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ICOSAHEDRAL SYMMETRY AND THE QUINTIC EQUATION

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Abstract—An algorithm has been implemented on a microcomputer for solving the general quintic equation. This algorithm is based on the isomorphism of the A_5 alternating Galois group of the general quintic equation to the symmetry group of the icosahedron, coupled with the ability to partition an object of icosahedral symmetry into five equivalent objects of tetrahedral or octahedral symmetry. A detailed discussion is presented of the portions of this algorithm relating to these symmetry properties of the icosahedron. In addition, the entire algorithm is summarized.

1. INTRODUCTION

One of the classical problems of algebra is the solution of polynomial equations using an algorithm which calculates the roots of the equation from its coefficients. Such a formula for the roots of a quadratic equation is well-known to even beginning algebra students and contains a single square root. An analogous but considerably more complicated algorithm containing both square and cube roots is known for solution of the cubic equation [1,2]. Appropriate combination of the algorithms for the quadratic and cubic equations leads to an algorithm for solution of the quartic equation [1,2]. However, persistent efforts by numerous mathematicians before the early 1800's to find an algortitihm for solution of the general quintic equation using only radicals and elementary arithmetic operations were uniformly unsuccessful. The reason for the futility of these efforts was uncovered by Galois [3–5], who proved the impossibility of solving a general quintic equation using only radicals and arithmetic operations. Galois' proof led to the birth of group theory and the concept of the symmetry of an algebraic equation relating to permutations of its roots [3–5].

The demonstrated impossibility of solving quintic equations using only radicals and simple arithmetic led the 19th century mathematicians following Galois to develop the mathematics of the more complicated functions required for an algebraic solution of the quintic equation. The necessary functions turn out to be Jacobi and/or Weierstrass elliptic functions [6] and the theta series used for their evaluation. The required relationships between such elliptic functions and the roots of a class of quintic equations derivable from the general quintic equation by Tschirnhaus transformations [7] were first established by Hermite [8,9], and developed subsequently by Gordan [10] into an algorithm for solution of the quintic equation. This algorithm was both simplified and clarified by Kiepert [11]. The nature of these algebraic efforts was clarified further by publication, in 1884, of Klein's classical book linking the symmetry of the icosahedron with algorithms for solution of the quintic equation [12]. This book attracted the attention of one of us (R.B.K), who for many years has been fascinated by the special features of icosahedral symmetry [13] in chemical contexts, such as chirality [14], boron hydrides [15] and quasicrystals [16].

These 19th-century algorithms for the solution of the general quintic equation were so complicated that there was no hope of implementing them in that precomputer era. Numerical,

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX rather than algebraic methods based on elliptic functions, evolved to satisfy practical needs for determining roots of quintic and higher degree equations [17]. However, such methods ignore the interesting symmetry aspects of such equations first discovered by Galois [3–5]. The concise presentation of an apparent algorithm for solution of a general quintic equation at the end of the 1878 paper by Kiepert [11] stimulated us to try this algorithm on a modern microcomputer. However, our initial attempts soon revealed a number of ambiguities and even errors of the expected type for an algorithm which had never been actually implemented because of the lack of the necessary computer technology at the time of its development. We therefore soon realized that, contrary to the impression given by Cole [18] as long ago as 1886, a really complete workable algorithm for solution of the general quintic equation by elliptic functions, rather than numerical methods, was not contained in any of the 19th century or more recent literature. We therefore undertook a deeper study of the underlying rather complicated mathematics. Useful surveys of much but not all of the necessary algebra for solution of the general quintic equation are presented in 20th century, but now rather old, algebra texts by Dickson [19] and Perron [20]. However, even these less ancient writeups lack much of the necessary information for a complete algorithm for solution of the general quintic equation. We have now filled in all of the gaps and have a complete computer-tested algorithm for solution of the general quintic equation by algebraic rather than numerical methods. The underlying mathematics of this successful algorithm is long and complicated. This paper focusses on the aspects of the algorithm based on the icosahedral symmetry of the general quintic equation. In addition, a summary of the complete algorithm is presented, so that it can be implemented by an interested reader on an appropriate microcomputer without referring to the literature. Future publications are planned to discuss other aspects of this algorithm.

2. THE SYMMETRY OF ALGEBRAIC EQUATIONS

Consider an algebraic equation of degree n

$$f(x) = a_0\, x^n + a_1\, x^{n-1} + \cdots + a_n = \sum_{i=1}^{n} a_{n-i}\, x^i = 0, \tag{2.1}$$

where a_k $(0 \leq k \leq n)$ is rational. Without loss of generality, a_0 can be taken to be unity since a_k $(1 \leq k \leq n)$ can be adjusted appropriately. The polynomial $f(x)$ is said to be factorable if it can be expressed as a product of polynomials of lower degrees with rational coefficients. The Galois group, G, of an algebraic equation is a permutation group of its roots [3–5]. If the polynomial of the equation is non-factorable, then the corresponding Galois group, G, is transitive, i.e., for any pair of roots x_i and x_k, there is an operation g in G such that x_i is permuted to x_k.

The structure of the Galois group, G, of an algebraic equation relates to the types of functions that are needed to determine its roots. Consider a group G and a subgroup H, with $g \in G$ and $h \in H$. If g is fixed and h is varied over the elements of H, then all elements of the form $g\,h\,g^{-1}$ form another subgroup of G which is said to be conjugate to H. A normal subgroup N with $n \in N$ is one which is self-conjugate, i.e., $g\,n\,g^{-1}$ generates N, and we write $N \triangleleft G$. Subgroups H induce an equivalence relation on the group G partitioning it into various equivalence classes, two elements g and g' of G being regarded as equivalent when $g\,(g')^{-1} \in H$. A normal subgroup N as defined above consists of entire conjugacy classes of G. The conjugacy class headings of the character table of G are thus useful for finding the normal subgroups of G. If N is a normal subgroup of G, the quotient group G/N is well-defined and relates to the type of functions needed to determine the roots of an equation with the Galois group G. A group G with only the identity group C_1 as a normal subgroup is called a simple group. Simple permutation groups tend to be groups of high order relative to the number of objects being permuted. For example, the alternating groups A_n, consisting of all $n!/2$ even permutations of n objects, are simple groups for $n \geq 5$. The smallest non-trivial simple permutation group is thus the alternating group A_5, which is isomorphic with the icosahedral pure rotation group I [13]. This is the general basis of the relationship of icosahedral symmetry to the solution of the general quintic equation, as summarized in Klein's book [12]. The icosahedral pure rotation group I is the only non-trivial simple group which can serve as a symmetry point group in three dimensions. This accounts for the unusual appearance of its character table [21].

Consider a group G that is not simple. Then a series of a maximum number of normal subgroups can be assembled, so that

$$C_1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_m = G, \tag{2.2}$$

and each factor group G_i/G_{i-1} is simple. The group G is said to be solvable when each of these factors is of prime order. In this case, each step from G_i to G_{i-1} in (2.2) generates radicals of the type $\sqrt[s]{a}$ in the solution of an equation with G as the Galois group, where s is the number of elements in the factor group G_i/G_{i-1}. Thus equations with solvable Galois groups are exactly those which are solvable using radicals. Conversely, general equations of degree 5 or higher, which have the simple alternating groups A_n $(n \geq 5)$ as their Galois groups, are not solvable using only radicals.

Some of these and other relevant ideas can be illustrated by the well-known algorithm for solution of the general cubic equation [2]

$$x^3 + a_1\, x^2 + a_2\, x + a_3 = 0. \tag{2.3}$$

The Galois group of Equation (2.3), S_3, is isomorphic to the D_{3h} point group of the equilateral triangle, and has the following normal series and associated factor groups:

$$G_0 = C_1 \triangleleft C_3 = A_3 \triangleleft S_3 = D_{3h}, \tag{2.4}$$

$$\frac{A_3}{C_1} = C_3, \tag{2.5a}$$

$$\frac{S_3}{A_3} = C_2. \tag{2.5b}$$

The solution of a general cubic equation based on this normal series is thus seen to involve a square root from the factor group $S_3/A_3 = C_2$, inside the cube root from the factor group $A_3/C_1 = C_3$.

In the actual algorithm to solve such a general cubic equation (2.3), the equation is first simplified by applying a Tschirnhaus transformation [7] to give the following "reduced" cubic equation with no quadratic term:

$$z^3 + b_2\, z + b_3 = 0, \tag{2.6}$$

where

$$x = z - \frac{1}{3}\, a_1, \tag{2.7a}$$

$$b_2 = \frac{1}{3}\,(3a_2 - a_1^2), \tag{2.7b}$$

$$b_3 = \frac{1}{27}\,(2a_1^3 - 9a_1\, a_2 + 27\, a_3). \tag{2.7c}$$

The roots, z_k $(k = 0, 1, 2)$, of the "reduced" cubic equation (2.6) can be expressed by the relations

$$z_k = \sqrt[3]{\frac{-b_3}{2} + \sqrt{\frac{b_3^2}{4} + \frac{b_2^3}{27}}} + \sqrt[3]{\frac{-b_3}{2} - \sqrt{\frac{b_3^2}{4} + \frac{b_2^3}{27}}}, \tag{2.8}$$

with the cube roots of the two terms being selected so that their product is always $-b_2/3$. If the discriminant $4\,b_2^3 + 27\,b_3^2$ is negative, then the corresponding cubic equations (2.6), and hence (2.3), have three distinct real roots, but the evaluation of these real roots requires taking cube roots of imaginary numbers [1]. In order to avoid this difficulty, the following trigonometric formulas are used to determine the roots of a cubic equation with a negative discriminant [1]:

$$\phi = \cos^{-1}\left(\frac{-b_3}{2\sqrt{-b_2^3/27}}\right), \tag{2.9a}$$

$$z_k = a\sqrt{-b_2/3}\,\cos\left(\frac{\phi + 2k\pi}{3}\right). \tag{2.9b}$$

Note that the sequence of steps (2.9a) and (2.9b) first involves taking the inverse cosine of one number followed by taking the cosine of another number, derived from the first inverse cosine. Both the logarithms needed in general to evaluate the radicals for Equation (2.8) and the inverse cosine needed for Equation (2.9a) can be expressed as integrals of the type

$$u(a) = \int_1^a \frac{dy}{\sqrt{g(y)}}, \tag{2.10}$$

where $g(y)$ is a quadratic polynomial, i.e., y^2 for $\log a$, and $1 - y^2$ for $-\cos^{-1} a$. Integrals of the type (2.10), but with cubic or still higher degree polynomials for $g(y)$, are used for solution of general equations of degree 5 or higher having simple Galois groups. Thus the solvability of an equation by radicals can be translated into the solvability using integrals of the type (2.10) with a quadratic polynomial $g(y)$ in the denominator, whereas the inability to solve an equation using only radicals means that the polynomial $g(y)$ in the denominator of integral (2.10) must be of degree 3 or higher. Inverses of integrals of the type (2.10) in which $g(y)$ is cubic or quartic correspond to the Weierstrass or Jacobi elliptic functions [22], respectively, which play key roles in the solution of the general quintic equation discussed in this paper.

Increasing the degree of an algebraic equation from 3 to 4 introduces no fundamentally new features. The Galois group of the general quartic, S_4, is isomorphic to the T_d full point group of the regular tetrahedron and has the following normal series and associated factor groups [2]:

$$G_0 = C_1 \triangleleft C_2 \triangleleft D_2 \triangleleft A_4 = T \triangleleft S_4 = T_d, \tag{2.11}$$

$$\frac{C_2}{C_1} = C_2, \tag{2.12a}$$

$$\frac{D_2}{C_2} = C_2, \tag{2.12b}$$

$$\frac{A_4}{D_2} = C_3, \tag{2.12c}$$

$$\frac{S_4}{A_4} = C_2. \tag{2.12d}$$

Thus, the solution of a quartic equation can be reduced to the solution of a resolvent cubic equation (using (2.8) or (2.9)) combined with two additional quadratic equations in accord with the standard algorithms [1,2].

The general quintic equation with the S_5 Galois group exhibits important fundamentally new features because of the simplicity of the A_5 group in its normal series and associated factor groups, namely,

$$G_0 = C_1 \triangleleft A_5 = I \triangleleft S_5, \tag{2.13}$$

$$\frac{A_5}{C_1} = A_5, \tag{2.14a}$$

$$\frac{S_5}{A_5} = C_2. \tag{2.14b}$$

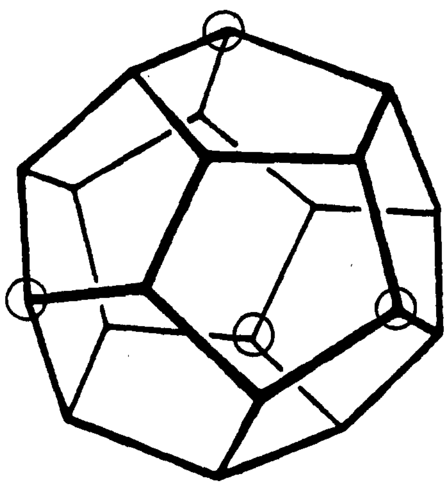

Figure 1. Partitioning the 20 vertices of a regular dodecahedron into five sets of four vertices, each corresponding to regular tetrahedra. The four vertices in one of these sets are circled.

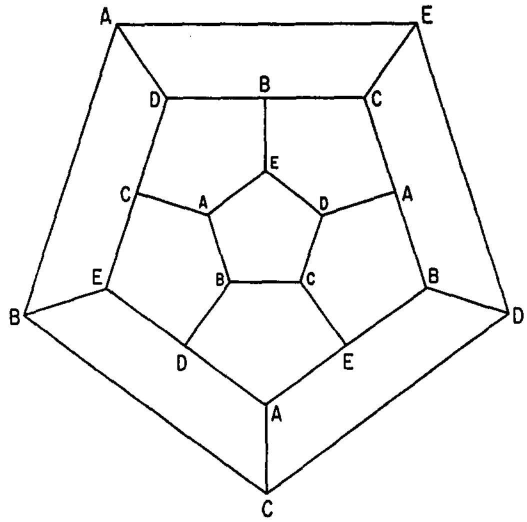

Figure 2. A Schlegel diagram of a regular dodecahedron, showing the relative positions of all five sets of tetrahedral vertices.

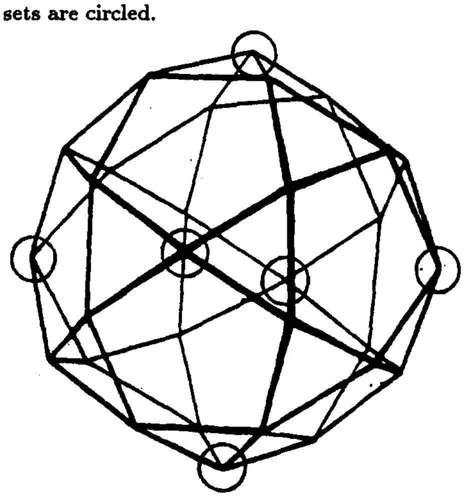

Figure 3. The icosidodecahedron formed from the 30 edge-midpoints of a regular icosahedron, an the partitioning of its 30 vertices into five sets of six vertices, each corresponding to regular octahedra. The six vertices in one of these sets are circled.

The simplicity of the A_5 factor group, which is isomorphic to the icosahedral pure rotation group, I, precludes purely radical solutions of the general quintic equation as noted above. However, the ability to partition an appropriate object of icosahedral symmetry into five equivalent objects of tetrahedral or octahedral symmetry provides the key to the algorithm for solution of the general quintic equation discussed in this paper.

There are several ways of visualizing this type of partition of an object of icosahedral symmetry into five equivalent objects of at least tetrahedral symmetry, thereby separating the effect of the five-fold axis of the icosahedron from that of its two-fold and three-fold symmetry elements. The 20 vertices of a regular dodecahedron can be partioned into five sets of four vertices, each corresponding to a regular tetrahedron. One of these tetrahedral sets of four vertices is circled on the dodecahedron in Figure 1 and all five sets of tetrahedral vertices are labeled in the Schlegel diagram of a regular dodecahedron in Figure 2. Klein [12] partitions the 30 edges of an icosahedron into five sets of six edges each by the following method:

(1) A straight line is drawn from the midpoint of each edge through the center of the icosahedron to the midpoint of the opposite edge;
(2) The resulting 15 straight lines are divided into five sets of three mutually perpendicular straight lines.

Each set of three mutually perpendicular straight lines resembles a set of Cartesian coordinates and defines a regular octahedron. This construction is the dual [23] of a construction depicted by DuVal [24] in color, in which a regular dodecahedron is partioned into five equivalent cubes. Alternatively, the 30 edge midpoints of an icosahedron can be used to form an icosidodecahedron (Figure 3) having 30 vertices, 60 edges, 32 faces, and retaining icosahedral symmetry. The 30 vertices of this icosidodecahedron can then be partitioned into five sets of six vertices, each corresponding to a regular octahedron. One of these sets of octahedral vertices is circled on the icosidodecahedron in Figure 3.

The approach of Kiepert [11] for solving the general monic quintic equation

$$x^5 + A\,x^4 + B\,x^3 + C\,x^2 + D\,x + E = 0, \tag{2.15}$$

which is the basis of our complete computer-tested algorithm, can be broken down into the following seven steps:

(1) Tschirnhaus transformation of the general quintic, into the principal quintic

$$z^5 + 5a\,z^2 + 5b\,z + c = 0. \tag{2.16}$$

This Tschirnhaus transformation requires only a single square root in addition to arithmetic operations.

(2) A second Tschirnhaus transformation of the principal quintic (2.16) into the Brioschi quintic

$$y^5 - 10\,Z\,y^3 + 45\,Z^2\,y - Z^2 = 0, \tag{2.17}$$

in which the coefficients are expressed in terms of a single parameter Z. This Tschirnhaus transformation also requires only a single square root in addition to simple arithmetic operations. This is the step of the Kiepert algorithm which uses the special symmetry properties of the icosahedron, including the ability to partition an object of icosahedral symmetry into five equivalent objects of at least tetrahedral symmetry, as noted above. The next section of this paper discusses this step in detail, since this is the critical step relating icosahedral symmetry to the quintic equation as already noted by Klein [12] in his 1884 book on the icosahedron.

(3) Transformation of the Brioschi quintic (2.17) into the Jacobi sextic

$$s^6 + \frac{10}{\Delta}\,s^3 - \frac{12\,g_2}{\Delta^2}\,s + \frac{5}{\Delta^2} = 0, \tag{2.18}$$

$$\text{where } Z = -\frac{1}{\Delta}. \tag{2.19}$$

The roots s_∞ and s_k $(0 \le k \le 4)$ of the Jacobi sextic can be used to calculate the roots, y_k $(0 \le k \le 4)$, of the Brioschi quintic (2.17), by the relationship

$$y_k = \left[\frac{1}{\sqrt{5}}\,(s_\infty - s_k)\,(s_{k+2} - s_{k+3})\,(s_{k+4} - s_{k+1})\right]^{1/2}, \tag{2.20}$$

with addition of indices modulo 5. The Galois group of the Jacobi sextic (2.18) has 60 elements, like the A_5 Galois group of the quintic equations (2.15), (2.16) and (2.17), but involves transitive permutations of the six roots s_∞ and s_k $(0 \le k \le 4)$ rather than only the five roots of the quintic equations [20].

(4) Solution of the Jacobi sextic (2.18) by the Weierstrass elliptic functions defined by the integral (2.10) with a cubic polynomial in the denominator, defined by

$$g(y) = 4p^3 - g_2\,p - g_3, \tag{2.21}$$

where g_2 is the same as the g_2 in (2.18), and g_3 is related to g_2 and Δ in (2.18).

(5) Evaluation of these Weierstrass elliptic functions using rapidly converging genus 1 theta functions [6,25].
(6) Evaluation of the periods of the theta functions corresponding to a particular Jacobi sextic (2.18). This requires solution of the cubic equation

$$g(y) = 0, \tag{2.22}$$

where $g(y)$ is defined by Equation (2.21) with the parameters g_2 and g_3 coming from the g_2 and Δ in the Jacobi sextic (2.18). A simple function of the three roots of the cubic (Equations (2.21) and (2.22)) is then substituted into a special infinite series called the Jacobi nome [26]. Substitution of these periods into appropriate theta series gives the roots s_∞ and s_k $(0 \le k \le 4)$ of the Jacobi sextic (2.18).
(7) Calculation of the five roots, y_k, of the Brioschi quintic (2.17), from the roots s_∞ and s_k $(0 \le k \le 4)$ of the Jacobi sextic (2.18), using (2.20) and then undoing the Tschirnhaus transformations of the quintic equations in the sequence

$$\text{Brioschi (2.17)} \longrightarrow \text{Principal (2.16)} \longrightarrow \text{General (2.15)}. \tag{2.23}$$

The next section presents details of the second step of this algorithm, which pertains to the icosahedral symmetry of the general quintic equation (2.15). The final section of this paper summarizes the entire algorithm in sufficient detail for programming on a computer.

3. THE ROLE OF ICOSAHEDRAL SETS OF OCTAHEDRA IN THE TSCHIRNHAUS TRANSFORMATION OF THE PRINCIPAL QUINTIC TO THE BRIOSCHI QUINTIC

The objective of this step of the algorithm for the solution of the general quintic equation (2.15) is a Tschirnhaus transformation of a principal quintic (Equation (2.16))

$$z^5 + 5a\,z^2 + 5b\,z + c = 0,$$

into a single parameter Brioschi quintic (Equation (2.17))

$$y^5 - 10\,Z\,y^3 + 45\,Z^2\,y - Z^2 = 0.$$

This transformation uses the following relationship between the variables in (2.16) and (2.17):

$$z = \frac{\lambda + \mu\,y}{(y^2/Z) - 3}. \tag{3.1}$$

This Tschirnhaus transformation is best described geometrically using polyhedral polynomials on the Riemann sphere [12,19]. The five roots of the Brioschi quintic (2.17) are thus the vertex polynomials of the five octahedra in an icosahedral set, where the single parameter Z in Equation (2.17) is obtained from the polyhedral polynomials of the underlying icosahedron. The genesis and properties of the relevant polyhedral polynomials are now presented in greater detail.

Consider a stereographic projection of the north pole of a unit sphere $p^2 + q^2 + r^2 = 1$ (the Riemann sphere) onto its equatorial plane as an Argand plane. A complex number $x = a + bi$ gives

$$a = \frac{p}{1-r}, \quad b = \frac{q}{1-r}, \quad a + bi = \frac{p + iq}{1-r}. \tag{3.2}$$

Solving for p, q and r gives

$$p = \frac{2a}{1 + a^2 + b^2}, \quad q = \frac{2b}{1 + a^2 + b^2}, \quad r = \frac{-1 + a^2 + b^2}{1 + a^2 + b^2}. \tag{3.3}$$

The north pole of the sphere corresponds to ∞ and the south pole to 0. Every rotation of this sphere around its center corresponds to a linear substitution

$$x' = \frac{\alpha\,x + \beta}{\gamma\,x + \delta}. \tag{3.4}$$

For a rotation of the sphere by an angle θ, where p, q, r and $-p, -q, -r$ remain constant,

$$x' = \frac{(v + iu)\,x - (t - is)}{(t + is)\,x + (v - iu)}, \tag{3.5}$$

where $s = p\sin(\theta/2),\ t = q\sin(\theta/2),\ u = r\sin(\theta/2)$ and $v = \cos(\theta/2)$, so that

$$s^2 + t^2 + u^2 + v^2 = 1. \tag{3.6}$$

For a rotation about the 0-∞ axis (south pole-north pole), this reduces to

$$x' = e^{i\theta} x. \tag{3.7}$$

Consider a regular octahedron and a regular icosahedron represented by points on the surface of such a Riemann sphere so that the north pole ($x = \infty$) is one of the vertices in each case. This leads to the following homogeneous polynomials for the regular octahedron and icosahedron in these orientations where x is taken to be u/v [12,19]:

(a) Octahedron (O_h symmetry):

Vertices: $\tau = u\,v\,(u^4 - v^4)$, (3.8a)

Edges: $\chi = u^{12} - 33\,u^8\,v^4 - 33\,u^4\,v^8 + v^{12}$, (3.8b)

Faces: $W = u^8 + 14\,u^4\,v^4 + v^8$. (3.8c)

(b) Icosahedron (I_h symmetry):

Vertices: $f = u\,v\,(u^{10} + 11\,u^5\,v^5 - v^{10})$, (3.9a)

Edges: $T = u^{30} + 522\,u^{25}\,v^{25} - 10,005\,u^{20}\,v^{10} - 10,005\,u^{10}\,v^{20} - 522\,u^5\,v^{25} + v^{30}$, (3.9b)

Faces: $H = -u^{20} + 228\,u^{15}\,v^5 - 494\,u^{10}\,v^{10} - 228\,u^5\,v^{15} - v^{20}$. (3.9c)

The polynomials in Equations (3.8) and (3.9) are conveniently called polyhedral polynomials. Their roots correspond to the locations of the vertices, edge midpoints and face midpoints on the Riemann sphere. Their degrees are equal to the numbers of corresponding elements (vertices, edges, or faces).

The special symmetries of the regular octahedron and icosahedron lead to the following identities:

$$\text{Octahedron:}\quad 108\,\tau^4 - W^3 + \chi^2 = 0,\ \text{(degree 24)}, \tag{3.10a}$$

$$\text{Icosahedron:}\quad 1728\,f^5 - H^3 - T^2 = 0,\ \text{(degree 60)}. \tag{3.10b}$$

Equation (3.10b) plays a role in the Tschirnhaus transformation of a principal quintic (2.16) into the corresponding Brioschi quintic (2.17). Note that the degrees of the left sides of the identities (3.10a) and (3.10b) in (u, v) (see (3.8) and (3.9)) correspond to the orders of the pure rotation groups of the corresponding polyhedra which are isomorphic to permutation groups relevant to the solution of algebraic equations, i.e., $O \approx S_4$ for the quartic equation, and $I \approx A_5$ for the quintic equation, respectively. The special symmetries of these regular polyhedra also lead to the vanishing of the fourth transvectants of the vertex polynomials $(\tau, \tau)^4$ and $(f, f)^4$, as determined by the methods of invariant theory [27,28].

Consider, further, the icosahedron on a Riemann sphere so that one vertex is at the north pole. The midpoints of its 30 edges define an icosahedral set of five octahedra with the following vertex polynomials, t_k, and face polynomials, W_k, $(0 \le k \le 4)$ [19]:

$$t_k = \epsilon^{3k}u^6 + 2\epsilon^{2k}u^5v - 5\epsilon^k u^4v^2 - 5\epsilon^{4k}u^2v^4 - 2\epsilon^{3k}uv^5 + \epsilon^{2k}v^6, \tag{3.11a}$$

$$W_k = -\epsilon^{4k}u^8 + \epsilon^{3k}u^7v - 7\epsilon^{2k}u^6v^2 - 7\epsilon^k u^5v^3 + 7\epsilon^{4k}u^3v^5 - 7\epsilon^{3k}u^2v^6 - \epsilon^{2k}uv^7 - \epsilon^k v^8. \tag{3.11b}$$

In Equations (3.11), $\epsilon = \exp(2\pi i/5)$ and $0 \le k \le 4$. Furthermore, the octahedral polynomials (3.11) are different from the octahedral polynomials (3.8) because of the different orientations of the octahedra relative to the 0-∞ axis.

The vertex polynomials t_k (Equation (3.11a)) of an icosahedral set of five octahedra play an important role in the Tschirnhaus transformation of a principal quintic (2.16) into the Brioschi quintic (2.17), by being the roots of a closely related Brioschi quintic

$$t^5 - 10\,f\,t^3 + 45\,f^2\,t - T = 0, \tag{3.12}$$

in which f and T are the vertex and edge functions of the underlying icosahedron defined by (3.9a) and (3.9b), respectively. Substituting $f^2 = T$ into (3.12) and taking $t = y\,T/f^2$ leads to the one-parameter Brioschi quintic (2.17).

Now consider the principal quintic (2.16). Express its roots, z_k, using the polyhedral polynomials (3.11) of the icosahedral set of five octahedra, in conjunction with the polyhedral polynomials (3.9) of the underlying icosahedron, to give

$$z_k = \left(\frac{\lambda f}{H}\right) W_k + \left(\frac{\mu f^3}{H\,T}\right) t_k\,W_k, \qquad (0 \le k \le 4). \tag{3.13}$$

In (3.13), f, H and T are the icosahedral polynomials (3.9), and t_k and W_k are the polynomials of the icosahedral set of five octahedra (Equations (3.11)). The right hand side of Equation (3.13) is thus homogeneous of degree zero in the variable pair (u, v). The ability to express the roots of any principal quintic (2.16) by (3.13) depends upon the following properties of the polyhedral polynomials, which lead to the vanishing of the z^4 and z^3 coefficients in the principal quintic (2.16):

$$\sum W_k = 0, \quad \text{(degree 8)}, \tag{3.14a}$$
$$\sum t_k\,W_k = 0, \quad \text{(degree 14)}, \tag{3.14b}$$
$$\sum W_k^2 = 0, \quad \text{(degree 16)}, \tag{3.14c}$$
$$\sum t_k\,W_k^2 = 0, \quad \text{(degree 22)}, \tag{3.14d}$$
$$\sum t_k^2 W_k^2 = 0, \quad \text{(degree 28)}. \tag{3.14e}$$

The vanishing of the sums in Equations (3.14) is a consequence of the fact that no product of the icosahedral polynomials f, H and T, of degrees 12, 20 and 30, respectively, can have degrees 8, 14, 16, 22 and 28.

The next step is the calculation of the coefficients a, b and c of the principal quintic (2.16) with roots z_k expressed by (3.13), in terms of two composite icosahedral functions Z and V where

$$Z = \frac{f^5}{T^2}, \tag{3.15a}$$
$$V = \frac{H^3}{f^5}, \tag{3.15b}$$

and Z and V are related by

$$\frac{1}{Z} + V = 1728, \tag{3.16}$$

derived from the identity (3.10b). This calculation uses the following powers of the octahedral polynomials t_k and W_k of which the first five terms are given:

$$t_k^2 = \epsilon^k\,u^{12} + 4u^{11}\,v - 6\epsilon^{4k}\,u^{10}\,v^2 - 20\,\epsilon^{3k}\,u^9\,v^3 + 15\,\epsilon^{2k}\,u^8\,v^4 + \cdots, \tag{3.17a}$$
$$t_k^3 = \epsilon^{4k}\,u^{18} + 6\epsilon^{3k}\,u^{17}\,v - 3\,\epsilon^{2k}\,u^{16}\,v^2 - 52\,\epsilon^k\,u^{15}\,v^3 + 84\,\epsilon^{4k}\,u^{13}\,v^5 + \cdots, \tag{3.17b}$$

(no $u^{14}v^4$ term)

$$t_k^4 = \epsilon^{2k}\,u^{24} + 8\,\epsilon^k\,u^{23}\,v + 4\,u^{22}\,v^2 - 88\,\epsilon^{4k}\,u^{21}\,v^3 - 94\,\epsilon^{3k}\,u^{20}\,v^4 + \cdots, \tag{3.17c}$$

$$W_k^2 = \epsilon^{3k}\,u^{16} - 2\,\epsilon^{2k}\,u^{15}\,v + 15\,\epsilon^{k}\,u^{14}\,v^2 + 35\,\epsilon^{4k}\,u^{12}\,v^4 + 84\,\epsilon^{3k}\,u^{11}\,v^5 + \cdots, \tag{3.18a}$$

$$W_k^3 = -\epsilon^{2k}\,u^{24} + 3\,\epsilon^{k}\,u^{23}\,v - 24\,u^{22}\,v^2 + 22\,\epsilon^{4k}\,u^{21}\,v^3 - 126\,\epsilon^{3k}\,u^{20}\,v^4 + \cdots, \tag{3.18b}$$

$$W_k^4 = \epsilon^{k}\,u^{32} - 4\,u^{31}\,v + 34\,\epsilon^{4k}\,u^{30}\,v^2 - 60\,\epsilon^{3k}\,u^{29}\,v^3 + 295\,\epsilon^{2k}\,u^{28}\,v^4 + \cdots. \tag{3.18c}$$

Also, note that if n is an integer not divisible by 5,

$$\sum_{k=1}^{5} \epsilon^{nk} = 0, \tag{3.19}$$

so that when taking sums of the type

$$\sum_{k=1}^{5} t_k^m\, W_k^n, \tag{3.20}$$

all of the terms with the ϵ^{jk} $(1 \le j \le 4)$ coefficients in (3.18) and (3.19) vanish. By comparing the indicated terms with the indicated products of the icosahedral polynomials f, H and T (Equations (3.9)), the Equations (3.21) and (3.22) can be derived as follows:

$$\begin{array}{lll}
u^{22}v^2 \text{ terms:} & \sum W_k^3 & = (5)(-24)\,f^2 = -120\,f^2, \qquad (3.21a)\\
u^{30} \text{ terms:} & \sum W_k^3 t_k & = (5)(-1)\,T = -5T, \qquad (3.21b)\\
u^{33}v^3 \text{ terms:} & \sum W_k^3\, t_k^2 & = (5)(20 - 18 - 96 + 22)\,u^{33}v^3\\
 & & = (5)(-72)\,f^3 = -360\,f^3, \qquad (3.21c)\\
u^{41}v \text{ terms:} & \sum W_k^3\, t_k^3 & = (5)\,(-6+3)\,u^{41}v = (5)\,(-3)\,fT = -15\,fT, \qquad (3.21d)\\
u^{31}v \text{ terms:} & \sum W_k^4 & = (5)\,(-4)\,u^{31}v = (5)\,(4)\,f\,H = 20\,f\,H, \qquad (3.22a)\\
 & \sum W_k^4 t_k & = 0, \text{ since no product of}\\
 & & \quad f \text{ (degree 12)}, H \text{ (degree 20)}\\
 & & \quad \text{and, } T \text{ (degree 30) can have degree 38}, \qquad (3.22b)\\
u^{42}v^2 \text{ terms:} & \sum W_k^4\, t_k^2 & = (5)\,(34 - 16 - 6)\,u^{42}v^2\\
 & & = (5)\,(-12)\,f^2 H = -60\,f^2 H, \qquad (3.22c)\\
u^{50} \text{ terms:} & \sum W_k^4\, t_k^3 & = (5)\,(-1)\,H\,T = -5\,H\,T, \qquad (3.22d)\\
u^{53}v^3 \text{ terms:} & \sum W_k^4\, t_k^4 & = (5)\,(-88 - 16 + 272 - 60)\,u^{53}v^3 = (5)\,(-108)\,f^3\,H\\
 & & = -540\,f^3\,H. \qquad (3.22e)
\end{array}$$

In addition,

$$\sum z_k^3 = -15\,a, \tag{3.23a}$$

$$\sum z_k^4 = -20\,b. \tag{3.23b}$$

Calculating $\sum z_k^3$ and $\sum z_k^4$ by expanding (3.13) and substituting the values from (3.15) and (3.21)–(3.23) gives

$$V\,a = 8\lambda^3 + \lambda^2\,\mu + (72\,\lambda\,\mu^2 + \mu^3)\,Z, \tag{3.24}$$

$$V\,b = -\lambda^4 + 18\,\lambda^2\,\mu^2\,Z + \lambda\,\mu^3\,Z + 27\,\mu^4\,Z^2. \tag{3.25}$$

Now consider Equation (3.2) and its five roots t_k $(0 \leq k \leq 4)$, to give

$$x^5 - 10\,f\,x^3 + 45\,f^2\,x - T = \prod(x - t_k), \tag{3.26}$$

for all values of x. Let $x = -\lambda\,T/\mu\,f^2$, which generates Equation (2.17) if $y = -\lambda/\mu$. Then,

$$\prod(\lambda\,T + \mu\,f^2\,t_k) = \lambda^5\,T^5 - 10\,\lambda^3\,\mu^2\,T^3\,f^5 + 45\,\lambda\,\mu^4\,T\,f^{10} + \mu^5\,f^{10}\,T. \tag{3.27}$$

However,

$$\prod W_k = -H^2. \tag{3.28}$$

Then, after substituting the values for V and Z from (3.15), we get

$$V c = \lambda^5 - 10\,\lambda^3\,\mu^2\,Z + 45\,\lambda\,\mu^4\,Z^2 + \mu^5\,Z^2. \tag{3.29}$$

Thus the values of a, b and c determined from (3.24), (3.25) and (3.29) give the coefficients of the principal quintic (2.16) corresponding to the Brioschi quintic (2.17), with the change of variables (3.1).

Equations (3.24), (3.25) and (3.29), obtained above from the relationship of the polyhedral polynomials of the icosahedral set of five octahedra to those of the underlying icosahedron can now be inverted using a procedure outlined by Dickson [19], so that the parameters λ, μ, Z and V can be calculated for the Brioschi quintic (2.17) corresponding to a given principal quintic (2.16) with coefficients a, b and c. First, add $\lambda\,V\,b$ obtained from (3.25) to $V\,c$ from (3.29) to give $\mu^2\,Z\,V\,a$ as determined by Equation (3.24), i.e.,

$$\mu^2 Z\,a = \lambda\,b + c. \tag{3.30}$$

Then, substract $\mu^2\,Z\,V\,b$ (obtained from (3.25)) from $\lambda\,V\,c$ (obtained from (3.29)), and use (3.30) to give

$$V = \frac{(\lambda^2 - 3\mu^2\,Z)^3}{(\lambda\,c - \mu^2\,Z\,b)} = \frac{(a\,\lambda^2 - 3b\,\lambda - 3c)^3}{a^2\,(\lambda\,a\,c - \lambda\,b^2 - b\,c)}. \tag{3.31}$$

Now, combine Equations (3.24) and (3.25) in the indicated manner to give

$$\frac{V\,(\lambda\,a + 8\,b)}{\mu} = \lambda^3 + 216\,\lambda^2\,\mu\,Z + 9\lambda\,\mu^2\,Z + 216\,\mu^3\,Z^2. \tag{3.32}$$

Divide the square of the right hand side of (3.32) by Z, substract the result from 27 times the square of the right hand side of (3.24), and combine pairs of terms reflected by the identity (3.16) to give

$$27\,a^2\,V - \frac{V\,(\lambda\,a + 8b)^2}{\mu^2\,Z} = (\lambda^2 - 3\mu^2\,Z)^3. \tag{3.33}$$

Now substitute (3.31) into (3.33), to give

$$27\,a^2 - \frac{(\lambda\,a + 8b)^2}{\mu^2\,Z} = \lambda\,c - \mu^2\,Z\,b. \tag{3.34}$$

Substitution of $(\lambda\,b + c)/a$ for $\mu^2\,Z$ from (3.30) now gives the following quadratic equation for λ, in terms only of the coefficients a, b and c of the principal quintic (2.16):

$$\lambda^2\,(a^4 + a\,b\,c - b^3) - \lambda\,(11\,a^3\,b - a\,c^2 + 2b^2\,c) + 64\,a^2\,b^2 - 27\,a^3\,c - b\,c^2 = 0. \tag{3.35}$$

After solving this quadratic equation for λ the value for V can be found by (3.31), and then the value for Z from (3.16). Equation (3.24) can then be rewritten as

$$(\lambda^2 + \mu^2\,Z)\,\mu = V\,a - 8\lambda^3 - 72\,\lambda\,\mu^2\,Z. \tag{3.36}$$

Substitution of (3.30) for $\mu^2 Z$ into Equation (3.36), and solving for μ then gives

$$\mu = \frac{V a^2 - 8\lambda^3 a - 72\lambda^2 b - 72\lambda c}{\lambda^2 a + \lambda b + c}. \tag{3.37}$$

In this way, the four parameters λ, μ, V and Z are obtained for a Brioschi quintic (2.17) corresponding to a principal quintic (2.16) with roots (3.13) and coefficients a, b and c.

The geometrical relationship between the five octahedra of an icosahedral set and the underlying icosahedron can be used to derive the Tschirnhaus transformation (3.1) relating the principal quintic (2.16) to the one-parameter Brioschi quintic (2.17). The octahedron face polynomials W_k, corresponding to the octahedron vertex functions t_k which are solutions of the Brioschi quintic (3.12), vanish at the midpoints of the faces of the octahedra which are located at the midpoints of the faces of the underlying icosahedron, where its face polynomial H also vanishes. Hence, each W_k is a factor of H. In addition, transposing the term T of the quintic (3.12), squaring both sides, and replacing t^2 by $3f$ gives

$$1728 f^5 = T^2. \tag{3.38}$$

Substituting the icosahedral identity (3.10b) into (3.38) gives $H = 0$ indicating that $t_k^2 - 3f$ is a factor of H for each k so that

$$H = W_k (t_k^2 - 3f), \qquad (0 \le k \le 4). \tag{3.39}$$

Equation (3.39) thus defines five ways of factoring the degree 20 polynomial H into a degree 8 polynomial W_k and a degree 12 polynomial $t_k^2 - 3f$. Substituting (3.39) into (3.13), suppressing subscripts, using (3.15a) for Z, and the relationship $t = yT/f^2$ to convert the two-parameter Brioschi quintic (3.12) into the one-parameter Brioschi quintic (2.17) gives the required relationship (3.1) between the variable z of the principal quintic (2.16) and the variable y of the Brioschi quintic (2.17).

4. THE ALGORITHM FOR SOLUTION OF THE GENERAL QUINTIC EQUATION

Consider a general monic quintic (Equation (2.15))

$$x^5 + A x^4 + B x^3 + C x^2 + D x + E = 0. \tag{4.1}$$

In order to apply the Tschirnhaus transformation

$$z = x^2 - u x + v, \tag{4.2}$$

to give the corresponding principal quintic (Equation (2.16))

$$z^5 + 5a z^2 + 5b z + c = 0, \tag{4.3}$$

first obtain u by solving the quadratic equation

$$(2A^2 - 5B) u^2 + (4A^3 - 13AB + 15C) u + (2A^4 - 8A^2 B + 10AC + 3B^2 - 10D) = 0. \tag{4.4}$$

Then,

$$v = \frac{1}{5}(-Au - A^2 + 2B), \tag{4.5}$$

$$a = \frac{1}{5}[-C(u^3 + Au^2 + Bu + C) + D(4u^2 + 3Au + 2B) - E(5u + 2A) - 10v^3], \tag{4.6}$$

$$b = \frac{1}{5}[D(u^4 + Au^3 + Bu^2 + Cu + D) - E(5u^3 + 4Au^2 + 3Bu + 2C) - 5v^4 - 10av], \tag{4.7}$$

$$c = \frac{1}{5} - E(u^5 + Au^4 + Bu^3 + Cu^2 + Du + E) - v^5 - 5av^2 - 5bv. \tag{4.8}$$

Next apply the Tschirnhaus transformation (Equation (3.1))

$$z = \frac{\lambda + \mu\, y}{(y^2/Z) - 3}, \tag{4.9}$$

to the principal quintic (4.3) to give the one-parameter Brioschi quintic (Equation (2.17))

$$y^5 - 10\, Z\, y^3 + 45\, Z^2\, y - Z^2 = 0, \tag{4.10}$$

by first obtaining λ, solving the quadratic equation (Equation (3.35))

$$(a^4 + a\, b\, c - b^3)\, \lambda^2 - (11\, a^3\, b - a\, c^2 + 2b^2\, c)\, \lambda + 64\, a^2\, b^2 - 27\, a^3\, c - b\, c^2 = 0. \tag{4.11}$$

Then (Equations (3.31), (3.16) and (3.37), respectively)

$$V = \frac{(a\, \lambda^2 - 3b\, \lambda - 3c)^3}{a^2\, (\lambda\, a\, c - \lambda\, b^2 - b\, c)}, \tag{4.12}$$

$$Z = \frac{1}{(1728 - V)}, \tag{4.13}$$

$$\mu = \frac{V\, a^2 - 8\lambda^3\, a - 72\, \lambda^2\, b - 72\, \lambda\, c}{\lambda^2\, a + \lambda\, b + c}. \tag{4.14}$$

Now calculate the elliptic invariants necessary to solve the corresponding Jacobi sextic (Equation (2.18))

$$s^6 + \frac{10}{\Delta}\, s^3 - \frac{12\, g_2}{\Delta^2}\, s + \frac{5}{\Delta^2} = 0, \tag{4.15}$$

by the relationships

$$\Delta = \frac{1}{Z}, \tag{4.16}$$

$$g_2 = \frac{\sqrt[3]{(1 - 1728Z)/Z^2}}{12}, \tag{4.17}$$

$$g_3 = \sqrt{\frac{(g_2^3 - \Delta)}{27}}. \tag{4.18}$$

In order to determine the periods of the theta functions needed to evaluate the relevant Weierstrass elliptic function, solve the cubic equation (Equation (2.21))

$$g(y) = 4p^3 - g_2\, p - g_3 = 0, \tag{4.19}$$

by defining

$$G = -\frac{1}{4}\, g_3, \tag{4.20a}$$

$$H = -\frac{1}{12}\, g_2. \tag{4.20b}$$

Then

$$U = \left[\frac{1}{2}\,(-G + \sqrt{G^2 + 4H^3})\right]^{1/3}, \tag{4.21}$$

$$V = -\frac{H}{U}, \tag{4.22}$$

$$\gamma = \exp\left(\frac{2\pi i}{3}\right). \tag{4.23}$$

The three roots of $g(y)$ (Equation (4.19)), designated as e_1, e_2 and e_3, are obtained from U, V and γ by

$$e_1 = U + V, \tag{4.24a}$$
$$e_2 = \gamma U + \gamma^2 V, \tag{4.24b}$$
$$e_3 = \gamma^2 U + \gamma V. \tag{4.24c}$$

Now assign the subscripts 1, 2 and 3 to a, b and c, in such manner that neither

$$k^2 = \frac{e_b - e_c}{e_a - e_c}, \tag{4.25a}$$

nor

$$(k')^2 = \frac{e_a - e_b}{e_a - e_c}, \tag{4.25b}$$

has a negative real part or a positive real part equal to or greater than unity. Now calculate

$$k' = \frac{\sqrt{e_a - e_b}}{\sqrt{e_a - e_c}}, \tag{4.26}$$

taking the square roots so that both k and k' have positive real parts. Next, calculate

$$L = \frac{(1 - \sqrt{k'})}{(1 + \sqrt{k'})}, \tag{4.27}$$

taking the principal value of the square root $\sqrt{k'}$. The parameter q for the theta function can then be calculated by the Jacobi nome [26]

$$\begin{aligned} q = {} & \left(\frac{L}{2}\right) + 2\left(\frac{L}{2}\right)^5 + 15\left(\frac{L}{2}\right)^9 + 150\left(\frac{L}{2}\right)^{13} + 1707\left(\frac{L}{2}\right)^{17} + 20{,}910\left(\frac{L}{2}\right)^{21} \\ & + 268{,}616\left(\frac{L}{2}\right)^{25} + 3{,}567{,}400\left(\frac{L}{2}\right)^{29} + \cdots . \end{aligned} \tag{4.28}$$

Define the theta functions

$$\mathrm{ths}(q) = \sum_{j=-\infty}^{\infty} (-1)^j \, q^{(6j+1)^2/12}, \tag{4.29}$$

and

$$B^2 = \Delta^{1/3} \, [\mathrm{ths}(q)]^2, \tag{4.30}$$

where Δ comes from (4.15) and (4.16). Then the roots of the Jacobi sextic (4.15) are

$$s_\infty = \frac{5\,[\mathrm{ths}(q^5)]^2}{B^2}, \tag{4.31a}$$
$$s_k = \frac{[\mathrm{ths}(\epsilon^{12k} q^{1/5})]^2}{B^2}, \qquad (0 \le k \le 4), \tag{4.31b}$$

where

$$\epsilon = \exp\left(\frac{2\pi i}{5}\right). \tag{4.32}$$

The roots of the Jacobi sextic obtained from equations (4.31) should be checked before proceeding further, in order to assure that the correct square root of k', the correct cube root of Δ and the correct fifth root of q have been taken for Equations (4.27), (4.30) and (4.31b), respectively. After the correct roots s_∞ and s_k $(0 \le k \le 4)$ for the Jacobi sextic (4.15) have been obtained, these are used to calculate the roots of the Brioschi quintic (4.10), by using the two equations

$$y_k^2 = \frac{1}{\sqrt{5}}(s_\infty - s_k)(s_{k+2} - s_{k+3})(s_{k+4} - s_{k+1}), \tag{4.33}$$

$$y_k = \frac{Z^2}{(y_k^2)^2 + (10/\Delta)\,y_k^2 + (45/\Delta^2)}, \tag{4.34}$$

adding indices modulo 5 in Equation (4.33) and using (4.34) in order to assure that the correct square root is taken in Equation (2.20). Undoing the Tschirnhaus transformations gives

$$z_k = \frac{\lambda + \mu\, y_k}{(y_k^2/Z) - 3}, \tag{4.35}$$

from (4.9), for the roots of the principal quintic (4.3) and then

$$x_k = -\frac{E + (z_k - v)(u^3 + A\,u^2 + B\,u + C) + (z_k - v)^2(2u + A)}{u^4 + A\,u^3 + B\,u^2 + C\,u + D + (z_k - v)(3u^2 + 2A\,u + B) + (z_k - v)^2}, \tag{4.36}$$

for the roots of the general quintic (4.1).

The above algorithm has been checked by implementing it as a complete Pascal program which executes on a personal microcomputer. This computer program also checks the ambiguities arising from the several radicals in the algorithm, by investigating whether the product

$$(s - s_\infty)\sum_{k=0}^{4}(s - s_k), \tag{4.37}$$

agrees with the original Jacobi sextic (Equation (4.15)) for all of the possible choices of $\sqrt{k'}$, $q^{1/5}$ and $\Delta^{1/3}$ in (4.27), (4.31b) and (4.30), respectively, before proceeding with the final steps of the algorithm involving the sequence of roots $s_k \longrightarrow y_k \longrightarrow z_k \longrightarrow x_k$ using (4.33)–(4.36) and the correct values of s_k.

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