Available online at www.sciencedirect.com
SciVerse ScienceDirect
Procedia Engineering 31 (2012) 1050 – 1059
Procedia Engineering
www.elsevier.com/locate/procedia

International Conference on Advances in Computational Modeling and Simulation

A splitting mixed space-time discontinuous Galerkin method for parabolic problems

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Abstract

A splitting mixed space-time discontinuous Galerkin method is formulated to solve a class of parabolic problems. This method, in which the stress equation is separated from displacement equation, is based on mixed method and space-time discontinuous finite element method which is discontinuous in time and continuous in space. By a splitting technique, the stress equation is separated from the stress-displacement coupled system. The finite element approximation of the stress is solved by time discontinuous Galerkin method with high accuracy. Then, if required, the discrete displacement function is also solved by the time discontinuous Galerkin method. The convergence of the scheme is analyzed by the technique of combining finite difference and finite element methods. The optimal priori error estimates in $L^\infty(L^2)$ norm for displacement and in $L^\infty(L^2)$ norm and $L^2(H(\mathrm{div}))$ norm for stress are derived, respectively. Numerical experiments are presented to confirm theoretical results.

Keywords: space-time discontinuous Galerkin method, splitting mixed method, parabolic problem, error estimates;

1. Introduction

Time discontinuous space-time Galerkin method (TDG), in which the spatial and temporal variables are considered simultaneously, is an important finite element method to solve the time dependent problems [1-4]. In some applications, one is interested in both a vector variable (e.g., stress or strain) and the solution itself (e.g., displacement). For this purpose, Yu [5] formulated a mixed space-time discontinuous Galerkin method (MTDG), based on TDG method and H^1-Galerkin mixed method, to study the linear Sobolev equation. By combining mixed finite element method with TDG method, Liu et

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doi:10.1016/j.proeng.2012.01.1141

al. [6] and He and Li [7] also discussed the convergence of the MTDG method for the convection diffusion problem and fourth-order parabolic equations, respectively.

MTDG method allows the solution itself and the additional variables are solved in their different finite element spaces simultaneously. However, there will be usually necessary to solve a coupled system at each time step in the discrete scheme of MTDG and thus the difficulty of solving the system is increased. Therefore, how to design an efficient algorithm to reduce computing cost is very significant in practice.

In this paper, we formulate a splitting mixed space-time discontinuous Galerkin procedure (SMTDG) by using a splitting technique [8, 9] to solve a class of parabolic initial boundary problem:

$$\begin{cases} (a)\ u_t - \nabla\cdot(A(x,t)\nabla u) = f(x,t), & (x,t)\in\Omega\times J, \\ (b)\ u(x,t) = 0, & (x,t)\in\partial\Omega\times J, \\ (c)\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

where $\Omega\in R^d(d=1$ or 2 is the space dimension) is a bounded convex polygon domain with a smooth boundary $\partial\Omega$ and $J=[0,T]$. Assume that the matrix $A(t,x)=[a_{k\ell}(t,x)]_{d\times d}$ is symmetric with the property that $\alpha_0||\xi||^2\le\xi^TA\xi\le M||\xi||^2, 0\ne\xi\in R^d$, for some positive constants α_0, M. Furthermore, assume that the coefficients $a_{k\ell}$ of matrix $A(x,t)$ and functions u_0, f are bounded and sufficiently smooth.

The SMTDG scheme is formulated as follows. Firstly, the equation (1(a)) is written as a coupled system of stress '$-A\nabla u$' and displacement 'u' by introducing an auxiliary vector variable '$\sigma=A\nabla u$' as in the classical mixed finite element methods. Secondly, a first-order mixed system, which is equivalent to the couple system, is obtained by differentiating both hand sides of equation $\widetilde{A}\sigma=\nabla u$, (where$\widetilde{A}=A^{-1}$) with respect to time. And then, one can solve the first-order mixed system through use of TDG method respectively. The SMTDG scheme of the parabolic problem is then formulated directly in view of the commuting property [0] of $\mathrm{div}(H(\mathrm{div};\Omega))=L^2(\Omega)$. SMTDG method takes full advantages of both the mixed finite element method and the TDG method. In this procedure, the subsystem of solving discrete stress function is separated from the coupled system of displacement-stress functions and the approximation function of the stress can be solved by TDG method with high accuracy. By comparison with MTDG method, the scale of the original problem and the difficulty of solving system are reduced to some extent and the finite element spaces can be selected flexibly in view of computing point in SMTDG method.

The article is organized as follows. Preliminaries needed in this paper are given in Section 2. A splitting mixed space-time discontinuous finite element procedure and its Radau quadrature form are formulated in Section 3. In Section 4, the convergence of the scheme is proved by using the technique developed in [2]. An optimal priori error estimate in $L^\infty(L^2)$norm for displacement (i.e.,u) and in $L^\infty(L^2)$ and $L^2(H(\mathrm{div}))$ norm for stress (i.e., $\sigma=A\nabla u$) are obtained, respectively. Numerical experiments are presented in Section 5 to confirm theoretical results and conclusion is given in Section 6.

2. Preliminaries

First, we discretize the space-time domain $Q=\Omega\times J$. Let $0=t^0<t^1<\cdots<t^N=T$ be a subdivision of J into intervals $I_n=(t^n,t^{n+1}]$, and $k_n=t^{n+1}-t^n, n=0,\cdots,N-1$. In space-time slab $Q_n=I_n\times\Omega$, let T_σ^n and T_u^n be two different quasi-uniform partitions of Ω, with h_σ and h_u be the maximum values of the diameters, respectively. In each space-time slab Q_n, the partitions may not be the same, that is, on the common interface $t=t^n$, $(1\le n\le N-1)$ of T_σ^n(or T_u^n) and T_σ^{n-1}(or T_u^{n-1}), the nodes can be chosen differently.

The standard notations for Sobolev spaces and their norms are used. We shall also use some other norms as follows: Denote by $||\phi||_{L^2(Y;X)}=||\ ||\phi(\cdot,t)||_X||_Y$ and $||\psi||_{L^\infty(Y;X)}=\sup_{t\in Y}||\psi(\cdot,t)||_X$, where $||\cdot||_X$ is the norm of Hilbert space X and $Y=J$ or I_n . For simplicity, set $|||\phi|||_n=||\phi||_{L^2(I_n;L^2(\Omega))}$ and

$(\phi,\psi)_n=\int_{I_n}(\phi,\psi)dt$ for $X=L^2(\Omega)$ and $Y=I_n$, and set $\max_{I_n}||\psi(t)||_s=\max_{t^n\le t\le t^{n+1}}||\psi(t)||_s$, for $X=H^s(\Omega)$ and $Y=I_n$. For $s,m=0,1,\cdots$, and $v\in H^m(\Omega)$ (or $\left(H^m(\Omega)\right)^d$), define discrete norm as

$$||h_\sigma v||^2_{m,h}=\sum_{K_\sigma\in T^n_\sigma}h_1^{2s}||v||^2_{m,K_\sigma},\quad ||h_u v||^2_{m,h}=\sum_{K_u\in T^n_u}h_2^{2s}||v||^2_{m,K_u},$$

Let $(X)^q$ be a finite dimensional Hilbert vector space equipped with the inner product

$$((\Phi,\Psi))=\sum_{i=1}^{q}(\phi_i,\psi_i),\quad \Phi=(\phi_1,\cdots,\phi_q)^T,\ \ \Psi=(\psi_1,\cdots,\psi_q)^T\in\left(X\right)^q,$$

and with norm $(\sum_{i=1}^q||\psi_i||^2)^{1/2}$ denoted by $|||\cdot|||$.

The Radau quadrature rule is employed to compute the time integrations for the discrete systems of TDG scheme. For each fixed integer $q\ge 1$, let $\{\ell_i(\tau)\}_{i=1}^q$ be the Lagrange polynomials associated with the abscissae $0<\tau_1<\cdots<\tau_q=1$, that is $\ell_i(\tau)=\Pi_{j=1,j\ne i}^q(\tau-\tau_j)/(\tau_i-\tau_j)$. Then the Radau quadrature rule can be cited as follows: for a given function $g(\tau),\tau\in(0,1]$

$$\int_0^1 g(\tau)d\tau\approx\sum_{j=1}^{q}\omega_j g(\tau_j),\quad 0<\tau_1<\tau_2<\cdots<\tau_q=1,$$

which is exact for all polynomials of degree $\le 2q-2$. Further, by the linear transformation $t=t^n+\tau k_n$ that maps interval $[0,1]$ onto the interval $\bar{I}_n$, we have

$$t^{n,i}=t^n+\tau_i k_n,\ \ t^{n,q}=t^{n+1},\ \ \ell_{n,i}(t)=\ell_i(\tau),\ \ t=t^n+\tau k_n, \tag{2}$$
$$w_{n,i}=\int_{t^n}^{t^{n+1}}\ell_{n,i}(t)dt=k_n\int_0^1\ell_i(\tau)d\tau=k_n\omega_i,\quad i=1,\cdots,q.$$

In the following part of this section, we will introduce a lemma, which is important for the convergence analysis in the later. Define $q\times q$ matrices $\mathscr{N},\mathscr{M}$ independent of k_n by $\mathscr{N}_{ij}=w_{n,j}\ell'_{n,i}(t^{n,j})=\omega_j\ell'_i(\tau_j)$, $\mathscr{M}=e_qe_q^T-\mathscr{N}, e_q^T=(0,\cdots,0,1)\in R^q$. Obviously, if $Y^T=(y^{n,1},y^{n,2},\cdots,y^{n,q})\in R^q$, then

$$Y^T\mathscr{M}Y=\sum_{i=1}^{q}\delta_{qi}y^{n,q}y^{n,i}-\sum_{i,j=1}^{q}w_{n,j}\ell'_{n,i}(t^{n,j})y^{n,j}y^{n,i},$$

and matrix $D^{-1/2}\mathscr{M}D^{1/2}$ is positive definite [2], where $D=\mathrm{diag}\{\tau_1,\tau_2,\cdots,\tau_q\}$.

Lemma 1 [2] Assume that $\widetilde{\mathscr{M}}=D^{-1/2}\mathscr{M}D^{1/2}$, then there exists $\lambda:=\frac{\alpha_0}{2}\min\{\frac{\omega_1}{\tau_1},\cdots,\frac{\omega_{q-1}}{\tau_{q-1}},1+\omega_q\}>0$, such that for any $Y^T=(y_1,y_2,\cdots,y_q)\in R^q$, $Y^T\widetilde{\mathscr{M}}Y\ge\lambda|Y|^2=\lambda\sum_{i=1}^q y_i^2$ holds.

In addition to the above notations, the letter M will denote a generic positive constant, independent of k_n, h_u and h_σ.

3. SMTDG scheme

In order to accurately approximate both displacement 'u' and stress '$-A\nabla u$', equation (1(a)) can be written by

$$(a)\ u_t+\nabla\cdot\sigma=f(x,t),\quad (x,t)\in Q,\quad (b)\ \widetilde{A}\sigma+\nabla u=0,\quad (x,t)\in Q, \tag{3}$$

where $\widetilde{A}=A^{-1}$, $\sigma=-A\nabla u$. Furthermore, from (3(b)) we can derive

$$(\widetilde{A}\sigma)_t+\nabla u_t=0,\quad (x,t)\in Q. \tag{4}$$

Then, the weak formulation of (3) is: find $\{\sigma, u\} \in H^1(I_n; H(\mathrm{div})) \times H^1(I_n; L^2(\Omega))$ such that

$$(a)\ -(\widetilde{A}\sigma, w_t)_n + (\nabla\cdot\sigma, \nabla\cdot w)_n + ((\widetilde{A}\sigma)^{n+1}, w^{n+1}) = ((\widetilde{A}\sigma)^n, w^{n+})+(f,\nabla\cdot w)_n, \quad w\in L^2(I_n;H(\mathrm{div})),$$

$$(b)\ -(u,v_t)_n + (\nabla\cdot\sigma, v)_n + (u^{n+1}, v^{n+1}) = (u^n, v^{n+}) + (f,v)_n, \quad v\in L^2(I_n;L^2(\Omega)). \tag{5}$$

where $u^{n+} = \lim_{\varepsilon\to 0+} u(\cdot, t^n+\varepsilon)$. Here, we used the relation that: taking $v=\nabla\cdot w$ in (5(b)) for $w\in H(\mathrm{div})$, we obtain

$$-(u,\nabla\cdot w_t)_n + (\nabla\cdot\sigma, \nabla\cdot w)_n + (u^{n+1}, \nabla\cdot w^{n+1}) = (u^n, \nabla\cdot w^{n+}) + (f,\nabla\cdot w)_n.$$

Assume that finite element spaces $V_\sigma^n \subset H(\mathrm{div})$ and $W_u^n \subset L^2(\Omega)$ are defined on T_σ^n and T_u^n respectively, and have the following properties: there exist some integers $r, r_1, s > 0$, such that for $\forall w\in H(\mathrm{div})\cap (H^{r+1}(\Omega))^d$,

$$\inf_{w_h\in V_\sigma^n} ||w-w_h|| \le Mh_\sigma^{r+1}||w||_{r+1}, \quad \inf_{w_h\in V_\sigma^n} ||\nabla\cdot(w-w_h)|| \le Mh_\sigma^{r_1}||w||_{r_1},$$
$$\inf_{v_h\in W_u^n} ||v-v_h|| \le Mh_u^{s+1}||v||_{s+1}, \quad \forall v\in L^2(\Omega)\cap H^{s+1}(\Omega),$$

where $r_1 = r$ in case of BDFM elements and BDM elements, or $r_1 = r+1$ in case of Nedelec elements and RT elements. Set

$$W_{uK}^n = \{\phi : \phi = \sum_{j=0}^{q-1} t^j \chi_j(x),\ \ \chi_j(x)\in W_u^n\},\ V_{\sigma K}^n = \{\psi : \psi = \sum_{j=0}^{q-1} t^j\chi_j(x),\ \ \chi_j(x)\in V_\sigma^n\},\ \ q\le 1,$$

$$W_{uK} = \{w : w|_{Q^n} \in W_{uK}^n\},\ V_{\sigma K} = \{v : v|_{Q^n}\in V_{\sigma K}^n\}.$$

Then for $\forall t\in I_n, w\in W_u^n$ and $v\in V_\sigma^n$, respectively, and for $\forall x\in\Omega$, functions w and v are piecewise polynomial functions in t of degree $q-1$ with possible discontinuities at the nodes t^n, $(1\le n\le N-1)$.

Replacing σ in the weak formulation (5) by a function $Z\in V_{\sigma K}$ and u in (5(b)) by $U\in W_{uK}$, we derive the SMTDG procedure of problem (1) as follows: find $\{U, Z\}\in W_{uK}\times V_{\sigma K}$ such that

$$(a)\ L(\widetilde{A}Z, w) + (\nabla\cdot Z, \nabla\cdot w)_n = (\widetilde{A}^n Z^n, w^{n+}) + (f,\nabla\cdot w)_n,$$

$$(b)\ L(U,v) + (\nabla\cdot Z, v)_n = (U^n, v^{n+}) + (f,v)_n, \tag{6}$$

where $\forall v\in W_{uK}^n,\ w\in V_{\sigma K}^n,\ (0\le n\le N-1)$. And we set $L(v,w) = (v^{n+1}, w^{n+1}) - (v, w_t)_n$.

Remark 1. Obviously, the discrete stress function Z is calculated independently from (6(a)) in each space-time slab Q_n. Then, if required, we obtain the discrete displacement function U from (6(b)) by using Z and U^n. From the viewpoint of computation, one can choose the usual continuous finite element spaces as $V_{\sigma K}^n$ in each time interval I_n and its construction of the finite element is not necessary to match with that of W_{uK}^n as in the classical mixed element methods.

We choose $\{\ell_{n,i}(t)\}_{i=1}^q$ as the basis functions for piecewise polynomial function space $P_{q-1}(I_n)$. Then $U|_{I_n}$ and $Z|_{I_n}$ are uniquely determined by the functions $U^{n,j} = U^{n,j}(x)\in W_u^n$ and $Z^{n,j} = Z^{n,j}(x)\in V_\sigma^n$, respectively, such that

$$U(x,t) = \sum_{j=1}^q \ell_{n,j}(t) U^{n,j},\ \ Z(x,t) = \sum_{j=1}^q \ell_{n,j}(t) Z^{n,j}. \tag{7}$$

Taking $v=\ell_{n,i}(t)\psi, \psi\in W_u^n$ and $w=\ell_{n,i}(t)\phi, \phi\in V_\sigma^n$ and then substituting (7) into (6), we obtain the Radau quadrature form of the SMTDG procedure as follows:

$$(a)\ L_R(\mathbf{Z}_A, \phi) + k_n\omega_i(\nabla\cdot Z^{n,i}, \nabla\cdot\phi) = \ell_{n,i}(t^n)(\widetilde{A}^n Z^n, \phi) + (\ell_{n,i} f, \nabla\cdot\phi)_n + \sum_{j=1}^q (R_n Z^{n,j}, \phi),$$

$$(b)\ L_R(\mathbf{U},\psi)+k_n\omega_i(\nabla\cdot Z^{n,i},\psi)=\ell_{n,i}(t^n)(U^n,\psi)+(\ell_{n,i}f,\psi)_n,\quad i=1,\cdots,q,\tag{8}$$

where R_n is the error of numerical integration and satisfies $|R_n|\le Mk_n^q$,and $\mathbf{U}=\left(U^{n,1},\cdots,U^{n,q}\right)^T$, $\mathbf{Z}_A=\left(\widetilde{A}^{n,1}Z^{n,1},\cdots,\widetilde{A}^{n,q}Z^{n,q}\right)^T$, $L_R(\mathbf{U},v)=\delta_{qi}(U^{n,q},v)-\sum_{j=1}^{q}w_{n,j}\ell'_{n,i}(t^{n,j})(U^{n,j},v)$.

4. Convergence analysis and error estimate

4.1. *Projection operator and basic error equations*

The classical mixed finite elements (i.e., RT, BDM, BDFM and Nedelec) spaces possesses the property [10] that there are projection operators $\Pi_h^n: H(\mathrm{div})\to V_\sigma^n$ and $P_h^n: L^2(\Omega)\to W_u^n$ such that

$$(\nabla\cdot(\sigma-\Pi_h^n\sigma),\psi)=0,\ \ (\nabla\cdot\phi,u-P_h^nu)=0,\quad \forall\psi\in W_u^n,\ \forall\phi\in V_\sigma^n,\tag{9}$$

and have error estimates:

$$(a)\ \ ||\sigma-\Pi_h^n\sigma||\le M||h_\sigma^{r+1}\sigma||_{r+1,h},\ \ (b)\ \ ||\nabla\cdot(\sigma-\Pi_h^n\sigma)||\le M||h_\sigma^{r_1}\sigma||_{r_1,h},$$
$$(c)\ \ ||u-P_h^nu||\le M||h_u^{s+1}u||_{s+1,h},\quad \forall u\in H^{s+1}(\Omega).\tag{10}$$

Define a Lagrange interpolation operator $\mathcal{I}_{q-1}^n: C(I_n)\to P_{q-1}(I_n)$ at the Radau points of $I_n=(t^n,t^{n+1}]$ such that $(\mathcal{I}_{q-1}^n y)(t^{n,j})=y(t^{n,j}),\ 1\le j\le q,$ where $t^{n,j},\ 1\le j\le q$ are defined in (2). Obviously, $\mathcal{I}_{q-1}^n v(t)\in P_{q-1}(I_n)$ and $\mathcal{I}_{q-1}^n v(t^{n+1})=v(t^{n+1})$. Define functions $W_0:(0,T]\to L^2(\Omega)$ and $W_1:(0,T]\to H(\mathrm{div})$ such that $W_0(x,t)=\mathcal{I}_{q-1}^nP_h^nu(x,t),\ W_1(x,t)=\mathcal{I}_{q-1}^n\Pi_h^n\sigma(x,t),\ (x,t)\in Q_n$, then $W_0\in W_{uK}$ and $W_1\in V_{\sigma K}$ denote the restrictions to I_n, $W_0|_{I_n}$ again by W_0 and $W_1|_{I_n}$ again by W_1. We will split the error as follows:

$$U-u=\theta_0-\rho_0,\ \theta_0=U-W_0,\quad \rho_0=u-W_0,\ \ Z-\sigma=\theta_1-\rho_1,\ \theta_1=Z-W_1,\ \rho_1=\sigma-W_1,\tag{11}$$

then standard approximation and stability results for the interpolation operator give[2,11]:

$$\begin{aligned}
&(a)\ \ |||\rho_0|||_n\le Mk_n^q|||u^{(q)}|||_n+Mk_n^{1/2}\max_{I_n}||h_u^{s+1}u||_{s+1,h},\\
&(b)\ \ |||\rho_1|||_n\le Mk_n^q|||\sigma^{(q)}|||_n+Mk_n^{1/2}\max_{I_n}||h_\sigma^{r+1}\sigma||_{r+1,h},\\
&(c)\ \ |||\nabla\cdot\rho_1|||_n\le Mk_n^q|||\nabla\cdot\sigma^{(q)}|||_n+Mk_n^{1/2}\max_{I_n}||h_\sigma^{r_1}\sigma|||_{r_1,h},\\
&(d)\ \ \max_{I_n}||\rho_0||\le Mk_n^q\max_{I_n}||u^{(q)}||+M\max_{I_n}||h_u^{s+1}u||_{s+1,h},\\
&(e)\ \ \max_{I_n}||\rho_1||\le Mk_n^q\max_{I_n}||\sigma^{(q)}||+M\max_{I_n}||h_\sigma^{r+1}\sigma||_{r+1,h},
\end{aligned}\tag{12}$$

From (6) and (11), we can derive the functions $\theta_0|_{I_n}$and $\theta_1|_{I_n}$ satisfy:

$$L(\widetilde{A}\theta_1,w)+(\nabla\cdot\theta_1,\nabla\cdot w)_n=(\widetilde{A}^n(\theta_1^n+W_1^n),w^{n+})-\{L(\widetilde{A}W_1,w)+(\nabla\cdot W_1,\nabla\cdot w)_n\}+(f,\nabla\cdot w)_n,\tag{13}$$

$$L(\theta_0,v)+(\nabla\cdot\theta_1,v)_n=(\theta_0^n,v^{n+})+(W_0^n,v^{n+})+(f,v)_n-\{L(W_0,v)+(\nabla\cdot W_1,v)_n\},\tag{14}$$

here we set $W_0^0=P_h^0u_0,\ W_1^0=\Pi_h^0\sigma_0,\ \theta_0^0=u_0-P_h^0u_0,\ \theta_1^0=\sigma_0-\Pi_h^0\sigma_0$.

Taking $w=\ell_{n,i}(t)\phi,\ \phi\in V_\sigma^n$ in (13) and $v=\ell_{n,i}(t)\psi,\ \psi\in W_u^n$ in (14) with $W_\kappa(x,t)=\sum_{j=1}^q\ell_{n,j}(t)\omega_\kappa^{n,j}(x),\ \theta_\kappa=\sum_{j=1}^{q}\ell_{n,j}(t)\theta_\kappa^{n,j}(x),\ (\kappa=0,1),\ \omega_0^{n,j}=P_h^nu(\cdot,t^{n,j}),\ \omega_1^{n,j}=\Pi_h^n\sigma(\cdot,t^{n,j}),\ \theta_0^{n,j}=U^{n,j}-\omega_0^{n,j},\ \theta_1^{n,j}=Z^{n,j}-\omega_1^{n,j}$, then

$$L_R(\Theta_{1A},\phi)+k_n\omega_i(\nabla\cdot\theta_1^{n,i},\nabla\cdot\phi)=\ell_{n,i}(t^n)(\widetilde{A}^n\theta_1^n,\phi)+(\Sigma_1+\Sigma_2,\phi)$$

$$+(\Sigma_3, \nabla\cdot\phi) - \ell_{n,i}(t^n)(\widetilde{A}^n J[\eta_1^n], \phi) + \sum_{j=1}^{q}(R_n\theta_1^{n,j}, \phi), \quad i = 1, \cdots, q \quad (15)$$

$$L_R(\Theta_0, \psi) + k_n\omega_i(\nabla\cdot\theta_1^{n,i}, \psi) = \ell_{n,i}(t^n)(\theta_0^n, \psi)$$
$$+(\Sigma_1' + \Sigma_2' + \Sigma_3, \psi) - \ell_{n,i}(t^n)(J[\eta_0^n], \psi), \quad i = 1, \cdots, q \quad (16)$$

where $\Theta_{1A} = (\widetilde{A}^{n,1}\theta_1^{n,1}, \cdots, \widetilde{A}^{n,q}\theta_1^{n,q})^T$, $\Theta_0 = (\theta_0^{n,1}, \cdots, \theta_0^{n,q})^T$, $\eta_0 = u - P_h^n u$, $\eta_1 = \sigma - \Pi_h^n\sigma$,

$$\Sigma_1 := \delta_{qi}\widetilde{A}^{n,q}\eta_1^{n,q} - \sum_{j=1}^{q} w_{n,j}\ell'_{n,i}(t^{n,j})\widetilde{A}^{n,j}\eta_1^{n,j} - \ell_{n,i}(t^n)\widetilde{A}^n\eta_1^{n+},$$
$$\Sigma_2 := \sum_{j=1}^{q} w_{n,j}\ell'_{n,i}(t^{n,j})\widetilde{A}^{n,j}\sigma^{n,j} - \int_{I_n}\ell'_{n,i}\widetilde{A}\sigma dt, \quad \Sigma_3 := \int_{I_n}\ell_{n,i}\nabla\cdot\sigma dt - k_n\omega_i\nabla\cdot\sigma^{n,i},$$
$$\Sigma_1' := \delta_{qi}\eta_0^{n,q} - \sum_{j=1}^{q} w_{n,j}\ell'_{n,i}(t^{n,j})\eta_0^{n,j} - \ell_{n,i}(t^n)\eta_0^{n+}, \quad \Sigma_2' := \sum_{j=1}^{q} w_{n,j}\ell'_{n,i}(t^{n,j})u^{n,j} - \int_{I_n}\ell'_{n,i}u dt,$$
$$J[\eta_1^n] := \eta_1^n - \eta_1^{n+} = (\Pi_h^n - \Pi_h^{n-1})\sigma(t^n), \quad J[\eta_0^n] := \eta_0^n - \eta_0^{n+} = (P_h^n - P_h^{n-1})u(t^n).$$

4.2 Optimal error estimates

Theorem 1 Assume that $\{u, \sigma\}$ is the pair of solution of (5) and $\{U, Z\}$ is the solution of SMTDG-scheme (8) with initial values $U^0 = u_0$ and $Z^0 = \sigma_0$, then it satisfies

$$\max_{t\in[0,T]}\{||u(t) - U(t)|| + ||\sigma(t) - Z(t)||\} \le M\Big\{\max_n[||J[\eta_1^n]|| + ||J[\eta_0^n]||] + \max_n k_n^q \max_{I_n}(\blacksquare) + \max_n\max_{I_n}(\blacklozenge)\Big\},$$

where

$\blacksquare = ||u^{(q)}|| + ||u^{(q+1)}|| + \sum_{\nu=0}^{q+1}||\sigma^{(\nu)}|| + ||\nabla\cdot\sigma^{(q)}||$, $\blacklozenge = ||h_{n_u}^{s+1}u||_{s+1,h} + ||h_{n_u}^{s+1}u_t||_{s+1,h} + ||h_{n_\sigma}^{r+1}\sigma||_{r+1,h} + ||h_{n_\sigma}^{r+1}\sigma_t||_{r+1,h}$.

Proof. Taking $\phi = \theta_1^{n,i}$ in the right hand side of (15) and then summing i from 1 to q. The resulting equation will be equal to the left hand side of (13) for $w = \theta_1$, then we get

$$||\theta_1^{n+1}||^2 + |||\nabla\cdot\theta_1|||_n^2 \le M||\theta_1^n||^2 + M||J[\eta_1^n]||^2 + M|||\theta_1|||_n^2 + Mk_n^{2q}\{\sum_{\nu=0}^{q+1}|||\sigma^{(\nu)}|||_n^2 + |||\nabla\cdot\sigma^{(q)}|||_n^2\}$$
$$+M\{\int_{I_n}||h_\sigma^{r+1}\sigma||_{r+1,h}^2 dt + \int_{I_n}||h_\sigma^{r+1}\sigma_t||_{r+1,h}^2 dt + k_n\max_{I_n}||h_\sigma^{r+1}\sigma||_{r+1,h}^2\}, \quad (17)$$

Similarly, taking $\psi = \theta_0^{n,i}$ in the right hand side of (16) and then summing i from 1 to q. The resulting equation will be equal to the left hand side of (14) for $v = \theta_0$, then we have

$$||\theta_0^{n+1}||^2 \le M||\theta_0^n||^2 + M||J[\eta_0^n]||^2 + M|||\theta_0|||_n^2 + \frac{1}{2}|||\nabla\cdot\theta_1|||_n^2$$
$$+Mk_n^{2q}\{|||u^{(q+1)}|||_n^2 + |||\nabla\cdot\sigma^{(q)}|||_n^2\} + M\int_{I_n}||h_u^{s+1}u_t||_{s+1,h}^2 dt, \quad (18)$$

By using Lemma 1, inequalities (10), (12) and relations: $|||\theta_\kappa|||_n^2 = \sum_{j=1}^{q} w_{n,j}||\theta_\kappa^{n,j}||^2 = k_n\sum_{j=1}^{q}\omega_j||\theta_\kappa^{n,j}||^2$, $(\kappa = 0, 1)$ we can obtain

$$|||\theta_1|||_n^2 + |||\theta_0|||_n^2 \le Mk_n\{||\theta_1^n||^2 + ||\theta_0^n||^2 + ||J[\eta_1^n]||^2 + ||J[\eta_0^n]||^2 + k_n^{2q}(\bigstar) + k_n^2\max_{I_n}||h_\sigma^{r+1}\sigma||_{r+1,h}^2\}$$
$$+Mk_n\{k_n\int_{I_n}||h_\sigma^{r+1}\sigma||_{r+1,h}^2 dt + k_n\int_{I_n}||h_\sigma^{r+1}\sigma_t||_{r+1,h}^2 dt + k_n\int_{I_n}||h_u^{s+1}u_t||_{s+1,h}^2 dt\} \quad (19)$$

where $\bigstar = \sum_{\nu=0}^{q+1}|||\sigma^{(\nu)}|||_n^2 + |||\nabla\cdot\sigma^{(q)}|||_n^2 + |||u^{(q+1)}|||_n^2$. From (17)-(19), using iteration, inverse inequality, we have

$$\max_{I_n}\{||\theta_1|| + ||\theta_0||\} \le M\{||h_\sigma^{r+1}\sigma_0||_{r+1,h} + ||h_u^{s+1}u_0||_{s+1,h}\} + M\Big(\sum_{m=0}^{n}\Xi^m\Big)^{1/2} + M\Big(\sum_{m=1}^{n}\{||J[\eta_1^m]||^2 + ||J[\eta_0^m]||^2\}\Big)^{1/2},$$

where

$$\Xi^n = \int_{I_n}\{||h_u^{s+1}u_t||_{s+1,h}^2 + ||h_\sigma^{r+1}\sigma||_{r+1,h}^2 + ||h_\sigma^{r+1}\sigma_t||_{r+1,h}^2\}dt + k_n\max_{I_n}||h_\sigma^{r+1}\sigma||_{r+1,h}^2 + k_n^{2q}(\bigstar).$$

Finally, using the triangle inequality and (12), the proof is completed.

We note that Eq. (6(a)) is independent of Eq. (6(b)). So, we can analyze convergence of scheme (6(a)) independently.

Theorem 2 Assume that σ and Z are the solutions of Eq. (3(b)) and scheme (6(a)) with initial value $Z^0 = \sigma_0$, respectively. Then, there has the a priori error estimates

$$\max_{t\in[0,T]}||\sigma(t) - Z(t)|| \le M\max_n||J\eta_1^n]|| + M\max_n k_n^q\max_{I_n}\{\sum_{\nu=0}^{q+1}||\sigma^{(\nu)}|| + ||\nabla\cdot\sigma^{(q)}||\} + M\max_n\max_{I_n}\{||h_\sigma^{r+1}\sigma||_{r+1,h} + ||h_\sigma^{r+1}\sigma_t||_{r+1,h}\},$$

and for any $n = 0,\cdots,N-1$,

$$||\sigma - Z||_{L^2(I_n;H(\mathrm{div}))} \le M\max_{0\le m\le n}k_m^q\max_{I_m}\{\sum_{\nu=0}^{q+1}||\sigma^{(\nu)}|| + ||\nabla\cdot\sigma^{(q)}||\} + M\max_{0\le m\le n}||J[\eta_1^m]|| + M\max_{0\le m\le n}\max_{I_m}\{||h_\sigma^{r+1}\sigma||_{r+1,h} + ||h_\sigma^{r+1}\sigma_t||_{r+1,h} + ||h_\sigma^{r_1}\sigma||_{r_1,h}\}.$$

Proof. Much of this proof is analogous to that of Theorem 1and will be omitted. However, there are a few things worth noting. For any $n, (0\le n\le N-1)$ and sufficiently small k_n, we have

$$|||\theta_1|||_n^2 \le Mk_n\{||\theta_1^n||^2 + ||J[\eta_1^n]||^2 + k_n^2\max_{I_n}||h_\sigma^{r+1}\sigma||_{r+1,h}^2\} + Mk_n\{k_n\int_{I_n}||h_\sigma^{r+1}\sigma||_{r+1,h}^2dt + k_n\int_{I_n}||h_\sigma^{r+1}\sigma_t||_{r+1,h}^2dt + k_n^{2q}\Big(\sum_{\nu=0}^{q+1}|||\sigma^{(\nu)}|||_n^2 + |||\nabla\cdot\sigma^{(q)}|||_n^2\Big)\}. \quad (20)$$

Combining (17) with (20) and then iteration of n, applying inverse inequality, we obtain

$$\max_{I_n}||\theta_1|| \le M\max_{0\le m\le n}k_m^q\max_{I_m}\{\sum_{\nu=0}^{q+1}||\sigma^{(\nu)}|| + ||\nabla\cdot\sigma^{(q)}||\} + M\max_{0\le m\le n}||J[\eta_1^m]|| + M\max_{0\le m\le n}\max_{I_m}\{||h_\sigma^{r+1}\sigma||_{r+1,h} + ||h_\sigma^{r+1}\sigma_t||_{r+1,h}\},$$

And then using triangle inequality and (12), we can obtain the first result.
Form (17) and (20) , we also have

$$|||\theta_1|||_n + |||\nabla\cdot\theta_1|||_n \le M\max_{0\le m\le n}k_m^q\max_{I_m}\{\sum_{\nu=0}^{q+1}||\sigma^{(\nu)}|| + ||\nabla\cdot\sigma^{(q)}||\} + M\max_{0\le m\le n}||J[\eta_1^m]|| + M\max_{0\le m\le n}\max_{I_m}\{||h_\sigma^{r+1}\sigma||_{r+1,h} + ||h_\sigma^{r+1}\sigma_t||_{r+1,h}\}.$$

Then, an appeal to triangle inequality and (12(c)-(b)) complete the proof of theorem 2.

5. Numerical experiments

To test the SMTDG method proposed in this paper and confirm our theoretical analysis, we consider the following parabolic problem:

$$\begin{cases} u_t - u_{xx} = f(x,t), & x \in [0,0.5],\ t \in [0,5], \\ u(t,0) = u(t,0.5) = 0, & t \in [0,5], \\ u(x,0) = \sin^2(\pi x), & x \in [0,0.5], \end{cases}$$

where $f(x,t) = -e^{-t}\sin^2(\pi x) - 2\pi^2 e^{-t}\cos(2\pi x)$, $u(x,t) = e^{-t}\sin^2(\pi x)$, $\sigma(x,t) := -u_x = -\pi e^{-t}\sin(2\pi x)$.

In the computation of the approximate value Z by SMTDG scheme, we have used four kinds of basis functions: (a) linear polynomials both in time and space $(q = 2,\ r = 1)$, (b) linear polynomial in time and quadratic polynomial in space $(q = 2,\ r = 2)$, (c) quadratic polynomial in time and cubic polynomial in space $(q = 3,\ r = 3)$, (d) quadratic polynomial in time and fourth-order polynomial in space $(q = 3,\ r = 4)$.

Let h be the diameter of K_σ which forms a uniform partition of Ω and k be the time step. In order to neatly identify both spatial and temporal contributions in the error estimate in Theorem 2, we have performed the numerical computations by varying either the time step or the space grid only, having chosen in each case the discretization step in the other variable sufficiently small that the corresponding error can be neglected.

Table 1. Errors and convergence rates for $r = 1,\ q = 2$ and $k = 0.01$

h	Norm 1	Rate	Norm 2	Rate	Norm 3	Rate	Norm 4	Rate
0.1	5.6130e-02		1.7793e-00		3.8596e-04		1.1989e-02	
0.05	1.4122e-02	1.9908	8.9371e-01	0.9934	9.7202e-05	1.9894	6.0219e-03	0.9935
0.025	3.5362e-03	1.9977	4.4737e-01	0.9984	2.4352e-05	1.9969	3.0143e-03	0.9984
0.0125	8.8441e-04	1.9994	2.2375e-01	0.9995	6.0984e-06	1.9976	1.5076e-03	0.9996
0.00625	2.2112e-04	2.0000	1.1188e-01	1.0000	1.5323e-06	1.9927	7.5385e-04	1.0000

Table 2. Errors and convergence rates for $r = 1,\ q = 2$ and $h = 0.000125$

k	Norm 1	Rate	Norm 2	Rate	Norm 3	Rate	Norm 4	Rate
0.5	4.0219e-02		2.5588e-01		4.0024e-04		2.5464e-03	
0.25	9.5560e-03	2.0734	6.0822e-02	2.0728	7.3590e-05	2.4433	4.6844e-04	2.4426
0.125	1.8246e-03	2.3888	1.1775e-02	2.3689	1.2810e-05	2.5223	8.2879e-05	2.4988
0.0625	2.6936e-04	2.7600	2.7118e-03	2.1184	2.0371e-06	2.6526	1.9880e-05	2.0597

Table 3. Errors and convergence rates for $r = 2,\ q = 2$ and $k = 0.005$

h	Norm 1	Rate	Norm 2	Rate	Norm 3	Rate	Norm 4	Rate
0.1	2.2282e-03		1.4444e-01		1.5017e-05		9.7302e-04	
0.05	2.7969e-04	2.9940	3.6254e-02	1.9942	1.8848e-06	2.9942	2.4427e-04	1.9940
0.025	3.4997e-05	2.9985	9.0725e-03	1.9986	2.3588e-07	2.9983	6.1130e-05	1.9985
0.0125	4.3758e-06	2.9996	2.2687e-03	1.9996	2.9545e-08	2.9970	1.5286e-05	1.9996
0.00625	5.5045e-07	2.9909	5.6721e-04	1.9999	3.9600e-09	2.8994	3.8218e-06	1.9999

Table 4. Errors and convergence rates for $r = 2,\ q = 2$ and $h = 0.001$

k	Norm 1	Rate	Norm 2	Rate	Norm 3	Rate	Norm 4	Rate
0.125	1.8245e-03		1.1608e-02		1.2809e-05		8.1496e-05	
0.0625	2.6928e-04	2.7603	1.7133e-03	2.7603	2.0366e-06	2.6530	1.2958e-05	2.6529
0.03125	3.9079e-05	2.7846	2.4898e-04	2.7826	2.9753e-07	2.7750	1.8955e-06	2.7732
0.015625	5.3290e-06	2.8744	3.6391e-05	2.7744	3.5578e-08	3.06395	2.4660e-07	2.9423

Table 5. Errors and convergence rates for $r=3,\ q=3$ and $k=0.025$

h	Norm 1	Rate	Norm 2	Rate	Norm 3	Rate	Norm 4	Rate
0.25	3.0024e-03		1.1899e-01		2.0806e-05		7.8855e-04	
0.125	1.9377e-04	3.9537	1.5174e-02	2.9711	1.3278e-06	3.9699	1.0074e-04	2.9969
0.0625	1.2240e-05	3.9847	1.9063e-03	2.9928	8.3422e-08	3.9925	1.2662e-05	2.9921
0.03125	7.6744e-07	3.9954	2.3859e-04	2.9982	5.2207e-09	3.9981	1.5849e-06	2.9980
0.010625	4.8324e-08	3.9892	2.9833e-05	2.9995	3.2704e-10	3.9967	1.9817e-07	2.9995

Table 6. Errors and convergence rates for $r=3,\ q=3$ and $h=0.0005$

k	Norm 1	Rate	Norm 2	Rate	Norm 3	Rate	Norm 4	Rate
0.5	7.9321e-04		5.0466e-03		9.8348e-06		6.2572e-05	
0.25	7.5718e-05	3.3890	4.8174e-04	3.3890	7.0118e-07	3.8100	4.4611e-06	3.8100
0.125	4.9585e-06	3.9327	3.1547e-05	3.9327	3.8978e-08	4.1691	2.4799e-07	4.1691
0.0625	2.1723e-07	4.5126	1.3821e-06	4.5126	1.6731e-09	4.5420	1.6449e-08	4.5420

Table 7. Errors and convergence rates for $r=4,\ q=3$ and $k=0.01$

h	Norm 1	Rate	Norm 2	Rate	Norm 3	Rate	Norm 4	Rate
0.25	1.8108e-04		1.2666e-02		1.2352e-06		7.8522e-05	
0.125	5.7220e-06	4.9840	8.0385e-04	3.9779	3.9298e-08	4.9741	4.9922e-06	3.9754
0.0625	1.7952e-07	4.9943	5.0433e-05	3.9945	1.2336e-09	4.9935	3.1334e-07	3.9939
0.03125	5.6511e-09	4.9895	3.1551e-06	3.9986	3.8594e-11	4.9984	1.9605e-08	3.9985
0.010625	1.7757e-10	4.9921	1.9724e-07	3.9997	1.2187e-12	4.9849	1.2256e-09	3.9996

Table 8. Errors and convergence rates for $r=4,\ q=3$ and $h=0.0025$

k	Norm 1	Rate	Norm 2	Rate	Norm 3	Rate	Norm 4	Rate
0.125	4.9588e-06		3.1549e-05		3.8980e-08		2.4800e-07	
0.0625	2.1742e-07	4.5114	1.3833e-06	4.5114	1.6744e-09	4.5410	1.0653e-08	4.5410
0.03125	7.9514e-09	4.7731	5.0589e-08	4.7731	6.0454e-11	4.7917	3.8463e-10	4.7917
0.015625	2.1435e-10	5.2132	1.3700e-09	5.2065	1.6613e-12	5.1854	1.0614e-11	5.1794

Table 1, 3, 5, 7 show the errors and convergence rates in Norm 1 (i.e., $L^\infty(L^2)$-norm), Norm 2 (i.e., $L^\infty(H(\mathrm{div}))$-norm), Norm 3 (i.e., L^2-norm at $t=5$) and Norm 4 (i.e., $H(\mathrm{div})$-norm at $t=5$) with respect to space, respectively. Table 2, 4, 6, 8 show the errors and convergence rates in Norm 1-4 with respect to time, respectively.

From Table 1, 3, 5, 7, we can see that errors in $L^\infty(L^2)$-norm, as a function of the space step converge at the rate $O(h^{r+1})$, which confirms the theoretical result in Theorem 2. Errors in Norm 2 and Norm 4 converge at the rate $O(h^r)$ and those in Norm 3 converge at the rate $O(h^{r+1})$. In Table 2, the convergence rate of error in $L^\infty(L^2)$-norm as a function of the time step is nearly two, which verifies the theoretical result in Theorem 2. The convergence rates of error in Norm 2-4 are also approximately two. These results indicate that the errors converge optimally. The convergence rates of error in Norm 1-4 as a function of the time step in Table 4 are nearly three and those in Table 6 and 8 are nearly four and five, respectively. These results show that time discontinuous Galerkin method has super-convergence property at time nodes [1, 2].

6. Conclusion

In this paper, we have introduced and analyzed a splitting mixed space-time discontinuous Galerkin method for a class of parabolic problems. By a splitting technique, the equation of stress is separated from the stress-displacement coupled system and is solved by time discontinuous Galerkin method with high accuracy. The optimal priori error estimates in $L^\infty(L^2)$ norm and $L^2(H(\mathrm{div}))$ norm are proved. Super-convergence property at time nodes of the time discontinuous Galerkin method is shown in numerical experiments. There exist more difficulties to use Radau quadrature rule to analyze the super-convergence property of space-time discontinuous Galerkin method. We will report on the research in a forthcoming paper.

Acknowledgements

The work of the authors is supported by the National Nature Science Foundation of China (grant no. 11061021) and "211 project" innovative talents training program of Inner Mongolia University.

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