Bi-continuous Valuations

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In a recent paper *Partial Metrics and Co-continuous Valuations*, M. A. Bukatin and S. Yu. Shorina consider valuations on topological spaces that are continuous, co-continuous and strongly non-degenerate [2]. They use these valuations in order to construct pseudo-metrics that are Scott continuous. The notions mentioned above mean the following:

Let X be a topological space and $\mathcal{O}$ the collection of all open subsets of X considered as a complete lattice under the inclusion ordering. A *valuation* on X is a function

$$\mu : \mathcal{O} \to \mathbb{R}_+$$

from the open subsets $\mathcal{O}$ to the non-negative reals $\mathbb{R}_+$ such that:

$$\mu(\emptyset) = 0,$$
$$\mu(U) \leq \mu(V) \text{ whenever } U \subseteq V \text{ in } \mathcal{O},$$
$$\mu(U) + \mu(V) = \mu(U \cup V) + \mu(U \cap V) \quad \text{for all } U, V \in \mathcal{O}.$$

As we exlude the value $+\infty$ such valuations are sometimes called bounded. A valuation is called *continuous* if it satisfies:

$$\mu(\bigcup_i U_i) = \sup_i \mu(U_i) \text{ for every directed family } (U_i)_{i\in I} \text{ in } \mathcal{O}.$$

It will be called *co-continuous* if it satisfies:

$$\mu(\mathrm{int}(\bigcap_i U_i)) = \inf_i \mu(U_i) \text{ for every filtered family } (U_i)_{i\in I} \text{ in } \mathcal{O}.$$

We shall say that a valuation is *bi-continuous* if it is continuous and co-continuous. A valuation is called *strongly non-degenerate* or equivalently *strictly increasing* if it satisfies:

$$\mu(U) < \mu(V) \text{ whenever } U \subset V \text{ in } \mathcal{O}.$$

The above definitions can be applied to any complete lattice L in place of the lattice $\mathcal{O}$ of open sets in a topological space. A map $\mu : L \to \mathbb{R}_+$ is called a *valuation* if

$$\mu(\bot) = 0 \text{ where } \bot \text{ is the smallest element of } L,$$
$$\mu(u) \leq \mu(v) \text{ whenever } u \leq v \text{ in } L,$$
$$\mu(u) + \mu(v) = \mu(u \vee v) + \mu(u \wedge v) \quad \text{for all} \quad u, v \in L.$$

A valuation on L is called *continuous*, respectively *co-continuous*, if

$$\mu(\bigvee_i u_i) = \sup_i \mu(u_i) \text{ for every directed family } (u_i)_{i \in I} \text{ in } L,$$

respectively

$$\mu(\operatorname{int}(\bigwedge_i u_i)) = \inf_i \mu(u_i) \text{ for every filtered family } (u_i)_{i \in I} \text{ in } L.$$

A valuation is strictly increasing if

$$\mu(u) < \mu(v) \text{ whenever } u < v \text{ in } L.$$

Bi-continuity seems to be a contradictory requirement if one considers classical Hausdorff spaces such as, for example, the unit interval $[0,1]$ with its usual topology. There are no bounded bi-continuous valuations in $[0,1]$, which can be demonstrated as follows: first let us note that every continuous valuation μ on the unit interval may be extended to a Borel measure in a unique way (see [5], [6]); we shall use this fact without further reference. Take any countable dense subset Q of the unit interval. Then $\operatorname{int}(\bigcap\{[0,1] \setminus F \mid F \text{ finite} \subseteq Q\} = \emptyset$, whence $\mu(\bigcap\{[0,1] \setminus F \mid F \text{ finite} \subseteq Q\} = 0$. This shows that $\mu([0,1] \setminus F)$ gets arbitrarily small, i.e. some points in Q have to carry point masses. As this holds for every countable dense subset of $[0,1]$, there must be uncountably many points in $[0,1]$ carrying positive masses, a contradiction. The same argument may be applied to most of the classical spaces.

Surprisingly, in a preliminary version of [2], Bukatin and Shorina were able to show that on certain non-classical spaces, namely on bounded complete continuous domains (in the sense of D. S. Scott) there are bi-continuous valuations. Their prime example stems from countable convex combinations of point valuations (= Dirac measures) concentrating their mass on compact elements in algebraic domains. For the continuous case, they gave quite a complicated construction.

The property of being non-degenerate seems even more difficult to achieve: if in a Hausdorff space X we delete any element, then the resulting set is open and has a value strictly smaller than the value of X. Thus any point has to carry a positive point mass. This leads to unbounded measures as soon as X has uncountably many points. In non-Hausdorff spaces, deleting single points does not yield open sets in general. In a directed complete poset, if we take a Scott open set U and a point $a \in U$, then $U \setminus \{a\}$ will only be open if a is a minimal element of U and compact. Thus, if we have a countably based algebraic poset, countable convex combinations of point valuations concentrated on compact elements yield non-degenerate valuations, as Bukatin and Shorina have shown.

Here, I want to show that the original result of Bukatin and Shorina can be generalised and simplified using quite standard results about continuous domains and completely distributive lattices. Bukatin and Shorina have used this result in the final version of their paper [2]. At the end of this Note, I shall indicate an alternative proof of the theorem which works with the basic definition of a continuous domain only, and which does not use the results about continuous domains and completely distributive lattices mentioned above. This second proof has been worked out in detail by R. Flagg [3].

Thus, let X be a dcpo (= directed complete partially ordered set), and let $\mathcal{O}$ denote its Scott topology. (For the basic facts about dcpo's and the Scott topology we refer to [4],[1]). We shall use the following characterization of those dcpo's that are called continuous in the sense of Scott (see the Compendium [4]):

Lemma 0.1 *A dcpo X is continuous if, and only if, its lattice $\mathcal{O}$ of Scott open sets is completely distributive.*

In the fifties, G. N. Raney [7] has proved a fundamental fact about completely distributive lattices that we quote in the next lemma (see also [4], p. 204). In the lemma we use the following terminology. We denote by [0,1] the unit interval with its usual total ordering. For any set J, we consider the product lattice $[0,1]^J$, i.e. the set of all functions $a : J \to [0,1]$ under pointwise ordering. Note that $[0,1]^J$ is a complete, completely distributive lattice. A lattice homomorphism is a mapping between lattices preserving finite meets and finite joins. A complete lattice homomorphism preserves arbitrary meets and arbitrary joins.

Lemma 0.2 *For a complete lattice L the following conditions are equivalent:*

(i) *L is completely distributive.*

(ii) *For any two elements $u \not\leq v$ in L, there is a complete lattice homomorphism $\pi_{u,v} : L \to [0,1]$ such that $\pi_{u,v}(v) < \pi_{u,v}(u)$.*

(iii) *There is an injective complete lattice homomorphism $\pi : L \to [0,1]^J$ for some set J.*

If L has a countable base, then J can be chosen to be countable in condition (iii).

We now note the following easy lemma for arbitrary complete lattices L:

Lemma 0.3 *Any lattice homomorphism $\pi : L \to [0,1]$ with $\pi(\bot) = 0$ is a valuation. If π is a complete lattice homomorphism, it is a bi-continuous valuation.*

Indeed, any lattice homomorphism π is order preserving. And for arbitrary $u, v \in L$, one has $\pi(u) \leq \pi(v)$ or vice-versa. In the fist case, $\pi(u \vee v) + \pi(u \wedge v) =$

$max(\pi(u), \pi(v)) + min(\pi(u), \pi(v)) = \pi(v) + \pi(u)$, and likewise in the second case.

The complete lattice homomorphisms π will replace the point valuations on compact elements.

Note, in particular, that the canonical projections $\pi_j : [0,1]^J \to [0,1]$ are complete lattice homomorphisms, and hence also bi-continuous valuations. One easily verifies the following:

Lemma 0.4 *If $(r_j)_{j\in J}$ is any family of non-negative real numbers such that $\sum_{j\in J} r_j < \infty$, then $\pi = \sum_{j\in J} r_j\pi_j : [0,1]^J \to \mathbb{R}_+$ is a bi-continuous valuation. If J is countable, one may choose $r_j > 0$ for all $j \in J$, and then the valuation π becomes strictly increasing.*

We now combine these lemmas as follows. Let X be a continuous dcpo. By the first lemma, the lattice $\mathcal{O}$ of Scott open sets in X is completely distributive. By Raney's result, there is a complete lattice embedding $p : \mathcal{O} \to [0,1]^J$ for some set J. Composition with the bi-continuous valuations $\pi : [0,1]^J \to \mathbb{R}$ from the last lemma yields (plenty of) bi-continuous valuations $\pi \circ p$ on X. If X has a countable base, then the preceding lemma yields valuations on X that in addition are strictly increasing. We have proved:

Theorem 0.5 *On any countably based continuous domain X, there are (many) bi-continuous, strictly increasing valuations.*

Some questions arise:

(i) Can one give a representation $\pi : \mathcal{O} \to [0,1]^J$ explicitly?

(ii) How rich is the set of all bi-continuous valuations in the 'probabilistic power domain' of all continuous valuations? Is every continuous valuation the least upper bound of a directed family of bi-continuous ones?

(iii) Is every Scott continuous function $f : L \to \mathbb{R}_+$ the least upper bound of a family of bi-continuous valuations whenever L is a completely distributive lattice?

We now turn to an alternative proof of the preceding Theorem. More precisely, we prove the following proposition which replaces the first two lemmas above by a direct argument:

Proposition 0.6 *Let U and V be Scott open subsets of a continuous dcpo X such that $U \not\subseteq V$. Then there is a bi-continuous valuation μ on the Scott topology $\mathcal{O}$ of X such that $\mu(V) < \mu(U)$.*

Proof: Let U and V be two Scott-open subsets of X such that $U \not\subseteq V$. Then there are elements $x_0 \ll x_1$ in U such that $x_1 \notin V$, where $\ll$ is the way-below relation (cf. [1], [4]). Using the interpolation property for the way-below relation, we may associate an element $x_q \in X$ to every rational number $q \in [0,1]$ such that $x_q \ll x_p$, whenever $q < p$. Define $m: \mathcal{O} \to [0,1]$ by $m(W) = \inf\{q \in \mathbb{Q} \cap [0,1] \mid x_q \in W\}$. Let $(W_i)_i$ be any family of Scott-open sets in X. Clearly,

$$m(\bigcup_i W_i) = \inf\{q \mid x_q \in \bigcup_i W_i\} = \inf_i(\inf\{q \mid x_q \in W_i\}) = \inf_i m(W_i).$$

The inequality $m(\mathrm{int}(\bigcap_i W_i)) \geq \sup_i m(W_i)$ is clear. For the reverse inequality, choose any $q > \sup_i m(W_i)$. Let p be such that $q > p > \sup_i m(W_i)$. Then $x_p \in W_i$ for every i, whence $x_p \in \bigcap_i W_i$. As $x_p \ll x_q$, we conclude that $x_q \in \mathrm{int}(\bigcap_i W_i)$, whence $q \geq m(\mathrm{int}(\bigcap_i W_i))$. Now define μ on the collection $\mathcal{O}$ of all Scott open subsets of X by $\mu(W) = 1 - m(W)$. Then μ is a bi-continuous valuation such that $\mu(U) = 1$ and $\mu(V) = 0$. A detailed version of this proof can be found in [3].

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