W^{1,p} versus C^1 local minimizers and multiplicity results for quasilinear elliptic equations

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Abstract
In this paper we show that the local minimizers of a class of functionals in the C^1-topology are still their local minimizers in W^{1,p}_0(Ω). Using this fact, we study the multiplicity of solutions for a class of quasilinear elliptic equations via critical point theory.

Keywords: Quasilinear eigenvalue problems; Multiple solutions

1. Introduction
Let Ω ⊂ R^N (N ≥ 3) be a bounded smooth domain and p > 1. Consider the functional

\[ J(u) = \frac{1}{p} \int_\Omega |Du|^p \, dx - \int_\Omega F(u) \, dx, \quad u \in W^{1,p}_0(\Omega), \quad (1.1) \]

where \( F(u) = \int_0^u f(s) \, ds \) and \( f \in C^1(-\infty, \infty) \) satisfies

\[ |f(s)| \leq \alpha_2 + \alpha_1 |s|^\beta, \quad |f'(s)| \leq \alpha_4 + \alpha_3 |s|^\beta - 1, \quad \text{where } \alpha_1, \alpha_3 \in [0, 1], \alpha_2, \alpha_4 \in (0, +\infty), p - 1 < \beta < Np/(N - p) - 1 \text{ for } 1 < p < N, \text{ and } |f(s)| \leq \alpha_6 + \alpha_5 |s|^{\beta_1}, \]

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In many cases we need to know the answer of the following question:

(Q) If $u_0 \in W^{1,p}_0(\Omega)$ is a local minimizer of $J$ in the $C^1$-topology, is it still a local minimizer of $J$ in $W^{1,p}_0(\Omega)$?

For $p = 2$, Brezis and Nirenberg [6] gave a positive answer. For $p > 1$ and $p \neq 2$, their method does not apply since we have a nonlinear operator. In this paper we shall first give a positive answer to the above question for $p > 2$. Then using this positive answer we study the structure of solutions of the quasilinear elliptic problems

$$-\Delta_p u = f_\lambda(u) \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega,$$

(1.2)

where $p > 2$; $f_\lambda : \mathbb{R} \to \mathbb{R}$ and $\lambda > 0$ is a real parameter.

The background of (1.2) can be found in [23]. The existence and uniqueness of, possibly multiple, solutions of (1.2) have been studied by the first author in previous papers (see [10, 16–24, 38]). Such problems have also been treated by many other authors, see, for example, [1–3, 7, 9, 13, 14, 25, 26, 28–34, 39–41] and references therein.

When $f_\lambda(u) = \lambda g(s)$, it was shown in [16] that there exist at least two positive solutions of (1.2) when $g$ is strictly increasing on $\mathbb{R}^+$, $g(0) = 0$, $\lim_{s \to 0^+} (g(s)/s^{p-1}) = 0$ and $g(s) \leq \alpha_1 + \alpha_2 s^\theta$, $0 < \theta < p - 1$. On the other hand, the structure of positive solutions of (1.2) with $f_\lambda(s) = \lambda g(s)$ and $g$ changing sign has also been studied in [19–21, 23].

A solution of (1.2) is a pair of $(\lambda, u) \in \mathbb{R}^+ \times (W^{1,p}_0(\Omega) \cap C^1_0(\Omega))$ which satisfies (1.2) in the weak sense.

When $f_\lambda(u) = \lambda u^q$, $0 < q < p - 1$, the sub- and supersolution argument as in [22] easily provides the existence of a unique positive solution of (1.2) for all $\lambda > 0$.

As an application of the answer of our question (Q), we shall study problem (1.2) when $f_\lambda$ is, roughly, the sum of two terms: one has the growth less than $p - 1$, another one has the growth larger than or same as $p - 1$. We consider two types of conditions on $f_\lambda$:

(i) $f_\lambda(s) = \lambda s^q + s^\omega$ for $s > 0$; here $0 < q < p - 1 < \omega$.

(ii) $f_\lambda(s) = \lambda |s|^{q-1} |s + g(s)|$ for $s \in (-\infty, +\infty)$; $q \in (0, p - 1)$; $g \in C^1(-\infty, +\infty)$ satisfies

$$(H_1) \quad g'(s) \geq 0 \quad \text{for} \quad s \in (-\infty, +\infty), \quad g(s)s \geq 0 \quad \text{for any} \quad s \in (-\infty, +\infty) \quad \text{and} \quad \lim_{|s| \to 0} g(s)/|s|^{p-1} = 0.$$

In Theorem 3.9 below we show that there exists a positive constant $\Lambda > 0$ such that if $\lambda \in (0, \Lambda)$, there exist at least two solutions of the problem

$$-\Delta_p u = \lambda u^q + u^\omega, \quad u > 0 \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega.$$

(1.3)

Such kind of problems has been studied in [35] by variational method and genus. We shall obtain Theorem 3.9 by different ideas and provide more information on the solutions. Using the assumption that $\omega$ is an arbitrary positive number, we can only get one positive
solution of (1.3). To get the second positive solution, we need a subcritical growth assumption on $\omega$. When $p = 2$, this problem has been treated in [5], but their methods cannot be easily used to deal with (1.2) here, since the linearization of the operator in (1.2) is difficult to handle. To overcome the difficulty arising from our operator, we use a scale argument instead.

In Theorem 4.1 below, we study structure of solutions of the problem

$$-\Delta_p u = \lambda |u|^{q-1} u + g(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \quad (1.4)$$

Problem (1.4) with $1 < p < N$ was discussed in [2] for $g$ with critical exponent.

After we submitted the manuscript, we found out that Theorems 2.1 and 3.9 of Sections 2 and 3 had been obtained in [4] with different proofs. In [4], the authors considered the problem

$$-\Delta_p u = f_\lambda(u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega,$$

where $f_\lambda(u) = |u|^{r-2}u + \lambda |u|^{q-2}u$ with $1 < q < p < r < p^*$, $p^* = Np/(N - p)$ ($p < N$), $p^* = \infty$ ($p \geq N$).

The main result of [4] is Theorem 1.1 (for $p > 1$): If $u_0 \in W^{1,p}_0(\Omega)$ is a local minimizer of $J$ in $C^1(\Omega)$, then $u_0$ is a local minimizer in $W^{1,p}_0(\Omega)$. Then they use Theorem 1.1 and the mountain pass theorem to prove the above equation has at least two positive solutions for all $\lambda \in (0, \Lambda)$; there is no positive solution for $\lambda > \Lambda$; there exist at least one positive solution for $\lambda = \Lambda$ (Theorem 1.3). From the proof of Theorem 1.1 of [4], we know that the main work is to prove the uniform $C^{1,\alpha}$ estimate of $v_\epsilon$: $\|v_\epsilon\|_{C^{1,\alpha}} \leq C$ (Theorem 1.2, $v_\epsilon$ is the same as that in the proof of Theorem 2.1 here).

We should point out that our proof for the principal result (Theorem 2.1) is much simpler than that of Theorem 1.1 in [4], although we only prove for $p > 2$ and there are some hypotheses on $f'$ (see hypothesis (F)). Theorem 3.9 here is similar to Theorem 1.3 of [4]. Theorem 3.1 ($\omega$ can be arbitrary for the existence of one solution) and the results of Section 4 are new.

2. $W^{1,p}_0(\Omega)$ versus $C^1_0(\Omega)$ local minimizers

In this section we shall give a positive answer to our question (Q) for $p > 2$. The main result in this section is

**Theorem 2.1.** Assume that $p > 2$ and (F) holds. Assume that $u_0 \in W^{1,p}_0(\Omega) \cap C^1_0(\Omega)$ is a local minimizer of $J$ in the $C^1$-topology; this means that there is some $r > 0$ such that

$$J(u_0) \leq J(u_0 + v) \quad \forall v \in C^1_0(\Omega) \quad \text{with } \|v\|_{C^1_0(\Omega)} \leq r. \quad (2.1)$$

Then $u_0$ is local minimizer of $J$ in $W^{1,p}_0(\Omega)$, i.e., there exists $\kappa > 0$ such that

$$J(u_0) \leq J(u_0 + v) \quad \forall v \in W^{1,p}_0(\Omega) \quad \text{with } \|v\|_{W^{1,p}_0(\Omega)} \leq \kappa. \quad (2.2)$$
Proof. We first consider the case of \(2 < p < N\). Recall that \(u_0\) satisfies in the weak sense the problem
\[
-\Delta_p u_0 = f(u_0) \quad \text{in } \Omega, \quad u_0 = 0 \quad \text{on } \partial \Omega. \tag{2.3}
\]

Then it follows from the condition \((F)\) and the regularity results in [28] and [36,37] (see the proof of Proposition 2.2 of [16]) that \(u_0 \in C^{1, \sigma}(\overline{\Omega})\) (\(0 < \sigma < 1\)). Here we use the growth conditions on \(f\).

Suppose the conclusion does not hold. Then
\[
\forall \epsilon > 0, \exists v_\epsilon \in B_\epsilon \text{ such that } J(u_0 + v_\epsilon) < J(u_0), \tag{2.4}
\]
where \(B_\epsilon = \{ v \in W^{1,p}_0(\Omega): \| v \|_{W^{1,p}_0(\Omega)} \leq \epsilon \} \). It is easily known that \(J\) is lower semicontinuous on the convex set \(B_\epsilon\).

Notice that \(B_\epsilon\) is weak sequence compact and weakly closed in \(W^{1,p}_0(\Omega)\). By the embedding of \(W^{1,p}_0(\Omega)\) to \(L^\gamma(\Omega)\) with \(p-1 < \gamma \leq Np/(N-p)-1\) and a standard lower semicontinuity argument, we know that \(J\) is bounded from below on \(B_\epsilon\) and \(\exists v_\epsilon \in B_\epsilon\) such that
\[
J(u_0 + v_\epsilon) = \inf_{v \in B_\epsilon} J(u_0 + v).
\]

We shall prove that \(v_\epsilon \rightharpoonup 0\) in \(C^1\) as \(\epsilon \to 0\), but then (2.1) and (2.4) are contradictory. The corresponding Euler equation for \(v_\epsilon\) involves a Lagrange multiplier \(\mu_\epsilon \leq 0\) (by Theorem 26.1 of [27] or [15]), namely, \(v_\epsilon\) satisfies
\[
J'(u_0 + v_\epsilon)(h) = \mu_\epsilon \int_\Omega |Dv_\epsilon|^{p-2} Dv_\epsilon Dh, \quad \forall h \in W^{1,p}_0(\Omega),
\]
i.e.,
\[
-\Delta_p (u_0 + v_\epsilon) - f(u_0 + v_\epsilon) = -\mu_\epsilon \Delta_p v_\epsilon. \tag{2.5}
\]
Thus,
\[
-\Delta_p u_0 - [\Delta_p (u_0 + v_\epsilon) - \Delta_p u_0] = f(u_0 + v_\epsilon) - \mu_\epsilon \Delta_p v_\epsilon
\]
and
\[
-\Delta_p (u_0 + v_\epsilon) - \Delta_p u_0 + \mu_\epsilon \Delta_p v_\epsilon = f(u_0 + v_\epsilon) - f(u_0). \tag{2.6}
\]
Writing (2.6) to the form
\[
-\text{div} (A(v_\epsilon)) := -\text{div} \left( |D(u_0 + v_\epsilon)|^{p-2} D(u_0 + v_\epsilon) - |Du_0|^{p-2} Du_0 \right)
- \mu_\epsilon |Dv_\epsilon|^{p-2} Dv_\epsilon
 = f(u_0 + v_\epsilon) - f(u_0) = f'(\xi)v_\epsilon,
\]
where \(\xi \in (\min(u_0, u_0 + v_\epsilon), \max(u_0, u_0 + v_\epsilon))\). We know from Lemma 2.1 of [7] that for \(p > 2\) there exists \(\rho > 0\) independent of \(u_0\) and \(v_\epsilon\) such that
\[
|D(u_0 + v_\epsilon)|^{p-2} D(u_0 + v_\epsilon) - |Du_0|^{p-2} Du_0 \cdot Dv_\epsilon \geq \rho |Dv_\epsilon|^p.
\]
Thus,
\[
A(v_\epsilon) \cdot Dv_\epsilon \geq (\rho - \mu_\epsilon)|Dv_\epsilon|^p \geq \rho|Dv_\epsilon|^p,
\]
since \( \mu_\epsilon \leq 0 \). On the other hand, using the growth condition \((F)\) on \( f'(s) \), we have that
\[
|f'(\xi)| \leq \alpha_4 + \alpha_3|\xi|^{\beta - 1} \leq \alpha_4 + C[|u_0|^{\beta - 1} + |v_\epsilon|^{\beta - 1}]
\]
since \( \beta - 1 > p - 2 > 0 \). Thus, by the regularity results obtained in [28] (see Theorem 7.1 in [28, pp. 286–287], and Theorem 1.1 in [28, p. 251]) we have that for some \( 0 < \sigma < 1 \), there exists \( C > 0 \) independent of \( \epsilon \) such that
\[
\|v_\epsilon\|_{C^\sigma(\Omega)} \leq C(\|v_\epsilon\|_{W_0^{1,p}(\Omega)}) \leq C.
\]
By the regularity results in [29] (see also [8]), we also have that
\[
\|v_\epsilon\|_{C^{1,\sigma}(\Omega)} \leq C^\sigma,
\]
where \( C^\sigma \) is determined by \( C \). This implies that \( v_\epsilon \to v_0 \) in \( C^1 \) as \( \epsilon \to 0 \). Since
\[
\|v_\epsilon\|_{W_0^{1,p}(\Omega)} \to 0 \text{ as } \epsilon \to 0,
\]
we have \( v_0 \equiv 0 \). This completes the proof of the case \( 2 < p < N \).

The proof of the case of \( p \geq N \) is similar. Note that in this case the embedding \( W_0^{1,p}(\Omega) \hookrightarrow C^{0,1}(\Omega) \) holds. By Theorem 1 of [29], we know that
\[
\|v_\epsilon\|_{C^{1,\sigma}(\Omega)} \leq C,
\]
where \( C \) is independent of \( \epsilon \). This completes the proof. \( \square \)

3. Multiplicity results for the problem (1.3)

We consider below the problem of finding solutions of the boundary value problem (1.3). To emphasize the dependence on \( \lambda \), the problem (1.3) often referred to as problem \((1.3)_\lambda \) (the subscript \( \lambda \) is omitted if no confusion arises). We also assume that \( \Omega \) has a good property as in [17,22,23] and \( \Omega_\delta \) is defined in [17,22]. Our first result is

**Theorem 3.1.** Let \( p > 1 \). For all \( 0 < q < p - 1 < \omega \) there exists \( \Lambda > 0 \) such that for \( \lambda \in (0, \Lambda) \), the problem \((1.3)_\lambda \) has a minimal solution \( u_\lambda \), which is increasing with respect to \( \lambda \) and \( \|u_\lambda\|_\infty \to 0 \) as \( \lambda \to 0 \). For all \( \lambda > \Lambda \), the problem \((1.3)_\lambda \) has no solution. Moreover, for \( \lambda = \Lambda \), the problem \((1.3)_\lambda \) has at least one weak solution \( u_\lambda \in W_0^{1,p}(\Omega) \cap L^{\omega+1}(\Omega) \).

**Remark 3.2.** The existence of a solution of \((1.3)_\lambda \) with \( \lambda > 0 \) sufficiently small has been obtained in [2,3].

To prove Theorem 3.1 we need the following lemmas.

**Lemma 3.3.** Let \( \Lambda = \sup\{\lambda > 0: (1.3)_\lambda \text{ has a solution}\} \). Then \( 0 < \Lambda < \infty \).
Proof. Let $e$ be the unique positive solution of
$$-\Delta_p e = 1 \quad \text{in } \Omega, \quad e = 0 \quad \text{on } \partial \Omega.$$ Since $0 < q < p - 1 < \omega$, we can find $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$ there exists $M = M(\lambda) > 0$ satisfying
$$M^{p-1} = \lambda M^q \|e\|_\infty^q + M^\omega \|e\|_\infty^\omega.$$ As a consequence, the function $M e$ satisfies
$$-\Delta_p (M e) = M^{p-1} \geq \lambda M^q \|e\|_\infty^q + M^\omega \|e\|_\infty^\omega$$ and hence it is a supersolution of (1.3). Moreover, let $\phi_1$ with $\|\phi_1\|_\infty = 1$ be the first eigenfunction corresponding to the first eigenvalue $\lambda_1 > 0$ of the problem
$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.$$ Then, any $\epsilon \phi_1$ is a subsolution of (1.3) provided
$$-\Delta_p (\epsilon \phi_1) = \epsilon^{p-1} \lambda_1 \phi_1^{p-1} \leq \lambda \epsilon^q \phi_1^q + \epsilon^\omega \phi_1^\omega,$$ which is satisfied for all $\epsilon > 0$ small enough and any fixed $\lambda$. Taking $\epsilon$ possibly smaller, we also have
$$\epsilon \phi_1 < M e.$$ It follows from the sub- and supersolution argument as in [22,23] that (1.3) has a solution $\epsilon \phi_1 \leq u \leq M e$ (here we use the monotonicity of the function $f_t(s) = \lambda t^q + s^\omega$) whenever $\lambda \leq \lambda_0$ and thus $A \geq \lambda_0$. Next, let $\lambda$ be such that
$$\lambda t^q + t^\omega > \lambda_1 t^{p-1} \quad \text{for } t > 0.$$ For fixed $\lambda > \lambda$, if there exists a positive solution $u$ of (1.3) (we omit the subscript here and below), then
$$-\Delta_p u = \lambda u^q + u^\omega > \lambda_1 u^{p-1}.$$ Then, let
$$\beta = \sup\{\mu \in \mathbb{R}: u - \mu \phi_1 > 0 \text{ in } \Omega\}.$$ We have that
$$u \geq \beta \phi_1 \quad \text{in } \Omega.$$ We claim that $0 < \beta < \infty$. It follows from Lemma 2.3 of [22] that there exist $\ell_i > 0$ ($i = 1, 2, 3, 4$) such that
$$\ell_1 d(x) \leq u(x) \leq \ell_2 d(x),$$
$$\ell_3 d(x) \leq \phi_1(x) \leq \ell_4 d(x),$$ where $d(x) = \text{dist}(x, \partial \Omega)$. These imply that
$$\frac{\ell_1}{\ell_4} \phi_1(x) \leq u(x) \leq \frac{\ell_2}{\ell_3} \phi_1(x).$$
Thus $\ell_1/\ell_2 \leqslant \beta \leqslant \ell_2/\ell_3$. This is our claim. Moreover,

$$-\Delta_p u - \left\{-\Delta_p (\beta \phi_1)\right\} > \lambda_1 \left[u^{p-1} - (\beta \phi_1)^{p-1}\right] \geqslant 0.$$  

By a scale argument similar to [22] (for convenience of the readers, we shall give the outline of the proof later), we show that there exists $\delta_1 > 0$ such that $u \equiv \beta \phi_1$ in $\Omega_{\delta_1}$, where $\Omega_{\delta_1} = \{x \in \Omega : d(x, \partial \Omega) < \delta_1\}$. This clearly implies that

$$-\Delta_p u = -\Delta_p (\beta \phi_1) = \lambda_1 (\beta \phi_1)^{p-1} = \lambda_1 u^{p-1} \text{ in } \Omega_{\delta_1}.$$  

This contradicts

$$-\Delta_p u > \lambda_1 u^{p-1} \text{ in } \Omega.$$  

This also implies that $\lambda < \Lambda$ and shows that $\Lambda \leqslant \Lambda$.

Now we prove that there exists $\delta_1 > 0$ such that $u \equiv \beta \phi_1$ in $\Omega_{\delta_1}$. We first show that there exists $\eta \in \Omega$ such that $u(\eta) = \beta \phi_1(\eta)$. On the contrary, we have that $u > \beta \phi_1$ in $\Omega$. Since $\partial u/\partial n_s < 0$ and $\partial \phi_1/\partial n_s < 0$ on $\partial \Omega$ (here $n_s$ is as defined in [22]) and $\partial \Omega$ is compact, we know that there exists $\delta_1 > 0$ and $\gamma > 0$ such that

$$\frac{\partial u}{\partial n_s(x)} < -\gamma < 0 \text{ and } \frac{\partial \phi_1}{\partial n_s(x)} < -\gamma < 0 \text{ in } \Omega_{\delta_1}.$$  

Thus,

$$t \frac{\partial u}{\partial n_s(x)} + (1-t) \frac{\partial (\beta \phi_1)}{\partial n_s(x)} \leqslant -\gamma \text{ for } x \in \Omega_{\delta_1} \text{ and all } t \in [0, 1]. \quad (3.2)$$  

Hence, using the mean value theorem, we obtain

$$0 \leqslant -\Delta_p u - \left\{-\Delta_p (\beta \phi_1)\right\} = -\sum_{i,j} \frac{\partial}{\partial x_i} \left[a^{ij}(x) \frac{\partial (u - \beta \phi_1)}{\partial x_j}\right] \text{ in } \Omega_{\delta_1},$$  

where

$$a^{ij}(x) = \frac{1}{t} \int_0^t \frac{\partial a^i}{\partial q_j} [t Du + (1-t)D(\beta \phi_1)] dt$$  

and $a^i = |q|^{p-2} q_i$ ($i = 1, 2, \ldots, N$) for $q = (q_1, q_2, \ldots, q_N) \in \mathbb{R}^N$. Put

$$L = \sum_{i,j} \frac{\partial}{\partial x_i} \left[a^{ij}(x) \frac{\partial}{\partial x_j}\right].$$  

Using (3.2), we see that $L$ is a uniformly elliptic operator on $\Omega_{\delta_1}$. Consequently, we have

$$-L(u - \beta \phi_1) \geqslant 0 \text{ in } \Omega_{\delta_1},$$  

$$u(x) > \beta \phi_1(x) \text{ in } \Omega_{\delta_1}, \text{ and } u - \beta \phi_1 = 0 \text{ on } \partial \Omega \text{ (part of } \partial \Omega_{\delta_1}). \quad (3.3)$$  

By Hopf’s boundary point lemma [12, Lemma 3.4] we obtain $\partial (u - \beta \phi_1)/\partial n_s < 0$ on $\partial \Omega$. By arguments similar to those in [22], we have that there exists $\theta > 0$ such that

$$u(x) \geqslant (\beta + \theta)\phi_1(x) \text{ for } x \in \Omega. \quad (3.4)$$  

(3.5)
This contradicts the definition of $\beta$. By the same argument as that in the proof of Theorem 3.1 in [22], we also obtain that there exists a point $z \in \Omega_{\delta_1}$ where $u - \beta \phi_1$ vanishes and therefore

$$u \equiv \beta \phi_1 \quad \text{in} \quad \Omega_{\delta_1}.$$  

This completes the proof. $\square$

**Lemma 3.4.** For all $0 < \lambda < \Lambda$, the problem $(1.3)_\lambda$ has a solution.

**Proof.** Given $0 < \lambda < \Lambda$, let $u_\mu$ be a solution of $(1.3)_\mu$ with $\lambda < \mu < \Lambda$. Plainly, such a $u_\mu$ is a supersolution of $(1.3)_\lambda$. Since $\epsilon \phi_1 < u_\mu$ provided $\epsilon > 0$ is sufficiently small, it follows that $(1.3)_\lambda$ has a solution. This completes the proof. $\square$

We next prove that $(1.3)_\lambda$ possesses a minimal solution. To this end we need the following lemma.

**Lemma 3.5.** Assume that $f$ is a nondecreasing $C^1$ function with $f(0) = 0$ such that $s^{1-p} f(s)$ is strictly decreasing for $s > 0$. Let $v, w \in W^{1,p}_0 \cap C^1(\overline{\Omega})$ satisfy

$$-\Delta_p v \leq f(v), \quad v > 0 \quad \text{in} \quad \Omega, \quad v = 0 \quad \text{on} \quad \partial\Omega \quad (3.6)$$

and

$$-\Delta_p w \geq f(w), \quad w > 0 \quad \text{in} \quad \Omega, \quad w = 0 \quad \text{on} \quad \partial\Omega. \quad (3.7)$$

Moreover, $\partial v / \partial n < 0$ on $\partial\Omega$, where $n$ is the outward norm vector of $\partial\Omega$. Then $w \geq v$ in $\Omega$.

**Proof.** We also use the scale argument to prove this lemma. Let

$$\beta = \sup \{ \mu \in \mathbb{R}; \ w - \mu v > 0 \ \text{in} \ \Omega \}. \quad \text{Then} \ 0 < \beta < \infty \ \text{and} \ w \geq \beta v \ \text{in} \ \Omega. \ \text{We shall prove that} \ \beta \geq 1. \ \text{Suppose that} \ \beta < 1, \ \text{then}$$

$$-\Delta_p w - \left( -\Delta_p (\beta v) \right) \geq f(w) - \beta^{p-1} f(v) > f(w) - f(\beta v) \geq 0 \ \text{in} \ \Omega, \quad (3.8)$$

where we use the facts that $f$ is nondecreasing and that $s^{1-p} f(s)$ is strictly decreasing for $s > 0$. Using arguments similar to those in the proof of Lemma 3.3 we have that there exists $\delta_2 > 0$ such that

$$w \equiv \beta v \ \text{in} \ \Omega_{\delta_2}. \quad \text{This clearly contradicts} \ (3.8). \ \text{This completes the proof.} \ \square$$

It should be noted that the scale argument used in the proof of Lemma 3.3 works with any positive $\beta$ because the function involved is $s^{p-1}$; while the same argument used in the proof of Lemma 3.5 works only for $0 < \beta < 1$ because the involved function $f(s)$ satisfies that $s^{1-p} f(s)$ is strictly decreasing for $s > 0$.

**Lemma 3.6.** For all $0 < \lambda < \Lambda$, the problem $(1.3)_\lambda$ has a minimal solution $\gamma_\lambda$ and $\| \gamma_\lambda \|_\infty \to 0$ as $\lambda \to 0$. 
Proof. Let \( z_\lambda \in C^1_0(\Omega) \) be the unique positive solution of 
\[-\Delta_p z = \lambda z^q \text{ in } \Omega, \quad z = 0 \text{ on } \partial \Omega.\]
We already know that there exists a solution \( u_\lambda > 0 \) of (1.3), for every \( \lambda \in (0, \Lambda) \). Since 
\[-\Delta_p u_\lambda \geq \lambda u_\lambda^q, \]
we can use Lemma 3.5 with \( w = u_\lambda \) and \( v = z_\lambda \) to deduce that any solution 
of (1.3), must satisfy \( u_\lambda \geq z_\lambda \). (It follows from Lemma 2.2 of [16] that \( \partial v_\lambda / \partial n < 0 \) on \( \partial \Omega \).) Clearly, \( z_\lambda \) is a subsolution of (1.3). The monotone iteration 
\[-\Delta_p u_{n+1} = \lambda u_{n+1}^q + u_n^p, \quad u_0 = z_\lambda \]
and the maximum principle [16] imply that \( u_n \uparrow y_\lambda \), with \( y_\lambda \) a solution of (1.3). It is easy to check that \( y_\lambda \) is a minimal solution of (1.3). Indeed, for any solution \( u_\lambda \) of (1.3), we have \( u_\lambda \geq z_\lambda \). Then the weak comparison principle implies that \( u_n \leq u_\lambda \) for any \( n \) and thus \( y_\lambda \leq u_\lambda \). Since \( M(\lambda) \to 0 \) as \( \lambda \to 0 \) (see the proof of Lemma 3.3), it follows that 
\( \|y_\lambda\|_{\infty} \to 0 \) as \( \lambda \to 0 \). \( \square \)

Now we show the existence of a positive solution of (1.3), for \( \lambda = \Lambda \). Let 
\[ f_\lambda(s) = \begin{cases} 
\lambda s^q + s^p, & s \geq 0, \\
0, & s < 0,
\end{cases} \]
and 
\[ F_\lambda(u) = \int_0^u f_\lambda(s) ds. \]
We may define the functional \( \tilde{T}_\lambda : W^{1,p}_0(\Omega) \to \mathbb{R} \) by setting 
\[ \tilde{T}_\lambda(u) = \frac{1}{p} \|u\|_{W^{1,p}(\Omega)}^p - \int_\Omega F_\lambda(u) \, dx. \]
It is well known that the critical points of \( \tilde{T}_\lambda \) correspond to the solutions of (1.3). Now we have the following lemma. (In the sequel \( \lambda \) is fixed.)

**Lemma 3.7.** For all \( \lambda \in (0, \Lambda) \), the problem (1.3) has a solution \( u_\lambda \) which is in addition a local minimizer of \( \tilde{T}_\lambda \) in the \( C^1 \)-topology. Moreover, there exists \( C = C(\Lambda) > 0 \) such that 
\[ \tilde{T}_\lambda(u_\lambda) < 0 \]
and 
\[ \|u_\lambda\|_{W^{1,p}_0(\Omega)}^p \leq C, \quad \|u_\lambda\|_{L^{p+1}(\Omega)}^{p+1} \leq C. \]

**Proof.** We fix \( \lambda_1 < \lambda < \lambda_2 < \Lambda \) and consider the minimal solutions \( u_1 := y_{\lambda_1} \) and \( u_2 := y_{\lambda_2} \) defined in Lemma 3.6. We first show that \( u_1 \leq u_2 \). In fact, we can show that for any \( \lambda^* \in (0, \Lambda) \) 
\[ y_\lambda \leq y_{\lambda^*} \quad \text{whenever } \lambda \leq \lambda^*. \]
Indeed, if \( \lambda < \lambda^* \) then \( y_{\lambda^*} \) is a supersolution of (1.3). Since, for \( \epsilon > 0 \) small, \( \epsilon \phi_1 \) is a subsolution of (1.3), and \( \epsilon \phi_1 < y_{\lambda^*} \), then (1.3) possesses a solution \( v_\lambda \) with
\[
(\epsilon \phi_1 \leq v_\lambda \leq y_{\lambda^*}).
\]
Since \( y_\lambda \) is the minimal solution of (1.3), we infer that
\[
y_\lambda \leq v_\lambda \leq y_{\lambda^*}.
\]
Now we show actually
\[
u_1 < u_2 \text{ in } \Omega, \quad \frac{\partial (u_2 - u_1)}{\partial n} < 0 \text{ on } \partial \Omega,
\]
where \( n \) is the outward normal vector on \( \partial \Omega \). Clearly, \( u_1 \), respectively \( u_2 \), is a subsolution, respectively supersolution, of (1.3). Moreover,
\[
-\Delta_P u_2 - [-\Delta_P u_1] = \hat{\lambda}_2 u_2^q + u_2^\rho - (\hat{\lambda}_1 u_1^q + u_1^\rho) \\
\geq \hat{\lambda}_1 u_2^q + u_2^\rho - (\hat{\lambda}_1 u_1^q + u_1^\rho) \geq 0 \text{ in } \Omega.
\]
Since \( u_1 \neq u_2 \) (because \( \hat{\lambda}_1 < \hat{\lambda}_2 \)), by arguments similar to those in the proof of Lemma 3.3, we have that there exists \( \delta_3 > 0 \) such that
\[
0 \leq -\Delta_P u_2 - [-\Delta_P u_1] = -L(u_2 - u_1) \quad \text{in } \Omega_{\delta_3},
\]
where \( L \) is a uniformly elliptic operator in \( \Omega_{\delta_3} \). Since \( u_2 \geq u_1 \) in \( \Omega \), if there exists an \( \eta \in \Omega \) such that \( u_2(\eta) = u_1(\eta) \), we can prove that there exists a \( \xi \in \Omega_{\delta_3} \) such that \( u_2(\xi) = u_1(\xi) \). On the contrary, we can find \( \Omega_1 \subset \subset \Omega \) such that \( \eta \in \Omega_1 \) and \( \partial \Omega_1 \subset \Omega_{\delta_3} \), and \( u_2 \geq u_1 + \xi \) on \( \partial \Omega_1 \) with \( \xi > 0 \). Let \( u_3 = u_1 + \xi \). Then
\[
-\Delta_P u_2 - [-\Delta_P u_3] \geq 0 \text{ in } \Omega_1
\]
and
\[
u_2 \geq u_3 \text{ on } \partial \Omega_1.
\]
The weak comparison principle implies that
\[
u_2 \geq u_3 \text{ in } \Omega_1.
\]
This contradicts \( u_2(\eta) = u_1(\eta) \). Since \( L \) is uniformly elliptic in \( \Omega_{\delta_3} \) and \( \xi \in \Omega_{\delta_3} \), we have that \( u_1 \equiv u_2 \) in \( \Omega_{\delta_3} \). This contradicts \( \hat{\lambda}_1 < \hat{\lambda}_2 \). Thus, \( u_2 > u_1 \) in \( \Omega \). Since \( L \) is uniformly elliptic in \( \Omega_{\delta_3} \) and \( -L(u_2 - u_1) \geq 0 \) in \( \Omega_{\delta_3} \) with \( u_2 - u_1 = 0 \) on \( \partial \Omega \), the Hopf’s boundary point lemma yields
\[
\frac{\partial}{\partial n}(u_2 - u_1) < 0 \text{ on } \partial \Omega.
\]
Now we set
\[
\tilde{f}_\lambda(x, s) = \begin{cases} 
    f_\lambda(u_1(x)), & s \leq u_1, \\
    f_\lambda(s), & u_1 < s < u_2, \\
    f_\lambda(u_2(x)), & s \geq u_2, 
\end{cases}
\]
and
\[
\tilde{F}_\lambda(x, u) = \int_0^u \tilde{f}_\lambda(x, s) \, ds.
\]
and \[ \tilde{I}_\lambda(u) = \frac{1}{p} \int_\Omega |Du|^p \, dx - \int_\Omega \tilde{f}_\lambda(x, u) \, dx. \]

By the standard way, one can prove that \( \tilde{I}_\lambda \) achieves its (global) minimum at some \( u_\lambda \in W^{1,p}_0(\Omega) \). Moreover,

\[ -\Delta_p u_\lambda = \tilde{f}_\lambda(x, u_\lambda) \quad \text{for} \quad x \in \Omega. \]

We also know from Proposition 2.2 of [16] that \( u_\lambda \in C^1(\Omega) \). It is clear that \( \tilde{f}_\lambda(x, u_\lambda) \geq f_\lambda(x, u_1) \) in \( \Omega \). Using the scale argument as above, we obtain that

\[ u_1 < u_\lambda < u_2 \quad \text{in} \quad \Omega, \]  
\[ \frac{\partial}{\partial n}(u_\lambda - u_1) < 0, \quad \frac{\partial}{\partial n}(u_\lambda - u_2) > 0 \quad \text{on} \quad \partial \Omega. \]

These imply that \( u_\lambda \) is a solution of (1.3). From (3.9)–(3.10) it follows that if

\[ \|v - u_\lambda\|_{C^1} = \epsilon \]

with \( \epsilon \) small, then \( u_1 \leq v \leq u_2 \). Moreover, \( \tilde{T}_\lambda(v) - \tilde{I}_\lambda(v) \) is constant for \( u_1 \leq v \leq u_2 \) and therefore \( u_\lambda \) is also a local minimizer for \( \tilde{T}_\lambda \) in the \( C^1 \)-topology. Let \( J_\lambda(\epsilon) = \tilde{T}_\lambda(u_\lambda + \epsilon h) \) for any \( h \in C^1(\Omega) \) and \( h > 0 \). Then \( J_\lambda \) attains a local minimum at \( \epsilon = 0 \). Thus,

\[ J_\lambda''(0) = (\tilde{T}_\lambda''(u_\lambda)h, h) \geq 0. \]

Setting \( h = u_\lambda \), we have that

\[ \int_\Omega |Du_\lambda|^p - \left( \lambda q/(p - 1) \right) \int_\Omega u_\lambda^{q+1} - \left( \omega/(p - 1) \right) \int_\Omega u_\lambda^{\omega+1} \geq 0. \]

This together with \( \tilde{T}_\lambda(u_\lambda) = \frac{1}{p} \int_\Omega |Du_\lambda|^p - \frac{\lambda}{q+1} \int_\Omega u_\lambda^{q+1} - \frac{1}{\omega+1} \int_\Omega u_\lambda^{\omega+1} \)

and

\[ \int_\Omega |Du_\lambda|^p = \lambda \int_\Omega u_\lambda^{q+1} + \int_\Omega u_\lambda^{\omega+1} \]

imply that

\[ \tilde{T}_\lambda(u_\lambda) < 0, \]

and there exists \( C = C(A) \) such that

\[ \|u_\lambda\|_{W^{1,p}(\Omega)} \leq C, \quad \|u_\lambda\|_{L^{q+1}(\Omega)} \leq C, \quad \|u_\lambda\|_{L^{\omega+1}(\Omega)} \leq C. \]

This completes the proof. \( \square \)
Lemma 3.8. There exists a solution \( u^* \in W_0^{1,p}(\Omega) \cap L^{\omega+1}(\Omega) \) of (1.3)\( \lambda \) for \( \lambda = \Lambda \).

Proof. In fact, let \( \{ \lambda_n \} \) be a sequence such that \( \lambda_n \uparrow \Lambda \). By Lemma 3.7, there exists a solution \( u_n \in W^{1,p}(\Omega) \cap C_0^1(\overline{\Omega}) \) of (1.3)\( \lambda_n \) such that \( \tilde{I}_{\lambda_n}(u_n) < 0 \) and (3.11)–(3.12) hold. Then there exists \( u^* \in W_0^{1,p}(\Omega) \cap L^{\omega+1}(\Omega) \) such that \( u_n \to u^* \) a.e. in \( \Omega \), weakly in \( W_0^{1,p}(\Omega) \) and \( L^{\omega+1}(\Omega) \). Such an \( u^* \) is thus a weak solution of (1.3)\( \lambda \) for \( \lambda = \Lambda \). When \( p-1 < \omega < [Np/(N-p)] - 1 \) for \( 1 < p < N \) and \( \omega > p-1 \) for \( p \geq N \), we know from the proof of Theorem 2.1 that \( u^* \in C_0^1(\overline{\Omega}) \). This completes the proof. \( \square \)

Proof of Theorem 3.1. The proof of Theorem 3.1 can be obtained directly from the lemmas above. \( \square \)

Now we are looking for a second positive solution of (1.3)\( \lambda \). Denote \( y \) the solution obtained in Lemma 3.6. We have the following theorem.

Theorem 3.9. Let \( 0 < q < (p-1) < \omega < [Np/(N-p)] - 1 \) for \( 2 < p < N \), \( 0 < q < (p-1) < \omega < +\infty \) for \( p \geq N \). Then for all \( \lambda \in (0, \Lambda) \), the problem (1.3)\( \lambda \) has another solution \( w_\lambda \) with \( w_\lambda \neq y_\lambda \) and \( w_\lambda > y_\lambda \).

Remark 3.10. When \( \omega = [Np/(N-p)] - 1 \) for \( 1 < p < N \), some existence results of the problem (1.3)\( \lambda \) have been obtained in [2].

Proof of Theorem 3.9. Let

\[
\tilde{f}_\lambda(x, s) = \begin{cases} 
  f_\lambda(s), & \text{if } s \geq y_\lambda(x), \\
  f_\lambda(y_\lambda(x)), & \text{if } s < y_\lambda(x).
\end{cases}
\]

Then \( \tilde{f}_\lambda \) is continuous on \( x \) and \( s \). By arguments similar to those in the proof of Lemma 3.7, we have that the functional

\[
\tilde{I}_\lambda(u) = \frac{1}{p} \int_\Omega |Du|^p - \int_\Omega \tilde{f}_\lambda(x, u),
\]

with \( \tilde{f}_\lambda(x, s) = \int_0^s \tilde{f}_\lambda(x, \xi) \, d\xi \) restricted to \( E = C_0^1(\overline{\Omega}) \), has a local minimizer \( \tilde{u}_\lambda \) in the interval \([\tilde{u}_1, \tilde{u}_2]\) in \( E \), where \( \tilde{u}_1 = y_{\lambda-\epsilon} \) and \( \tilde{u}_2 = y_{\lambda+\epsilon} \) for \( \epsilon > 0 \) sufficiently small. Theorem 2.1 implies that this local minimizer is also a local minimizer of \( \tilde{I}_\lambda \) in \( W_0^{1,p}(\Omega) \). We assume that \( y_\lambda \) is the unique solution of the problem

\[-\Delta_p u = \tilde{f}_\lambda(x, u) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega \]

in \([\tilde{u}_1, \tilde{u}_2]\), otherwise we have obtained our conclusion. Thus, \( \tilde{u}_\lambda \equiv y_\lambda \) in \( \Omega \) and \( y_\lambda \) is the only local minimizer of \( \tilde{I}_\lambda \) in \( W_0^{1,p}(\Omega) \); this implies that \( y_\lambda \) is a strictly local minimizer of \( \tilde{I}_\lambda \) in \( W_0^{1,p}(\Omega) \). One easily checks that \( \tilde{I}_\lambda(t\phi_1) \to -\infty \) as \( t \to +\infty \). Moreover, if \( \tilde{I}_\lambda \) satisfies (PS) condition, then it is easy to show that for some \( \epsilon > 0 \) small,

\[
\inf\{ \tilde{I}_\lambda(u) : \|u - y_\lambda\|_{W_0^{1,p}(\Omega)} = \epsilon \} > \tilde{I}_\lambda(y_\lambda).
\]
Hence one can use the mountain pass lemma to obtain a critical point $w_\lambda \in W_{0}^{1,p}(\Omega)$ of $I_\lambda$ such that $w_\lambda \neq y_\lambda$. We also know that $w_\lambda \in C_0(\overline{\Omega})$ and thus $w_\lambda \geq y_\lambda$. We can show $w_\lambda > y_\lambda$ by the scale argument as above. This implies that $w_\lambda$ is another solution of $(1.3)_\lambda$.

We still have to show that $I_\lambda$ satisfies (PS) condition. This can be done by arguments similar to those in the proof of Theorem 4.6 of [17]. (We need to use the embedding $W_{0}^{1,p}(\Omega) \rightarrow C^0(\Omega)$ for $p \geq N$.) This completes the proof. □

4. Multiplicity results for the problem (1.4)

In this section we study problem $(1.4)_\lambda$ with $f_\lambda$ satisfying (ii) and $g$ satisfying (H1). It is clear that the nonlinearity of problem $(1.3)$ is a special case of the $f_\lambda$ discussed here. To obtain our multiplicity results for $(1.4)_\lambda$, we first use sub- and supersolution arguments like those in Sections 2 and 3 to obtain a minimal positive solution and a maximal negative solution for $(1.4)_\lambda$. Using Theorem 2.1, we easily know that the minimal positive solution and the maximal negative solution are strictly local minimizers of a corresponding functional of $(1.4)_\lambda$ with some extra conditions on $g$. Finally, using the mountain pass lemma we can obtain another positive solution and another negative solution of $(1.4)_\lambda$ by arguments similar to those in Sections 2 and 3. Moreover, we can also provide a sign-changing solution for $(1.4)_\lambda$.

In the following, $\lambda_1 > 0$ denotes the first eigenvalue of the problem

$$-\Delta_p u = \lambda |u|^{p-2} u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial \Omega. \quad (4.1)$$

It is well-known that $\lambda_1$ is simple. Our main result of this section is the following theorem.

**Theorem 4.1.** Suppose $p > 2$, $g$ satisfies (H1) and either

(H2) $\lim_{|u| \rightarrow \infty} g(u)/|u|^{p-2} u = a > \lambda_1$, or
(H3) $\lim_{|u| \rightarrow \infty} g(u)/|u|^{p-1} u = b > 0$, where $p - 1 < \gamma < [Np/(N - p)] - 1$ for $2 < p < N$; $\gamma > p - 1$ for $p \geq N$.

Then there exist $\Lambda^+, \Lambda^- > 0$ such that

(i) for $\lambda > \Lambda^+$ (resp. $\Lambda^-$), $(1.4)_\lambda$ has no positive (resp. negative) solution; 
(ii) for $0 < \lambda < \Lambda^+$ (resp. $\Lambda^-$), $(1.4)_\lambda$ has at least two positive (resp. negative) solutions; 
(iii) for $\lambda = \Lambda^+$ (resp. $\Lambda^-$), $(1.4)_\lambda$ has at least one positive (resp. negative) solution; 
(iv) when $g$ satisfies (H2), for $0 < \lambda < \min[\Lambda^-, \Lambda^+]$, $(1.4)_\lambda$ has at least one sign-changing solution; 
(v) when $g$ satisfies (H3) with $2 < p < N$, $p - 1 < \gamma < [Np/(N - p)] - 1$ and $\Omega$ is an $N$-ball, for $0 < \lambda < \min[\Lambda^-, \Lambda^+]$, $(1.4)_\lambda$ has at least one sign-changing radial solution.

**Remark 4.2.** We expect that (v) is true for any bounded smooth domain $\Omega$ and $p > 2$. The key point here is how to get the upper bound of the solutions of $(1.4)_\lambda$. Normally, we use a blow up argument to obtain this (see [23]). We need to know the structure of the
corresponding equation in $\mathbb{R}^N$. When $\Omega$ is a ball and $p - 1 < \gamma < \lfloor Np/(N - p) \rfloor - 1$ for $1 < p < N$, we know from [13] that there is no bounded positive radial solution for the equation $-\Delta_p u = u^\gamma$ in $\mathbb{R}^N$. Then we can find the upper bound of the positive radial solutions of (1.4)$_\lambda$ in this case by a blow up argument.

The proof of Theorem 4.1 can be obtained from the following lemmas and theorems.

**Lemma 4.3.** Suppose $g$ satisfies (H$_1$) and (H$_2$). Then there exist $\Lambda^+, \Lambda^- \in (0, \infty)$ such that

(i) for $\lambda > \Lambda^+$ (resp. $\Lambda^-$), (1.4)$_\lambda$ has no positive (resp. negative) solution;

(ii) for $0 < \lambda < \Lambda^+$ (resp. $\Lambda^-$), (1.4)$_\lambda$ has at least one positive (resp. negative) solution.

**Proof.** (i) Define

$$
\Lambda^+ = \sup \{ \lambda > 0 : (1.4)_\lambda \text{ has a positive solution} \},$

$$
\Lambda^- = \sup \{ \lambda > 0 : (1.4)_\lambda \text{ has a negative solution} \}.
$$

Then the conclusion follows from a simple variation of the proof of Lemma 3.3.

(ii) The same sub- and supersolution arguments as in the proofs of Lemmas 2.3–2.5 show that (1.4)$_\lambda$ has a minimal positive solution $y_\lambda$ for $0 < \lambda < \Lambda^+$, and a maximal negative solution $\tilde{y}_\lambda$ for $0 < \lambda < \Lambda^-$. This completes the proof. □

**Theorem 4.4.** Suppose that $g$ satisfies (H$_1$) and (H$_2$), $\Lambda^+, \Lambda^-$ are as in Lemma 4.3. Then

(i) for $\lambda = \Lambda^+$ (resp. $\Lambda^-$), (1.4)$_\lambda$ has at least one positive (resp. negative) solution;

(ii) for $0 < \lambda < \Lambda^+$ (resp. $\Lambda^-$), (1.4)$_\lambda$ has at least two positive (resp. negative) solutions;

(iii) for $0 < \lambda < \min(\Lambda^-, \Lambda^+)$, (1.4)$_\lambda$ has at least one sign-changing solution.

**Proof.** (i) We carry out the proof for $\lambda = \Lambda^+$ only; another case can be proved analogously. By Lemma 3.6, (1.4)$_\lambda$ has a minimal positive solution $y_\lambda$ for any $\lambda \in (0, \Lambda^+)$. Since

$$
\lim_{u \to +\infty} \frac{(\lambda u^q + g(u))}{u^{p-1}} = a > \lambda_1
$$

uniformly for $\lambda \in (0, \Lambda^+)$, we shall prove that there exists $C > 0$ independent of $\lambda$ such that

$$
\|y_\lambda\|_\infty \leq C
$$

for all $\lambda \in (0, \Lambda^+)$. Indeed, suppose that there exists a sequence $\{\lambda_n\} \subset (0, \Lambda^+)$ such that $\{y_n\} \equiv \{y_{\lambda_n}\}$ with $\|y_n\|_\infty \to \infty$ as $n \to \infty$. Then let $v_n = y_n/\|y_n\|_\infty$, we have

$$
-\Delta_p v_n = \lambda_n \|y_n\|_\infty^{-(p-q-1)} y_n^q + g(\|y_n\|_\infty v_n) \|y_n\|_\infty^{p-1} \quad \text{in } \Omega,
$$

$$
v_n = 0 \quad \text{on } \partial \Omega.
$$

The regularity of $-\Delta_p$ (see [16]) implies that $v_n \to v$ in $C^1_0(\overline{\Omega})$ with $v \geq 0$, $\|v\|_\infty = 1$ and $v$ satisfies

$$
-\Delta_p v = av^{p-1} \quad \text{in } \Omega, \quad v = 0 \quad \text{on } \partial \Omega.
$$
The scale argument as in the proof of Lemma 3.3 implies that there exists $\delta_4 > 0$ such that

$$v \equiv \beta \phi_1 \quad \text{in} \quad \Omega_{\delta_4}.$$  

This contradicts $a > \lambda_1$. This shows that $\|y_\lambda\|_\infty \leq C$. It follows from the regularity of the $p$-Laplacian that $\{y_\lambda : 0 < \lambda < A^+\}$ is precompact in $C^1$. Hence for some sequence $\lambda_n \to A^+$, $y_{\lambda_n}$ converges to a solution $y_{A^+}$ of (1.4)$_{A^+}$. By arguments similar to those in the proof of Lemma 3.6, $u_{A^+} \geq z_{A^+}$, where $z_{A^+}$ is the unique positive solution of

$$-\Delta_p z = \lambda z^q \quad \text{in} \quad \Omega, \quad z = 0 \quad \text{on} \quad \partial\Omega,$$

which is always a subsolution to (1.4)$_{A^+}$. Therefore, $y_{A^+}$ must be a positive solution of (1.4)$_{A^+}$ and $y_{A^+} \geq z_{A^+}$. This implies that (1.4)$_{A^+}$ has a minimal positive solution $y_\lambda$ for $\lambda = A^+$ as well.

(ii) Again we consider the case $0 < \lambda < A^+$ only. Another case is similar. Choose $\lambda' \in (\lambda, A^+)$ and define $\overline{y}_\lambda = y_{\lambda'}$. Then $\overline{y}_\lambda$ is an supersolution to (1.4)$_{\lambda'}$. Let $u^*$ be the minimal solution of (1.4)$_{\lambda'}$ between $z_{\lambda'}$ and $\overline{y}_\lambda$. Then the proof of Theorem 3.9 shows that (1.4)$_{\lambda'}$ has a second solution $u \geq u^*$.

(iii) We may assume that $A^+ < A^+$. The proofs for other cases are similar. We need only to show that (1.4)$_{A^+}$ has a sign-changing solution between the maximal negative solution $\overline{y}_{\lambda}$ and the minimal positive solution $y_{\lambda}$ for any $0 < \lambda < A := \min\{A^+, A^-\} = A^+$. By truncating $f_{\lambda}(u) = \lambda |u|^{q-1}u + g(u)$ as

$$\tilde{f}_{\lambda}(x, s) = \begin{cases} f_{\lambda}(\tilde{y}_{\lambda}(x)), & \text{if } s < \tilde{y}_{\lambda}(x), \\ f_{\lambda}(s), & \text{if } \tilde{y}_{\lambda}(x) \leq s \leq y_{\lambda}(x), \\ f_{\lambda}(y_{\lambda}(x)), & \text{if } s > y_{\lambda}(x), \end{cases}$$

we get the functional

$$\tilde{J}_{\lambda}(u) = \frac{1}{p} \int_{\Omega} |Du|^p - \int_{\Omega} \tilde{F}_{\lambda}(x, u)$$

with $\tilde{F}_{\lambda}(x, s) = \int_{0}^{s} \tilde{f}_{\lambda}(x, \xi) d\xi$ which satisfies (PS) condition in $W_{0}^{1,p}(\Omega)$. Define

$$\mathcal{D} = [\overline{y}_{\lambda}, y_{\lambda}] := \{ w \in C_{0}^{1}(\overline{\Omega}) : \overline{y}_{\lambda} \leq w \leq y_{\lambda} \}.$$

The proof of Lemma 3.7 implies that $\tilde{J}_{\lambda}$ has a positive and a negative minimizers in $W^{1,p}_{0}(\Omega)$ and they are $y_{\lambda}$ and $\overline{y}_{\lambda}$, respectively. We only show that $y_{\lambda}$ is the positive minimizer. The proof of $\overline{y}_{\lambda}$ is similar. Setting $\tilde{u}_1 = y_{\lambda'}$ with $0 < \lambda' < \lambda$, $\tilde{u}_2 = y_{\lambda}$ and

$$\tilde{f}_{\lambda}(x, s) = \begin{cases} f_{\lambda}(\tilde{u}_2(x)), & \text{if } s > \tilde{u}_2(x), \\ f_{\lambda}(s), & \text{if } \tilde{u}_1(x) \leq s \leq \tilde{u}_2(x), \\ f_{\lambda}(\tilde{u}_1(x)), & \text{if } s < \tilde{u}_1(x), \end{cases}$$

we obtain by arguments similar to those in the proof of Lemma 3.7 that the functional

$$\tilde{I}_{\lambda}(u) = \frac{1}{p} \int_{\Omega} |Du|^p - \int_{\Omega} \tilde{F}_{\lambda}(x, u)$$
with \( \tilde{F}_\lambda(x, s) = \int_0^s \tilde{f}_\lambda(x, \xi) \, d\xi \) has a local minimizer \( \tilde{u}_\lambda \) in \( W^{1,p}_0(\Omega) \) and \( \tilde{u}_\lambda \in [\tilde{u}_1, \tilde{u}_2]_E \). Since \( \gamma_\lambda \) is the minimal positive solution of \( (1.4)_\lambda \), we easily know that \( \tilde{u}_\lambda = y_\lambda \). Arguments similar to those in the proof of Theorem 3.9 imply that \( y_\lambda \) and \( y_\lambda \) are strictly local minimizers of \( J_\lambda \) in \( W^{1,p}_0(\Omega) \). Therefore, there is a mountain pass critical point \( w \) of \( J_\lambda \) in \( W^{1,p}_0(\Omega) \). It is clear that \( 0 \) is not a mountain pass critical point of \( J_\lambda \), so \( w \neq 0 \). Now we show that \( w \) must be a sign-changing critical point. On the contrary, we have \( w \geq 0 \) or \( w \leq 0 \), but \( w \neq 0 \). We shall only derive a contradiction for the first case. The proof of the second case is similar. Since the regularity of \( -\Delta_p \) implies \( w \in C^{1,0}(\Omega) \), by the fact \( g(w) \geq 0 \), we easily know \( -\Delta_p w \geq 0 \). The strong maximum principle in [16] implies \( w > 0 \). Now we show \( w \in [0, y_\lambda]_E \) and thus \( w \) is a positive solution of \( (1.4)_\lambda \), which is a contradiction since \( \gamma_\lambda \) is the minimal positive solution of \( (1.4)_\lambda \). Indeed, we know that

\[
-\Delta_p w = \tilde{f}_\lambda(w) \leq f_\lambda(y_\lambda) = -\Delta_p y_\lambda,
\]

where we use the monotonicity of \( g \). Then the weak comparison principle of \( -\Delta_p \) implies \( w \leq y_\lambda \) and thus \( w \in [0, y_\lambda]_E \). The analysis above shows that \( w \) must be a sign-changing critical point. We also need to show that \( w \in D \).

We only show that \( w \leq y_\lambda \). On the contrary, suppose that there exists \( \Omega_1 \subset \Omega \) such that \( w > y_\lambda \) on \( \Omega_1 \). Then

\[
-\Delta_p w = \tilde{f}_\lambda(w) \equiv f_\lambda(y_\lambda) = -\Delta_p y_\lambda \quad \text{in} \ \Omega_1.
\]

Defining \( \psi = (w - y_\lambda)^+ \), we have that

\[
\Omega_1 \int (|Dw|^{p-2} Dw - |Dy_\lambda|^{p-2} Dy_\lambda) (Dw - Dy_\lambda) \, dx = 0.
\]

On the other hand, we know from [7] that there exists \( \gamma_0 > 0 \) independent of \( p \) such that

\[
\Omega_1 \int (|Dw|^{p-2} Dw - |Dy_\lambda|^{p-2} Dy_\lambda) (Dw - Dy_\lambda) \, dx \geq \gamma_0 \int_{\Omega_1} (1 + |Dw| + |Dy_\lambda|)^{p-2} |Dw - Dy_\lambda|^2 \, dx \quad \text{if} \ 1 < p < 2,
\]

\[
\int_{\Omega_1} |Dw - Dy_\lambda|^p \, dx \quad \text{if} \ p \geq 2,
\]

and hence we derive a contradiction. Therefore, \( w \in D \) and hence \( w \) is a sign-changing solution of \( (1.4)_\lambda \). This completes the proof. \( \Box \)

**Theorem 4.5.** Suppose \( p > 2 \) and \( g \) satisfies \((H_1)\) and

\[
(H_3) \quad \lim_{|u| \to \infty} g(u)/|u|^{\gamma-1}u = b > 0, \quad \text{where} \ p - 1 < \gamma < [Np/(N - p)] - 1 \ \text{for} \ 2 < p < N \ \text{and} \ \gamma > p - 1 \ \text{for} \ p \geq N.
\]

Then there exists \( \Lambda^+, \Lambda^- > 0 \) such that
(i) for $\lambda > \Lambda^+$ (resp. $\Lambda^-$), $(1.4)_\lambda$ has no positive (resp. negative) solution;
(ii) for $\lambda = \Lambda^+$ (resp. $\Lambda^-$), $(1.4)_\lambda$ has at least one positive (resp. negative) solution;
(iii) for $0 < \lambda < \Lambda^+$ (resp. $\Lambda^-$), $(1.4)_\lambda$ has at least two positive (resp. negative) solutions;
(iv) when $\Omega$ is an $N$-ball and $g$ satisfies (H3) for $2 < p < N$, $p - 1 < \gamma < [Np/(N - p)] - 1$, $(1.4)_\lambda$ has at least one sign-changing radial solution for $0 < \lambda < \min\{\Lambda^+, \Lambda^\prime\}$.

**Proof.** (i) can be obtained by the idea similar to that in the proof of Lemma 4.3.

(ii) We consider the case $\lambda = \Lambda^+$ only. By the idea similar to that in the proof of Lemma 3.6, $(1.4)_\lambda$ has a minimal positive solution $y_\lambda$ for any $\lambda \in (0, \Lambda^+)$. As above, one easily sees that $y_{\lambda'} \leq y_{\lambda''}$ if $0 < \lambda' \leq \lambda'' < \Lambda^+$. Also, if $z_\lambda$ is defined as in the proof of Lemma 3.6, then $z_\lambda \leq y_\lambda$ and $z_\lambda$ is a subsolution of $(1.4)_\lambda$ for any $\lambda \in (0, \Lambda^+)$. Thus, as in the proof of Lemma 3.6, one can conclude that $(1.4)_\lambda$ has a solution $\tilde{u}_\lambda$ between $z_\lambda$ and $y_\lambda$ for $\lambda' \in (\lambda, \Lambda^+)$ which minimizes $J_\lambda$ on $[z_\lambda, y_\lambda]_{C^1}$, where

$$J_\lambda(u) = \frac{1}{p} \int_{\Omega} |Du|^p - \frac{1}{q + 1} \int_{\Omega} |u|^{q+1} - \int_{\partial \Omega} G(u), \quad G(u) = \int_0^u g(s) \, ds.$$ 

In particular, $J_\lambda(\tilde{u}_\lambda) \leq J_\lambda(z_\lambda)$. This implies that

$$J_\lambda(\tilde{u}_\lambda) \leq M = \max_{\Lambda^+/2 \leq \xi \leq \Lambda^+} J_\xi(z_\xi) \quad \text{for any } \lambda \in (\Lambda^+/2, \Lambda^+).$$

Since $\tilde{u}_\lambda$ is also a local minimizer of $J_\lambda$ in $W^{1, p}_0(\Omega)$, then (H3) (actually, we have that $J_\lambda$ satisfies the (PS) condition) implies that

$$\sup\{\|\tilde{u}_\lambda\|_{W^{1, p}_0(\Omega)}^\lambda; \Lambda^+/2 < \lambda < \Lambda^+\} < \infty.$$ 

The proof of Proposition 2.2 of [16] implies that $\tilde{u}_\lambda \in C^{1, \alpha}_0(\Omega)$ and $\|\tilde{u}_\lambda\|_{C^{1, \alpha}(\Omega)} \leq C$, where $C$ is independent of $\lambda$. Now choosing a sequence $\lambda_n \to \Lambda^+$ such that $\tilde{u}_{\lambda_n} \to u_{\Lambda^+}$, and passing to the limit in $(1.4)_\lambda$ with $(\lambda, u) = (\lambda_n, \tilde{u}_{\lambda_n})$, using (H1), we conclude that $u_{\Lambda^+}$ is a positive solution of $(1.4)_\lambda$ with $\lambda = \Lambda^+$. $u_{\Lambda^+}$ must be a positive solution since $u_{\Lambda^+} \geq z_{\Lambda^+}$. This implies that $(1.4)_\lambda$ has a minimal positive solution at $\lambda = \Lambda^+$.

(iii) can be obtained by a simple variation of the proof of Theorem 4.4. Note that (H1) and (H3) guarantee that we still have the (PS) condition and the mountain pass lemma applies as before.

(iv) can be obtained by arguments similar to those in the proof of Theorem 4.4. Now since we only consider the radial solutions of $(1.4)_\lambda$, we can write $(1.4)_\lambda$ to the form

$$-(r^{N-1}u')' + \lambda r^{N-1}|u|^{q-2}u + r^{N-1}g(u)$$

under the corresponding boundary conditions. Writing the operator $-\Delta_p$ in Theorem 4.4 in the radial form, we know the compactness of $-\Delta_p$ from [18]. The existence of sub- and supersolutions of $(1.4)_\lambda$ can also be obtained by arguments similar to those in the proof of Theorem 4.4. Now we only need to prove that all the positive and negative radial solutions of $(1.4)_\lambda$ are uniformly bounded in $C^0(\Omega)$. We only find the boundedness of the positive
solutions of (1.4). This can be obtained by a blow up argument as in [11,23] since we know from [13] that there is no bounded positive radial solution of the equation
\[-\Delta_p u = u^\gamma \quad \text{in } \mathbb{R}^N,\]
where \(\gamma\) is as in the assumption of (iv).

\[\square\]

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References

[32] M.A. del Pino, M. Elgueta, R. Manasevich, A homotopic deformation along $p$ of a Leray–Schauder degree result and existence for $(|u'|^{p−2}u')'+f(t,u)=0$, $u(0)=u(T)=0$, $p>1$, J. Differential Equations 80 (1989) 1–13.