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A Markov chain on the symmetric group and Jack symmetric functions

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Abstract

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Diaconis and Shahshahani studied a Markov chain $W_f(1)$ whose states are the elements of the symmetric group S_f. In $W_f(1)$, you move from a permutation π to any permutation of the form $\pi(i, j)$ with equal probability. In this paper we study a deformation $W_f(\alpha)$ of this Markov chain which is obtained by applying the Metropolis algorithm to $W_f(1)$. The stable distribution of $W_f(\alpha)$ is $\alpha^{f-c(\pi)}$ where $c(\pi)$ denotes the number of cycles of π. Our main result is that the eigenvectors of the transition matrix of $W_f(\alpha)$ are the Jack symmetric functions. We use facts about the Jack symmetric functions due to Macdonald and Stanley to obtain precise estimates for the rate of convergence of $W_f(\alpha)$ to its stable distribution.

1. A Markov chain

A number of mathematical and statistical problems lead to the considerations of random walks where the set of states is a finite or continuous group (see Diaconis [1, Chapter 3], for a thorough and entertaining discussion with references). Usually, in these random walks on groups, the transitional probability $\tau(x, y)$ of going from y to x depends only on the group element xy^{-1} and in most cases only on the conjugacy class of xy^{-1}. Diaconis and Shahshahani [2] analyze a proposed card-shuffling procedure using a random walk on the symmetric group S_f where the transitional probability $t_1(\sigma, \pi)$ is $\binom{f}{2}^{-1}$ if $\sigma\pi^{-1}$ is a transposition and 0 otherwise. In their random walk, you move from a permutation π to any permutation of the form $\pi(i, j)$ with equal probability. We will denote this random walk by $W_f(1)$.

Let $c(\pi)$ denote the number of cycles in the disjoint cycle decomposition of π.

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It is easy to see that

$$c(\pi(i,j)) = c(\pi) \pm 1.$$

So at each step of the random walk $W_f(1)$ the value of the function c changes by 1.

In this paper we consider a variant $W_f(\alpha)$ of $W_f(1)$ where α is a real number greater than or equal to 1. $W_f(\alpha)$ is a Markov chain on S_f and the only allowable moves away from a permutation π are to permutations of the form $\pi(i,j)$ (it will be possible to stay at π). However in $W_f(\alpha)$ you do *not* move to each $\pi(i,j)$ with equal probability. Instead, the probability of moving from π to $\pi(i,j)$ depends only on whether $c(\pi(i,j))$ is $c(\pi)+1$ or $c(\pi)-1$. The exact rule is that you are α times as likely to move to $\pi(i,j)$ if $c(\pi(i,j)) = c(\pi) - 1$ than if $c(\pi(i,j)) = c(\pi)+1$.

For λ a partition, let λ' denote the conjugate partition and let $n(\lambda)$ be the function

$$n(\lambda) = \sum_i (i-1)\lambda_i = \sum_j \binom{\lambda_j'}{2}.$$

For σ a permutation let $n(\sigma)$ denote $n(\lambda_\sigma)$ where λ_σ is the partition whose *columns* are the cycle lengths of σ. So for example $n(\mathrm{id}) = 0$.

Definition 1.1. Let α be a real number with $\alpha \geqslant 1$. Define the random walk $W_f(\alpha)$ on S_f by saying that the probability $t_\alpha(\sigma,\pi)$ of moving from π to σ is

$$t_\alpha(\sigma,\pi) = \begin{cases} \dfrac{(\alpha - 1n)(\pi)}{\alpha\binom{f}{2}} & \text{if } \sigma = \pi, \\[2ex] \dfrac{1}{\binom{f}{2}} & \text{if } \sigma = \pi(i,j) \text{ and } c(\sigma) = c(\pi) - 1, \\[2ex] \dfrac{1}{\alpha\binom{f}{2}} & \text{if } \sigma = \pi(i,j) \text{ and } c(\sigma) = c(\pi) + 1, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

This matrix of transition probabilites for the cases $f = 3$ appears below.

$$\begin{array}{c} \\ \mathrm{id} \\ (1,2) \\ (1,3) \\ (2,3) \\ (1,2,3) \\ (1,3,2) \end{array} \begin{array}{c} \begin{array}{cccccc} \mathrm{id} & (1,2) & (1,3) & (2,3) & (1,2,3) & (1,3,2) \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 \\ \alpha & \alpha-1 & 0 & 0 & 1 & 1 \\ \alpha & 0 & \alpha-1 & 0 & 1 & 1 \\ \alpha & 0 & 0 & \alpha-1 & 1 & 1 \\ 0 & \alpha & \alpha & \alpha & 3\alpha-3 & 0 \\ 0 & \alpha & \alpha & \alpha & 0 & 3\alpha-3 \end{bmatrix} \end{array} \cdot (3\alpha)^{-1}$$

We denote the matrix of transition probabilities for this Markov chain on S_f by $T_f(\alpha)$. The reader is warned that our matrix of transition probabilities is the transpose of the one that usually used.

It is clear that $T_f(\alpha)$ is a nonnegative matrix and it is straightforward to check that the rows and columns of $T_f(\alpha)$ sum to 1. Note that the transition probability from σ to $\sigma(i, j)$ is a factor of α larger if $c(\sigma(i, j)) = c(\sigma) - 1$ than if $c(\sigma(i, j)) = c(\sigma) + 1$. Note also that $T_f(1)$ gives the transition probabilities for the random walk $W_f(1)$ considered by Diaconis and Shahshahani.

Definition 1.2. For $\sigma, \pi \in S_f$ and $n \in \mathbb{N}$ define $P_n^{(\alpha)}(\sigma, \pi)$ to be the probability that a random walk of length n in $W_f(\alpha)$ which begins at π ends at σ.

Suppose $\alpha > 1$. We will show that

$$P^{(\alpha)}(\sigma, \pi) = \lim_{n \to \infty} P_n^{(\alpha)}(\sigma, \pi)$$

exists for all σ and π. At any step in a random walk we are more likely to move to a permutation with fewer cycles than to a permutation with more cycles. So we might expect that

$$P^{(\alpha)}(\sigma, \pi_1) > P^{(\alpha)}(\sigma, \pi_2)$$

wherever $c(\pi_1) < c(\pi_2)$. We will show that

$$P^{(\alpha)}(\sigma, \pi) = \alpha^{-c(\pi)} \left\{ \prod_{i=0}^{f-1} \frac{\alpha}{1 + i\alpha} \right\}$$

and that the error term

$$E_n(\sigma, \pi) = |P_n^{(\alpha)}(\sigma, \pi) - P^{(\alpha)}(\sigma, \pi)|$$

is exponentially decreasing with n. We will find some lower bounds for the error term.

In the last section we will consider the special case where π is the identity in S_f. In this case we can get precise estimates for the asymptotic value of $E_n(\sigma, \pi)$. Quite surprisingly these estimates come from results proved recently by Macdonald [8] and Stanley [11] concerning the Jack symmetric functions.

2. A simple estimate for $P^{(\alpha)}(\sigma, \pi)$

Let $\{\ ,\ \}_\alpha$ be the form on $\mathbb{R}S_f$ given by

$$\{\sigma, \pi\}_\alpha = \begin{cases} \dfrac{\alpha^{c(\pi)}}{f!} & \text{if } \sigma = \pi, \\ 0 & \text{if } \sigma \neq \pi. \end{cases}$$

and let $Q_f(\alpha)$ be the matrix of this form.

Lemma 2.1. *For all $u, v \in \mathbb{R}S_f$ we have*

$$\{T_f(\alpha)u, v\}_\alpha = \{u, T_f(\alpha)v\}_\alpha.$$

Proof. We may assume $u, v \in S_f$. If $u = v$ then

$$\{T_f(\alpha)u, u\}_\alpha = \frac{(\alpha-1)n(u)}{\alpha\binom{f}{2}}\alpha^{c(u)} = \{u, T_f(\alpha)u\}_\alpha.$$

If $u \neq v$ and u is not of the form $v(i, j)$ then

$$\{T_f(\alpha)u, v\}_\alpha = 0 = \{u, T_f(\alpha)v\}_\alpha.$$

Suppose $u = v(i, j)$ and $c(u) = c(v) - 1$. Then

$$\{T_f(\alpha)u, v\}_\alpha = \{v, v\}_\alpha = \alpha^{c(v)},$$

and

$$\{u, T_f(\alpha)v\}_\alpha = \{u, \alpha u\}_\alpha = \alpha^{c(u)+1} = \alpha^{c(v)}.$$

This completes the proof. □

We will be interested in the eigenvalues and eigenvectors of $T_f(\alpha)$. The previous lemma shows that

$$T_f(\alpha) = T_f(\alpha)^{\mathrm{t}}$$

where the transpose is taken with respect to the form $\{\ ,\ \}_\alpha$. In particular $T_f(\alpha)$ is diagonalizable and all its eigenvalues are real. Below we see the eigenvalues and corresponding eigenvectors for the matrix $T_3(\alpha)$.

Eigenvalue	Eigenvectors
1	$(1, \alpha, \alpha, \alpha, \alpha^2, \alpha^2)$
$1 - 1/\alpha$	$(0, 0, 0, 0, 1, -1)$
$\frac{1}{3} - \frac{1}{3\alpha}$	$(6, 2\alpha - 2, 2\alpha - 2, 2\alpha - 2, -3\alpha, -3\alpha)$
	$(0, 1, -1, 0, 0, 0)$
	$(0, 0, 1, -1, 0, 0)$
$-1/\alpha$	$(1, -1, -1, -1, 1, 1)$

Lemma 2.2. *The matrix $T_f(\alpha)$ has 1 as an eigenvalue of multiplicity 1. Moreover, if $\alpha > 1$ then all other eigenvalues of $T_f(\alpha)$ have absolute values less than 1.*

Proof. Define $\mathcal{J}_f(\alpha)$ be the vector with entries indexed by S_f whose σth entry is $\alpha^{f-c(\sigma)}$. It is straightforward to verify that

$$T_f(\alpha)\mathcal{J}_f(\alpha) = \mathcal{J}_f(\alpha)$$

(we leave this computation to the reader because we will prove something more general in the next section).

Now suppose $\alpha > 1$. In this case $T_f(\alpha)$ is a primitive nonnegative matrix (i.e., $T_f(\alpha)$ has nonnegative entries and some power of $T_f(\alpha)$ has all positive entries). This follows because every permutation can be written as a product of transpositions and because at least one diagonal entry of $T_f(\alpha)$ is positive. The last assertion in Lemma 2.2 is a consequence of the Perron–Frobenius theory (see [12]). □

It is well known that

$$\sum_{\sigma \in S_f} \alpha^{f-c(\sigma)} = \prod_{i=0}^{f-1} (1 - i\alpha). \tag{$*$}$$

We let K denote the inverse of the above quantity $(*)$. The next theorem follows immediately from Lemma 2.2 and the ergodic theorem for Markov chains.

Theorem 2.3. *Let α be a real number greater than* 1. *For any σ, $\pi \in S_f$ we have*

$$P_n^{(\alpha)}(\sigma, \pi) = K\alpha^{f-c(\sigma)}(1 + \mathrm{O}(\epsilon_f(\alpha)^n))$$

where $0 < \epsilon_f(a) < 1$.

Remark. Diaconis points out that the Markov chain $W_f(\alpha)$ is an example of a Metropolis chain for $\alpha > 1$. The Metropolis algorithm is an algorithm for creating a Markov chain on a finite set X whose stationary distribution agrees with a given probability distribution. The actual algorithm, which was first announced in [9], has the following description (see also [3, Chapter 9]). Let X be a finite set and let $f: X \to \mathbb{R}$ be any function. The problem solved by the Metropolis algorithm is to create a Markov chain $P(x, y)$ on X having stationary distribution $\Pi(x) = \mathrm{e}^{-\beta f(x)} K(\beta)$ where $K(\beta)$ is the normalizing constant

$$K(\beta) = \left\{ \sum_{x \in X} \mathrm{e}^{-\beta f(x)} \right\}^{-1}.$$

To run the Metropolis algorithm, one begins with any symmetric Markov chain $P^*(x, y)$. One defines the new Markov chain $p(x, y)$ by

$$p(x, y) = \begin{cases} P^*(x, y)\left(\dfrac{\pi(y)}{\pi(x)}\right) & \text{if } \pi(y) < \pi(x), \\ P^*(x, y) & \text{if } y \neq x \text{ and } \pi(y) \geqslant \pi(x), \\ P^*(x, x) + \sum_y P^*(x, y)\left(1 - \dfrac{\pi(x)}{\pi(x)}\right) & \text{if } x = y, \end{cases}$$

where the last sum is over y with $\pi(y) < \pi(x)$.

To see that our Markov chain $W_f(\alpha)$ arises according to this algorithm let

$X = S_f$, let $f(\pi)$ be the number of cycles in π and let P^* be

$$P^*(\sigma, \tau) = \begin{cases} \binom{f}{2}^{-1} & \text{if } \sigma\tau^{-1} \text{ is a transposition,} \\ 0 & \text{otherwise.} \end{cases}$$

Also let $\beta = \log \alpha$. Then

$$\pi(\sigma) = \alpha^{-c(\sigma)} K(\beta)$$

where

$$K(\beta) = \sum_{\tau \in S_f} \alpha^{-c(\tau)} = K\alpha^{-f}.$$

So

$$\pi(\sigma) = \alpha^{f-c(\sigma)} K$$

which we know to be the stationary distribution of $W_f(\alpha)$.

For $\sigma \neq \tau$ we have that $\Pi(\tau) < \Pi(\sigma)$ iff $c(\sigma) < c(\tau)$. So for $\sigma \neq \tau$,

$$P(\sigma, \tau) = \begin{cases} \dfrac{K\alpha^{f-c(\sigma)-1}}{K\alpha^{f-c(\sigma)}} \binom{f}{2}^{-1} & \text{if } \tau = \sigma(i, j) \text{ and } c(\tau) = c(\sigma) + 1, \\ \binom{f}{2}^{-1} & \text{if } \tau = \sigma(i, j) \text{ and } c(\tau) = c(\sigma) - 1, \\ 0 & \text{otherwise.} \end{cases}$$

So $P(\sigma, \tau) = t_\alpha(\tau, \sigma)$ hence the Metropolis chain agrees with our Markov chain $W_f(\alpha)$.

Let $1 = \Lambda_1(\alpha), \Lambda_2(\alpha), \ldots, \Lambda_{f!}(\alpha)$ be the eigenvalues of $T_f(\alpha)$ ordered by absolute value. So

$$1 = \Lambda_1(\alpha) > |\Lambda_2(\alpha)| > \cdots > |\Lambda_{f!}(\alpha)| \geqslant 0.$$

At this point it is natural to ask about $|\Lambda_2(\alpha)|$ since we can take $\epsilon_f(\alpha) = |\Lambda_2(\alpha)|$ in Theorem 2.3. In the next section we will show that

$$(n(\lambda')\alpha - n(\lambda))/\alpha\binom{f}{2}$$

is an eigenvalue of $T_f(\alpha)$ for any partition λ of f. In particular this shows that

$$\left(\binom{f-1}{2}\alpha - 1\right)\Big/\alpha\binom{f}{2} \quad \text{and} \quad -\binom{f}{2}\Big/\alpha\binom{f}{2}.$$

are eigenvalues. We end this section by finding some other eigenvalues of $T_f(\alpha)$ which are not of the form $(\alpha n(\lambda') - n(\lambda))/\alpha(\frac{f}{2})$.

Definition 2.7. Let D_f denote the set of partitions of f into distinct odd parts. For $\lambda \in D_f$ let $v_\lambda = v_\lambda(\alpha)$ be the vector in $\mathbb{R}S_f$ given by

$$v_\lambda = \sum_{\tau \in S_f} \operatorname{sgn}(\tau)\sigma^\tau$$

where σ is an arbitrarily chosen permutation with cycle type λ (v_λ depends up to sign on the choice of σ).

Since λ has distinct odd parts, the centralizer of any σ with cycle type λ lies in the alternating group A_f. So v_λ is nonzero. In fact the v_λ, $\lambda \in D_f$, are a basis for the sgn-isotopic component of the conjugation action of S_f on itself (see Kostant [7]).

Theorem 2.8. *Let $\lambda \in D_f$. Then v_λ is an eigenvector of $T_f(\alpha)$ with eigenvalue*

$$(\alpha - 1)n(\lambda')/\alpha\binom{f}{2}$$

Proof. Write $T_f(\alpha) = D + E$ where D is a diagonal matrix and E has diagonal entries 0. It is obvious that

$$Dv_\lambda = \left((\alpha - 1)n(\lambda')/\alpha\binom{f}{2}\right)v_\lambda$$

(since v_λ is supported on permutations of cycle type λ). So it is enough to show that $Ev_\lambda = 0$.

It is straightforward to check that if σ has cycle type λ(for $\lambda \in D_f$) then $\sigma(i, j)$ has exactly one length for all (i, j). So

$$Ev_\lambda = \sum_\tau b_\tau \tau$$

where the sum is over permutations with exactly one cycle of even length.

Let τ_0 be a permutation with exactly one cycle C_0 of even length. We will show that $b_{\tau_0} = 0$. In the next section we will show that $T_f(\alpha)$ commutes with the conjugation action of $\mathbb{R}S_f$ on itself. So E also commutes with this conjugation action. Hence

$$\begin{aligned} b_{\tau_0}\tau_0 + \sum_{\tau \neq \tau_0} b_\tau \tau^{C_0} &= \Big(\sum_\tau b_\tau \tau\Big)^{C_0} = (Ev_\lambda)^{C_0} = Ev_\lambda^{C_0} \\ &= E(-v_\lambda) = -b_{\tau_0}\tau_0 + \sum_{\tau \neq \tau_0} b_\tau \tau. \end{aligned}$$

Since $\tau^{C_0} \neq \tau_0$ for $\tau \neq \tau_0$, we have $b_{\tau_0} = -b_{\tau_0}$ which completes the proof. □

Corollary 2.9. *If f is odd then $(\alpha - 1)/\alpha$ is an eigenvalue of $T_f(\alpha)$. If f is even then $(\alpha - 1)\binom{f-1}{2}/\alpha\binom{f}{2}$ is an eigenvalue of $T_f(\alpha)$.*

We now know three eigenvalues of $T_f(\alpha)$, namely

$$\begin{cases} 1 - 1/\alpha,\ \binom{f-2}{f} - 1/\alpha\binom{f}{2},\ -1/\alpha & f \text{ odd}, \\ \binom{f-2}{f}(1 - 1/\alpha),\ \binom{f-2}{f} - 1/\alpha\binom{f}{2},\ -1/\alpha & f \text{ even}. \end{cases} \tag{2.10}$$

Their relative absolute values depend on the parameter α. For fixed α, the largest of the three gives a lower bound for $\Lambda_2(\alpha)$. In general this is not a good lower bound as can be seen by considering the case $f=4$. The 24 eigenvalues of $T_4(\alpha)$ are given with multiplicities in the chart

Eigenvalue	Multiplicity
1	1
$\frac{1}{2}-1/6\alpha$	4
$-1/\alpha$	1
$\frac{1}{2}-1/2\alpha$	1
$\frac{1}{6}-1/2\alpha$	3
$\dfrac{9(\alpha-1)+\sqrt{9\alpha^2-2\alpha+9}}{12\alpha}$	3
$\dfrac{9(\alpha-1)-\sqrt{9\alpha^2-2\alpha+9}}{12\alpha}$	3
r_1, r_2, r_3	2 each

where r_1, r_2, r_3 are the three roots of the equation

$$\lambda^3-(9\alpha-9)\lambda^2+(20\alpha^2-44\alpha+20)\lambda-(12\alpha^3-50\alpha^2+50\alpha-12)=0.$$

The lower bound for $\Lambda_2(\alpha)$ given by (2.10) is less than $\frac{1}{2}$. However for large values of α the eigenvalue

$$\frac{9(\alpha-1)+\sqrt{9\alpha^2-2\alpha+9}}{12\alpha}$$

is arbitrarily close to 1. It would be interesting to have more information about the absolute value of $\Lambda_2(\alpha)$.

3. Random walks from the identity

The goal of this section is to get precise estimates for $P_n^{(\alpha)}(\pi, e)$ where e denotes the identity element of S_f. More generally we will obtain estimates for the average probability of a random walk of length n going from a permutation of cycle type λ to a permutation of cycle type μ.

Definition 3.1. Let λ and μ be partitions of f and let $\mathscr{C}_\lambda$ and $\mathscr{C}_\mu$ denote the conjugacy classes of permutations having cycle type λ and μ respectively. Define $P_n^{(\alpha)}(\mu, \lambda)$ to be

$$P_n^{(\alpha)}(\mu,\lambda)=(|\mathscr{C}_\mu|\,|\mathscr{C}_\lambda|)^{-1}\sum_{\sigma\in\mathscr{C}_\mu,\pi\in\mathscr{C}_\lambda}P_n^{(\alpha)}(\sigma,\pi)$$

Lemma 3.2. *The matrix $T_f(\alpha)$ commutes with the conjugation action of S_f on itself.*

Proof. This follows from three observations which hold for all $\sigma, \pi, \tau \in S_f$ and all $1 \leq i < j \leq f$.

(1) $\sigma = \pi(i, j)$ iff $\sigma^\tau = \pi^\tau(\tau i, \tau j)$

(2) $c(\sigma) = c(\sigma^\tau)$ and $c(\pi) = c(\pi^\tau)$.

(3) $n(\sigma^\tau) = n(\sigma)$. □

Lemma 3.2 has the following important corollary.

Corollary 3.3. *Let λ and μ be partitions of f and let σ be any permutation of cycle type μ. Then*

$$P_n^{(\alpha)}(\mu, \lambda) = |\mathscr{C}_\lambda|^{-1} \sum_{\pi \in \mathscr{C}_\lambda} P_n^{(\alpha)}(\sigma, \pi).$$

In particular

$$P_n^{(\alpha)}(\sigma, e) = P_n^{(\alpha)}(\mu, 1^f).$$

Proof. Let σ_1 and σ_2 be in $\mathscr{C}_\mu$ with $\sigma_2 = \sigma_1^\tau$. Then

$$P_n(\sigma_2, \pi) = (T_f(\alpha)^n)_{\sigma_2, \pi} = (T_f(\alpha)^n)_{\sigma_2^\tau, \pi^\tau} = P_n(\sigma_1, \pi^\tau).$$

So

$$|\mathscr{C}_\lambda|^{-1} \sum_{\pi \in \mathscr{C}_\lambda} P_n(\sigma_1, \pi) = |\mathscr{C}_\lambda|^{-1} \sum_{\pi \in \mathscr{C}_\lambda} P_n(\sigma_2, \pi)$$

and the first assertion follows. The second assertion is an immediate consequence of the first. □

In what follows we will obtain precise estimates for the $P_n(\mu, \lambda)$ and so in particular for the $P_n(\mu, 1^f)$. Thus, using Corollary 3.3, we will obtain precise estimates for $P_n(\sigma, e)$. To estimate the $P_n(\mu, \lambda)$ we must examine the restriction of $T_f(\alpha)$ to the center of $\mathbb{R}S_f$. Note that $T_f(\alpha)$ acts on the center because the center is an isotypic component for the conjugation action.

For each partition λ of f let $\mathscr{P}_\lambda$ denote the element of the center of $\mathbb{R}S_f$ given by

$$\mathscr{P}_\lambda = \frac{1}{|\mathscr{C}_\lambda|} \sum_{\sigma \in \mathscr{C}_\lambda} \sigma \tag{3.4}$$

It is well known that the $\mathscr{P}_\lambda$ are a basis for the center of $\mathbb{R}S_f$. We order the basis in the reverse lexicographic order of λ. In particular the first basis element is

$$\mathscr{P}_{1^f} = e.$$

By Lemma 3.2 there exists a matrix $L_f(\alpha) = (l_{\mu\lambda}(\alpha))$ such that

$$T_f(\alpha)\mathscr{P}_\lambda = \sum_\mu l_{\mu\lambda}(\alpha)\mathscr{P}_\mu.$$

Quite remarkably, this matrix $L_f(\alpha)$ has appeared in an entirely different context as the next theorem relates. Before stating this result we recall some notation from Macdonald [8].

Fix an integer $m \geqslant f$ and let Λ_m^f denote the vector space of homogeneous polynomials of degree f which are symmetric in variables $x_1, \ldots, x_m$. For each partition λ of f let $p_\lambda(x) = p_\lambda(x_1, \ldots, x_m)$ be the power sum symmetric function indexed by λ.

Theorem 3.5. *Let $D(\alpha)$ be the endomorphism of Λ_m^f given by*

$$D(\alpha) = \frac{\alpha}{2}\sum_{i=1} x_i^2 \frac{\partial}{\partial x_i^2} + \sum_{i \neq j} \frac{x_i^2}{x_i - x_j}\frac{\partial}{\partial x_i}.$$

Then

$$D(\alpha)p_\lambda(x) = (m-1)fp_\lambda(x) + \alpha\binom{f}{2}\sum_{\mu} l_{\mu\lambda}(\alpha)p_\mu(x).$$

Proof. Our proof will rely on the following observation which we state as a lemma. □

Lemma. *Let $M = (m_{\sigma\pi})$ be a linear transformation from $\mathbb{R}S_f$ to $\mathbb{R}S_f$ which commutes with the conjugation action. Define $M^\circ = (m^\circ_{\mu\lambda})$ by*

$$M\mathcal{P}_\lambda = \sum_{\mu} m^\circ_{\mu\lambda}\mathcal{P}_\mu.$$

Then we can compute the entries $m^\circ_{\mu\lambda}$ by the following method. Choose any permutation π of cycle type λ. Then

$$m^\circ_{\mu\lambda} = \sum_{\sigma \in \mathscr{C}_\mu} m_{\sigma\pi}.$$

Proof. The key observation is that for any λ and any $\pi \in \mathscr{C}_\lambda$ we have

$$\mathcal{P}_\lambda = \frac{1}{f!}\sum_{\tau \in S_f} \pi^\tau.$$

Now fix $\pi \in \mathscr{C}_\lambda$. Then

$$M\mathcal{P}_\lambda = \frac{1}{f!}M\Big(\sum_{\tau \in S_f} \pi^\tau\Big) = \frac{1}{f!}\sum_{\tau \in S_f}(M\pi)^\tau$$
$$= \frac{1}{f!}\sum_{\tau \in S_f}\sum_{\sigma \in S_f} m_{\sigma\pi}\sigma^\tau = \sum_{\mu}\mathcal{P}_\mu\Big\{\sum_{\sigma \in \mathscr{C}_\mu} m_{\sigma\pi}\Big\}$$

which proves the lemma. □

The content of this lemma is that we can compute the entry $m^\circ_{\mu\lambda}$ by considering the effect of M on just one permutation π in $\mathscr{C}_\lambda$. We will apply this lemma to compute the entries $L_{\mu\lambda}(\alpha)$. Fix a partition λ and a permutation π of cycle type λ. The above lemma gives immediately that

$$L_{\lambda\lambda}(\alpha) = (\alpha - 1)n(\lambda')/\alpha\binom{f}{2}.$$

It remains to compute the off-diagonal entries.

Let $\Gamma = (\gamma_1, \ldots, \gamma_r)$ and $\Delta = (\delta_1, \ldots, \delta_s)$ be an r-cycle and an s-cycle of π and consider the σ, π entry of $T_f(\alpha)$ where

$$\sigma = \pi(\gamma_i, \delta_j).$$

The permutation σ has exactly the same cycles as π except that the two cycles Γ and Δ in π are replaced by a single cycle

$$(\gamma_1, \gamma_2, \ldots, \gamma_i, \delta_{j+1}, \ldots, \delta_s, \delta_1, \ldots, \delta_j, \gamma_{i+1}, \ldots, \gamma_r).$$

So the σ, π entry of $T_f(\alpha)$ is $\alpha/\alpha\binom{f}{2}$ and the cycle type of σ is $\mu = \lambda[r, s \leftarrow r + s]$ which means the partition obtained from λ by replacing the parts r and s by their sum $r + s$. This accounts for all entries $m_{\sigma\pi}$ where $\sigma = \pi(u, v)$ and u, v come from different cycles of π.

Next let $\Gamma = (\gamma_1, \ldots, \gamma_r)$ be an r-cycle of π and consider the σ, π entry of $T_f(\alpha)$ where

$$\sigma = \pi(\gamma_i, \gamma_j) \quad 1 \leq i < j \leq r.$$

The permutation σ has exactly the same cycles as π except that the cycle Γ in π is replaced by two cycles

$$(\gamma_1, \ldots, \gamma_i, \gamma_{j+1}, \ldots, \gamma_r) \quad \text{and} \quad (\gamma_{i+1}, \ldots, \gamma_j).$$

So the σ, π entry of $T_f(\alpha)$ is $1/\alpha\binom{f}{2}$ and the cycle type of σ is $\mu = \lambda[r \leftarrow (j - i), r + i - j]$ which as above means the partition obtained from λ by replacing r by $j - i$ and $r + i - j$. This accounts for all entries $m_{\sigma\pi}$ where $\sigma = \pi(u, v)$ and u, v come from the same cycle of π. So

$$\alpha\binom{f}{2}T_f(\alpha)\mathcal{P}_\lambda = (\alpha - 1)n(\lambda')\mathcal{P}_\lambda + \alpha \sum_{u<v} \lambda_u\lambda_v \mathcal{P}_{\lambda[\lambda_u, \lambda_v \leftarrow \lambda_u + \lambda_v]} + \frac{1}{2}\sum_k \lambda_k \sum_{j=1}^{\lambda_k - 1} \mathcal{P}_{\lambda[\lambda_k \leftarrow j, \lambda_k - j]}. \tag{3.6}$$

In the formula (3.6) the second sum accounts for σ of the form $\pi(i, j)$ where i and j come from a λ_u-cycle Γ and a λ_v-cycle Δ. The factor $\lambda_u\lambda_v$ accounts for the fact that there are $\lambda_u\lambda_v$ many choices for i and j from Γ and Δ. The third sum accounts for those σ of the form $\pi(r, s)$ where r and s come from the same λ_k-cycle Γ. These pairs are chosen by first picking r (this can be done in λ_k-ways) and then choosing $s = r + j \pmod{\lambda_k}$.

Note that

$$(\alpha-1)n(\lambda')=(\alpha-1)\sum_k\binom{\lambda_k}{2}$$
$$=\frac{1}{2}\left\{\alpha\sum_k\lambda_k(\lambda_k-1)+\sum_k\lambda_k(2n-2-(\lambda_k-1)\right\}$$
$$-f(m-1). \tag{3.7}$$

Comparing (3.6) and (3.7) with the first formula in the proof of Theorem 3.1 of [11] gives the result.

The operator $D(\alpha)$ is the so-called Laplace–Beltrami operator from the theory of the Jack symmetric functions. There has been an abundance of work on the Jack symmetric functions in recent years (see [5, 6, 8, 11]). The Jack symmetric functions $J_\lambda(x;\alpha)$ are the eigenfunctions of the operator $D(\alpha)$ hence by the theorem above their expansions in terms of the power sum symmetric functions give us the entries in the eigenvectors of $L_f(\alpha)$. Our immediate goal is to read off information about the eigenvalues and eigenvectors of $L_f(\alpha)$ from information available about the $J_\lambda(x;\alpha)$.

Definition 3.8. Let $\langle\ ,\ \rangle_\alpha$ be the symmetric bilinear form on the space of symmetric polynomials in $x_1,\ldots,x_m$ defined by

$$\langle p_\lambda, p_\mu\rangle=\delta_{\lambda\mu}z_\lambda\alpha^{l(\lambda)}.$$

The next result is due to Macdonald (see [8]).

Theorem 3.9 (Macdonald). *Suppose $m\geqslant f$. Then there are unique symmetric polynomials $J_\lambda(x_1,\ldots,x_m;\alpha)$ where λ ranges over the partitions of f which satisfy the following three conditions*:

(1) $\langle J_\lambda, J_\mu\rangle_\alpha=0$ *if λ is different than μ.*

(2) *Write $J_\lambda=\sum_\mu v_{\lambda\mu}(\alpha)m_\mu$ where m_μ is the μth monomial symmetric function. Then $v_{\lambda\mu}(\alpha)=0$ unless λ (weakly) dominates μ.*

(3) *The coefficient $v_{\lambda,1^f}$ is $f!$*

Moreover, each $J_\lambda(x;\alpha)$ is an eigenfunction of $D(\alpha)$ with eigenvalue $e_\lambda(m;\alpha)$ given by

$$e_\lambda(m;\alpha)=\alpha n(\lambda')-n(\lambda)+f(m-1).$$

For each $\lambda\vdash f$ define $E_\lambda(\alpha)$ to be

$$E_\lambda(\alpha)=(e_\lambda(m;\alpha)-f(m-1))/\alpha\binom{f}{2}.$$

Note that $E_\lambda(\alpha)$ does not depend on m.

Definition 3.10. Let λ be a partition of f. For each square $s = (i, j)$ in the Ferrer's diagram of λ define $a_\lambda(s)$ and $l_\lambda(s)$ by

$$a_\lambda(s) = \lambda_i - j \quad \text{and} \quad l_\lambda(s) = \lambda_j' - i.$$

Lemma 3.11 (Stanley [9, p.36]). *For each partition λ let $j_\lambda(\alpha)$ denote $\langle J_\lambda(x;\alpha), J_\lambda(x,\alpha)\rangle_\alpha$. Then*

$$j_\lambda(\alpha) = \prod_{s\in\lambda} (l_\lambda(s) + \alpha(a_\lambda(s)+1))((l_\lambda(s)+1) + \alpha a_\lambda(s)).$$

In this paper we will need only the following three values of j_λ, each of which is easily computed using Lemma 3.11 above.

$$j_f(\alpha) = \alpha^f f! \prod_{i=0}^{f-1} (1 + i\alpha) \tag{3.12a}$$

$$j_{1^f} = f! \prod_{i=0}^{f-1} (i + \alpha) \tag{3.12b}$$

$$j_{f-1,1}(\alpha) = (1 + \alpha(f-1))(2 + \alpha(f-2))\alpha^{f-1}(f-2)! \prod_{i=0}^{f-3} (1 + i\alpha) \tag{3.12c}$$

Definition 3.12. For λ a partition of f define $\mathcal{J}_\lambda \in \mathbb{R}S_f$ by

$$\mathcal{J}_\lambda = (j_\lambda^{-\frac{1}{2}}) \sum_\mu c_{\lambda\mu}(\alpha)\mathcal{P}_\mu$$

where the constants $c_{\lambda,\mu}(\alpha)$ are defined by

$$J_\lambda(x;\alpha) = \sum_\mu c_{\lambda\mu}(\alpha)p_\mu(\alpha).$$

It is straightforward to check that

$$\{\mathcal{P}_\lambda, \mathcal{P}_\mu\} = \langle p_\lambda(x), p_\mu(x)\rangle_\alpha \quad \text{for all } \lambda, \mu.$$

Hence the set of $\mathcal{J}_\lambda$ is an orthonormal basis of eigenvectors for $L_f(\alpha)$ (orthonormal with respect to $\{\ \}_\alpha$. Moreover, the eigenvalue associated with $\mathcal{J}_\lambda$ is $E_\lambda(\alpha)$.

Before stating the main result we need to know certain coefficients $c_{\lambda\mu}(\alpha)$. The formulas below are due to either Macdonald [8] or Stanley [11].

Lemma 3.13. *For $\lambda = f$, 1^f and $f-1, 1$ the coefficients $c_{\lambda\mu}(\alpha)$ have the following values:*

$$c_{f,\mu} = \alpha^{f-l(\mu)}(f!/z_\mu) \tag{3.13a}$$

$$c_{1^f,\mu} = \operatorname{sgn}(\mu)(f!/z_\mu) = \Big(\prod_i (-1)^{\mu_i - 1}\Big)(f!/z_\mu) \tag{3.13b}$$

$$c_{(f-1,1),\mu} = (f!/z_\mu)\alpha^{f-l(\mu)}(-f + (1-(f-1)\alpha)m_1(\mu))/(f-1) \tag{3.13c}$$

where $m_1(\mu)$ is the number of parts of μ equal to 1.

We can now state the main result of this section.

Theorem 3.14. *Let λ and μ be partitions of f. Define constants $A(\mu, \lambda)$ and $B(\mu, \lambda)$ by*

$$A(\mu, \lambda) = \frac{\operatorname{sgn}(\mu)\operatorname{sgn}(\lambda)\alpha^{l(\lambda)}}{\prod_{i=0}^{f-1}(i+\alpha)}$$

$$B(\mu, \lambda) = \left(\frac{f\alpha^{f+1-l(\mu)}}{(f-1)\prod_{i=0}^{f-3}(1+i\alpha)}\right) \times \left(\frac{(f+((f-1)\alpha-1)m_1(\mu))(f+((f-1)\alpha-1)m_1(\lambda))}{(1+\alpha(f-1))(2+\alpha(f-2))}\right)$$

Then we have the following asymptotic expansions for $P_n^{(\alpha)}(\mu, \lambda)$ which depend on the size of α relative to f:

(1) *If $1 < \alpha < (f^2-f+2)/(f^2-3f+2)$ then*

$$P_n^{(\alpha)}(\mu, \lambda) = K\alpha^{f-l(\mu)} + A(\mu, \lambda)\alpha^{-n} + \mathrm{O}\left(\left(\left(\frac{f-2}{f}\right)\frac{2\alpha^{-1}}{f(f-1)}\right)^n\right).$$

(2) *If $\alpha > (f^2-f+2)/(f^2-3f+2)$ then*

$$P_n^{(\alpha)}(\mu, \lambda) = K\alpha^{f-l(\mu)} + B(\mu, \lambda)\left(\left(\left(\frac{f-2}{f}\right)+\frac{2\alpha^{-1}}{f(f-1)}\right)^n\right) + \mathrm{O}(\alpha^{-n}).$$

(3) *If $\alpha = (f^2-f+2)/(f^2-3f+2)$ then*

$$P_n^{(\alpha)}(\mu, \lambda) = K\alpha^{f-l(\mu)} + (A(\mu, \lambda)+B(\mu, \lambda))\alpha^{-n} + \mathrm{O}\left(\left(\frac{f-2}{f\alpha}\right)^n\right).$$

Proof. Fix λ and μ partitions of f. We have

$$\begin{aligned} P_n^{(\alpha)}(\mu, \lambda) &= |\mathscr{C}_\mu|^{-1}|\mathscr{C}_\lambda|^{-1} \sum_{\sigma\in\mathscr{C}_\mu, \pi\in\mathscr{C}_\lambda} (T_f(\alpha)^n)_{\sigma,\pi} = \mathscr{P}_\mu T_f(\alpha)^n \mathscr{P}_\lambda \\ &= \mathscr{P}_\mu \cdot \left(\sum_\beta (L_f(\alpha))^n_{\beta\lambda}\mathscr{P}_\beta\right) \quad \text{here } \cdot \text{ is ordinary dot product of vectors} \\ &= \frac{1}{|\mathscr{C}_\mu|}(L_f(\alpha))^n_{\mu\lambda}. \end{aligned}$$

The last equality holds as $\mathscr{P}_\mu \cdot \mathscr{P}_\beta = |\mathscr{C}_\mu|\,\delta_{\mu\beta}$.

Let $C_f(\alpha)$ be the matrix whose λth column contains the coefficients in the expansion of $J_{\lambda'}(x;\alpha)/j_{\lambda'}^{\frac{1}{2}}$ in terms of power sums. To be precise, $C_f(\alpha)$ is the matrix whose β, λ entry is

$$C_f(\alpha) = c_{\beta\lambda'}(\alpha)/\lambda_{\lambda'}^{\frac{1}{2}}.$$

Since the $\mathcal{J}_\lambda$ are a set of orthonormal eigenvectors for $L_f(\alpha)$ we have

$$L_f(\alpha)C_f(\alpha) = C_f(\alpha)E_{\beta'}(\alpha) \tag{3.16a}$$

and

$$(C_f(\alpha)^{-1})_{\beta\gamma} = (C_f(\alpha))_{\gamma\beta}\alpha^{l(\gamma)}z_\gamma. \tag{3.16b}$$

By (3.16a) we have

$$L_f(\alpha)^n = C_f(\alpha)\mathrm{diag}(E_{\beta'}(\alpha)^n)C_f(\alpha)^{-1}$$

hence

$$\begin{aligned}\mathcal{P}_n^{(\alpha)}(\mu, \lambda) &= \frac{1}{|\mathscr{C}_\mu|}\sum_\beta (C_f(\alpha))_{\mu\beta}E_{\beta'}(\alpha)^n(C_f(\alpha))_{\lambda\beta}\alpha^{l(\lambda)}z_\lambda \\ &= \frac{z_\mu z_\lambda \alpha^{l(\lambda)}}{f!}\sum_\beta c_{\mu\beta'}(\alpha)E_{\beta'}(\alpha)^n c_{\lambda\beta'}(\alpha) \\ &= \frac{z_\mu z_\lambda \alpha^{l(\lambda)}}{f!}\sum_\beta \frac{c_{\mu\beta}(\alpha)c_{\lambda\beta}(\alpha)E_\beta(\alpha)^n}{j_\beta}.\end{aligned} \tag{3.17}$$

We get an asymptotic expansion for $\mathcal{P}_n^{(\alpha)}(\mu, \lambda)$ by taking those terms on the right hand side of (3.17) where $E_\beta(\alpha)$ is maximum in absolute value. The following chart gives the three largest values of $|E_\beta(\alpha)|$ together with the corresponding partitions β:

α	Largest	2nd Largest	3rd Largest
$1 < \alpha < \dfrac{f^2-f+2}{f^2-3f+2}$	1 $\beta = f$	α^{-1} $\beta = 1^f$	$\dfrac{f-2}{f} - \dfrac{2\alpha^{-1}}{f(f-1)}$ $\beta = f-1, 1$
$\alpha = \dfrac{f^2-f+2}{f^2-3f+2}$	1 $\beta = f$	α^{-1} $\beta = 1^f$ and $\beta = f-1, 1$	
$\alpha > \dfrac{f^2-f+2}{f^2-3f+2}$	1 $\beta = f$	$\dfrac{f-2}{f} - \dfrac{2\alpha^{-1}}{f(f-1)}$ $\beta = f-1, 1$	α^{-1} $\beta = 1^f$

We should point out that the α^{-1} which appears above is actually the absolute value of $E_{1^f}(\alpha) = -\alpha^{-1}$. From (3.17) and the chart above we have the following asymptotic expansions:

(1) If $1 < \alpha < (f^2 - f + 2)/(f^2 - 3f + 2)$ then

$$\begin{aligned}\mathcal{P}_n^{(\alpha)}(\mu, \lambda) = \frac{z_\mu z_\lambda \alpha^{l(\lambda)}}{f!}\bigg\{&\left(\frac{f!}{z_\mu}\alpha^{f-l(\mu)}\right)\left(\frac{f!}{z_\lambda}\alpha^{f-l(\mu)}\right)j_f^{-1} \\ &+ \left(\frac{f!}{z_\lambda}\mathrm{sgn}(\mu)\right)\left(\frac{f!}{z_\mu}\mathrm{sgn}(\lambda)\right)j_{1^f}^{-1}\alpha^{-n}\bigg\} \\ &+ \mathrm{O}\left(\left(\frac{f-2}{f} - \frac{2\alpha^{-1}}{f(f-1)}\right)^n\right)\end{aligned}$$

Substituting the values of j_f and j_{1^f} given in (3.12) we have

$$\mathcal{P}_n^{(\alpha)}(\mu,\lambda)=\left(\frac{\alpha^{f-l(\mu)}}{\prod_{i=0}^{f-1}(1+i\alpha)}\right)+\left(\frac{\mathrm{sgn}(\mu)\mathrm{sgn}(\lambda)\alpha^{l(\lambda)}}{\prod_{i=0}^{f-1}(i+\alpha)}\right)\alpha^{-n} + \mathrm{O}\left(\left(\frac{f-2}{f}-\frac{2\alpha^{-1}}{f(f-1)}\right)^n\right) \tag{3.18}$$

(2) If $\alpha>(f^2-f+2)/(f^2-3f+2)$ then

$$\begin{aligned}\mathcal{P}_n^{(\alpha)}(\mu,\lambda)&=\frac{z_\mu z_\lambda\alpha^{l(\lambda)}}{f!}\left\{\left(\frac{f!}{z_\mu}\alpha^{f-l(\mu)}\right)\left(\frac{f!}{z_\lambda}\alpha^{f-l(\lambda)}\right)j_f^{-1}\right.\\&\quad+\left(\frac{(f!)^2}{z_\mu z_\lambda}\right)\frac{\alpha^{2f-l(\lambda)-l(\mu)}}{(f-1)^2}(-f+(1-(f-1)\alpha)m_1(\mu))\\&\quad\left.\cdot(-f+(1-(f-1)\alpha)m_1(\lambda))\left(\left(\frac{f-2}{f}-\frac{2\alpha^{-1}}{f(f-1)}\right)^n\right)j_{f-1,1}^{-1}\right\}\\&\quad+\mathrm{O}(\alpha^{-n})\\&=\left(\frac{\alpha^{f-l(\mu)}}{\prod_{i=0}^{f-1}(1+i\alpha)}\right)\\&\quad+\left(\frac{f\alpha^{f+1-l(\mu)}(-f+(1-(f-1)\alpha)m_1(\mu))(-f+(1-(f-1)\alpha)m_1(\lambda))}{(f-1)\prod_{i=0}^{f-3}(1+i\alpha)(1+\alpha(f-1))(2+\alpha(f-2))}\right)\\&\quad\cdot\left(\left(\frac{f-2}{f}\right)-\frac{2\alpha^{-1}}{f(f-1)}\right)^n+\mathrm{O}(\alpha^{-n}).\end{aligned}$$

(3) If $\alpha=(f^2-f+2)/(f^2-3f+2)$ then the two eigenvalues α^{-1} and $(f-2)/f-2\alpha^{-1}/f(f-1)$ are equal. So one has

$$\begin{aligned}\mathcal{P}_n^{(\alpha)}(\mu,\lambda)&=\left(\frac{\alpha^{f-l(\mu)}}{\prod_{i=0}^{f-1}(1+i\alpha)}\right)+\left\{\left(\frac{\mathrm{sgn}(\mu)\mathrm{sgn}(\lambda)\alpha^{l(\lambda)}}{\prod_{i=0}^{f-1}(i+\alpha)}\right)\right.\\&\quad\left.+\left(\frac{f\alpha^{f+1-l(\mu)}(-f+(1-(f-1)\alpha)m_1(\mu))(-f+(1-(f-1)\alpha)m_1(\lambda))}{(f-1)\prod_{i=0}^{f-3}(1+i\alpha)(1+\alpha(f-1))(2+\alpha(f-2))}\right)\right\}\alpha^{-1}\\&\quad+\mathrm{O}(\epsilon^n)\end{aligned}$$

where ϵ is the next largest absolute value amongst the $E_\beta(\alpha)$ after 1 and α^{-1}. The actual size of ϵ depends on the relative size of α and f. It is easy to check that in all cases one has $\epsilon<((f-2)/f)\alpha^{-1}$. The theorem follows. □

It is interesting that the probabilities in the primary distribution $k\alpha^{f-l(\mu)}$ for $\mathcal{P}_n^{(\alpha)}(\mu,\lambda)$ do not depend on λ. Intuitively this says that if you walk long enough then the probability of ending at a conjugacy class μ does not depend on where you started. However Theroem 3.14 shows that the starting point λ comes into the secondary distributions. If we restrict attention to walks which begin at the identity then we derive the following corollary from Theorem 3.14 and Corollary 3.3.

Corollary 3.19. *Define $A(\mu, \lambda)$ and $B(\mu, \lambda)$ as in Theorem 3.14 and let σ be a permutation of cycle type μ.*

(1) *If $1 < \alpha < (f^2 - f + 2)/(f^2 - 3f + 2)$ then*

$$\mathcal{P}_n^{(\alpha)}(\sigma, e) = K\alpha^{f-l(\mu)} + A(\mu, f)\alpha^{-n} + \mathrm{O}\left(\left(\frac{f-2}{f} + \frac{2\alpha^{-1}}{f(f-1)}\right)^n\right).$$

(2) *If $\alpha > (f^2 - f + 2)/(f^2 - 3f + 2)$ then*

$$\mathcal{P}_n^{(\alpha)}(\sigma, e) = K\alpha^{f-l(\mu)} + B(\mu, f)\left(\left(\frac{f-2}{f} + \frac{2\alpha^{-1}}{f(f-1)}\right)^n\right) + \mathrm{O}(\alpha^{-n}).$$

(3) *If $\alpha = (f^2 - f + 2)/(f^2 - 3f + 2)$ then*

$$\mathcal{P}_n^{(\alpha)}(\sigma, e) = K\alpha^{f-l(\mu)} + (A(\mu, f)) + B(\mu, f))\alpha^{-n} + \mathrm{O}\left(\left(\frac{f-2}{f\alpha}\right)^n\right).$$

4. Other problems

There are several other problems suggested by this work. It would be interesting to find a more conceptual proof of the connection between Jack polynomials and the random walk $W_f(\alpha)$. Also, there is a generalization $L_\Phi(\alpha)$ of the Laplace–Beltrami operator for Jack polynomials to arbitrary roots systems Φ. The Jack case corresponds to the root systems of type A_f. This generalized Laplacian is due to Heckemann and has been studied at length by Heckemann and Opdam (see [5, 10]). Is there a random walk on the Weyl group W of Φ which is connected to $L_\Phi(\alpha)$ as happens in this paper for the root systems of type A_f?

There are numerous questions about the random walk $W_f(\alpha)$ which are still unanswered. For example, if we start at a permutation π, how long do we expect to walk in $W_f(\alpha)$ before we reach the identity for the first time? This question is answered in [2] for the random walk $W_f(1)$ but the representation-theoretic techniques used there do not apply in the case $\alpha > 1$.

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