text{in } \Omega, \\ \nabla\chi_\infty \cdot \mathbf{n} = \nabla(\Delta\chi_\infty) \cdot \mathbf{n} = 0, \quad \text{on } \partial\Omega. \end{cases} \tag{1.6}$$

This result is obtained by exploiting a well-known technique originated from some works of S. Łojasiewicz [37,38] and then refined by L. Simon [48]. We recall that, in more than one spatial dimension, the structure of the set of solutions to (1.6) may contain a continuum of solutions if Ω is a ball or an annulus (cf., e.g., [31] and references therein). If this is the case, it is nontrivial to decide whether or not a given trajectory converges to a single stationary state. Moreover, this might not happen even for finite-dimensional dynamical systems (cf. [5]) and there are negative results for semilinear parabolic equations with smooth nonlinearities (see [42,43]).

During the last years, the Łojasiewicz–Simon technique has been modified and used by many authors (cf., e.g., [9,10,12,18,32–36,53]) to investigate a number of parabolic and hyperbolic semilinear equations with variational structure. More recently, this technique has also been used for problems with only a partial variational structure, like the phase-field systems. More precisely, nonconserved models (with or without memory effects) have been analyzed in [1,2,17, 24,54], while the case of a hyperbolic dynamics for the order parameter has been examined in [25,51]. There are also results for nonlocal models (see [16,26]). Concerning the standard Cahn–Hilliard equation, convergence to stationary states has been examined in [11,19,44,46,52], while the nonconstant temperature case, namely (1.1) with (1.5), has been first analyzed in [15] and then in [45] in the case of dynamic boundary conditions. The memory effects in the heat flux have been treated in [3,4] for the Coleman–Gurtin law and, recently, in [39] for a generalization of the Maxwell–Cattaneo law. As we shall see, here we need a particular Łojasiewicz–Simon type inequality which is a refinement of the one proved in [19] (see Lemma 4.1 and its proof in Appendix A).

2. Well-posedness and uniform bounds

Let $H = L^2(\Omega)$ and $\mathbf{H} = (L^2(\Omega))^3$. These spaces are endowed with the natural inner product $\langle\cdot,\cdot\rangle$ and the induced norm $\|\cdot\|$. For the sake of simplicity, we will assume $|\Omega| = 1$ and $\varepsilon = 1$. Then, we set $V = H^1(\Omega)$, $\mathbf{V} = (H^1(\Omega))^3$ and $W = H^2(\Omega)$, both endowed with their standard inner products, and we define the subspace of H of the null mean functions

$$H_0 = \{v \in H\colon \langle v, 1\rangle = 0\}.$$

We also introduce the linear nonnegative operator $A = -\Delta : \mathcal{D}(A) \subset H \to H_0$ with domain

$$\mathcal{D}(A) = \{v \in W\colon \nabla v \cdot \mathbf{n} = 0, \text{ on } \partial\Omega\},$$

and denote by A_0 its restriction to H_0. Note that A_0 is a positive linear operator; hence, for any $r \in \mathbb{R}$, we can define its powers A^r and, consequently, set $V_0^r = \mathcal{D}(A_0^{r/2})$ endowed with the inner product

$$\langle v_1, v_2\rangle_{V_0^r} = \langle A_0^{r/2} v_1, A_0^{r/2} v_2\rangle.$$

Clearly, we have $V_0^0 \equiv H_0$. In addition, we need to use the Hilbert spaces

$$\mathbf{V}_0 = \{\mathbf{v} \in \mathbf{V}\colon \mathbf{v} \cdot \mathbf{n} = 0, \text{ on } \partial\Omega\},$$

and

$$\mathcal{H}_\sigma = H \times \mathbf{H} \times V \times V^*, \qquad \mathcal{V}_\sigma = V \times \mathbf{V}_0 \times \mathcal{D}(A) \times H,$$

endowed with the following norms, respectively,

$$\begin{aligned} \|(z^1, \mathbf{z}^2, z^3, z^4)\|^2_{\mathcal{H}_\sigma} &= \|z^1\|^2 + \sigma\|\mathbf{z}^2\|^2 + \|z^3\|^2_V + \|z^4\|^2_{V^*}, \\ \|(z^1, \mathbf{z}^2, z^3, z^4)\|^2_{\mathcal{V}_\sigma} &= \|z^1\|^2_V + \sigma\|\mathbf{z}^2\|^2_{\mathbf{V}} + \|z^3\|^2_W + \|z^4\|^2, \end{aligned}$$

if $\sigma > 0$. Otherwise, we simply set

$$\mathcal{H}_0 = H \times V \times V^*, \qquad \mathcal{V}_0 = V \times \mathcal{D}(A) \times H.$$

Our assumptions on the function ϕ and on the potential Φ, defined by

$$\Phi(y) = \int_0^y \phi(\xi)\, d\xi, \quad \forall y \in \mathbb{R},$$

are the following:

$$\Phi \in C^3(\mathbb{R}) \quad \text{such that} \quad \Phi(y) \geqslant -c_0, \quad \forall y \in \mathbb{R}; \tag{2.1}$$

$$\left|\phi''(y)\right| \leqslant c_1\left(1+|y|\right), \quad \forall y \in \mathbb{R}; \tag{2.2}$$

$$\forall \epsilon > 0, \quad \text{there exists } c_\epsilon > 0 \quad \text{such that}$$

$$\left|\phi(y)\right| \leqslant \epsilon \Phi(y) + c_\epsilon, \quad \forall y \in \mathbb{R}; \tag{2.3}$$

$$\forall \varsigma \in \mathbb{R}, \quad \text{there exist } c_2 > 0 \text{ and } c_3 \geqslant 0 \quad \text{such that}$$

$$(y-\varsigma)\phi(y) \geqslant c_2\Phi(y) - c_3, \quad \forall y \in \mathbb{R}; \tag{2.4}$$

$$\phi'(y) \geqslant -c_4, \quad \forall y \in \mathbb{R}, \tag{2.5}$$

for some positive constants c_0, c_1, c_4. Here c_2 and c_3 continuously depend on ς.

We now rewrite system (1.3) together with (1.5) in the following form

$$\begin{cases} \left\langle(\vartheta+\chi)_t, v\right\rangle - \langle \mathbf{q}, \nabla v\rangle = 0, & \text{in } (0,\infty), \\ \langle \sigma \mathbf{q}_t + \mathbf{q}, \mathbf{v}\rangle = \langle \vartheta, \nabla\cdot\mathbf{v}\rangle, & \text{in } (0,\infty), \\ \langle \chi_{tt} + \chi_t, w\rangle + \left\langle A\chi + \phi(\chi) + \alpha\chi_t - \vartheta, Aw\right\rangle = 0, & \text{in } (0,\infty), \end{cases} \tag{2.6}$$

for all $v \in V$, $\mathbf{v} \in \mathbf{V}_0$, and $w \in D(A)$, endowed with initial conditions (1.4).

Let us prove

Theorem 2.1. *Let* (2.1)–(2.5) *hold. Then, for any* $(\vartheta_0, \mathbf{q}_0, \chi_0, \chi_1)$ *such that*

$$\vartheta_0 \in H, \tag{2.7}$$

$$\sigma\mathbf{q}_0 \in \mathbf{H}, \tag{2.8}$$

$$\chi_0 \in V, \tag{2.9}$$

$$\chi_1 \in V^*, \tag{2.10}$$

the Cauchy problem (2.6)–(1.4) *has a* (*weak*) *solution* (θ, χ) *with the following properties*:

$$\vartheta \in C^0\left([0,\infty), H\right), \tag{2.11}$$

$$\sigma\mathbf{q} \in C^0\left([0,\infty); \mathbf{H}\right), \quad \mathbf{q} \in L^2(0,\infty; \mathbf{H}), \tag{2.12}$$

$$\chi \in C^0\left([0,\infty), V\right), \tag{2.13}$$

$$\chi_t \in C^0\left([0,\infty), V^*\right) \cap L^2(0,\infty, V^*), \tag{2.14}$$

$$\alpha\chi_t \in L^2(0,\infty, H), \tag{2.15}$$

and there exists a positive constant C*, depending on the norms of the initial data and on* ϕ*, such that, for all* $t \geqslant 0$*,*

$$\left\|\left(\vartheta(t), \mathbf{q}(t), \chi(t), \chi_t(t)\right)\right\|^2_{\mathcal{H}_\sigma} + \int_t^\infty \left(\left\|\vartheta(\tau) - \left\langle\vartheta(\tau), 1\right\rangle\right\|^2 + \left\|\mathbf{q}(\tau)\right\|^2 + \left\|\chi_t(\tau)\right\|^2_{V^*} + \alpha\left\|\chi_t(\tau)\right\|^2\right) d\tau \leqslant C, \tag{2.16}$$

and

$$\big\langle(\vartheta+\chi)(t),1\big\rangle=\langle\vartheta_0+\chi_0,1\rangle,\qquad \big\langle\chi(t),1\big\rangle=\langle\chi_0+\chi_1,1\rangle-\big\langle\chi_1,e^{-t}\big\rangle. \tag{2.17}$$

If $\alpha>0$*, then the solution is unique and the following bound holds*

$$\sup_{t\geqslant 0}\int_t^{t+1}\big\|A\chi(\tau)\big\|^2\,d\tau\leqslant C. \tag{2.18}$$

Moreover, for any fixed $T>0$*, if* $(\vartheta_{0i},\mathbf{q}_{0i},\chi_{0i},\chi_{1i})\in\mathcal{H}_\sigma$*,* $i=1,2$*, then the corresponding solutions* $(\vartheta^i,\mathbf{q}^i,\chi^i,\chi^i_t)$ *satisfy*

$$\begin{aligned}&\big\|\big((\vartheta^1-\vartheta^2)(t),\big(\mathbf{q}^1-\mathbf{q}^2\big)(t),\big(\chi^1-\chi^2\big)(t),\big(\chi^1-\chi^2\big)_t(t)\big)\big\|^2_{\mathcal{H}_\sigma}\\&\quad\leqslant C(R)e^{KT}\big\|(\vartheta_{01}-\vartheta_{02},\mathbf{q}_{01}-\mathbf{q}_{02},\chi_{01}-\chi_{02},\chi_{11}-\chi_{12})\big\|^2_{\mathcal{H}_\sigma},\quad \forall t\in[0,T],\end{aligned} \tag{2.19}$$

for some positive constants $C(R)$ *and* K*, both independent of* T*, where*

$$\big\|(\vartheta_{0i},\mathbf{q}_{0i},\chi_{0i},\chi_{1i})\big\|_{\mathcal{H}_\sigma}\leqslant R,\quad i=1,2.$$

Proof. We first show inequality (2.16) arguing formally. This argument can be made rigorous within a Faedo–Galerkin scheme and it suffices to prove the existence of a solution for all $\alpha\geqslant 0$. From now on C will denote a generic positive constant which depends on ϕ and on the spatial averages of $\vartheta_0+\chi_0$ and $\chi_0+\chi_1$ (cf. (2.17), see also (2.37) below), at most. If a solution exists, then it is easy to show the validity of (2.17), due to the boundary conditions (1.5). Moreover, we have

$$\big\langle\chi_t(t),1\big\rangle=\langle\chi_1,1\rangle e^{-t}. \tag{2.20}$$

Let us set now

$$\tilde\vartheta=\vartheta-\langle\vartheta,1\rangle,\qquad \tilde\chi=\chi-\langle\chi,1\rangle, \tag{2.21}$$

and rewrite problem (2.6) in the form

$$\begin{cases}\big\langle(\tilde\vartheta+\tilde\chi)_t,v\big\rangle-\langle\mathbf{q},\nabla v\rangle=0, & \text{in } (0,\infty),\\ \langle\sigma\mathbf{q}_t+\mathbf{q},\mathbf{v}\rangle=\langle\tilde\vartheta,\nabla\cdot\mathbf{v}\rangle, & \text{in } (0,\infty),\\ \langle\tilde\chi_{tt}+\tilde\chi_t,w\rangle+\big\langle A\tilde\chi+\phi(\chi)+\alpha\tilde\chi_t-\tilde\vartheta,Aw\big\rangle=0, & \text{in } (0,\infty),\end{cases} \tag{2.22}$$

for all $v\in V$, $\mathbf{v}\in\mathbf{V}_0$, and $w\in D(A)$.

Let us take $v=\tilde\vartheta$ in the first equation, $\mathbf{v}=\mathbf{q}$ in the second equation, and $w=A_0^{-1}(\tilde\chi_t+\beta\tilde\chi)$, where $\beta>0$ will be chosen small enough. Adding together the resulting identities, we get

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\tilde\vartheta\|^2+\sigma\|\mathbf{q}\|^2+\big\|A_0^{-1/2}\tilde\chi_t\big\|^2+\|\nabla\tilde\chi\|^2+2\beta\big\langle A_0^{-1/2}\tilde\chi_t,A_0^{-1/2}\tilde\chi\big\rangle\\
&\quad+\beta\big\|A_0^{-1/2}\tilde\chi\big\|^2+\alpha\beta\|\tilde\chi\|^2+2\big\langle\Phi(\chi),1\big\rangle\big)\\
&\quad+2\|\mathbf{q}\|^2+2(1-\beta)\big\|A_0^{-1/2}\tilde\chi_t\big\|^2+2\alpha\|\tilde\chi_t\|^2-2\big\langle\phi(\chi),\big\langle\chi_t(t),1\big\rangle\big\rangle\\
&\quad+2\beta\|\nabla\tilde\chi\|^2+2\beta\big\langle\phi(\chi),\tilde\chi\big\rangle-2\beta\langle\tilde\vartheta,\tilde\chi\rangle=0.
\end{aligned}\tag{2.23}
$$

Observe that, using (2.4) with $\varsigma=\langle\chi,1\rangle$, we deduce

$$
\big\langle\phi(\chi),\tilde\chi\big\rangle\geqslant C_1\big\langle\Phi(\chi),1\big\rangle-C_2,\tag{2.24}
$$

for some $C_1>0$, while, on account of (2.3), we infer

$$
-\big\langle\phi(\chi),\langle\chi_t,1\rangle\big\rangle=-\big\langle\phi(\chi),1\big\rangle\langle\chi_t,1\rangle\geqslant-\frac{\beta C_1}{2}\big\langle\Phi(\chi),1\big\rangle-\frac{c_\beta}{2}e^{-t}.\tag{2.25}
$$

Hence, using (2.1), we have

$$
\begin{aligned}
&-2\big\langle\phi(\chi),\langle\chi_t,1\rangle\big\rangle+2\beta\big\langle\phi(\chi),\tilde\chi\big\rangle\\
&\quad\geqslant\beta C_1\big\langle\Phi(\chi),1\big\rangle-2\beta C_2-c_\beta e^{-t}\geqslant-C\big(\beta+c_\beta e^{-t}\big).
\end{aligned}\tag{2.26}
$$

Then, taking (2.17) and (2.20) into account, from (2.23) we deduce

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\tilde\vartheta\|^2+\sigma\|\mathbf{q}\|^2+\big\|A_0^{-1/2}\tilde\chi_t\big\|^2+\|\nabla\tilde\chi\|^2+2\beta\big\langle A_0^{-1/2}\tilde\chi_t,A_0^{-1/2}\tilde\chi\big\rangle\\
&\quad+\beta\big\|A_0^{-1/2}\tilde\chi\big\|^2+\alpha\beta\|\tilde\chi\|^2+2\big\langle\Phi(\chi),1\big\rangle\big)\\
&\quad+2\|\mathbf{q}\|^2+2(1-\beta)\big\|A_0^{-1/2}\tilde\chi_t\big\|^2+2\alpha\|\tilde\chi_t\|^2+2\beta\|\nabla\tilde\chi\|^2-2\beta\langle\tilde\vartheta,\tilde\chi\rangle\\
&\leqslant C\big(\beta+c_\beta e^{-t}\big).
\end{aligned}\tag{2.27}
$$

Let us now test the third equation of (2.22) with $\tilde\chi$. We obtain

$$
\begin{aligned}
&\frac{d}{dt}\big(2\langle\tilde\chi_t,\tilde\chi\rangle+\|\tilde\chi\|^2+\alpha\|\nabla\tilde\chi\|^2\big)\\
&\quad-2\|\tilde\chi_t\|^2+2\|A\tilde\chi\|^2+2\big\langle\phi'(\chi)\nabla\tilde\chi,\nabla\tilde\chi\big\rangle-2\langle\tilde\vartheta,A\tilde\chi\rangle=0.
\end{aligned}\tag{2.28}
$$

Moreover, in the case $\sigma>0$, using the first two equations of (2.22), we have

$$
\begin{aligned}
\frac{d}{dt}\big\langle\mathbf{q},\nabla A_0^{-1}\tilde\vartheta\big\rangle&=\big\langle\mathbf{q}_t,\nabla A_0^{-1}\tilde\vartheta\big\rangle+\big\langle\mathbf{q},\nabla A_0^{-1}\tilde\vartheta_t\big\rangle\\
&=-\sigma^{-1}\big\langle\mathbf{q},\nabla A_0^{-1}\tilde\vartheta\big\rangle-\sigma^{-1}\|\tilde\vartheta\|^2-\big\langle\mathbf{q},\nabla A_0^{-1}\tilde\chi_t\big\rangle+\big\|A_0^{-1/2}\nabla\cdot\mathbf{q}\big\|^2.
\end{aligned}\tag{2.29}
$$

Let us discuss first the case $\sigma>0$. Then, multiply (2.28) by γ_1 and (2.29) by γ_2, $\gamma_1>0$ and $\gamma_2>0$ to be chosen later, and sum both the obtained expressions to (2.27). Note also that, by the Poincaré inequality and (2.5), for some $\kappa_1>0$ depending only on Ω, we have

$$-2\beta\langle\tilde{\vartheta},\tilde{\chi}\rangle+2\gamma_1\big\langle\phi'(\chi)\nabla\tilde{\chi},\nabla\tilde{\chi}\big\rangle-2\gamma_1\langle\tilde{\vartheta},A\tilde{\chi}\rangle$$
$$\geqslant-(\beta+2\gamma_1c_4)\|\nabla\tilde{\chi}\|^2-\gamma_1\|A\tilde{\chi}\|^2-(\beta\kappa_1+\gamma_1)\|\tilde{\vartheta}\|^2. \tag{2.30}$$

Additionally, for some $\kappa_2>0$ depending also only on Ω, we get

$$-\sigma^{-1}\big\langle\mathbf{q},\nabla A_0^{-1}\tilde{\vartheta}\big\rangle-\big\langle\mathbf{q},\nabla A_0^{-1}\tilde{\chi}_t\big\rangle\leqslant\frac{\sigma^{-1}}{2}\|\tilde{\vartheta}\|^2+\big\|A_0^{-1/2}\tilde{\chi}_t\big\|^2+\kappa_2\big(1+\sigma^{-1}\big)\|\mathbf{q}\|^2. \tag{2.31}$$

Then, let us introduce the functional

$$\begin{aligned}\Psi_\sigma(\tilde{\vartheta},\mathbf{q},\tilde{\chi},\tilde{\chi}_t)&=\|\tilde{\vartheta}\|^2+\sigma\|\mathbf{q}\|^2+\big\|A_0^{-1/2}\tilde{\chi}_t\big\|^2+\|\nabla\tilde{\chi}\|^2\\&\quad+2\beta\big\langle A_0^{-1/2}\tilde{\chi}_t,A_0^{-1/2}\tilde{\chi}\big\rangle+\beta\big\|A_0^{-1/2}\tilde{\chi}\big\|^2+\alpha\beta\|\tilde{\chi}\|^2+2\big\langle\Phi(\chi),1\big\rangle\\&\quad+\gamma_1\big(2\langle\tilde{\chi}_t,\tilde{\chi}\rangle+\|\tilde{\chi}\|^2+\alpha\|\nabla\tilde{\chi}\|^2\big)+\gamma_2\big\langle\mathbf{q},\nabla A_0^{-1}\tilde{\vartheta}\big\rangle,\end{aligned} \tag{2.32}$$

and, recalling (2.30) and (2.31), let us choose, in turn, γ_2 so small that

$$\max\big\{2\gamma_2,\gamma_2\kappa_2\big(1+\sigma^{-1}\big)\big\}\leqslant 1,$$

and then β and γ_1 so small that $\beta\leqslant 1/2$, $\gamma_1c_4\leqslant\beta/4$, and $(\beta\kappa_1+\gamma_1)\leqslant\gamma_2\sigma^{-1}/4$. Then, Ψ_σ fulfills the inequality

$$\begin{aligned}&\frac{d}{dt}\Psi_\sigma(\tilde{\vartheta},\mathbf{q},\tilde{\chi},\tilde{\chi}_t)+c\big(\sigma^{-1}\gamma_2\|\tilde{\vartheta}\|^2+\|\mathbf{q}\|^2+\big\|A_0^{-1/2}\tilde{\chi}_t\big\|^2+\alpha\|\tilde{\chi}_t\|^2+\beta\|\nabla\tilde{\chi}\|^2\big)+\gamma_1\|A\tilde{\chi}\|^2\\&\quad\leqslant C\big(\beta+c_\beta e^{-t}\big).\end{aligned} \tag{2.33}$$

Moreover, possibly choosing a smaller γ_2 (and consequently smaller β and γ_1), we find

$$\begin{aligned}&\Psi_\sigma\big(\tilde{\vartheta}(t),\mathbf{q}(t),\tilde{\chi}(t),\tilde{\chi}_t(t)\big)\\&\quad\geqslant C_\beta\big(\big\|\vartheta(t)\big\|^2+\sigma\big\|\mathbf{q}(t)\big\|^2+\big\|\chi(t)\big\|_V^2+\big\|\chi_t(t)\big\|_{V^*}^2\big)-C,\quad\forall t\geqslant 0,\end{aligned} \tag{2.34}$$

where we stress once more that the constants C are allowed to depend on the spatial averages of the initial data specified in (2.17). On the other hand, on account of (2.1), (2.2) and (2.7)–(2.10), and recalling notation (2.21), we find $R_0>0$ such that

$$\Psi_\sigma(\tilde{\vartheta}_0,\mathbf{q}_0,\tilde{\chi}_0,\tilde{\chi}_1)\leqslant R_0.$$

Using then [22, Lemma 2.1], we deduce that there exists $t_0=t_0(R_0)>0$ such that, for all $t\geqslant t_0$,

$$\Psi_\sigma\big(\tilde{\vartheta}(t),\mathbf{q}(t),\tilde{\chi}(t),\tilde{\chi}_t(t)\big)\leqslant R,$$

where R is independent of R_0. Thus, recalling (2.34), we deduce that

$$\big\|\big(\vartheta(t),\mathbf{q}(t),\chi(t),\chi_t(t)\big)\big\|_{\mathcal{H}_\sigma}^2\leqslant C(R_0), \tag{2.35}$$

for all $t \in [0, \infty)$. On account of (2.20) and (2.35), taking $\beta = 0$ in (2.23), integrating from t to T and letting T go to ∞ we also get the integral control

$$\int_t^\infty \left(\|\mathbf{q}(\tau)\|^2 + \|\chi_t(\tau)\|_{V^*}^2 + \alpha\|\chi_t(\tau)\|^2\right) d\tau \leqslant C(R_0). \tag{2.36}$$

Then, using (2.35) and (2.36), from (2.29) we deduce

$$\int_t^\infty \|\tilde{\vartheta}(\tau)\|^2 \, d\tau \leqslant C(R_0),$$

so that (2.16) is proved. In addition, integrating (2.33) from t to $t+1$, we then find (2.18).

The case $\sigma = 0$ is simpler. We can take the functional

$$\begin{aligned}\Psi_0(\tilde{\vartheta}, \tilde{\chi}, \tilde{\chi}_t) = {} & \|\tilde{\vartheta}\|^2 + \|A_0^{-1/2}\tilde{\chi}_t\|^2 + \|\nabla\tilde{\chi}\|^2 + 2\beta\langle A_0^{-1/2}\tilde{\chi}_t, A_0^{-1/2}\tilde{\chi}\rangle \\ & + \beta\|A_0^{-1/2}\tilde{\chi}\|^2 + \alpha\beta\|\tilde{\chi}\|^2 + 2\langle\Phi(\chi), 1\rangle + \gamma_1\left(2\langle\tilde{\chi}_t, \tilde{\chi}\rangle + \|\tilde{\chi}\|^2 + \alpha\|\nabla\tilde{\chi}\|^2\right),\end{aligned}$$

and observe that

$$\begin{aligned}&\frac{d}{dt}\Psi_0(\tilde{\vartheta}, \tilde{\chi}, \tilde{\chi}_t) + c\left(\|\nabla\tilde{\vartheta}\|^2 + \|A_0^{-1/2}\tilde{\chi}_t\|^2 + \alpha\|\tilde{\chi}_t\|^2 + \beta\|\nabla\tilde{\chi}\|^2\right) + \gamma_1\|A\tilde{\chi}\|^2 \\ &\quad \leqslant C\left(\beta + c_\beta e^{-t}\right).\end{aligned}$$

Then we can argue as above.

Estimate (2.19) is standard, provided that $\alpha > 0$. Indeed, it suffices to write down problem (2.6) for the difference of two solutions $(\vartheta_i, \mathbf{q}_i, \chi_i)$, $i = 1, 2$, and then multiply the first equation by $\vartheta_1 - \vartheta_2$, the second one by $\mathbf{q}_1 - \mathbf{q}_2$, and the third one by $A_0^{-1}(\tilde{\chi}_1 - \tilde{\chi}_2)_t$. Using the Gronwall Lemma and taking (2.2) into account, one easily gets the wanted estimate (see, e.g., [6] or [22] for the isothermal case). □

From Theorem 2.1 and its proof we deduce that, letting

$$X_\sigma^\delta = \left\{\left(z^1, \mathbf{z}^2, z^3, z^4\right) \in \mathcal{H}_\sigma\colon \left|\langle z^1 + z^3, 1\rangle\right| + \left|\langle z^3 + z^4, 1\rangle\right| \leqslant \delta\right\} \tag{2.37}$$

for some $\delta \geqslant 0$, endowed with the metric induced by the norm of $\mathcal{H}_\sigma$, and setting

$$\left(\vartheta(t), \mathbf{q}(t), \chi(t), \chi_t(t)\right) =: S_\sigma(t)(\vartheta_0, \mathbf{q}_0, \chi_0, \chi_1), \quad \forall t \geqslant 0,$$

and for $(\vartheta_0, \mathbf{q}_0, \chi_0, \chi_1) \in \mathcal{H}_\sigma$, we have that $S_\sigma(t)$ is a strongly continuous semigroup on $\mathcal{H}_\sigma$. Moreover, its restriction $S_\sigma^\delta(t)$ to X_σ^δ admits a bounded absorbing set. Similarly, we can define a strongly continuous dissipative semigroup $S_0^\delta(t)$ on X_0^δ. Summing up, we have

Corollary 2.2. *Let* (2.1)–(2.5) *hold. For any given* $\sigma \in [0, 1]$ *and* $\delta \geqslant 0$, *the semigroup* $S_\sigma^\delta(t)$ *acting on* X_σ^δ *has a bounded absorbing set.*

3. Precompactness of trajectories and global attractor

Here we prove

Theorem 3.1. *Let* (2.1)–(2.5) *hold and suppose* $\alpha > 0$. *If* $\sigma \in (0, 1]$ *and* $(\vartheta_0, \mathbf{q}_0, \chi_0, \chi_1)$ *satisfies* (2.7)–(2.10)*, then, indicating by* $(\vartheta, \mathbf{q}, \chi)$ *the corresponding solution to* (2.6)–(1.4) *given by Theorem* 2.1*, the orbit* $\bigcup_{t \geqslant 0}(\vartheta(t), \mathbf{q}(t), \chi(t), \chi_t(t))$ *is precompact in* $\mathcal{H}_\sigma$. *Moreover, there holds*

$$\|\vartheta(t) - \langle\vartheta_0 - \chi_1, 1\rangle\| \to 0, \tag{3.1}$$

$$\|\mathbf{q}(t)\| \to 0, \tag{3.2}$$

$$\|\chi_t(t)\|_{V^*} \to 0, \tag{3.3}$$

as t *goes to* ∞*, and the* ω*-limit set* $\omega(\vartheta_0, \mathbf{q}_0, \chi_0, \chi_1)$ *consists only of equilibrium points of the form* $(\vartheta_\infty, 0, \chi_\infty, 0)$ *where* $(\vartheta_\infty, \chi_\infty)$ *satisfies* (1.6). *Similar results hold when* $\sigma = 0$.

Proof. On account of [41], observe first that, thanks to (2.2), (2.5), and (2.16), we can choose $\ell > c_4$ large enough, and depending on the norms of the initial data, such that

$$\frac{1}{2}\|\nabla z\|^2 + (\ell - 2c_4)\|z\|^2 - \langle\phi'(\chi(t))z, z\rangle \geqslant 0, \tag{3.4}$$

for all $z \in V$ and every $t \geqslant 0$. Consequently, we set

$$\psi(r) = \phi(r) + \ell r, \quad \forall r \in \mathbb{R}.$$

Then, we split the solution to (2.6) in this way

$$(\vartheta, \mathbf{q}, \chi) = (\vartheta^d, \mathbf{q}^d, \chi^d) + (\vartheta^c, \mathbf{q}^c, \chi^c),$$

where

$$\begin{cases} \langle(\vartheta^d + \chi^d)_t, v\rangle - \langle\mathbf{q}^d, \nabla v\rangle = 0, & \text{in } (0, \infty), \\ \langle\sigma\mathbf{q}_t^d + \mathbf{q}^d, \mathbf{v}\rangle = \langle\vartheta_d, \nabla\cdot\mathbf{v}\rangle, & \text{in } (0, \infty), \\ \langle\chi_{tt}^d + \chi_t^d, w\rangle + \langle A\chi^d + \psi(\chi) - \psi(\chi^c) + \alpha\chi_t^d - \vartheta^d, Aw\rangle = 0, & \text{in } (0, \infty), \\ \vartheta^d(0) = \tilde{\vartheta}_0, \quad \sigma\mathbf{q}^d(0) = \sigma\mathbf{q}_0, \quad \chi^d(0) = \tilde{\chi}_0, \quad \chi_t^d(0) = \tilde{\chi}_1, & \text{in } \Omega, \end{cases} \tag{3.5}$$

and

$$\begin{cases} \langle(\vartheta^c + \chi^c)_t, v\rangle - \langle\mathbf{q}^c, \nabla v\rangle = 0, & \text{in } (0, \infty), \\ \langle\sigma\mathbf{q}_t^c + \mathbf{q}^c, \mathbf{v}\rangle = \langle\vartheta_c, \nabla\cdot\mathbf{v}\rangle, & \text{in } (0, \infty), \\ \langle\chi_{tt}^c + \chi_t^c, w\rangle + \langle A\chi^c + \psi(\chi^c) + \alpha\chi_t^c - \vartheta^c, Aw\rangle = \langle\ell\chi, Aw\rangle, & \text{in } (0, \infty), \\ \vartheta^c(0) = \langle\vartheta_0, 1\rangle, \quad \sigma\mathbf{q}^c(0) = \mathbf{0}, \quad \chi^c(0) = \langle\chi_0, 1\rangle, \quad \chi_t^c(0) = \langle\chi_1, 1\rangle, & \text{in } \Omega, \end{cases} \tag{3.6}$$

for all $v \in V$, $\mathbf{v} \in \mathbf{V}_0$, and $w \in D(A)$.

We shall prove that $(\vartheta^d(t), \mathbf{q}^d(t), \chi^d(t), \chi_t^d(t))$ exponentially decays at 0 in $\mathcal{H}_\sigma$ as t goes to ∞, while $(\vartheta^c, \mathbf{q}^c, \chi^c, \chi_t^c)$ is bounded in a space which is compactly embedded in $\mathcal{H}_\sigma$, uniformly in time.

Let us prove first that, for any $t \geqslant s \geqslant 0$ and every $\varpi > 0$, there holds

$$\alpha \int_s^t \|\chi_t^c(\tau)\|^2 \, d\tau \leqslant \varpi(t-s) + \frac{C}{\varpi}. \tag{3.7}$$

This estimate combined with (2.16) will allow us to use a suitable version of the Gronwall Lemma.

Let us take $v = \tilde{\vartheta}^c$ in the first equation of (3.6), $\mathbf{v} = \mathbf{q}^c$ in the second equation, and $w = A_0^{-1}\tilde{\chi}_t^c$ in the third one. Then we obtain

$$\begin{aligned} &\frac{d}{dt}\big(\|\tilde{\vartheta}^c\|^2 + \sigma\|\mathbf{q}^c\|^2 + \|A_0^{-1/2}\tilde{\chi}_t^c\|^2 + \|\nabla\tilde{\chi}^c\|^2 + 2\langle\Psi(\chi^c), 1\rangle - 2\ell\langle\chi, \tilde{\chi}^c\rangle\big) \\ &\qquad + 2\|\mathbf{q}^c\|^2 + 2\|A_0^{-1/2}\tilde{\chi}_t^c\| + 2\alpha\|\tilde{\chi}_t^c\|^2 \\ &\quad = 2\langle\psi(\chi^c), \langle\chi_t, 1\rangle\rangle - 2\ell\langle\chi_t, \tilde{\chi}^c\rangle. \end{aligned} \tag{3.8}$$

Here Ψ is a primitive of ψ. Observe first that it is not difficult to realize that an estimate similar to (2.16) holds for $(\vartheta^c, \mathbf{q}^c, \chi^c, \chi_t^c)$ as well. Therefore, on account of (2.2) and (2.20), we have, for any $\varpi > 0$ and any $t \geqslant 0$,

$$2\langle\psi(\chi^c(t)), \langle\chi_t(t), 1\rangle\rangle - 2\ell\langle\chi_t(t), \tilde{\chi}^c(t)\rangle \leqslant 2\varpi + \frac{C}{\varpi}\big(\|\chi_t(t)\|^2 + \varpi e^{-t}\big).$$

Therefore, (3.7) follows from integrating (3.8) with respect to time from s to t, using the above inequality and (2.16), recalling that $\alpha > 0$, and observing that (cf. (2.20))

$$\langle\chi_t^c(t), 1\rangle = \langle\chi_1, 1\rangle e^{-t}.$$

In order to prove the exponential decay of $(\vartheta^d, \mathbf{q}^d, \chi^d, \chi_t^d)$, we first note that (cf. (2.17))

$$\langle\vartheta^d(t), 1\rangle = \langle\chi^d(t), 1\rangle = 0, \quad \forall t \geqslant 0, \tag{3.9}$$

so that $\vartheta^d = \tilde{\vartheta}^d$ and $\chi^d = \tilde{\chi}^d$.

We now argue as to get (2.32), namely, we take $v = \vartheta^d$ in the first equation, $\mathbf{v} = \mathbf{q}^d$ in the second equation, and $w = A_0^{-1}(\chi_t^d + \beta\chi^d)$, with some $\beta > 0$ which will be chosen later on. Then, we add the functional $\gamma\langle\mathbf{q}^d, \nabla A^{-1}\vartheta^d\rangle$ with $\gamma > 0$. Thus, recalling (2.29), we obtain

$$\begin{aligned} &\frac{d}{dt}\big(\|\vartheta^d\|^2 + \sigma\|\mathbf{q}^d\|^2 + \|A_0^{-1/2}\chi_t^d\|^2 + \|\nabla\chi^d\|^2 + 2\beta\langle A_0^{-1/2}\chi_t^d, A_0^{-1/2}\chi^d\rangle \\ &\qquad + \beta\|A_0^{-1/2}\chi^d\|^2 + \alpha\beta\|\chi^d\|^2 + \gamma\langle\mathbf{q}^d, \nabla A^{-1}\vartheta^d\rangle + 2\langle\psi(\chi) - \psi(\chi^c), \chi^d\rangle \\ &\qquad - \langle\psi'(\chi)\chi^d, \chi^d\rangle\big) + 2\|\mathbf{q}^d\|^2 + 2(1-\beta)\|A_0^{-1/2}\chi_t^d\|^2 + 2\alpha\|\chi_t^d\|^2 + \frac{\gamma}{\sigma}\langle\mathbf{q}^d, \nabla A_0^{-1}\vartheta^d\rangle \\ &\qquad + \frac{\gamma}{\sigma}\|\vartheta^d\|^2 + \gamma\langle\mathbf{q}^d, \nabla A_0^{-1}\chi_t^d\rangle - \gamma\|A_0^{-1/2}\nabla\cdot\mathbf{q}^d\|^2 + 2\beta\|\nabla\chi^d\|^2 \end{aligned}$$

$$+2\beta\langle\psi(\chi)-\psi(\chi^c),\chi^d\rangle-2\beta\langle\vartheta^d,A_0^{-1}\chi^d\rangle$$
$$=2\langle(\psi'(\chi)-\psi'(\chi^c))\chi_t^c,\chi^d\rangle-\langle\psi''(\chi)\chi_t,(\chi^d)^2\rangle. \tag{3.10}$$

Observe that, owing to (2.1), (2.16), and (2.20), we have

$$\begin{aligned}&2\langle(\psi'(\chi)-\psi'(\chi^c))\chi_t^c,\chi^d\rangle-\langle\psi''(\chi)\chi_t,(\chi^d)^2\rangle\\&\quad\leqslant C(\|\chi_t\|+\|\chi_t^c\|)\|\nabla\chi^d\|^2\\&\quad\leqslant\beta\|\nabla\chi^d\|^2+C_\beta(\|\chi_t\|^2+\|\chi_t^c\|^2)\|\nabla\chi^d\|^2.\end{aligned} \tag{3.11}$$

On the other hand, setting

$$\begin{aligned}\Lambda_d={}&\|\vartheta^d\|^2+\sigma\|\mathbf{q}^d\|^2+\|A_0^{-1/2}\chi_t^d\|^2+\|\nabla\chi^d\|^2+2\beta\langle A_0^{-1/2}\chi_t^d,A_0^{-1/2}\chi^d\rangle\\&+\beta\|A_0^{-1/2}\chi^d\|^2+\alpha\beta\|\chi^d\|^2+\gamma\langle\mathbf{q}^d,\nabla A^{-1}\vartheta^d\rangle\\&+2\langle\psi(\chi)-\psi(\chi^c),\chi^d\rangle-\langle\psi'(\chi)\chi^d,\chi^d\rangle,\end{aligned}$$

and observing that (cf. (2.5) and (3.4))

$$2\langle\psi(\chi)-\psi(\chi^c),\chi^d\rangle-\langle\psi'(\chi)\chi^d,\chi^d\rangle\geqslant(\ell-2c_4)\|\chi^d\|^2-\langle\phi'(\chi)\chi^d,\chi^d\rangle\geqslant-\frac{1}{2}\|\nabla\chi^d\|^2,$$

we have that, for β and γ small enough,

$$\frac{1}{4}\|(\vartheta^d,\mathbf{q}^d,\chi^d,\chi_t^d)\|_{\mathcal{H}_\sigma}^2\leqslant\Lambda_d\leqslant C\|(\vartheta^d,\mathbf{q}^d,\chi^d,\chi_t^d)\|_{\mathcal{H}_\sigma}^2. \tag{3.12}$$

Moreover, possibly choosing β and γ smaller than before, and using (3.11), from (3.10) we infer

$$\frac{d}{dt}\Lambda_d+c_{\beta,\gamma}\Lambda_d\leqslant C_{\beta,\gamma}(\|\chi_t\|^2+\|\chi_t^c\|^2)\Lambda_d.$$

Thus, on account of (2.16) and (3.7), we can apply the Gronwall Lemma reported, e.g., in [13, Lemma 2.1] and deduce the exponential decay of Λ_d, so that (cf. (2.20) and (3.12))

$$\|(\vartheta^d(t),\mathbf{q}^d(t),\chi^d(t),\chi_t^d(t))\|_{\mathcal{H}_\sigma}\leqslant C(R)e^{-ct}, \tag{3.13}$$

provided that $\|(\vartheta_0,\mathbf{q}_0,\chi_0,\chi_1)\|_{\mathcal{H}_\sigma}\leqslant R$.

Moreover, taking $w=\chi^d$ in the third equation of (3.5), we obtain (cf. (2.28))

$$\begin{aligned}&\frac{d}{dt}(2\langle\chi_t^d,\chi^d\rangle+\|\chi^d\|^2+\alpha\|\nabla\chi^d\|^2)\\&\quad-2\|\chi_t^d\|^2+2\|A\chi^d\|^2+2\langle\psi(\chi)-\psi(\chi^c),A\chi^d\rangle-2\langle\vartheta^d,A\chi^d\rangle=0,\end{aligned}$$

which yields, using the Young inequality, (2.23), and (3.13),

$$\frac{d}{dt}\big(2\langle\chi_t^d,\chi^d\rangle+\|\chi^d\|^2+\alpha\|\nabla\chi^d\|^2\big)+\|A\chi^d\|^2\leqslant C\big(1+\|\chi_t^d\|^2\big).$$

On account of (2.36) and (3.7), we have

$$\sup_{t\geqslant 0}\int_t^{t+1}\|\chi_t^d(\tau)\|^2\,d\tau\leqslant C,$$

so that there holds the additional bound

$$\sup_{t\geqslant 0}\int_t^{t+1}\|A\chi^d(\tau)\|^2\,d\tau\leqslant C. \tag{3.14}$$

We now consider (3.6). Taking $v=A\vartheta^c$ in the first equation, $\mathbf{v}=-\nabla\nabla\cdot\mathbf{q}^c$ in the second one (this procedure is just formal at this stage, but could be made rigorous through an approximation), and adding together the resulting identities, we obtain

$$\frac{d}{dt}\big(\|\nabla\vartheta^c\|^2+\sigma\|\nabla\cdot\mathbf{q}^c\|^2\big)+2\langle\chi_t^c,A\vartheta^c\rangle+2\|\nabla\cdot\mathbf{q}^c\|^2=0. \tag{3.15}$$

We also have (cf. (2.29))

$$\begin{aligned}\frac{d}{dt}\langle\mathbf{q}^c,\nabla\vartheta^c\rangle&=\langle\mathbf{q}_t^c,\nabla\vartheta^c\rangle+\langle\mathbf{q}^c,\nabla\vartheta_t^c\rangle\\&=-\sigma^{-1}\langle\mathbf{q}^c,\nabla\vartheta^c\rangle-\sigma^{-1}\|\nabla\vartheta^c\|^2-\langle\mathbf{q}^c,\nabla\chi_t^c\rangle+\|\nabla\cdot\mathbf{q}^c\|^2.\end{aligned} \tag{3.16}$$

Let us now take $w=\chi_t^c+\beta\chi^c$ in the third equation. We find

$$\begin{aligned}&\frac{d}{dt}\big(\|\chi_t^c\|^2+\|A\chi^c\|^2+2\beta\langle\chi_t^c,\chi^c\rangle+\beta\|\chi^c\|^2+\alpha\beta\|\nabla\chi^c\|^2+2\langle\psi(\chi^c),A\chi^c\rangle\big)\\&\quad+2(1-\beta)\|\chi_t^c\|^2+2\alpha\|\nabla\chi_t^c\|^2+2\beta\|A\chi^c\|^2+2\beta\langle\psi(\chi^c),A\chi^c\rangle-2\langle\vartheta^c,A\chi_t^c\rangle\\&\quad-2\beta\langle\vartheta^c,A\chi^c\rangle-2\langle\psi'(\chi^c)\chi_t^c,A\chi^c\rangle=\langle\ell\nabla\chi,\nabla(\chi_t^c+\beta\chi^c)\rangle.\end{aligned} \tag{3.17}$$

Observe that, on account of (2.16),

$$\begin{aligned}\langle\psi'(\chi^c)\chi_t^c,A\chi^c\rangle&\leqslant C\big(1+\|\chi^c\|^2_{L^6(\Omega)}\big)\|\chi_t^c\|_{L^6(\Omega)}\|A\chi^c\|\\&\leqslant C\|\chi_t^c\|_V\|A\chi^c\|.\end{aligned} \tag{3.18}$$

Therefore, setting

$$\begin{aligned}\Lambda_c=&\|\nabla\vartheta^c\|^2+\sigma\|\nabla\cdot\mathbf{q}^c\|^2+\|\chi_t^c\|^2+\|A\chi^c\|^2+2\beta\langle\chi_t^c,\chi^c\rangle\\&+\beta\|\chi^c\|^2+\alpha\beta\|\nabla\chi^c\|^2+2\langle\psi(\chi^c),A\chi^c\rangle+\gamma\langle\mathbf{q}^c,\nabla\vartheta^c\rangle,\end{aligned}$$

for some $\gamma > 0$, using the Young inequality, we can choose β and γ small enough so that

$$\frac{d}{dt}\Lambda_c + c_{\beta,\gamma}\Lambda_c \leqslant C_{\beta,\gamma}\big(1 + \|A\chi^c\|^2\big).$$

Then, on account of (2.18) and (3.14), we obtain the uniform boundedness of Λ_c which implies

$$\big\|\nabla\vartheta^c(t)\big\|^2 + \sigma\big\|\nabla\cdot\mathbf{q}^c(t)\big\|^2 + \big\|\chi_t^c(t)\big\|^2 + \big\|A\chi^c(t)\big\|^2 \leqslant C, \quad \forall t \geqslant 0. \tag{3.19}$$

The second equation of (3.6) can now be written in the strong form, namely,

$$\sigma\mathbf{q}_t^c + \mathbf{q}^c = -\nabla\vartheta_c, \quad \text{a.e. in } \Omega\times(0,\infty),$$

so that

$$\sigma\big(\nabla\times\mathbf{q}^c\big)_t + \nabla\times\mathbf{q}^c = \mathbf{0}, \quad \text{a.e. in } \Omega\times(0,\infty),$$

and, since $(\nabla\times\mathbf{q}^c)(0) = \mathbf{0}$, we have $(\nabla\times\mathbf{q}^c)(t) = \mathbf{0}$ for any $t \geqslant 0$. Consequently, thanks to (3.19), $\|\mathbf{q}^c(t)\|_{\mathbf{V}}$ is uniformly bounded as well.

Summing up, we have shown that a given trajectory originating from $\mathcal{H}_\sigma$ is a sum of an exponentially decaying part and a term which belongs to a closed bounded subset of $\mathcal{V}_\sigma$. Therefore the trajectory is precompact in $\mathcal{H}_\sigma$ and, due to the integral controls of (2.16) and to (2.17), we infer (3.1)–(3.3). Finally, it is not difficult to prove that

$$\omega(\vartheta_0, \mathbf{q}_0, \chi_0, \chi_1) \subseteq \big\{(\vartheta_\infty, 0, \chi_\infty, 0)\colon\ (\vartheta_\infty, \chi_\infty) \text{ satisfies (1.6)}\big\}.$$

The case $\sigma = 0$ is easier. In fact, arguing as in the isothermal case (see [6]), we can prove the bound

$$\big\|\big(\vartheta(t), \chi(t), \chi_t(t)\big)\big\|_{\mathcal{V}_0}^2 \leqslant C, \quad \forall t \geqslant t_1 = t_1(R) > 0,$$

provided that $\|(\vartheta_0, \chi_0, \chi_1)\|_{\mathcal{H}_0} \leqslant R$. Hence the trajectory is precompact in $\mathcal{H}_0$ and we can conclude as above. □

From the proof of Theorem 3.1, we deduce that the semigroup $S_\sigma^\delta(t)$ has a bounded attracting set in $\mathcal{V}_\sigma$, for any $\sigma \in (0,1]$, while $S_0^\delta(t)$ has a compact absorbing set. Therefore we have (see, e.g., [30,49])

Corollary 3.2. *For each $\sigma \in [0,1]$ and $\delta \geqslant 0$, the semigroup $S_\sigma^\delta(t)$ has a connected global attractor $\mathcal{A}_\sigma^\delta$ which is bounded in $\mathcal{V}_\sigma$.*

Remark 3.3. The above result is a first, but essential, step toward the construction of a family of exponential attractors which is stable (robust) with respect to σ and, possibly, to ε (see [22] for the isothermal case). This might be the subject of a future investigation.

4. Convergence to stationary states

Let us set

$$E(v)=\frac{1}{2}\|\nabla v\|^2+\langle\tilde{\Phi}(v),1\rangle$$

for any $v\in V_0^1$, where

$$\tilde{\Phi}(y)=\int_0^y\tilde{\phi}(\xi)\,d\xi,\quad \forall y\in\mathbb{R},$$

and

$$\tilde{\phi}(y)=\phi\big(y+\langle\chi_0+\chi_1,1\rangle\big),\quad \forall y\in\mathbb{R}. \tag{4.1}$$

The version of the Łojasiewicz–Simon inequality we need is the following (see Appendix A)

Lemma 4.1. *Suppose that ϕ is real analytic and assume* (2.2) *and* (2.5)*. Let $v_\infty\in V_0^2$ be such that*

$$A\big(A_0v_\infty+\tilde{\phi}(v_\infty)\big)=0. \tag{4.2}$$

Then there exist $\rho\in(0,\frac{1}{2})$, $\eta>0$, and a positive constant L such that

$$\big|E(v)-E(v_\infty)\big|^{1-\rho}\leqslant L\big\|A_0v+\tilde{\phi}(v)-\langle\tilde{\phi}(v),1\rangle\big\|_{V_0^{-1}}, \tag{4.3}$$

for all $v\in V_0^1$ such that $\|v-v_\infty\|_{V_0^1}\leqslant\eta$.

Then we prove

Theorem 4.2. *Let the assumptions of Lemma* 4.1 *hold and let $\alpha>0$ and $\sigma>0$ be fixed. If $(\vartheta_0,\mathbf{q}_0,\chi_0,\chi_1)$ satisfies* (2.7)–(2.10)*, then the trajectory $(\vartheta(t),\mathbf{q}(t),\chi(t),\chi_t(t))$ originated from $(\vartheta_0,\mathbf{q}_0,\chi_0,\chi_1)$ is such that*

$$\omega(\vartheta_0,\mathbf{q}_0,\chi_0,\chi_1)=\big\{(\vartheta_\infty,\mathbf{0},\chi_\infty,0)\big\}, \tag{4.4}$$

where $(\vartheta_\infty,\chi_\infty)$ satisfies

$$\begin{cases}\vartheta_\infty=|\Omega|^{-1}\displaystyle\int_\Omega(\vartheta_0-\chi_1),\\ \displaystyle\int_\Omega\chi_\infty=\int_\Omega(\chi_1+\chi_0),\\ A\big(A\chi_\infty+\phi(\chi_\infty)\big)=0.\end{cases} \tag{4.5}$$

Moreover,

$$\lim_{t\to\infty}\|\chi(t)-\chi_\infty\|_V=0, \tag{4.6}$$

and there exist $t^*>0$ and a positive constant C such that

$$\|\vartheta(t)-\vartheta_\infty\|_{V^*}+\|\chi(t)-\chi_\infty\|_{V^*}\leqslant Ct^{-\frac{\rho}{1-2\rho}},\quad \forall t\geqslant t^*. \tag{4.7}$$

If $\sigma=0$ a similar result hold.

Proof. Let us set $\sigma=1$ for simplicity. On account of Theorem 3.1, we consider

$$(\vartheta_\infty,0,\chi_\infty,0)\in\omega(\vartheta_0,\mathbf{q}_0,\chi_0,\chi_1),$$

and we observe first that (3.1)–(3.3) hold and $(\vartheta_\infty,\chi_\infty)$ fulfills (4.5).

On account of (2.17), we can rewrite (2.22) in the form

$$\begin{cases}\langle(\tilde\vartheta+\tilde\chi)_t,v\rangle-\langle\mathbf{q},\nabla v\rangle=0, & \text{in } (0,\infty),\\ \langle\sigma\mathbf{q}_t+\mathbf{q},\mathbf{v}\rangle=\langle\tilde\vartheta,\nabla\cdot\mathbf{v}\rangle, & \text{in } (0,\infty),\\ \langle\tilde\chi_{tt}+\tilde\chi_t,w\rangle+\langle A_0\tilde\chi+\tilde\phi(\tilde\chi)+\alpha\tilde\chi_t-\tilde\vartheta,Aw\rangle=\langle h(\tilde\chi),Aw\rangle, & \text{in } (0,\infty),\end{cases} \tag{4.8}$$

for all $v\in V$, $\mathbf{v}\in\mathbf{V}_0$, and $w\in\mathcal{D}(A)$. Here we have set (cf. (4.1))

$$h(\tilde\chi)=\tilde\phi(\tilde\chi)-\tilde\phi(\tilde\chi-\langle\chi_1,e^{-t}\rangle),\quad \forall r\in\mathbb{R},\ \forall t\geqslant 0. \tag{4.9}$$

Arguing as in the proof of Theorem 2.1 (cf. (2.23)), we find

$$\frac{d}{dt}\mathcal{L}(\tilde\vartheta(t),\mathbf{q}(t),\tilde\chi(t),\tilde\chi_t(t))=-\|\mathbf{q}(t)\|^2-\|\tilde\chi_t(t)\|_{V*}^2-\alpha\|\tilde\chi_t(t)\|^2+\langle h(\tilde\chi(t)),\tilde\chi_t(t)\rangle, \tag{4.10}$$

where

$$\begin{aligned}&\mathcal{L}(\tilde\vartheta(t),\mathbf{q}(t),\tilde\chi(t),\tilde\chi_t(t))\\ &\quad=\frac12\left(\|\tilde\vartheta(t)\|^2+\|\mathbf{q}(t)\|^2+\|\nabla\tilde\chi(t)\|^2+2\langle\tilde\Phi(\tilde\chi(t)),1\rangle+\|\tilde\chi_t(t)\|_{V*}^2\right).\end{aligned} \tag{4.11}$$

Note that, due to (2.1), (2.2), (2.16) and (4.9), there holds

$$\langle h(\tilde\chi(t)),\tilde\chi_t(t)\rangle\leqslant C_\alpha e^{-2t}+\frac{\alpha}{2}\|\tilde\chi_t(t)\|^2,$$

using also the Young inequality. Therefore, from (4.10) we deduce

$$\begin{aligned}&\frac{d}{dt}\mathcal{L}(\tilde\vartheta(t),\mathbf{q}(t),\tilde\chi(t),\tilde\chi_t(t))\\ &\quad\leqslant-\|\mathbf{q}(t)\|^2-\|\tilde\chi_t(t)\|_{V*}^2-\frac{\alpha}{2}\|\tilde\chi_t(t)\|^2+C_\alpha e^{-2t},\end{aligned} \tag{4.12}$$

for all $t\geqslant 0$.

Then, combining (2.29) with (4.10), we obtain

$$\frac{d}{dt}\big(\mathcal{L}+\mu\langle\mathbf{q},\nabla A_0^{-1}\tilde\vartheta\rangle\big)+\|\mathbf{q}\|^2+\|\tilde\chi_t\|^2_{V*}+\alpha\|\tilde\chi_t\|^2+\mu\|\tilde\vartheta\|^2$$
$$+\mu\langle\mathbf{q},\nabla A_0^{-1}\tilde\vartheta\rangle+\mu\langle\mathbf{q},\nabla A_0^{-1}\tilde\chi_t\rangle-\mu\big\|A_0^{-1/2}\nabla\cdot\mathbf{q}\big\|^2=\langle h(\tilde\chi),\tilde\chi_t\rangle, \tag{4.13}$$

for some $\mu>0$ to be chosen below.

Following a well-known strategy (see, e.g., [12,36]) we consider the functional

$$\mathcal{G}(t)=\big\langle A_0^{-1}\tilde\chi_t,A_0^{-1}\big(A_0\tilde\chi+\tilde\phi(\tilde\chi)-\overline{\tilde\phi(\tilde\chi)}\big)\big\rangle,\quad t\geqslant 0,$$

where

$$\overline{\tilde\phi(\tilde\chi)}=\langle\tilde\phi(\tilde\chi),1\rangle,$$

and we observe that

$$\begin{aligned}\frac{d}{dt}\mathcal{G}&=\big\langle A_0^{-1}\tilde\chi_{tt},A_0^{-1}\big(A_0\tilde\chi+\tilde\phi(\tilde\chi)-\overline{\tilde\phi(\tilde\chi)}\big)\big\rangle+\big\langle A_0^{-1}\tilde\chi_t,A_0^{-1}\big(A_0\tilde\chi_t+\tilde\phi'(\tilde\chi)\tilde\chi_t-\overline{\tilde\phi'(\tilde\chi)\tilde\chi_t}\big)\big\rangle\\&=-\big\langle A_0^{-1}\tilde\chi_t,A_0^{-1}\big(A_0\tilde\chi+\tilde\phi(\tilde\chi)-\overline{\tilde\phi(\tilde\chi)}\big)\big\rangle-\big\|A_0^{-1/2}\big(A_0\tilde\chi+\tilde\phi(\tilde\chi)-\overline{\tilde\phi(\tilde\chi)}\big)\big\|^2\\&\quad-\alpha\big\langle\tilde\chi_t,A_0^{-1}\big(A_0\tilde\chi+\tilde\phi(\tilde\chi)-\overline{\tilde\phi(\tilde\chi)}\big)\big\rangle+\big\langle\tilde\vartheta,A_0^{-1}\big(A_0\tilde\chi+\tilde\phi(\tilde\chi)-\overline{\tilde\phi(\tilde\chi)}\big)\big\rangle\\&\quad+\big\langle h(\tilde\chi),A_0^{-1}\big(A_0\tilde\chi+\tilde\phi(\tilde\chi)-\overline{\tilde\phi(\tilde\chi)}\big)\big\rangle+\big\|A_0^{-1/2}\tilde\chi_t\big\|^2\\&\quad+\big\langle A_0^{-1}\tilde\chi_t,A_0^{-1}\big(\tilde\phi'(\tilde\chi)\tilde\chi_t-\overline{\tilde\phi'(\chi)\tilde\chi_t}\big)\big\rangle.\end{aligned} \tag{4.14}$$

Observe that (cf. (2.2))

$$\big\langle A_0^{-1}\tilde\chi_t,A_0^{-1}\big(\tilde\phi'(\tilde\chi)\tilde\chi_t-\overline{\tilde\phi'(\tilde\chi)\tilde\chi_t}\big)\big\rangle\leqslant C\big\|A_0^{-1/2}\tilde\chi_t\big\|^2. \tag{4.15}$$

Then, from (4.13) and (4.14), using the Young inequality, we find (cf. also (4.9), (4.11), and (4.15))

$$\frac{d}{dt}\mathcal{M}+C_{\mu,\nu}\mathcal{N}^2\leqslant 0, \tag{4.16}$$

for $\mu>0$ and $\nu>0$ sufficiently small, where

$$\mathcal{M}(t)=\frac{1}{2}\big(\|\tilde\vartheta\|^2+\|\mathbf{q}\|^2+\|\tilde\chi_t\|^2_{V^*}\big)+E(\tilde\chi)-E(\tilde\chi_\infty)+\mu\langle\mathbf{q},\nabla A_0^{-1}\tilde\vartheta\rangle$$
$$+\nu\mathcal{G}+C_{\alpha,\nu}e^{-2t}, \tag{4.17}$$

$$\mathcal{N}^2(t)=\big\|\mathbf{q}(t)\big\|^2+\big\|\tilde\chi_t(t)\big\|^2_{V^*}+\mu\big\|\tilde\vartheta(t)\big\|^2+\nu\big\|A_0\tilde\chi(t)+\tilde\phi\big(\tilde\chi(t)\big)-\overline{\tilde\phi\big(\tilde\chi(t)\big)}\big\|^2_{V_0^{-1}}, \tag{4.18}$$

for all $t\geqslant 0$.

Let us introduce the unbounded set

$$\Sigma = \left\{ t \geqslant 0\colon \ \left\| \tilde{\chi}(t) - \tilde{\chi}_\infty \right\|_{V_0^1} \leqslant \frac{\eta}{3} \right\}$$

where η is given by Lemma 4.1. Then, for every $t \in \Sigma$, define

$$\tau(t) = \sup\left\{ t' \geqslant t\colon \sup_{s\in[t,t']} \left\| \tilde{\chi}(s) - \tilde{\chi}_\infty \right\|_{V_0^1} \leqslant \eta \right\},$$

and observe that $\tau(t) > t$, for every $t \in \Sigma$.

Recalling (3.1)–(3.3), let $t_0 \in \Sigma$ be large enough such that

$$\left\| \tilde{\vartheta}(t) \right\| + \left\| \mathbf{q}(t) \right\| + \left\| \tilde{\chi}_t(t) \right\|_{V^*} \leqslant 1, \quad \forall t \geqslant t_0, \tag{4.19}$$

and set

$$\begin{aligned} J &= \left[t_0, \tau(t_0)\right), \\ J_1 &= \left\{ t \in J\colon \mathcal{N}(t) > e^{-2(1-\rho)t} \right\}, \\ J_2 &= J \setminus J_1. \end{aligned}$$

From (4.16), we have that $\mathcal{M}$ is decreasing, therefore it is constant on $\omega(\vartheta_0, \mathbf{q}_0, \chi_0, \chi_1)$. In addition, there holds

$$\frac{d}{dt}\left(\left| \mathcal{M}(t) \right|^\rho \operatorname{sgn} \mathcal{M}(t) \right) = \rho \left| \mathcal{M}(t) \right|^{\rho-1} \frac{d}{dt} \mathcal{M}(t), \quad \forall t \geqslant 0, \tag{4.20}$$

so that $|\mathcal{M}|^\rho \operatorname{sgn} \mathcal{M}$ is decreasing as well.

Observe now that, for every $t \in J_1$, thanks to (4.3) and (4.19), we have

$$\left| \mathcal{M}(t) \right|^{1-\rho} \leqslant C\mathcal{N}(t),$$

possibly choosing μ and ν even smaller than before. Then, on account of (4.16) and (4.20), we infer

$$\begin{aligned} \int_{J_1} \mathcal{N}(t)\, dt &\leqslant -C \int_{t_0}^{\tau(t_0)} \frac{d}{dt}\left(\left| \mathcal{M}(t) \right|^\rho \operatorname{sgn} \mathcal{M}(t) \right) dt \\ &\leqslant C\left(\left| \mathcal{M}(t_0) \right|^\rho + \left| \mathcal{M}\left(\tau(t_0)\right) \right|^\rho \right), \end{aligned} \tag{4.21}$$

where we mean that $|\mathcal{M}(\tau(t_0))| = 0$ if $\tau(t_0) = \infty$. On the other hand, we easily get

$$\int_{J_2} \mathcal{N}(t)\, dt \leqslant Ce^{-2(1-\rho)t_0}.$$

Therefore $\|\tilde{\chi}_t(\cdot)\|_{V^*}$ is integrable over J and

$$
\begin{aligned}
0 &\leqslant \limsup_{t_0\in\Sigma,\, t_0\to\infty} \int_{t_0}^{\tau(t_0)} \big\|\tilde{\chi}_t(t)\big\|_{V^*}\,dt \\
&\leqslant c \limsup_{t_0\in\Sigma,\, t_0\to\infty} \big(\big|\mathcal{M}(t_0)\big|^\rho + \big|\mathcal{M}\big(\tau(t_0)\big)\big|^\rho + Ce^{-2(1-\rho)t_0}\big) = 0.
\end{aligned}
\tag{4.22}
$$

Notice that, for every $t \in J$,

$$
\big\|\tilde{\chi}(t) - \tilde{\chi}_\infty\big\|_{V^*} \leqslant \int_{t_0}^{t} \big\|\tilde{\chi}_t(s)\big\|_{V^*}\,ds + \big\|\tilde{\chi}(t_0) - \tilde{\chi}_\infty\big\|_{V^*}. \tag{4.23}
$$

Suppose now that $\tau(t_0) < \infty$ for any $t_0 \in \Sigma$. Then, by definition,

$$
\big\|\tilde{\chi}\big(\tau(t_0)\big) - \tilde{\chi}_\infty\big\|_{V_0^1} = \eta, \quad \forall t_0 \in \Sigma.
$$

Consider an unbounded sequence $\{t_n\}_{n\in\mathbb{N}} \subset \Sigma$ with the property

$$
\lim_{n\to\infty}\big\|\tilde{\chi}(t_n) - \tilde{\chi}_\infty\big\|_{V_0^1} = 0.
$$

By compactness, we can find a subsequence $\{t_{n_k}\}_{k\in\mathbb{N}}$ and an element $v_\infty \in \mathcal{D}(A)$ such that $(\vartheta_\infty, \mathbf{0}, v_\infty, 0) \in \omega(\vartheta_0, \mathbf{q}_0, \chi_0, \chi_1)$, $\|\tilde{v}_\infty - \tilde{\chi}_\infty\|_{V_0^1} = \eta$, and

$$
\lim_{k\to\infty}\big\|\tilde{\chi}\big(\tau(t_{n_k})\big) - \tilde{v}_\infty\big\|_{V_0^1} = 0.
$$

Then, owing to (4.22) and (4.23), we deduce the contradiction

$$
0 < \|\tilde{v}_\infty - \tilde{\chi}_\infty\|_{V^*} \leqslant \limsup_{k\to\infty}\left(\int_{t_{n_k}}^{\tau(t_{n_k})} \big\|\tilde{\chi}_t(s)\big\|_{V^*}\,ds + \big\|\tilde{\chi}(t_{n_k}) - \tilde{\chi}_\infty\big\|_{V^*}\right) = 0.
$$

Hence, $\tau(t_0) = \infty$ for some $t_0 > 0$ large enough and, recalling (2.20), we can deduce that $\|\chi_t(\cdot)\|_{V^*}$ is indeed integrable over (t_0, ∞). This yields (4.6) by precompactness. On the other hand, on account of (3.1)–(3.3), (4.4) holds as well. Finally, arguing as in [25], we can prove that

$$
\int_t^\infty \mathcal{N}(\tau)\,d\tau \leqslant Ct^{-\frac{\rho}{1-2\rho}}, \quad \forall t \geqslant t^*, \tag{4.24}
$$

for some $t^* > 0$. This entails (cf. (2.20) and (4.18))

$$
\int_t^\infty \big\|\chi_t(\tau)\big\|_{V^*}\,d\tau \leqslant Ct^{-\frac{\rho}{1-2\rho}}, \quad \forall t \geqslant t^*.
$$

Thus we have

$$\left\|\chi(t)-\chi_\infty\right\|_{V^*}\leqslant Ct^{-\frac{\rho}{1-2\rho}},\quad \forall t\geqslant t^*. \tag{4.25}$$

Recalling now (2.20), setting $v=1$ in the first equation of (2.6), and integrating from t to ∞, we obtain

$$\left\langle\vartheta(t),1\right\rangle-\vartheta_\infty=\langle\chi_1,1\rangle e^{-t}. \tag{4.26}$$

Therefore, integrating the first equation of (2.22) with respect to time from $t\geqslant t^*$ to ∞, we deduce

$$\left\langle-\tilde\vartheta(t)+\tilde\chi_\infty-\tilde\chi(t),v\right\rangle=\int_t^\infty\left\langle\mathbf{q}(\tau),\nabla v\right\rangle d\tau,$$

so that, on account of (4.18), (4.24), and (4.25), we infer

$$\left\|\tilde\vartheta(t)\right\|_{V^*}\leqslant Ct^{-\frac{\rho}{1-2\rho}},\quad \forall t\geqslant t^*. \tag{4.27}$$

Therefore, rate estimate (4.7) is a consequence of (4.25)–(4.27). In the case $\sigma=0$ we can proceed in a similar (actually, simpler) way, noting that $\mathbf{q}=-\nabla\vartheta$. $\square$

Remark 4.3. The decay estimate (4.7) for ϑ can be slightly improved. Actually, using the decomposition

$$\left\|\vartheta(t)-\vartheta_\infty\right\|^2\leqslant 2\left\|\vartheta^d(t)\right\|^2+2\left\|\vartheta^c(t)-\vartheta_\infty\right\|^2, \tag{4.28}$$

we see that, by (3.13), the first term decays exponentially. Concerning the latter, one can use (3.19), (4.27) and the interpolation inequality $\|v\|^2\leqslant c\|v\|_V\|v\|_{V^*}$, holding for all $v\in V$. Thus, (4.28) eventually gives

$$\left\|\vartheta(t)-\vartheta_\infty\right\|\leqslant Ct^{-\frac{\rho}{2-4\rho}},\quad \forall t\geqslant t^*. \tag{4.29}$$

Appendix A

This section is devoted to demonstrate Lemma 4.1. Let us introduce the functional

$$E(v)=\int_\Omega\left(\frac12|\nabla v|^2+\tilde\Phi(v)\right)dx,$$

defined on the space V_0^1. As before, we assume $|\Omega|=1$. The differential operator associated with the gradient ∂E does not conserve null mean functions. Hence the version of the Łojasiewicz theorem given in [32] is not applicable directly. This problem was solved in [19], but [19, Assumption 5] is not exactly satisfied here. Our proof, essentially given for the reader's convenience, follows the lines of [11] based on a general version of the Łojasiewicz–Simon theorem obtained in [9].

Proof of Lemma 4.1. We begin to observe that v_∞ satisfying (4.2) is a solution to

$$Av_\infty + \tilde{\phi}(v_\infty) - \overline{\tilde{\phi}(v_\infty)} = 0. \tag{A.1}$$

Moreover, v_∞ is a critical point of E on V_0^1. Indeed, it is easy to check that, owing to our hypotheses, E is continuously differentiable on V_0^1, and

$$\begin{aligned} \partial E(v_\infty)h &= \int_\Omega \big(\nabla v_\infty \cdot \nabla h + \tilde{\phi}(v_\infty)h\big)\,dx \\ &= \int_\Omega \big(\nabla v_\infty \cdot \nabla h + \tilde{\phi}(v_\infty)h - \overline{\tilde{\phi}(v_\infty)}h\big)\,dx, \end{aligned} \tag{A.2}$$

for all $h \in V_0^1$.

We recall that the dual space $(V_0^1)^*$ is the space of classes

$$[f] = \big\{f + g;\ g \in V^*, \langle\!\langle g, V_0^1\rangle\!\rangle = 0\big\}, \quad f \in V^*,$$

where $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ stands for the duality between V^* and V, endowed with the norm

$$\big\|[f]\big\|_{(V_0^1)^*} = \inf_{g\in V^*,\, \langle g, V_0^1\rangle = 0} \|f+g\|_{V^*} = \inf_{c\in\mathbb{R}} \|f+c\|_{V^*} = \|f - \overline{f}\|_{V^*}.$$

Consider the mapping $F : V_0^2 \to H_0$, $F = \partial E|_{V_0^2}$ defined by

$$F(v) = A_0 v + \tilde{\phi}(v) - \overline{\tilde{\phi}(v)}.$$

By virtue a well-known Sobolev embedding theorem, we have that $v_\infty \in V_0^2 \subset L^\infty(\Omega)$, and, due to our assumptions, we can find a neighborhood $\mathcal{U}(v_\infty)$ in the space V_0^2 such that $F : \mathcal{U}(v_\infty) \to H_0$ is analytic. Further, $A_0 : V_0^1 \to (V_0^1)^*$ and $A_0 : V_0^2 \to H_0$ have compact resolvents. Observe now that, when $\partial^2 E : V_0^1 \to \mathcal{L}in[V_0^1, (V_0^1)^*]$, then

$$\big\langle\!\big\langle \partial^2 E(v)[w], z\big\rangle\!\big\rangle = \int_\Omega \tilde{\phi}'(v) w z\,dx + \int_\Omega \nabla w \cdot \nabla z\,dx. \tag{A.3}$$

In addition, we have

$$\partial^2 E = \partial F : V_0^2 \to \mathcal{L}in\big[V_0^2, H_0\big], \qquad \partial F(v)[w] = A_0 w + \tilde{\phi}(v) w - \overline{\tilde{\phi}(v) w}.$$

Hence, in both cases, $\partial^2 E(v_\infty)$ can be viewed as a bounded perturbation of A_0 restricted to the respective spaces. It follows that $\operatorname{Ker}\partial^2 E(v_\infty) \subset V_0^2$ and its range is closed in $(V_0^1)^*$ and H_0, respectively. Moreover, there holds

$$\big(V_0^1\big)^* = \operatorname{Ker}\big(\partial^2 E(v_\infty)\big) \oplus \operatorname{Ran}\big(\partial^2 E(v_\infty)\big), \qquad H_0 = \operatorname{Ker}\big(\partial^2 E(v_\infty)\big) \oplus \operatorname{Ran}\big(\partial F(v_\infty)\big).$$

Now, we can apply [9, Theorem 3.10] and [9, Corollary 3.11] to obtain

$$\left|E(v)-E(v_\infty)\right|^{1-\rho}\leqslant L\left\|\partial E(v)\right\|_{(V_0^1)^*},$$

and, consequently, (4.3). □

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