Relativistic Dissipative Cosmological Models and Abel Differential Equation

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Abstract—We analyze the second-order strongly nonlinear differential equation describing the relativistic evolution of a causal dissipative cosmological fluid in a conformally flat space-time. By means of appropriate transformations, the evolution equation can be reduced to a first-order second-type Abel differential equation. The general solution of the Abel equation is obtained in an exact form and, as a consequence, it is shown that the general solution of the second-order evolution differential equation can be represented in an exact parametric form. © 2003 Elsevier Ltd. All rights reserved.

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1. INTRODUCTION

Dissipative processes are supposed to play a fundamental role in the evolution and dynamics of the early universe. Bulk viscosity may arise in different contexts during the evolution of the early universe. Some physical processes involving viscous effects are the evolution of cosmic strings, the classical description of the (quantum) particle production, interaction between matter and radiation, quark and gluon plasma, interaction between different components of dark matter, etc. [1].

The first attempts at creating a theory of relativistic dissipative fluids were those of Eckart [2] and Landau and Lifshitz [3]. These theories are now known to be pathological in several respects. Regardless of the choice of equation of state, all equilibrium states in these theories are unstable, and in addition, signals may be propagated through the fluid at velocities exceeding the speed of light. Conventional theory is thus applicable only to phenomena that are "quasistationary", i.e., slowly varying on space and time scales. This is inadequate for many situations in high-energy astrophysics and relativistic cosmology, involving steep gradients or rapid variations.

A relativistic second-order theory was found by Israel [4], and developed by Israel and Stewart [5] into what is called ‘transient’ or ‘extended’ irreversible thermodynamics. Causal bulk viscous thermodynamics has been extensively used for describing some astrophysical processes and the dynamics and evolution of the early universe [6–20].

Because of technical reasons, most investigations of dissipative causal cosmological models...
have assumed Friedmann-Robertson-Walker symmetry (i.e., homogeneity and isotropy), or small
coupling to background radiation. The Einstein field equations for homogeneous models with dis-
sipative fluids can be decoupled, and therefore, they are reduced to an autonomous system of
first-order ordinary differential equations, which can be analyzed qualitatively [22,23]. Roman0
and Pavon [24] have performed a qualitative analysis of the Bianchi type cosmological models,
under the assumption of the plane symmetry.

It is the purpose of the present paper to consider the general solution of the Einstein gravita-
tional field equations, describing the evolution of a bulk viscous relativistic fluid in a conformally
flat geometry.

The line element conformal to a flat Minkowskian space-time with coordinates $x^0 = t$, $x^1 = x$,
$x^2 = y$, $x^3 = z$, and with metric tensor $\eta_{ik}$, with diagonal elements equals to $(1, -1, -1, -1)$
and other components vanishing, is of the form [25]

$$ds^2 = e^{\Gamma(t,x,y,z)} \eta_{ik} dx^i dx^k.\quad (1)$$

For the energy-momentum tensor of the fluid and for the velocity field, we assume the forms
given by Tauber [25,26]. In order to close the gravitational field equations, the equations of
state for the pressure $p$, bulk viscosity coefficient $\xi$, temperature $T$, and relaxation time $\tau$
are also needed. As usual, we follow the standard approach [27], by taking $p = (\gamma - 1)\rho$, $\xi = \alpha p^{\gamma - 1}$,
$T = \beta p^{\gamma - 1}$, $\tau = \xi / \rho = \alpha p^{\gamma - 1}$, where $1 \leq \gamma \leq 2$, $\alpha \geq 0$, $\beta \geq 0$, $R = (\gamma - 1) / \gamma \geq 0$, and $m \geq 0$ are
constants. $\rho$ is the energy density of the cosmological fluid.

With the use of these physical assumptions, the basic equation describing the evolution of the
conformally flat causal bulk viscous cosmological model can be obtained in the form [19]

$$\frac{d^2w}{du^2} + \left[ \frac{3^{1-m}}{\alpha} \frac{w^{1-m}}{\sqrt{w + e^{-2u}}} + \frac{3}{2} \left( 1 + \gamma \left( 1 - R \right) \right) \right] \frac{dw}{du} - \frac{(1 + R)}{2w} \left( \frac{dw}{du} \right)^2
+ 9 \left( \frac{\gamma}{2} - 1 \right) w + \frac{3^{2-m} \gamma}{\alpha} \frac{w^{2-m}}{\sqrt{w + e^{-2u}}} = 0,\quad (2)$$

where $w$ is a function related to the Hubble parameter and the independent variable $u$ is propor-
tional to the metric function $\Gamma$.

In the next sections, we shall show that the dynamics of the conformally flat cosmological
model, described by (2), can be reduced to an Abel type equation, whose general solution is
expressed in an exact parametric form.

The present paper is organized as follows. The evolution equation is reduced to an Abel type
differential equation in Section 2. In Section 3, the general solution of the associated first-order
equation is obtained. We conclude our results in Section 4.

2. REDUCTION TO ABEL DIFFERENTIAL EQUATION

By introducing a new variable $v$ by means of the transformation $w = v^{2/(1 - R)}$, the evolution
equation (2) takes the form

$$\frac{d^2v}{du^2} + \left[ \frac{3^{1-m}}{\alpha} \frac{v^{2(1-m)/(1-R)}}{\sqrt{v^{2/(1-R)} + e^{-2u}}} + \frac{3}{2} \left( 1 + \gamma \left( 1 - R \right) \right) \right] \frac{dv}{du}
+ \frac{(1 - R)}{2} \left( \frac{\gamma}{2} - 1 \right) v + \frac{3^{2-m} \gamma}{2\alpha} \frac{v^{2(1-m)/(1-R)+1}}{\sqrt{v^{2/(1-R)} + e^{-2u}}} = 0.\quad (3)$$

Of particular importance is the value of $m = 1/2$, corresponding to the high-matter density
limit. For this value of $m$, (3) takes the form

$$\frac{d^2v}{du^2} + \left[ \frac{3^{1/2}}{\alpha} \frac{v^{1/(1-R)}}{\sqrt{v^{2/(1-R)} + e^{-2u}}} + \frac{3}{2} \left( 1 + \gamma \left( 1 - R \right) \right) \right] \frac{dv}{du}
+ \frac{(1 - R)}{2} \left( \frac{\gamma}{2} - 1 \right) v + \frac{3^{3/2} \gamma(1 - R)}{2\alpha} \frac{v^{1/(1-R)+1}}{\sqrt{v^{2/(1-R)} + e^{-2u}}} - 0.\quad (4)$$
We shall start our study of (4) by proving the following lemma.

**Lemma 1.** The second-order nonlinear differential equation (4) is equivalent to a first-order second-kind nonlinear Abel type differential equation.

**Proof.** To transform the equation to a simpler form, we introduce first a new dependent variable $f(u)$ by means of the definition

$$f(u) = \tanh^{-1}\left(\frac{\eta^{1/(1-R)}}{\sqrt{\eta^{2/(1-R)} + e^{-2u}}}\right),$$

or, equivalently,

$$v = \sinh^{1-R}(f)e^{-(1-R)u}$$

In the new variable, (4) becomes

$$\frac{d^2 f}{du^2} + [a \tanh(f) + b] \frac{df}{du} + \left(\tanh(f) - \frac{R}{\tanh(f)}\right) \left(\frac{df}{du}\right)^2 + \tanh(f) (c + d \tanh(f)) = 0,$$

where we have denoted

$$a = \frac{3^{1/2}}{\alpha}, \quad b = \frac{3}{2} \left(1 + \gamma(1-R)\right) - 2(1-R) = \frac{3\gamma}{2} \left(1 - \frac{R}{2}\right) + 2R - \frac{1}{2},$$

$$c = \frac{9}{2} \left(\frac{\gamma}{2} - 1\right) - \frac{3}{2} \left(1 + \gamma(1-R)\right) + 1 - R = \frac{3\gamma}{2} \left(\frac{1}{2} - R\right) - 5 - R, \quad d = \frac{3^{1/2}}{\alpha} \left(\frac{3\gamma}{2} - 1\right).$$

By means of the substitution

$$\frac{df}{du} = X(f) \frac{\sinh R(f)}{\cosh(f)},$$

equation (7) is transformed into

$$X \frac{dX}{df} + (a \tanh(f) + b) \frac{\cosh(f)}{\sinh R(f)} X + \tanh(f) (c + d \tanh(f)) \frac{\cosh^2(f)}{\sinh^{2R}(f)} = 0.$$  

Defining two new variables $Y = 1/X$ and $\omega = \tanh(f)$, takes the form

$$\frac{dY}{d\omega} - A(\omega)Y^2 - B(\omega)Y^3 = 0,$$

with

$$A(\omega) = \omega^{-R}(a\omega + b) \left(1 - \omega^2\right)^{(R-3)/2}, \quad B(\omega) = \omega^{1-2R}(d\omega + c) \left(1 - \omega^2\right)^{R-2},$$

and $\omega = \tanh(f)$. Therefore, we have proved that the second-order nonlinear differential equation (4) can be transformed into a second-kind first-order Abel type differential equation.

### 3. General Solution of the Abel Differential Equation

In order to obtain the general solutions of (4), we also need to use the following lemma.

**Lemma 2.** The Abel type differential equation (10) can be exactly integrated if and only if the functions $A(\omega)$ and $B(\omega)$ satisfy the condition

$$\frac{d}{d\omega}\left(\frac{B(\omega)}{A(\omega)}\right) = kA(\omega),$$

where $k$ is a constant [28].
For (10), condition (12) is satisfied only for some particular values of the coefficients $a$, $b$, $c$, $d$, and $k$, so that

$$a = b, \quad d = 3c, \quad k = \frac{ac - bd}{a^3} = -\frac{2d}{3ab} = -\frac{bd + 3ac}{3ab^2} = -\frac{2c}{b^2}. \quad (13)$$

For the parameters $R$, $\alpha$, and $\gamma$ of the initial equation (4), the conditions (13), relating the coefficients in (10), lead to

$$R = 3, \quad \alpha = \frac{2\sqrt{3}}{(11 - 6\gamma)}, \quad k = \frac{2(32 - 21\gamma)}{(11 - 6\gamma)^2}. \quad (14)$$

$\gamma$ is given by the roots of the quadratic equation

$$9\gamma^2 + 9\gamma - 37 = 0. \quad (15)$$

The roots of (15) are

$$\gamma_1 = \frac{-3 + \sqrt{157}}{6} \approx 1.588 \quad \text{and} \quad \gamma_2 = \frac{-3 - \sqrt{157}}{6} \approx -2.588,$$

respectively. Hence, the values of $R$ and $\gamma$ violate the condition $R = (\gamma - 1)/\gamma$. This aspect has been discussed in [14]. Using Lemmas 1 and 2, we are now able to prove the following theorem.

**Theorem.** For the values of the constants $a$, $b$, $c$, and $d$ satisfying conditions (13), the general solution of the nonlinear differential equation (4) can be expressed in an exact closed parametric form.

**Proof.** For values of the numerical constants given by (14), (15), and by means of the substitution $\phi(\omega) = (B(\omega)/A(\omega)) Y(\omega)$, (10) is transformed into the following first-order differential equation:

$$\frac{d\phi}{d\omega} = \frac{A^2(\omega)}{B(\omega)} \left( \phi^3 + \phi^2 + k\phi \right) = \frac{b^2}{c} \frac{1 + \omega}{\omega(1 - \omega)(3\omega + 1)} \left( \phi^3 + \phi^2 + k\phi \right). \quad (16)$$

With $k = -(2c/b^2)$, (16) has the general solution

$$\omega(\phi) = \frac{-1 \pm \sqrt{4 - e^{2G(\phi)}}}{e^{G(\phi)} - 3}, \quad (17)$$

where we have denoted $G(\phi) = \int d\phi/(\phi^2 + \phi + k)$. The function $G(\phi)$ can be represented as

$$G(\phi) = \frac{1}{k} \ln \left| \phi \left( \frac{\phi}{\phi_1} \right)^{(1/2)(1/\sqrt{3}+1)} \left( \frac{\phi}{\phi_2} \right)^{(1/2)(1/\sqrt{3}-1)} \right|, \quad \text{if} \ k < \frac{1}{4}, \quad (18)$$

$$G(\phi) = \frac{1}{k} \left[ \ln \left| \frac{\phi}{\sqrt{\phi - \phi_1}(\phi - \phi_2)} \right| - \frac{1}{\sqrt{-\Delta}} \tan^{-1} \left( \frac{2^k + 1}{\sqrt{-\Delta}} \right) \right], \quad \text{if} \ k > \frac{1}{4}, \quad (19)$$

$$G(\phi) = \frac{1}{k} \left[ \ln \left| \frac{\phi}{\sqrt{\phi - \phi_1}(\phi - \phi_2)} \right| + \frac{1}{2\phi + 1} \right], \quad \text{if} \ k = \frac{1}{4}, \quad (20')$$

with $\Delta = 1 - 4k$ and $\phi_{1,2} = (-1 \pm \sqrt{\Delta})/2$.

From (8), we obtain

$$u - u_0 = \frac{1}{b} \int \frac{e^{kG(\phi)} - 3}{(\phi^2 + \phi + k) \left[ e^{kG(\phi)} - 4 \pm \sqrt{4 - e^{2kG(\phi)}} \right]} d\phi = \int D(\phi) d\phi \quad (21)$$

and

$$w(\phi) = \frac{\phi^{2/(1-R)}}{v_0} = \frac{5 - e^{kG(\phi)} \mp \sqrt{4 - e^{2kG(\phi)}}}{4 + e^{2kG(\phi)} - 5e^{kG(\phi)} \pm 2\sqrt{4 - e^{2kG(\phi)}}} e^{-2 \int D(\phi) d\phi} = E(\phi) e^{-2 \int D(\phi) d\phi}, \quad (22)$$

with $u_0$ a constant of integration and $e^{-2u_0}$ has been absorbed in $E(\phi)$.

Equations (21), (22) represent the exact parametric representation of the solution of (4).
4. CONCLUDING REMARKS

In the present paper, we have shown that the second-order nonlinear differential equation (4), describing the cosmological evolution of a conformally flat bulk viscous general relativistic universe, is associated with a first-order Abel type nonlinear equation of the second kind. The general solution of equation (4) has been obtained in an exact parametric representation. The existence of an exact solution critically depends on the numerical values of the coefficients of the Abel equation. Other mathematical properties of (4) will be the subject of future investigations.

REFERENCES