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Marubayashi–Krull Orders and Strongly Graded Rings

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1. INTRODUCTION

The aim of this note is to give sufficient conditions for a strongly $\mathbb{Z}$-graded ring to be a Marubayashi–Krull order (cf. Theorem 3.10). Concrete examples of strongly $\mathbb{Z}$-graded rings are skew group rings over $\mathbb{Z}$ and generalized Rees rings. We also prove that under the same conditions the positive part of a strongly $\mathbb{Z}$-graded ring is a Marubayashi–Krull order (cf. Theorem 4.1). Finally, we examine when these conditions may be weakened.

2. PRELIMINARIES

A ring R is said to be a strongly $\mathbb{Z}$-graded ring if there exists a family of additive subgroups R_n $(n \in \mathbb{Z})$ of R such that $R = \bigoplus_{n\in\mathbb{Z}} R_n$ and $R_n R_m = R_{n+m}$ for all $n, m \in \mathbb{Z}$. The elements of $h(R) = \bigcup_{n\in\mathbb{Z}} R_n$ are called the homogeneous elements of R. If $x \in R_n$, then x is said to be homogeneous of degree n. If $x = \sum x_n$ $(x_n \in R_n)$, then the elements x_n are called the homogeneous components of x. Let I be an ideal of R. Denote by I^g the ideal generated by the homogeneous elements contained in I. I is said to be graded if $I = I^g$. In this case $I = RI_0 = I_0R$, where $I_0 = I \cap R_0$. Details about strongly graded rings may be found in [10].

If I is an ideal of R, then $C(I)$ denotes the set of elements regular modulo I. If $C(I)$ is a left and right Ore set in R, then $R_{C(I)}$ denotes the left and right localization of R towards this Ore set $C(I)$.

If R is a prime Goldie ring, then $Q(R)$, the classical ring of quotients of R, is a simple Artinian ring. The Asano overring of R is defined as $S(R) = \{x \in Q(R) \mid Ix \subset R$ for some nonzero ideal I of $R\}$. A prime Goldie ring R is said to be a *Marubayashi–Krull order* (or for short an M–*Krull order*) if:

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(1) there is a family of rings R_i ($i \in I$) such that $R = \bigcap_{i \in I} R_i \cap S(R)$;

(2) each R_i and $S(R)$ are essential overrings of R in the sense of [8];

(3) for all $i \in I$ R_i is a local Noetherian Asano order and $S(R)$ is a simple Noetherian ring;

(4) for every regular element c in R, $R_i c \neq R_i$ (resp. $cR_k \neq R_k$) for only finitely many i (resp. k) in I.

Another kind of noncommutative Krull ring is defined by M. Chamarie, which we will call a C–Krull ring. For the definition, we refer to [2] and [3]. In particular, if R is a C–Krull ring, then $R = (\bigcap_P R_{C(P)}) \cap S(R)$, where the first intersection runs through all prime divisorial ideals P of R, and each $R_{C(P)}$ is a local Noetherian Asano order (cf. [2]). We also mention

PROPOSITION 2.1. *Let R be a prime Goldie ring. The following are equivalent*:

(1) *R is an M–Krull order*;

(2) *R is a C–Krull ring, $S(R)$ is Noetherian and $S(R) = S(R)I = IS(R)$ for every ideal I of R.*

Proof. Cf. [3].

Finally, a prime Goldie ring R is said to be a maximal order if $(I :_l I) = (I :_r I) = R$. (If A and B are subsets of $Q(R)$, then $(A :_l B) = \{x \in Q(R) \mid xB \subset A\}$ and $(A :_r B) = \{x \in Q(R) \mid Bx \subset A\}$.) In this case, $(R :_l I) = (R :_r I)$. An ideal I is said to be divisorial if $I = I^*$, where $I^* = (R : (R : I))$. If R is a maximal order, then the set $\mathbb{D}(R)$ of nonzero divisorial ideals is a commutative group under $*$ where $A * B = (AB)^*$ if $A, B \in \mathbb{D}(R)$. In particular, if R satisfies the ascending chain condition on divisorial ideals contained in R, then $\mathbb{D}(R)$ is a free abelian group generated by the prime divisorial ideals of R.

3.

Let R be a strongly $\mathbb{Z}$-graded ring, and suppose that R_0 is a prime Goldie ring. Then it is clear that R is also a prime ring. Denote by $C_0 = \{x \in R_0 \mid x$ is regular in $R_0\}$. Then $C_0 = \{x \in R_0 \mid x$ is regular in $R\}$. For let $x \in R_0$ be regular in R_0 and suppose that $ax = 0$, $a \in R$. Write $a = \sum a_n$ ($a_n \in R_n$), then $a_n x = 0$ for all n. Fix n, then $R_{-n} a_n x = 0$ implies $R_{-n} a_n = 0$ and therefore $a_n \in R_0 a_n = R_n R_{-n} a_n = 0$. Since R_0 is a prime Goldie ring, C_0 is an Ore set in R_0. The next lemma proves that C_0 is also an Ore set in R.

LEMMA 3.1. *R satisfies the left (and right) Ore condition with respect to C_0.*

Proof. We need to prove that $Ra \cap C_0 r \neq \phi$ for all $a \in C_0$ and $r \in R$. Suppose first that r is homogeneous. Put $I = \{x \in R_0 \mid xr \in Ra\}$. It suffices to prove that I is an essential left ideal of R_0. Let $0 \neq J$ be a left ideal of R_0. If $Jr = 0$, then $J \subset I$. If $Jr \neq 0$ and $r \in R_n$, then $R_{-n}Jr$ is a nonzero left ideal of R_0. So $R_{-n}Jr \cap R_0 a \neq 0$. Let $0 \neq \alpha r = xa \in R_{-n}Jr \cap R_0 a$ $(\alpha \in R_{-n}J,$ $x \in R_0)$. Then $R_n \alpha \subset J$ and $R_n \alpha r \subset Ra$ whence $0 \neq R_n \alpha \subset I \cap J$. Therefore $I \cap C_0 \neq \phi$ and $Ra \cap C_0 r \neq \phi$.

Suppose now that $r \in R$ is arbitrary. Write $r = \sum_i r_i$. For all i, there exist $b_i \in R$ and $c_i \in C_0$ such that $b_i a = c_i r_i$ (by the first part of the proof). Since C_0 is an Ore set in R_0, all elements c_i have a common multiple c $(c \in C_0)$ so that $b_i' a = c r_i$ for all i and $b_i' \in R$. Then $(\sum b_i')a = cr \in Ra \cap C_0 r$.

In particular, the left and right rings of quotients of R with respect to C_0 exist and they coincide. Denote this ring by $Q^g = Q^g(R)$. Every element of Q^g can be written as $c^{-1}(\sum_i r_i) = \sum_i c^{-1} r_i$, $c \in C_0$, $r_i \in h(R)$ for all i. $Q^g(R)$ is again a strongly $\mathbb{Z}$-graded ring in the obvious way and its part of degree 0 equals $Q(R_0)$, the classical rings of quotients of R_0. Since $Q(R_0)$ is Artinian, hence Noetherian, $Q^g(R)$ is also Noetherian by Lemma 5.4 of [10]. Moreover, $Q^g(R)$ is a gr-Artinian ring, i.e., Artinian for graded left and right ideals. Therefore, all homogeneous regular elements of R are invertible in $Q^g(R)$. It follows that $Q^g(R)$ is also the quotient ring of R with respect to all the homogeneous regular elements of R. Finally, note that ideals of R extend to ideals of $Q^g(R)$ because $Q^g(R)$ is a Noetherian ring which is a localization of R towards an Ore set (cf. Theorem 1.31 of [4]).

LEMMA 3.2. *Let R be a strongly $\mathbb{Z}$-graded ring such that for every nonzero ideal I of R_0 $(I \neq R_0)$, we have that $R_1 I R_{-1} \neq I$. Then every ideal A of R can be written as $A = Ra$ and $a \in Z(R)$.*

Proof. Let A be a nonzero ideal of R. Let $0 \neq \alpha = \sum_{i=n}^{m} \alpha_i \in A$. Since $R_{-m}\alpha_m \neq 0$ there will exist an element $b \in R_{-m}$ such that $0 \neq b\alpha = \sum_{i=-p}^{0} b\alpha_i \in A$ $(p = m - n)$. Let p be the minimal positive integer such that there exists an $\alpha = \sum_{i=-p}^{0} \alpha_i \in A$. Denote by $\mathscr{A}$ the set of elements α_0 of R_0 such that there exists an $\alpha \in A$ with $\alpha = \sum_{i=-p}^{0} \alpha_i$. Then $\mathscr{A}$ is an ideal of R_0 such that $R_1 \mathscr{A} R_{-1} = \mathscr{A}$. By the hypothesis on R_0, $\mathscr{A} = R_0$, i.e., there exists an element $\alpha = \sum_{i=-p}^{0} \alpha_i \in A$ and $\alpha_0 = 1$. Let $\beta \in A$ be arbitrary. Write $\beta = \beta_l + \cdots + \beta_k$, β_i homogeneous. Because $\alpha_0 = 1$, we can devide β by α and we have $\beta = \gamma\alpha + r$, $\gamma \in R$, $r = \sum_{i=n'}^{m'} r_i$ and $m' - n' < p$. Therefore $r = 0$ and $A = R\alpha$. Finally, because $\alpha_0 = 1$, $x\alpha - \alpha x = 0$ for all $x \in R_m$ and all $m \in \mathbb{N}$, whence $\alpha \in Z(R)$.

Remark. The imposed condition on R_0 in Lemma 3.2 is clearly equivalent to saying that R is a gr-simple ring, i.e., R has only trivial graded ideals. In particular, if R_0 is a simple ring, this condition is fulfilled.

COROLLARY 3.3. *Under the conditions of Lemma 3.2 and assuming that R is a prime, left and right Noetherian ring, then R is an Asano order.*

COROLLARY 3.4. *If Q is a simple Noetherian ring, then every strongly $\mathbb{Z}$-graded ring R with $R_0 = Q$ is an Asano order. Moreover, every ideal is centrally generated. In particular, this holds for the skew group ring* $Q[X, X^{-1}, \sigma]$.

LEMMA 3.5. *Let R be a strongly $\mathbb{Z}$-graded ring such that R_0 and R are prime, left and right Noetherian, maximal orders.*

(1) *If P is a prime divisorial ideal of R such that $P^g = 0$, then* $R_{C(P)} = Q^g(R)_{C(Q^gP)}$.

(2) *Conversely, if P is a prime ideal of $Q^g(R)$, then* $Q^g(R)_{C(P)} = R_{C(P \cap R)}$.

Proof. (1) Since R is a prime Noetherian maximal order, R is a C–Krull ring. Let P be a prime divisorial ideal of R such that $P^g = 0$. By Proposition 4.1.4 of [1] Q^gP is a prime ideal of Q^g and $P = Q^gP \cap R$. Obviously, $C(P) = C(Q^gP) \cap R$. Since R is a C–Krull ring, $C(P)$ is an Ore set in R (cf. [2]). $C(Q^gP)$ is also an Ore set in Q^g because Q^g is an Asano order by Corollary 3.4 (cf. [6]). Then it is straightforward to check that $R_{C(P)} = Q^g_{C(Q^gP)}$.

(2) Conversely, let P be a prime ideal of Q^g, then P is divisorial (Q^g is an Asano order). Then $P \cap R$ is a prime ideal of R such that $(P \cap R)^g = 0$. We claim that $(Q^g : P) = Q^g(R : (P \cap R))$. One inclusion is clear. Let $\alpha P \subset Q^g$ whence $\alpha(P \cap R) \subset Q^g$. Write $P \cap R = a_1R + \cdots + a_nR$. Then $c\alpha a_i \subset R$ for all i and for some homogeneous regular element c in R. Hence $c\alpha(P \cap R) \subset R$ and $\alpha \in Q^g(R : (P \cap R))$. Therefore $(P \cap R)^* \subset Q^g(R : (R : (P \cap R))) \subset (Q^g : Q^g(R : (P \cap R))) = (Q^g : (Q^g : P)) = P$ whence $P \cap R$ is a divisorial ideal. The statement now follows from (1).

LEMMA 3.6. *Let R be a strongly $\mathbb{Z}$-graded ring such that R_0 is a prime Goldie ring which is a maximal order. Then R is a maximal order.*

Proof. Let I be an ideal of R and $\alpha \in Q(R)$ such that $\alpha I \subset I$. Then $\alpha IQ^g \subset IQ^g$. Since IQ^g is an ideal of Q^g and Q^g is an Asano order, we have $\alpha \in (IQ^g)(IQ^g)^{-1} = Q^g$. Write $\alpha = \sum_{i=-m}^{n} \alpha_i$, α_i homogeneous. Denote by $C(I)$ the (graded) ideal of R generated by the homogeneous components of highest degree of elements of I. Then $\alpha_n C(I) \subset C(I)$. Put $J = R\alpha_n R$. Then $JC(I) \subset C(I)$, $J_0 C(I)_0 \subset C(I)_0$ whence $J_0 \subset R_0$ and $\alpha_n \in J = RJ_0 \subset R$. By induction, $\alpha \in R$.

LEMMA 3.7. *Under the same conditions as in Lemma 3.6, if R_0 satisfies* *the ACC on divisorial ideals, then R satisfies the ACC on divisorial ideals contained in R.*

Proof. Similar to the proof of Proposition V.2.5 of [9].

The proof of the following lemma is adapted from the proof of a lemma of [11].

LEMMA 3.8. *Let R be a strongly $\mathbb{Z}$-graded ring, and R_0 and R semiprime Goldie rings. Let I be an essential left ideal of R. Then there exists an element $\alpha \in I$, $\alpha = \sum_{i=-m}^{n} \alpha_i$, such that α_n is regular.*

Proof. Put $J = \sum R_{-n}\alpha_n$ where the summation is taken over all α_n occurring as the homogeneous component of highest degree of an element $\alpha \in I$ and $R_{-n}\alpha_n \subset R_0$. Then J is a left ideal of R_0. In fact, J is the left ideal of R_0 consisting of the homogeneous components α_0 of highest degree of elements α of I and $\alpha_0 \in R_0$. We prove that J contains a regular element, which is equivalent to proving that J is an essential left ideal of R_0. Let A be a nonzero left ideal of R_0, then $I \cap RA \neq 0$ by the hypothesis. Let $0 \neq \alpha = \sum_{i=-m}^{n} \alpha_i \in I \cap RA$. Then $0 \neq R_{-n}\alpha_n \subset J$ and $R_{-n}\alpha_n \subset R_{-n}R_nA = A$ whence $J \cap A \neq 0$.

LEMMA 3.9. *Let R be a strongly $\mathbb{Z}$-graded ring such that for every ideal I of R_0 there exists a natural number $n > 0$ with $R_nIR_{-n} = I$. If R_0 is a bounded prime Goldie ring, then $S(R) = S(Q^g(R))$ is a simple Noetherian ring.*

Proof. If $\alpha \in S(R)$, then $I\alpha \subset R$ for some nonzero ideal I of R. Hence $Q^gI\alpha \subset Q^g$ and Q^gI is an ideal of Q^g. Therefore $\alpha \in S(Q^g)$. Conversely, let $I\alpha \subset Q^g$ and I an ideal of Q^g. By Lemma 3.2 and Corollary 3.4, $I = Q^gc$ for some $c \in Z(Q^g)$. So $c\alpha \in Q^g$. Then $J_1c\alpha = cJ_1\alpha \subset R$ for some ideal J_1 of R_0. Since $c \in Q^g$, $J_2c \subset R$ for some ideal J_2 of R_0. Therefore $Jc\alpha \subset R$ and $Jc \subset R$, where $J = J_2J_1$. Denote $J' = \bigcap_{n\in\mathbb{Z}} R_nJR_{-n}$. This is a finite intersection by hypothesis. Then $J' \subset J$ and $R_nJ'R_{-n} = J'$ for all $n \in \mathbb{Z}$. The reason for taking J' instead of J is that J' extends to an ideal RJ' of R. So $(RJ'c)\alpha \subset R$ and $RJ'c$ is an ideal of R whence $\alpha \in S(R)$. Finally, since Q^g is an Asano order, $S(Q^g)$ is a simple Noetherian ring by [6].

THEOREM 3.10. *Let R be a strongly $\mathbb{Z}$-graded ring such that for every ideal I of R_0 there exists a natural number $n > 0$ with $R_nIR_{-n} = I$. If R_0 is a bounded M–Krull order, then R is an M–Krull order.*

Proof. Since R_0 is a bounded M–Krull order, we have $R_0 = \bigcap_{P\in X^1(R_0)} R_{0,C(P)}$, where $X^1(R_0)$ stands for the set of height one prime ideals of R_0. If $P_1 \in X^1(R_0)$, then $R_nP_1R_{-n}$ is also a height one prime ideal of R_0 for all $n \in \mathbb{Z}$. By the hypothesis, P_1 has only finitely many "conjugates" P_1, $R_1P_1R_{-1} = P_2$, $R_2P_1R_{-2} = P_3, \ldots, R_nP_1R_{-n} = P_1$.

Consider the ring $\bigcap_{i=1}^{n} R_{0,C(P_i)}$. By Lemma 3.1 of [7], $\bigcap_{i=1}^{n} R_{0,C(P_i)} = R_{0,C(A)} = \{x \in Q(R_0) \mid Bx \subset R$ for some ideal B of R_0 such that $B \not\subset P_i$ for all i with $1 \leqslant i \leqslant n\}$ and $C(A) = C(P_1 \cap \cdots \cap P_n) = C(P_1) \cap \cdots \cap C(P_n)$. Moreover, by Lemma 3.3 of [7], $R_{0,C(A)}$ is a bounded Dedekind prime ring with maximal ideals $R_{0,C(A)}P_1, ..., R_{0,C(A)}P_n$. Then it is easy to check that for all $m \in \mathbb{N}$, $R_m R_{0,C(A)} R_{-m} = R_{0,C(A)}$.

For each $P = P_1 \in X^1(R_0)$, consider all the conjugate prime ieals $R_k P R_{-k} = P_{k+1} (1 \leqslant k \leqslant n-1)$ and form the intersection $\bigcap_{i=1}^{n} R_{0,C(P_i)}$. Denote this ring by $L_{0,P}$ $(=L_{0,P_1} = \cdots = L_{0,P_n})$. By the foregoing, $RL_{0,P} = L_{0,P}R$ is a strongly graded ring contained in $Q^g(R)$. We denote this ring by L_P. Moreover, $R = \bigcap_P L_P$. Note that L_P is a maximal order by Lemma 3.6 and L_P is Noetherian by Lemma 5.4 of [10]. Therefore L_P is a C–Krull ring. We have $L_P = \bigcap_k L_{P,k} \cap S(L_P)$, where $L_{P,k}$ is the localization of L_P towards a prime divisorial L_P-ideal and the intersection is taken over all prime divisorial ideals of L_P. So $R = \bigcap_{P,k} L_{P,k} \cap (\bigcap_P S(L_P))$. Since $L_{P,k}$ (resp. L_P) is a localization of L_P (resp. R) towards an Ore set, $L_{P,k}$ is also a localization of R towards an Ore set, in particular $L_{P,k}$ is an essential overring of R. Moreover, $S(R) = S(L_P) = S(Q^g(R))$ for all P by Lemma 3.9. If I is an ideal of R, then $S(R)I = S(Q^g)\, Q^g I = S(Q^g)$ because $Q^g I$ is an ideal of Q^g and Q^g is an Asano order. Then Lemma 2.2 of [8] proves that $S(R)$ is an essential overring of R.

Finally, we still need to prove the finite character property, i.e., if c is a regular element of R, then $cL_{P,k} = L_{P,k}$ and $L_{P,k}c = L_{P,k}$ for almost all P, k (i.e., except finitely many). For this purpose, we will rewrite the rings L_P in a somewhat different form. First, recall that L_P is a C–Krull ring. If P'' is a prime divisorial ideal of $L_P = RL_{0,P}$, then either $P''^g = 0$ or $P'' = P''^g$. Suppose that $P'' = P''^g$. Then $P'' = L_P P_0''$, where $P_0'' = P'' \cap L_{0,P}$. Clearly $R_m P_0'' R_{-m} = P_0''$ for all $m \in \mathbb{Z}$. Since $P_1' = L_{0,P}P_1, ..., P_n' = L_{0,P}P_n$ are the maximal ideals of $L_{0,P}$ (cf. the first part of the proof, or Lemma 3.3 of [7]), $P_0'' = P_1'^{k_1} \cdots P_n'^{k_n}$, $k_i \in \mathbb{N}$. From the fact that $P_1', ..., P_n'$ are conjugate and $R_m P_0'' R_{-m} = P_0''$ for all m, we derive that $k_1 = \cdots = k_n$. Moreover, $k_1 = 1$ since P'' is a prime ideal of L_P. Therefore $P'' = L_P P_1' \cdots P_n' = L_P P_1 \cdots P_n$. In particular, each L_P has only one prime divisorial ideal which is graded. If $P''^g = 0$, then $(L_P)_{C(P'')} = Q^g_{C(Q^g P'')}$ by Lemma 3.5. Therefore, by Lemma 3.5 and Lemma 3.9, $L_P = (L_P)_{C(P'')} \cap Q^g$, where P'' is the only graded prime divisorial ideal of L_P. So $R = \bigcap (L_P)_{C(P'')} \cap Q^g$. Now, let α be a regular element in R. By Lemma 3.8, there is a $\gamma = \beta\alpha \in R$, $\gamma = \sum_{i=-n}^{0} \gamma_i$ with γ_0 regular. Since R_0 is an M–Krull order, $\gamma_0 \in C(P_i)$ for almost all $P_i \in X^1(R_0)$. Clearly $\gamma_0 \in C(P_1 \cap \cdots \cap P_n)$ for almost all $P_1, ..., P_n$, where $P_1 \cap \cdots \cap P_n$ is the intersection of P_1 and its conjugates. We also have that $\gamma_0 \in C(P_1' \cap \cdots \cap P_n') = C(P_1' \cdots P_n')$ (note that $P_1' \cap \cdots \cap P_n' = P_1' \cdots P_n'$ because $L_{0,P}$ is an Asano order). We claim that $\gamma \in C(L_P P_1' \cdots P_n')$. Let $\delta = \sum_{i=-p}^{q} \delta_i \in L_P$ and $\gamma\delta \in L_P P_1' \cdots P_n'$. Then $\gamma_0 \delta_q R_{-q} \subset P_1' \cdots P_n'$ whence

$\delta_q R_{-q} \subset P'_1 \cdots P'_n$ and $\delta_q \in L_P P'_1 \cdots P'_n$. By induction, $\delta \in L_P P'_1 \cdots P'_n$. This proves that $\gamma \in C(L_P P'_1 \cdots P'_n)$. Therefore $(L_P)_{C(P'')} \alpha = (L_P)_{C(P'')}$ for almost all graded prime divisorial ideals. Finally, since Q^g is an Asano order, the finite character property holds.

Remark. In the above Theorem it suffices to assume that every divisorial ideal I (instead of every ideal) has only finitely many conjugates, i.e., there exists a natural number n with $R_n I R_{-n} = I$. This is because in the proof of Theorem 3.10 we only require that a prime divisorial ideal has finitely many conjugates. Also the proof of Lemma 3.9 needs to be slightly adapted: when $Jca \subset R$ then also $J^*ca \subset R$ and $J^* = (R_0 : (R_0 : J))$ is divisorial. The rest of this proof remains the same.

Before giving some concrete examples we will concern ourselves with the question whether the hypotheses of Theorem 3.10 may be weakened. First we will show that the condition that every nonzero ideal of R_0 has only finitely many conjugates cannot be removed.

EXAMPLE 3.11. Let K be a field. Consider the polynomial ring in infinitely many variables $R = K[(Y_i)_{i \in \mathbb{Z}}]$. Let L denote the field of quotients of R. Denote by σ the automorphism of R which leaves K elementwise fixed and sends Y_i to Y_{i+1} for all $i \in \mathbb{Z}$. Clearly σ is an automorphism of infinite order. We will prove that the skew group ring $A = R[X, X^{-1}, \sigma]$ is not an M–Krull order.

Since R is a unique factorization domain, every height one prime ideal of R is a principal ideal. If $P = Rf(Y) \in X^1(R)$ $(Y = (Y_i)_{i \in \mathbb{Z}})$, then it is clear that $\sigma^n(P) \neq P$ for all $n \in \mathbb{N}$.

Since R is a Krull domain, A is a C–Krull ring by Lemma 2.3 and Proposition 3.3 of [2]. Therefore $A = \bigcap A_{C(P)} \cap S(A)$, where the first intersection is taken over all prime divisorial ideals of A. Let P be a prime divisorial ideal of A. Then P^g is also a prime divisorial ideal contained in P. Hence $P^g = 0$ or $P = P^g$. Suppose first that $P = P^g$. Then $P = P'[X, X^{-1}, \sigma]$, where $P' = P \cap R$ is a divisorial ideal of R and $\sigma(P') = P'$. So $P' = \prod_* Q_i^{n_i}$ $(Q_i \in X^1(R),\ n_i \in \mathbb{Z})$ because R is a Krull domain. We have $\sigma(P') = P' = \prod_* \sigma(Q_i)^{n_i}$. By the unique decomposition in height one prime ideals, all the finitely many Q_i are conjugated under each other by σ. Since $\sigma^n(P') = P'$ for all n, there will exist a natural number m_i (for each i) such that $\sigma^{m_i}(Q_i) = Q_i$. This is a contradiction by the foregoing. Therefore, A has no prime divisorial ideals P such that $P = P^g$. Suppose now that $P^g = 0$. We will first study the ideals of $S = L[X, X^{-1}, \sigma]$. By Corollary 3.4 (or by results of [1]), every ideal of S is a principal ideal, generated by a central element. A straightforward computation shows that the center of S is L^σ, the fixed field of L under σ. It follows that S is a simple Noetherian ring. Now, A/P is a prime Goldie ring by Lemma 1.1 of [2]. An easy adaption of

Proposition 4.1.4 of [1] shows that SP is an ideal of S such that $P = SP \cap A$. Hence $P = A$. Therefore A has no prime divisorial ideals. Since $A = \bigcap_P A_{C(P)} \cap S(A)$, we derive that $A = S(A)$. But $A = S(A)$ is not Noetherian, since R is not Noetherian. This proves that A is not an M–Krull order.

Problem 1. Can the boundedness condition be omitted in Theorem 3.10? The next proposition proves that this is related to

Problem 2. If R is a strongly $\mathbb{Z}$-graded ring and R_0 a C–Krull ring, is R a C–Krull ring?

PROPOSITION 3.12. *Let R be a strongly $\mathbb{Z}$-graded ring such that for every ideal I of R_0 there exists a natural number $n > 0$ such that $R_n I R_{-n} = I$. Suppose that R_0 is an M–Krull order. If R is a C–Krull ring, then R is an M–Krull ring.*

Proof. Define $\bar{S}_0 = \bigcup (R : I)$, where the union is taken over all nonzero ideals of R_0 such that $R_n I R_{-n} = I$ for all $n \in \mathbb{Z}$. Clearly $\bar{S}_0 \subset S_0$, where $S_0 = S(R_0)$. Conversely, let $\alpha \in S_0$. Then $I\alpha \subset R_0$ for some nonzero ideal I of R_0. Put $J = \bigcap_{n \in \mathbb{Z}} R_n I R_{-n}$. Then $R_n J R_{-n} = J$ for all $n \in \mathbb{Z}$ and $J\alpha \subset R_0$, i.e., $\alpha \in \bar{S}_0$, hence $S_0 = \bar{S}_0$. In particular, $R_n S_0 = S_0 R_n$ for all $n > 0$ and $RS_0 = S_0 R$ is a strongly $\mathbb{Z}$-graded ring. By Corollary 3.4, RS_0 is an Asano order since S_0 is a simple Noetherian ring. It is also easy to check that $RS_0 = S^g(R) = S^g = \{\alpha \in Q^g(R) \mid I\alpha \subset R$ for some ideal I of R with $I = I^g\}$.

We will now prove that $S(R) = S(S^g(R))$. First, let P be a prime divisorial ideal of R. If $P = P^g$, then $P = RP_0 = P_0 R$ whence $S^g P = RS_0 P_0 R = RS_0 = S^g$. If $P^g = 0$, then $P' = Q^g P = PQ^g$ is a prime ideal of Q^g and $P = P' \cap R$. This is proved as in Example 3.11. Put $P'' = P' \cap S^g$. Then $P'' \cap R = P$ and $P'' = S^g P = PS^g$ since S^g is an essential overring of R. Let $\alpha \in S(R)$, so that $I\alpha \subset R$ for some divisorial ideal I of R. Write $I = (\prod_* P_i^{n_i}) * (\prod_* P_j'^{m_j})$, where $P_i = P_i^g$ and $P_j'^g = 0$. Then $S^g(\prod P_i^{n_i})(\prod P_j'^{m_j})\alpha \subset S^g$ whence $(\prod S^g P_j'^{m_j})\alpha \subset S^g$ and $\alpha \in S(S^g(R))$. Conversely, let $\alpha \in S(S^g)$, so that $(I \cap R)\alpha \subset S^g$ for some nonzero ideal I of S^g. Since R satisfies the ACC on left divisorial ideals, we may write $J^* = (Ra_1 + \cdots + Ra_n)^*$, where $J = I \cap R$. So $a_i \alpha \in S^g$ for all i. As before $Ca_i \alpha \subset R$ for some ideal C of R_0 with $R_n C R_{-n} = C$ for all $n \in \mathbb{N}$. Therefore $(RC)(I \cap R)\alpha \subset R$ and RC is an ideal of R. This proves that $S(R) = S(S^g(R))$, which is a simple Noetherian ring because $S^g(R)$ is an Asano order. Moreover, $S(R)$ is an essential overring of R: $S(R) = Q_\kappa^l(R)$, where $\mathscr{L}(\kappa)$ is the filter of left ideals of R generated by the ideals $C(A \cap R)$, where C is a graded ideal of R and A is an ideal of S^g. Then $\mathscr{L}(\kappa)$ is the set of left ideals of R such that $S(R)I = S(R)$ (and $IS(R) = S(R)$). By Proposition 2.1

of [8] ideals of R extend to ideals of $S(R)$. Finally, Proposition 2.1 yields that R is an M–Krull order.

COROLLARY 3.13. *Let R be an M–Krull order and σ an automorphism of R such that some power of the extension of σ to $Q(R)$ is an inner automorphism of $Q(R)$. Then $R[X, X^{-1}, \sigma]$ is an M–Krull order.*

Proof. We already know that $R[X, X^{-1}, \sigma]$ is a C–Krull ring by Lemma 2.3 and Proposition 3.3 of [2]. By the foregoing proposition and the remark after Theorem 3.10, it remains to prove that if I is a divisorial R-ideal, then $\sigma^n(I) = I$ for some $n > 0$. Clearly, σ can be extended to $Q(R)$. Denote this extension again by σ. By hypothesis, there exists for some $n > 0$ an element $u \in U(Q(R))$ such that for all $x \in Q(R)$ $\sigma^n(x) = uxu^{-1}$. In particular, $R = \sigma^n(R) = uRu^{-1}$, which implies that $Ru = uR$ is an invertible R-ideal. If I is a divisorial R-ideal, then $I^*(Ru) = Iu = (uR)^*I = uI$. Therefore $\sigma^n(I) = uIu^{-1} = I$.

Remarks. (1) Note that we did not need to assume that R is bounded in Corollary 3.13.

(2) In general, Example 3.11 shows that some finiteness condition on σ is needed in order to show that a skew group ring $R[X, X^{-1}, \sigma]$ is an M–Krull order. This does not imply that σ cannot be of infinite order if $R[X, X^{-1}, \sigma]$ is an M–Krull order. For instance, if K is a field and σ an automorphism of K of infinite order, then as shown in Example 3.11, $K[X, X^{-1}, \sigma]$ is a simple Noetherian ring, in particular an M–Krull ring.

EXAMPLE 3.14. Let R be a bounded M–Krull order and let I be an invertible ideal of R. If J is a divisorial ideal of R, then $IJ = JI$, hence $IJI^{-1} = J$. Therefore the generalized Rees ring $\check{R}(I) = \oplus_{n \in \mathbb{Z}} I^n X^n \subset Q(R)[X, X^{-1}]$ is an M–Krull order.

4.

Finally, we will prove that under certain conditions the positive part of a strongly graded M–Krull order is again an M–Krull order. We start with a few observations.

(1) If R is a strongly $\mathbb{Z}$-graded ring such that R_0 is Artinian, then R is a crossed product $R_0 * \mathbb{Z}$ by Corollary 3.4 of [10]. Since $\mathbb{Z}$ is a free group, it follows that R is a skew group ring $R[X, X^{-1}, \sigma]$.

(2) Let R be a strongly $\mathbb{Z}$-graded ring such that R_0 is a prime Goldie ring. By Lemma 3.1 R is embedded in $Q^g(R) = \{c^{-1}r \mid r \in R,\ c \in C_0\}$, where $C_0 = \{x \in R_0 \mid x \text{ regular}\}$. Moreover, by (1), $Q^g(R) = RQ(R_0) =$

$Q(R_0)[X, X^{-1}, \sigma]$ for some automorphism σ of $Q(R_0)$. So $Q^g(R)$ contains a regular element X^n for every degree $n \in \mathbb{Z}$. By multiplying each X^n with an appropriate homogeneous regular element of degree 0, we obtain that for all $n \in \mathbb{Z}$ R_n contains a regular element.

THEOREM 4.1. *Let R be a strongly $\mathbb{Z}$-graded ring such that for every ideal I of R_0 there exists a natural number $n > 0$ with $R_n I R_{-n} = I$. If R_0 is a bounded M–Krull order, then $R_+ = \oplus_{n \geq 0} R_n$ is an M–Krull order.*

Proof. We already know that $Q^g(R) = Q(R_0)[X, X^{-1}, \sigma]$. Denote by $Q^g(R)_+ = Q^g_+ = \oplus_{n \geq 0} Q(R_0)R = Q(R_0)[X, \sigma]$. Then $R_+ = Q^g_+ \cap R$. Clearly Q^g_+ equals the localization of R_+ towards the Ore set C_0 (C_0 is an Ore set in R, and hence also in R_+). Moreover, $Q^g_+ = Q(R_0)[X, \sigma]$ is an Asano order by [1]. Therefore Q^g_+ is equal to an intersection of local Noetherian Asano orders (which are localizations of Q^g_+ and hence also of R_+) and $S(Q^g_+)$.

By Theorem 3.10 R is an M–Krull order. Write $R = \bigcap_P R_{C(P)} \cap S(R)$, where the first intersection is taken over all prime divisorial ideals P of R. We prove that each $R_{C(P)}$ is an essential overring of R_+. Clearly $R_{C(P)} = Q^l_\sigma(R)$, where $\mathscr{L}(\sigma) = \{I \mid I$ a left ideal of R such that $(I : r) \cap C(P) \neq \phi$ for all $r \in R\}$. Define $\mathscr{L}(\sigma_+) = \{I \mid I$ is a left ideal of R_+ such that $Q_\sigma(R)I = Q_\sigma(R)\}$. We claim that σ_+ is an idempotent kernel functor (in the sense of Goldman, cf. [5]; the notation $Q_\sigma(R)$ is also meant as in [5]). Let $I \in \mathscr{L}(\sigma_+)$ so that $Q_\sigma(R)I = Q_\sigma(R)$. Write $1 = \sum \alpha_j a_j$, $\alpha_j \in Q_\sigma(R)$, $a_j \in I$. Then $\alpha_j = q^{-1}\beta_j$, $\beta_j \in R$, $q \in C(P)$, whence $q = \sum \beta_j a_j$. For some $n > 0$, $R_n \beta_j \subset R_+$ for all j so that $R_+(R_n q)$ is a left ideal of R_+ contained in I. It follows from $Q_\sigma(R) R_+ R_n q = Q_\sigma(R)$ that $R_+(R_n q) \in \mathscr{L}(\sigma_+)$. This proves that $\mathscr{L}(\sigma_+)$ is generated by the left ideals $R_+(R_n q)$, where $q \in C(P)$ and $R_n q \subset R_+$. It is now easy to deduce that if $I \in \mathscr{L}(\sigma_+)$ and $I \subset J$ then $J \in \mathscr{L}(\sigma_+)$; if $I, J \in \mathscr{L}(\sigma_+)$ then $I \cap J \in \mathscr{L}(\sigma_+)$; if $I \in \mathscr{L}(\sigma_+)$ and $r \in R_+$ then $(I : r) = \{x \in R_+ \mid xr \in I\} \in \mathscr{L}(\sigma_+)$. Finally, we have to check that if $I \subset K$ are left ideals of R_+, $K \in \mathscr{L}(\sigma_+)$ and K/I is σ_+-torsion, i.e., for all $k \in K$ there is a $J_k \in \mathscr{L}(\sigma_+)$ such that $J_k \cdot k \subset I$, then $I \in \mathscr{L}(\sigma_+)$. It is clear that $RK \in \mathscr{L}(\sigma)$ and RK/RI is σ-torsion. Therefore $RI \in \mathscr{L}(\sigma)$ because σ is an idempotent kernel functor. So $Q_\sigma(R)RI = Q_\sigma(R)I = Q_\sigma(R)$ whence $I \in \mathscr{L}(\sigma_+)$. Moreover, it is also clear that $Q_\sigma(R) = Q_{\sigma_+}(R_+)$.

From the equation $R_+ = Q^g_+ \cap R$ and the foregoing, we know that R_+ is an intersection of local Noetherian Asano orders (which are essential overrings of R_+) intersected with $S(Q^g_+)$ and $S(R)$. Therefore it remains to prove that $S(R_+) = S(Q^g_+) = S(R)$.

(1) $S(R) = S(Q^g(R))$: this is already proved in Theorem 3.10.

(2) $S(Q^g) = S(Q^g_+)$: let $\alpha \in S(Q^g_+)$, i.e, $I\alpha \subset Q^g_+$, I an ideal of $Q^g_+ = Q(R_0)[X, \sigma]$. By [1], $I = Q^g_+ X^m f(X)$, $f(X) \in Z(Q^g_+)$. Therefore

$Q^gI = Q^gf(X)$ is an ideal of Q^g and $Q^gIa \subset Q^g$. So $a \in S(Q^g)$. Conversely, let $a \in S(Q^g)$, i.e., $Ia \subset Q^g$ for some ideal I of Q^g. Hence $(I \cap Q^g_+)a \subset Q^g$. By [1], $I \cap Q^g_+ = Q^g_+ X^m g(X)$, $g(X) \in Z(Q^g_+)$. So $X^m g(X)a \subset Q^g$. For some $n > 0$, $Q^g(X^n X^m g(X))a \subset Q^g_+$ whence $a \in S(Q^g_+)$.

(3) $S(R_+) = S(Q^g_+)$: if $a \in S(R_+)$, then $Ia \subset R_+$ for some ideal I of R_+. Since Q^g_+ is the localization of R_+ towards an Ore set and Q^g_+ is Noetherian, $Q^g_+ I$ is an ideal of Q^g_+ whence $Q^g_+ Ia \subset Q^g_+$ and $a \in S(Q^g_+)$. Conversely, let $a \in S(Q^g_+) = S(Q^g)$, then $Ia \subset Q^g$ for some ideal I of Q^g. Since every ideal of Q^g is centrally generated (Corollary 3.4), $aa \in Q^g$ for some $a \in Z(Q^g)$. Then $caa \in R$ and $ca \in R$ for some $c \in C_0$. Because R_0 is bounded, $Jaa \subset R$ and $Ja \subset R$ for some ideal J of R_0. As before, by hypothesis, we may assume that $R_n J = JR_n$ for all $n \in \mathbb{Z}$. Then $R_+ R_n Jaa \subset R_+$ and $R_+(R_n Ja) \subset R_+$ for some $n > 0$. Since $R_n J = JR_n$ for all n and a is central, $R_+(R_n Ja)$ is an ideal of R_+. This proves that $S(R_+) = S(Q^g_+)$.

Finally, let I be an ideal of R_+. Because $Q^g_+ I$ is an ideal of Q^g_+ and Q^g_+ is an Asano order, we have that $S(R_+)I = S(Q^g_+)\, Q^g_+ I = S(Q^g_+) = S(R_+)$. Then Lemma 2.2 of [8] proves that $S(R_+)$ is an essential overring of R_+ and $S(R_+)$ is a simple Noetherian ring.

Example 4.2. Let R be a bounded M–Krull order and I an invertible ideal of R. Then the ring $\oplus_{n \geqslant 0} I^n X^n$ ($\subset R[X]$) is an M–Krull order.

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