Computability on computable metric spaces

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Abstract


In previous papers [e.g. Weihrauch (1987)], "Type 2 theory of effectivity" (TTE) has been shown to be a very general and powerful computer-oriented theory for studying effectivity in mathematics. In this contribution computability on certain "computable" separable metric spaces is studied in detail by applying the framework of TTE.

Computationally admissible representations of metric spaces are introduced after showing that four different "effective" representations are computationally equivalent. For extending computability to the set of continuous functions, several effective namings which are related to definitions of continuity are introduced and compared. The definitions are used to prove effective \( G^r \)-extension theorems for continuous functions and an effective \( G^r \)-characterization of the domains of "strongly continuous" functions which generalize the known properties of real functions [Kreitz (1984)].

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1. Introduction

Metric spaces are not only important in analysis and functional analysis but recently they have also been used for defining the semantics of concurrency of programming languages [1]. While the abstract theory is far developed, not much is known about effectivity for metric spaces. In this paper basic definitions for a general computability theory on metric spaces are discussed in detail. Their applicability is demonstrated by proving effective theorems on domains of continuous functions.

In the past, several different approaches for studying effectivity in analysis have been proposed. Some of them not only consider real numbers and functions but also suggest general principles for introducing effectivity on metric spaces. Among others, such principles must clarify the following questions: Which metric spaces shall be called “effective”? Which are the effective elements of an effective metric space? How can elements be handled effectively (“effective functions”)? How can effective functions on effective metric spaces be handled effectively? Lacombe [11] suggests to define a recursive metric space as a triple \((M, D, v)\), where \(M\) is a metric space, \(D\) is a dense subset of \(M\), and \(v: \mathbb{N} \to D\) is a bijective numbering of \(D\) such that \(d(v(i), v(j)) < v_Q(k)\) and \(d(v(i), v(j)) > v_Q(k)\) are recursively enumerable in \(i, j, k\) (where \(v_Q\) is a standard numbering of the rational numbers).

Moschovakis [12] considers only denumerable metric spaces (e.g. the “computable” real numbers). In this sense, a recursive metric space is a pair \((M, v)\), where \(M\) is a metric space and \(v: \mathbb{N} \to M\) (partial, onto) is a numbering such that, for some recursively enumerable sets \(X\) and \(Y\), \(\{(i, j, k) | d(v(i), v(j)) < v_Q(k)\} = X \cap D\) and \(\{(i, j, k) | d(v(i), v(j)) > v_Q(k)\} = Y \cap D\), where \(D = \text{dom}(v) \times \text{dom}(v) \times \mathbb{N}\). Also Cieitin [3] and Kushner [10] define constructive metric spaces in essentially the same way. Central in these works is the proof that, under certain conditions, computable functions on metric spaces are continuous. Bishop and Bridges [2] present another elaborated approach to effective metric spaces based on very restricted constructive reasoning. Finally, it is possible to reduce computability on metric spaces to computability on complete partial orders (cpo’s) (see [6, 23]). All of these approaches have shortages. They are too restricted, not sufficiently far developed or not sufficiently close to ordinary classical analysis or computability theory.

Several years ago, Weihrauch and Kreitz [5–8, 17–19, 21] have introduced a general Type 2 theory of effectivity (TTE), which especially supplies a basis for a computer-oriented constructive and computable analysis including computational complexity [4, 13–15, 20, 22]. It has been shown that the theory of topological \(T_0\)-spaces (which include the separable metric spaces) can be embedded adequately to TTE by means of (topologically admissible) \((t\)-admissible\) representations [7, 18]. TTE already provides the essential parts of a constructivity theory for metric spaces.

In this paper we make an attempt to specialize this constructivity theory to a computability theory by adding appropriate computability requirements. We define computable metric spaces and compare five \(t\)-admissible representations of such spaces. This leads to the definition of computationally admissible (c-admissible)
representations. In section 3 we introduce four naming systems of continuous functions, three of which turn out to be essentially equivalent. In Section 4 we prove effective versions of the (classical) $G_δ$-extension theorem for continuous functions on complete metric spaces. Finally, we prove an effective version of the $G_δ$-characterization of the domains of "strongly continuous" functions. All the theorems not only state existence but also guarantee computable procedures for determining the objects under consideration.

In this paper we shall use the notations from [18]. By $f: X \to Y$ we shall denote a partial function from $X$ to $Y$. $B := \mathbb{N}^\mathbb{N}$ is the Baire space with metric $d(p, q) = 2^{-\eta}$, where $n := \mu i[p(i) \neq q(i)]$ if $p \neq q$. The open balls $[w] := \{w | p \in B\}$, with $w \in \mathbb{N}^*$, are a basis of the corresponding topology. $\langle \ldots \rangle$ denotes standard pairing and tupling functions on $\mathbb{N}$ as well as on $B$. For numbers, $\pi_i^{[k]}(x_k) := x_i$, where $1 \leq i \leq k$, $v_0: \mathbb{N} \to \mathbb{Q}$ is a standard numbering of the rational numbers, and $v^*: \mathbb{N} \to \mathbb{N}^*$ is a standard numbering of the finite sequences of natural numbers. Finally, $\text{En}: B \to 2^\mathbb{N}$ is the enumeration representation of the subsets of $\mathbb{N}$ defined by $\text{En}(p) = \{i | (\exists k) p(k) = i + 1\}$.

2. Computational admissible representations of computable metric spaces

In TTE, constructivity (i.e. continuity) and computability is introduced for a set $M$ be defining a numbering $\nu: \subseteq \mathbb{N} \to M$ or a representation $\delta: \subseteq B \to M$. This definition is crucial and has to be discussed thoroughly. We shall consider the case of representations here. For separable metric spaces the $\tau$-admissible representations yield a reasonable constructivity theory [7, 18]. Here we shall start one step behind this. Only metric spaces $(M, d)$ with $\text{card}(M) \leq \text{card}(B)$ can be treated in TTE. For greater spaces no effectivity theory is available until today. "Effectivity" should be reasonably related to the metric structure $d$. As a most elementary property of an "effective" representation $\delta: \subseteq B \to M$, one should require that distances can be approximated effectively from above by rational numbers arbitrarily precisely. This has already important consequences. Let $\rho_\succ: \subseteq B \to \mathbb{R}$ be the "recursively enumerable (r.e.)-right cut" representation of the real numbers (cf. [21]), defined by $\rho_\succ(p) = x$ iff $(\forall i \in \text{En}(p)) x < v_0(i)$ and $x = \inf\{v_0(i) | i \in \text{En}(p)\}$. Then in TTE ($\langle \delta, \tau \rangle$, $\rho_\succ$)-continuity formalizes the above informal requirement.

Lemma 2.1. Let $(M, d)$ be a metric space and let $\delta: \subseteq B \to M$ be a representation such that $d$ is $((\delta, \tilde{\tau}), \rho_\succ)$-continuous. Then $(M, d)$ is separable and $\sigma$ is continuous.

Proof. By assumption, there is a continuous function $\delta: \subseteq B \to B$ such that $d(\delta(p), \delta(q)) = \rho_\succ(\Gamma(p, q))$ whenever $p, q \in \text{dom}(\delta)$. Let $\langle p_i \rangle_{i \in \mathbb{N}}$ be a sequence in $\text{dom}(\delta)$ which converges w.r.t. the Baire metric to $p \in \text{dom}(\delta)$. Since $\Gamma$ is continuous, $q_i := \Gamma(p, p_i)$ converges to $q := \Gamma(p, p)$. We have $0 = d(\delta(p), \delta(q)) = \rho_\succ(\Gamma(p, q)) = \rho_\succ(q)$. Assume $\varepsilon > 0$. Then for some $k$, $0 < v_0(q(k) - 1) < \varepsilon$. Since $q_i \to q$, there is some $m$ such that $q(k) = q_l(k)$
for all \( i \geq m \); hence,
\[
d(\delta(p), \delta(p_i)) = \rho_\succ \Gamma(p, p_i) = \rho_\succ (q_i) < \nu_q(q_i(k) - 1) < \varepsilon \quad \text{for all } i \geq m.
\]
This shows that \( \delta(p_i) \) converges to \( \delta(p) \). Therefore, \( \delta : \mathbb{B} \to M \) is continuous. Let \( V := \{ \nu \in \mathbb{N}^* | \nu \in \text{dom}(\delta) \text{ for some } p \in \mathbb{B} \} \). Obviously, \( V \) is denumerable. Let \( h : V \to \mathbb{B} \) be a function with \( vh(\nu) \in \text{dom}(\delta) \) for all \( \nu \in V \) and define \( X := \{ vh(\nu) | \nu \in V \} \) and \( Y = \{ \delta(x) | x \in X \} \). Then \( X \) is dense in \( \text{dom}(\delta) \), and \( Y \) is dense in \( M \) by continuity of \( \delta \).

Therefore, only for separable metric spaces can a reasonable effectivity theory be defined in TTE. We know already that separable metric spaces have t-admissible representations [7, 18], which induce the natural constructivity theory. Topologically admissible representations are continuous, but continuity does not imply t-admissibility. As a counterexample, consider the decimal representation \( \delta_0 : \mathbb{B} \to \mathbb{R} \) of the real line which is not admissible [18]. But \( \delta_0 \) is continuous; even more, the distance on the real line is \((\delta_0, \delta_0, \rho_\succ)\)-continuous. For obtaining a natural computability theory on a metric space \((M, d)\), c-admissible representations must be chosen from the t-admissible ones. This is not possible without some additional computability property for \((M, d)\). Similar to our first requirement, we shall assume that distances on some denumerable dense subset can be approximated computationally from above by rational numbers arbitrarily precisely.

**Definition 2.2.** A computable metric space is a 4-tuple \( \tilde{M} = (M, d, A, x) \) such that

1. \((M, d)\) is a metric space,
2. \(A \subseteq M\), \(A\) is a dense subset,
3. \(x : \mathbb{N} \to A\) is a (total) numbering of \(A\),
4. \(D_< := \{ \langle i, j, k \rangle | d(x(i), x(j)) < \nu_q(k) \}\) is recursively enumerable.

Admitting \(x\) to be a partial numbering seems to be too general. We do not require that \(D_< := \{ \langle i, j, k \rangle | d(x(i), x(j)) > \nu_q(k) \}\) is r.e., i.e. we do not require that distances on \(M\) can be approximated arbitrarily precisely also from below. Thus, distinctness (apartness) of points cannot be proved in general. Many important metric spaces \((M, d)\) become computable metric spaces by adding appropriate natural dense sets \(A\) and numberings \(x\).

**Example 2.3 (Computable metric spaces).** (1) **Discrete spaces.** Let \((M, d)\) be a discrete metric space (i.e. \(d(x, y) = 1 \iff x \neq y\)) with denumerable set \(M\). Let \(x\) be any bijective numbering of \(M\). Then \((M, d, M, x)\) is a computable metric space.

2. **The real line.** The metric space \((\mathbb{R}, d, \mathbb{Q}, \nu_q)\), where \(d\) is the distance on the real line, is computable.

3. **The Baire space.** Let \((\mathbb{B}, d)\) be the Baire space [18], where \(d(p, q) := 2^{-\mu}\), with \(n = \mu [p(k) g q(k)]\) if \(p \neq q\). Let \(A := \{ p \in \mathbb{B} | p(n) = 0 \text{ for almost all } n \}\). Let \(v^* : \mathbb{N} \to \mathbb{N}^*\) be the standard numbering of the finite sequences over \(\mathbb{N}\), and define \(x : \mathbb{N} \to A\) by \(x(i) := v^*(i)00\ldots\). Then \((\mathbb{B}, d, A, x)\) is a computable metric space.
(4) Space of continuous functions. Let $C[0; 1]$ be the set of continuous functions $f: [0; 1] \to \mathbb{R}$. For $f, g \in C[0; 1]$ define $d(f, g) := \max \{ |f(x) - g(x)| : x \in [0; 1]\}$. Let $A$ be the (denumerable) set of polynomial functions with rational coefficients, restricted to $[0; 1]$, and let $\alpha$ be a standard numbering of $A$. Then $(C[0; 1], d, A, \alpha)$ is a computable metric space.

(5) $L^p$-spaces. Consider the interval $[0; 1] \subseteq \mathbb{R}$. A simple step function is a partial function $s: \subseteq [0; 1] \to \mathbb{R}$ such that there are real numbers $a, b, c \in \mathbb{R}$ with $1 < a < b$ such that

$$s(x) = \begin{cases} c & \text{if } a < x < b, \\ \text{div} & \text{if } x = a \text{ or } x = b, \\ 0 & \text{otherwise.} \end{cases}$$

Define the integral by $\int s \, dx := (b - a) \cdot c$. A step function is a finite sum of simple step functions, i.e. $s = s_1 + \cdots + s_n$. Define $\int s \, dx := \int s_1 \, dx + \cdots + \int s_n \, dx$. A step function is rational if it is a finite sum of simple step functions $s_i$ defined by rational points $a_i, b_i, c_i$. Let $SF$ be the set of rational step functions. Let $p \in \mathbb{R}$, $1 \leq p < \infty$.

By $d(s, s') := (\int |s(x) - s'(x)|^p \, dx)^{1/p}$ a pseudometric space $(SF, d)$ is defined. Let $\nu$ be a standard numbering of $SF$ and let $(M_p, d_p)$ be the metric completion of $(SF, d)$. Then $(M_p, d_p, SF, \nu)$ is a computable metric space if $p$ is a computable real number (see [18]), especially if $p \in \mathbb{Q}$.

(6) Let $(A, d_A)$ be a metric space and $\alpha$ be a numbering of $A$ such that Definition 2.2(4) holds. Let $(M', d)$ be a metric completion of $(A, d_A)$ and $A \subseteq M \subseteq M'$. Then, obviously, $(M, d, A, \alpha)$ is a computable metric space.

(7) Let $M_i = (M_i, d_i, A_i, \alpha_i)$ be computable metric spaces for $i = 1, 2$. Define $M := M_1 \times M_2$, $A := A_1 \times A_2$, $d((x_1, x_2), (y_1, y_2)) := \max(d_1(x_1, y_1), d_2(x_2, y_2))$. Then $(M, d, A, \alpha)$ is a computable metric space. The same holds for the distance $d_1(x_1, y_1) + d_2(x_2, y_2)$ and for the distance $(d_1(x_1, y_1))^2 + (d_2(x_2, y_2))^2)^{1/2}$.

(8) From (2) and (7), for each $n > 1$, we obtain a computable Euclidean $n$-space $(\mathbb{R}^n, d_n, A_n, \alpha_n)$.

Let $M = (M, d, A, \alpha)$ be a computable metric space. Let $U$ be a numbering of open balls defined by $U(i, j) := U_{i, j} := \{x \in M : d(x, \alpha(i)) < 2^{-j}\}$ and define $\text{rad}(i, j) := 2^{-j}$. Obviously, range($U$) is a basis of the topology induced by the metric. Note that $\text{rad}(i, j)$ is not the radius of $U(i, j)$ in general (e.g. if $\alpha(i)$ is an isolated point and $U(i, j) = \{\alpha(i)\}$). The inclusion $U_a \subseteq U_b$ depends on the "global" structure of $A$ and may be a very difficult property. As a substitute, we define a ("formal inclusion") relation $< \subseteq \mathbb{N}^2$ by $i < k, m) := d(\alpha(i), \alpha(k)) + 2^{-i} < 2^{-m}$. The relation $<$ is r.e. by Definition 2.2(4) and transitive. Note that $a < b$ implies $\text{cls}(U_a) \subseteq U_b$; the converse, however, may be false ($\text{cls}(X) :=$ the metric closure of $x$ in $M$). We shall now introduce and compare four representations of a computable metric space.

**Definition 2.4.** Let $M = (M, d, A, \alpha)$ be a computable metric space and let $U$ be the numbering defined above. Define partial representations $\delta_1, \ldots, \delta_4$ of $M$ as follows.
For all $x \in M$ and $p \in \mathcal{B}$ let

$$\delta_1(p) - x \quad := \quad \text{En}(p) = \{ i \mid x \in U_i \},$$

$$\delta_2(p) - x \quad := \quad \{ U_i \mid i \in \text{En}(p) \} \text{ is a basis of the neighbourhoods of } x \text{ and}$$

$$(\forall \varepsilon > 0)(\exists k \in \text{En}(p)) \text{rad}(k) < \varepsilon,$$

$$\delta_3(p) - x \quad := \quad (\forall j > i) d(xp(i), xp(j)) \leq 2^{-i} \text{ and } x = \lim xp(i),$$

$$\delta_4(p) - x \quad := \quad (\forall k)x \in U_{pk}, \text{ and } (\forall k)p(k + 1) < p(k)$$

and $x = \lim_k xp(k)$.

The representation $\delta_1$ is from [7], $\delta_2$ is the normed Cauchy representation [7, 18], and $\delta_4$ is the representation by means of nested closed balls. From TTE we know that $\delta_1$ is t-admissible; hence, it induces the natural constructivity on the metric space. The above representations are computationally equivalent.

**Lemma 2.5.**

$$\delta_1 \equiv^e \delta_2 \equiv^e \delta_3 \equiv^e \delta_4.$$

**Proof.** $\delta_1 \leq^e \delta_2$. Obviously, $\delta_1(p) = \delta_2(p)$ for all $p \in \text{dom}(\delta_1)$; hence, the identity function translates $\delta_1$ to $\delta_2$.

$\delta_2 \leq^e \delta_3$. Define a computable function $\Gamma : \mathcal{B} \rightarrow \mathcal{B}$ by

$$\Gamma(p)(n) := \pi^{(3)}_1(i, j, k) \quad [p(k) = \langle i, j \rangle + 1 \text{ and } j > n].$$

If $p \in \text{dom}(\delta_2)$ then $\Gamma(p)(n)$ exists for all $n$, and $d(z\Gamma(p)(n), \delta_2(p)) < \frac{1}{2^{n}} \cdot 2^{-n}$. The triangle inequality yields $\delta_3(p) = \delta_3 \Gamma(p)$. Hence, $\delta_2 \leq^e \delta_3$.

$\delta_3 \leq^e \delta_4$. Define a computable function $\Gamma : \mathcal{B} \rightarrow \mathcal{B}$ by

$$\Gamma(p)(n) := \langle p(n + 2), n \rangle$$

for all $p, n$. Suppose $\delta_3(p) = x$ and $n \in \mathbb{N}$. Then $d(zp(n + 2), x) \leq 2^{-(n + 2)} < 2^{-n}$; hence, $x \in U(\Gamma(p)(n))$, and $d(zp(n + 2), xp(n + 3)) + 2^{-n+1} \leq 2^{-n+2} + 2^{-n+1} < 2^{-n}$. Hence, $\Gamma(p)(n + 1) < \Gamma(p)(n)$. Finally, $\Gamma(p)(n) < \varepsilon$. Therefore, $\delta_3(p) = \delta_4 \Gamma(p)$.

$\delta_4 \leq^e \delta_1$. Since $\prec$ is recursively enumerable, there is a computable function $h : \mathbb{N} \rightarrow \mathbb{N}$ with range($h$) = $\{ \langle i, j \rangle \mid i < j \}$. Define a computable function $\Gamma : \mathcal{B} \rightarrow \mathcal{B}$ by

$$\Gamma(p)(n, k, m) := \begin{cases} n + 1 & \text{if } h(k) = \langle p(m), n \rangle, \\ 0 & \text{otherwise}, \end{cases}$$

for all $p \in \mathcal{B}, n \in \mathbb{N}$. Then $\Gamma(p)$ enumerates the set of all $i$ with $\delta_4(p) \in U_i$; hence, $\delta_1 \Gamma(p) = \delta_4(p)$, whenever $p \in \text{dom}(\delta_4)$.

The first condition for $\delta_2(p) = x$ already yields a representation $\delta_0$ (see [18], Definition 3.4.17). However, in general the corresponding names carry less computable information than those w.r.t. $\delta_1$. If sufficient information about isolated points is computationally available, $\delta_0$ becomes equivalent to $\delta_2$. 


Lemma 2.6. Let $M$ and $U$ be as in Definition 2.4. Define a representation $\delta_0 : \subseteq B \rightarrow M$ by $\delta_0(p) = x \iff \{ U_i | i \in \text{En}(p) \}$ is a basis of the neighbourhoods of $x$.

Let $X := \{ \langle m, n \rangle \mid \{ x(m) \} = U \langle m, n \rangle \}$, Then

1. $\delta_2 \leq_c \delta_0$.
2. $\delta_0 \leq_c \delta_2$.
3. $\delta_0 \leq_c \delta_2$ if $X$ is recursively enumerable.

Proof. Define a computable function $\Sigma : B \rightarrow B$ by

$$\Sigma(p, q) = \begin{cases} \langle m, n \rangle + 1 & \text{if } (k = 0 \land p(a) = \langle m, n \rangle + 1) \\ 0 & \text{otherwise} \end{cases}$$

If $\delta_0(p) = x$ and $\text{En}(q) = X$ then $\delta_2 \Sigma(p, q) = x$. By the smn-theorem for $\psi$, there is a computable function $\Gamma : B \rightarrow B$ with $\Sigma(p, q) = \psi_{\Gamma(p)}(p)$. We obtain $\delta_0 \leq_c \delta_2$. If $X$ is r.e. then $X = \text{En}(q)$ for some computable $q \in B$; hence, $\Gamma(q)$ is computable and $\delta_0 \leq_c \delta_2$.

On the other hand, the identity function reduces $\delta_2$ to $\delta_0$. □

Obviously, if $\bar{M}$ has no isolated points then $\delta_0$ is computationally equivalent to $\delta_2$. Lemma 2.5 suggests the following definition, which fixes the computability theory on computable metric spaces.

Definition 2.7. Let $\bar{M} = (M, d, A, \alpha)$ be a computable metric space.

1. A representation $\delta$ of $\bar{M}$ is called $c$-admissible if $\delta \equiv_c \delta_2$, where $\delta_2(p) = x \iff (\forall j > i)d(\alpha p(i), \alpha p(j)) \leq 2^{-i}$ and $x = \lim \alpha p(i)$.
2. An element $x \in M$ is called computable iff $x = \delta_2(p)$ for some computable function $p \in B$.
3. A function between two computable metric spaces with $c$-admissible representations $\delta$ and $\delta'$ is computable iff it is $(\delta, \delta')$-computable.

Note that representations $\delta$ and $\delta'$ of a set $M$ induce the same computability theory on $M$ iff they are $c$-equivalent [18]. In Definition 2.2 the set $\mathbb{Q}$ of rational numbers is chosen as a denumerable dense subset of $\mathbb{R}$, and the sequence $(2^{-i})_{i \in \mathbb{N}}$ is used for defining the numbering $U$ and the representations $\delta_0, \ldots, \delta_4$. If $\mathbb{Q}$ is replaced by some other appropriate set, e.g. by $\mathbb{Q}_2 := \{ (i-j)/2^k | i, j, k \in \mathbb{N} \}$, and $2^{-i}$ is replaced e.g. by $1/(1+i)$, then representations equivalent to $\delta_0, \ldots, \delta_4$ are obtained, especially the definition of $c$-admissibility is invariant under these changes. We do not discuss more details here.

3. Naming systems of continuous functions

For handling functions between metric spaces effectively, we need effective representations of sets of functions. In the spirit of TTE, we intend to consider all the
continuous functions. Although the set $X$ of all continuous partial functions $f: \subseteq B \to B$ is too large for having a representation, it is essentially represented by $\psi: B \to [B \to B]$, since every $f \in X$ has an extension in $[B \to B]$; see [17, 18]. A corresponding extension theorem does not hold in general for functions $f: \subseteq M_1 \to M_2$ between metric spaces. Therefore, we shall consider sets of functions associated with names $p \in B$ rather than single functions. Below, four naming systems for functions between metric spaces are introduced.

**Definition 3.1.** For $i = 1, 2$, let $\bar{M}_i = (M_i, d_i, A_i, \alpha_i)$ be a computable metric space with associated numbering $U_i$ of basic neighbourhoods, with formal inclusion relation $<_i$ and $\delta_i := \delta_{\alpha_i}$ (see Definition 2.7). Define representations $\delta_5, \ldots, \delta_8$ of sets of subsets of partial functions from $M_1$ to $M_2$ as follows. For all $p \in B$, $f: \subseteq M_1 \to M_2$ and $X := \text{dom}(f)$, let

\[
\begin{align*}
  f \in \delta_5(p) &\iff (\forall q \in \text{dom}(f \delta_1)) f \delta_1(q) = \delta_2 \psi_p(q), \\
  f \in \delta_6(p) &\iff (\forall j) f^{-1} U_j^f = X \cap \bigcup \{ U_i^f | \langle i, j \rangle \in E_n(p) \}, \\
  f \in \delta_7(p) &\iff (\forall \langle i, j \rangle \in E_n(p)) f U_j^f \subseteq U_i^f \text{ and } \\
  &\quad (\forall x \in X)(\forall \epsilon > 0)(\exists \langle i, j \rangle \in E_n(p)) (x \in U_i^f \land \text{rad}(j) < \epsilon), \\
  f \in \delta_8(p) &\iff f \in \delta_5(p) \text{ and } (\forall B \subseteq E_n(p), B \text{ finite}) \\
  &\quad \left[ \cap \{ U^f_a | a \in \text{pr}_1(B) \} \neq \emptyset \quad \Leftrightarrow \quad \cap \{ U^f_a | b \in \text{pr}_2(B) \} \neq \emptyset \right].
\end{align*}
\]

For $i = 5, \ldots, 8$ $f$ is called $\delta_i$-computable iff $f \in \delta_i(p)$ for some computable $p \in B$.

Above, $\bar{V} := \text{cls}(V)$ is the metric closure of a set. $\text{pr}_1(B) = \{ a | (\exists b) \langle a, b \rangle \in B \}$, and $\text{pr}_2(B) = \{ b | (\exists a) \langle a, b \rangle \in B \}$. The representation $\delta_5$ is derived from $(\delta_1, \delta_2)$-continuity in TTE (see [18]), and $\delta_6$ corresponds to the definition of continuity in topology ($f^{-1}(\text{open})$ is open). The representation $\delta_7$ is derived from the $(\epsilon, \delta)$-continuity definition for functions on metric spaces. If $f \in \delta_8(p)$, then $p$ enumerates how (even closed) formal balls can be mapped into open formal balls. Note that for $\langle i, j \rangle, \langle k, m \rangle \in E_n(p)$, $U_j^f \cap U_m^f$ may be empty even if $Y := U_i^f \cap U_k^f \neq \emptyset$, namely, in the case of $Y \cap \text{dom}(f) = \emptyset$. If $f \in \delta_8(p)$, then $E_n(p)$ must satisfy a finite consistency property which excludes such situations. First we show that the definition of $\delta_5$ is essentially independent of the special c-admissible representations $\delta_1$ and $\delta_2$.

**Lemma 3.2.** Let $\delta_i$ be a c-admissible representation of $\bar{M}_i$ ($i = 1, 2$) and define $\bar{\delta}_5$ according to $\delta_5$ by $f \in \bar{\delta}_5(p) = \iff (\forall q \in \text{dom}(f \delta_1)) f \delta_1(q) = \delta_2 \psi_p(q)$. Then there are computable functions $\Gamma_1, \Gamma_2: B \to B$, with

\[
\delta_5(p) \subseteq \bar{\delta}_5 \Gamma_1(p) \quad \text{and} \quad \bar{\delta}_5(p) \subseteq \delta_5 \Gamma_2(p),
\]

for all $p \in B$.

**Proof.** Since $\delta_1 \leq_c \delta_1$ and $\delta_2 \leq_c \delta_2$, there are computable functions $\Sigma_1, \Sigma_2: \subseteq B \to B$, with $\delta_1(p) = \delta_1 \Sigma_1(p)$ for all $p \in \text{dom}(\delta_1)$ and $\delta_2(p) = \delta_2 \Sigma_2(p)$ for all $p \in \text{dom}(\delta_2)$. By the
utm-theorem and the smn-theorem for $\psi$, there is a computable function $\Gamma : \mathcal{B} \to \mathcal{B}$, with $\psi_{f(p)} = \Sigma \psi_{p} \Sigma_{1}$ for all $p \in \mathcal{B}$. Consider $f \in \delta_{5}(p)$. Then, for all $q \in \text{dom} f_{\delta_{1}},$

$$f_{\delta_{1}}(q) = \delta_{1} \psi_{p} \Sigma_{1}(q) = \delta_{2} \Sigma_{2} \Sigma_{p} \Sigma_{1}(q) = \delta_{2} \psi_{p} \Sigma_{1}(q).$$

This proves $f \in \delta_{5}\Gamma(p)$; therefore, $\delta_{5}(p) \subseteq \delta_{5}\Gamma(p)$. The second statement is proved accordingly. $\square$

Since c-admissible representations are t-admissible, $f \in \delta_{5}(p)$ for some $p$ if $f$ is continuous [18, Theorem 3.4.11]. Therefore, $\delta_{5}$ associates names with exactly all partial continuous functions from $\hat{M}_{1}$ to $\hat{M}_{2}$. By Theorem 3.3, $\delta_{5}, \delta_{6}$ and $\delta_{7}$ are essentially computationally equivalent.

**Theorem 3.3.** There are computable functions $\Gamma_{1}, \Gamma_{2}, \Gamma_{3} : \mathcal{B} \to \mathcal{B}$, with

1. $\delta_{5}(p) \subseteq \delta_{6}\Gamma_{1}(p),$
2. $\delta_{6}(p) \subseteq \delta_{7}\Gamma_{2}(p),$
3. $\delta_{7}(p) \subseteq \delta_{5}\Gamma_{3}(p),$

for all $p \in \mathcal{B}$.

**Proof.** (1) By Lemma 3.2, it suffices to consider the definition of $\delta_{5}$ with $\delta_{4}$ from Definition 2.4 substituted for $\delta_{1}$ in Definition 3.1 and $\delta_{4}$ from Definition 2.4 substituted for $\delta_{2}$ in Definition 3.1. By the utm-theorem, there is a computable function $\Gamma : \subseteq \mathcal{B} \to \mathcal{B}$, with $\Gamma \langle p, q \rangle = \psi_{p}(q)$.

Let $M$ be a Type 2 machine which computes $\Gamma$. For $i_{1}, \ldots, i_{k}, j_{1}, \ldots, j_{k} \in \mathbb{N}$ define $[i_{1} \ldots i_{k}, j_{1} \ldots j_{k}] = i_{1} j_{1} \ldots i_{k} j_{k}$. Since $\preceq_{1}$ is recursively enumerable, there are a total recursive function $h$ and a computable function $\Gamma_{1}$, with \{v^{*}h(n) | n \in \mathbb{N}\} = \{i_{0} \ldots i_{k} | k, i_{1}, \ldots, i_{k} \in \mathbb{N} \text{ and } i_{m} < i_{m-1} \text{ for } m = 1, \ldots, k \}$ and

$$\Gamma_{1}(p)\langle a, b, i, j, n, t \rangle = \begin{cases} 1 + \langle i, j \rangle & \text{if } v^{*}(a) \text{ is a prefix of } p, \\ \text{lg}(v^{*}(a)) = \text{lg}(v^{*}h(b)), i \text{ is the last number of } v^{*}h(b), \text{ and } M \text{ with input } \{v^{*}(a), v^{*}h(b)\} \text{ within } t \text{ steps writes} \\ \text{the number } j \text{ on position } n \\ \text{of the output tape,} \\ 0 & \text{otherwise.} \end{cases}$$

Now assume $q \in \delta_{5}(p)$. Then $f_{\delta_{1}}(q) = \delta_{2} \psi_{p}(q) = \delta_{2}\Gamma \langle p, q \rangle$ for all $q \in \text{dom}(f_{\delta_{1}})$. Let $\langle i, j \rangle \in \text{En} \Gamma_{1}(p)$ and $x \in U_{1} \cap \text{dom}(f)$. There are $a, b \in \mathbb{N}$ such that $v^{*}(a)$ is a prefix of $p$, $\text{lg}(v^{*}(a)) = \text{lg}(v^{*}h(b)), i$ is the last number of $v^{*}h(b)$, and $M$ with input $\{v^{*}a, v^{*}h(b)\}$ writes $j$ on the output tape. Since $x \in U_{1}^{1}$, there is some $q \in \mathcal{B}$, with $\delta_{1}(q) = x$ and $v^{*}(h(b))$ a prefix of $q$. Therefore, $\Gamma$ with input $\langle p, q \rangle$ writes $j$ on the output tape, i.e. $f(x) \in U_{2}^{2}$. This shows $f^{-1}U_{2}^{2} \supseteq X \cap \bigcup \ldots$. On the other hand, let $x \in f^{-1}U_{2}^{2}$. Then $x = \delta_{3}(q)$ for some $q \in \mathcal{B}$ and $M$ with input $\langle p, q \rangle$ writes $j$ on the output tape. Hence, there are
a, b, i, n, t such that \( v^*(a) \) is a prefix of \( p, \ldots \); hence, \( \langle i, j \rangle \in \text{En} \Gamma_1(p) \) for some \( i \). This proves \( f^{-1} U \subseteq \cdots \); hence, \( \delta \in \delta \Gamma_1(p) \).

(2) There is a computable function \( h: \mathbb{N} \rightarrow \mathbb{N} \) with \( \text{range}(h) = \{ \langle k, i \rangle \mid k \leq i \} \).

Define a computable function \( \Gamma_2: \mathbb{B} \rightarrow \mathbb{B} \) by

\[
\Gamma_2(p)(i, j, k, a, b) := \begin{cases} 
1 + \langle k, j \rangle & \text{if } h(b) = \langle k, i \rangle \text{ and } p(a) = 1 + \langle i, j \rangle, \\
0 & \text{otherwise.}
\end{cases}
\]

Assume \( f \in \delta(p) \). If \( \langle k, j \rangle \in \text{En} \Gamma_2(p) \) then, for some \( i, k \leq i \) and \( \langle i, j \rangle \in \text{En}(p) \); hence, \( f \subseteq U \subseteq U \). Assume \( x \in X \) and \( \varepsilon > 0 \). Then \( f(x) \in f(U) \subseteq U \). Assume \( x \in X \) and \( \varepsilon > 0 \). Then \( f(x) \in f(U) \subseteq U \). Since \( f^{-1} U = X \setminus \{ U \mid \langle i, j \rangle \in \text{En}(p) \} \), there is some \( i \), with \( x \in U \) and \( \langle i, j \rangle \in \text{En}(p) \). Hence, there is some \( k \), with \( x \in U \) and \( \langle k, j \rangle \in \text{En}(p) \). This proves \( f \in \delta \Gamma_2(p) \).

(3) By Lemma 3.2, it suffices to consider for \( \delta_1 \) and \( \delta_2 \) in Definition 3.1 the representation \( \delta_2 \) from Definition 2.4 for \( M_1 \) and \( M_2 \), respectively. Let \( h: \mathbb{N} \rightarrow \mathbb{N} \) be a computable function with range(h) = \{ \langle i, k \rangle \mid i \leq k \}. Define a computable function \( \Gamma : \mathbb{B} \rightarrow \mathbb{B} \) by

\[
\Gamma(p, q)(i, k, m, a, b, c) := \begin{cases} 
1 + m & \text{if } q(a) = 1 + i, \ h(b) = \langle i, k \rangle \text{ and } p(c) = 1 + \langle k, m \rangle, \\
0 & \text{otherwise.}
\end{cases}
\]

Consider \( f \in \delta_1(p) \). Assume \( q \in \text{dom}(f \delta_1(q), x = \delta_1(q), \varepsilon > 0 \). Then there is \( \langle k, m \rangle \in \text{En}(p) \), with \( x \in U \) and \( \varepsilon > 0 \). Then \( f(x) \in f(U) \subseteq U \). There is some \( i \in \text{En}(q) \), with \( i \leq k \) and \( x \in U \); hence, \( m \in \text{En}(\langle i, j \rangle \subseteq \text{En}(p) \). This shows \( f(x) = \delta f_\Gamma(p, q) \). By the smn-theorem, there is a computable function \( \Gamma_3: \mathbb{B} \rightarrow \mathbb{B} \), with \( f \delta_1(q) = \delta 2 \psi_\Gamma(p, q) \) for all \( f \in \delta_1(p) \) and \( q \in \text{dom}(f \delta_2(q) \delta \Gamma_3(p)) \). Hence, \( f \in \delta \Gamma_3(p) \).

The theorem is an effective version of the equivalence of the three underlying continuity definitions. As usual, we shall consider this equivalence of three definitions as a strong argument that they are "natural". By the following theorem, \( \delta \)-names can be obtained continuously from \( \delta \)-names. They can be obtained computably if distances on \( M_1 \) can be estimated computably also from below.

**Theorem 3.4.** Let \( D_\geq := \{ \langle i, j, k \rangle \mid d(x(i), x(j)) > v(q) \} \). Then

1. \( (\forall p \in \text{dom}(\delta_\geq)) \delta_\geq(p) = \delta_\geq(p) \);
2. there is a computable function \( \Gamma : \mathbb{B} \rightarrow \mathbb{B} \), with \( \delta_\geq(p) \subseteq \delta_\geq \Gamma(p, q) \) for all \( p, q \in \mathbb{B} \) such that \( D_\geq = \text{En}(q) \);
3. there is a computable function \( \Sigma : \mathbb{B} \rightarrow \mathbb{B} \), with \( \delta_\geq(p) \subseteq \delta_\geq \Sigma(p) \) for all \( p \in \mathbb{B} \) if \( D_\geq \) is recursively enumerable.

**Proof.** (1) is trivial and (3) follows from (2). We prove (2). Define the "formal distance for formal balls" on \( M_1 \) by \( d_\geq(a, b, c, d) := \max\{d(x(a), x(b)) - 2^{-b} - 2^{-d}\} \).
There is a computable function \( A: \mathbb{B} \to \mathbb{B} \) such that \( E_nA(q) = \{ (m, n) \mid ds(m, n) > 0 \} \) for all \( q \) with \( En(q) = D_\succ \).

Define \( En(p, n) := \{ i \mid (3j < n)p(j) = 1 + i \} \) for all \( p \in \mathbb{B}, \, n \in \mathbb{N} \). Define \( I: \mathbb{B} \to \mathbb{B} \) as follows:

\[
\Gamma \langle p, q \rangle \langle c, n, a', b', t \rangle := \begin{cases} 
1 + \langle c, b' \rangle & \text{if } (1)p(n) = 1 + \langle a', b' \rangle \text{ and } (2) \text{ } \quad c \triangleleft a' \text{ can be proved in at most } t \text{ steps, and } (3) \text{ } \quad (\exists i)(\forall (a, b) \in J \cup \{ \langle c, b' \rangle \})x_2(i) \in U^2_b \\
0 & \text{can be proved in at most } t \text{ steps, otherwise.} 
\end{cases}
\]

For a complete specification of \( I \), concrete enumeration procedures must be defined for which steps can be counted. Note that \( \triangleleft \) is r.e., \( x_2(i) \in U^2_b \) is r.e. and \( \langle a, c \rangle \in EnA(q) \) is r.e. in \( q \). We omit further details here.

Assume now \( f \in \delta_\succ(p) \) and \( D_\succ = En(q) \).

**Proposition 3.5.** \( (\forall (c, b') \in En \Gamma \langle p, q \rangle) f\tilde{U}_c \subseteq U^2_b \).

**Proof.** \( (c, b') \in En \Gamma \langle p, q \rangle \) implies \( \langle a', b' \rangle \in En(p) \) and \( c \triangleleft a' \) for some \( a' \); hence, \( f\tilde{U}_c \subseteq U^2_b \).

**Proposition 3.6.** Assume \( x \in \text{dom}(f) \) and \( \varepsilon > 0 \). Then there are \( c, b' \in \mathbb{N} \), with \( \langle c, b' \rangle \in En \Gamma \langle p, q \rangle \), \( x \in U^1_c \) and \( \text{rad}(b') < \varepsilon \).

**Proof.** Since \( f \in \delta_\succ(p) \), there are \( a', b', n \), with \( p(n) = 1 + \langle a', b' \rangle \), \( x \in U^1_a \) and \( \text{rad}(b') < \varepsilon \).

We show indirectly that \( \langle c, b' \rangle \in En \Gamma \langle p, q \rangle \) for some \( c \) with \( x \in U^1_c \) and \( c \triangleleft a' \).

Assumption: For all \( c \) with \( c \triangleleft a' \) and \( x \in U^1_c \) there is a set \( J_c \subseteq En(p, n) \) with

\[
(\exists i)(\forall (a, b) \in J_c \cup \{ \langle c, b' \rangle \})x_2(i) \in U^2_b \quad \text{and} \quad (\forall (a, b) \in J_c) f(a, c) = 0.
\]

For every \( m \in \mathbb{N} \), there is some \( c \) with \( c \triangleleft a' \), \( x \in U^1_c \) and \( \text{rad}(c) < 2^{-m} \). Since there are only finitely many \( J \subseteq En(p, n) \), there is some \( J' \subseteq En(p, n) \) such that there are arbitrarily small \( c \) with \( x \in U^1_c \), \( c \triangleleft a' \), and

\[
(\exists i)(\forall (a, b) \in J' \cup \{ \langle c, b' \rangle \})x_2(i) \in U^2_b \quad \text{and} \quad (\forall (a, b) \in J') f(a, c) = 0.
\]

For all \( \langle a, b \rangle \in J' \), there are arbitrarily small \( c \), with \( ds(a, c) = 0 \) and \( x \in U^1_c \); hence, \( x \in U^1_a \). Since \( J' \subseteq En(p) \), \( f(x) \in f\tilde{U}_a \subseteq U^2_b \) for all \( \langle a, b \rangle \in J' \). Since \( x \in U^1_c \) and \( c \triangleleft a' \) for some \( c \) and \( f\tilde{U}_c \subseteq U^2_b \), we obtain \( f(x) \in U^2_b \). Therefore, \( (\forall (a, b) \in J' \cup \{ \langle c, b' \rangle \}) f(x) \in U^2_b \). Hence, there is some \( i \), with \( x_2(i) \in U^2_b \) for all those \( \langle a, b \rangle \), which contradicts our assumption. We conclude that there are some \( c, t \) such that \( \Gamma \langle a, b \rangle \langle c, n, a', b', t \rangle = 1 + \langle c, b' \rangle \) and \( x \in U^1_c \). This proves Proposition 3.6.

Propositions 3.5 and 3.6 imply immediately that \( f \in \delta_\succ \Gamma \langle p, q \rangle \). It remains to show finite consistency.
Proposition 3.7. $\text{En} \Gamma(p, q)$ is finitely consistent.

Proof. Let $B \subseteq \text{En} \Gamma(p, q)$ be nonempty and finite. Assume $\bigcap \{U^i \mid c \in \text{pr}_1 B\} \neq \emptyset$. For each $m \in B$ there are numbers $c_m, n_m, a'_m, b'_m, t_m$ with $m = \langle c_m, b'_m \rangle$ and

$$\Gamma(p, q) \langle c_m, n_m, a'_m, b'_m, t_m \rangle = 1 + \langle c_m, b'_m \rangle.$$

Let $r \in B$ such that $n_r = \max \{n_m \mid m \in B\}$. Consider the determination of

$$\Gamma(p, q) \langle c_r, n_r, a'_r, b'_r, t_r \rangle = 1 + \langle c_r, b'_r \rangle = 1 + r.$$

Choose $J := \{\langle a'_m, b'_m \rangle \mid m \in B, n_m < n_r\}$. Then $J \subseteq \text{En}(p, n_r)$; hence, (3) of the definition of $\Gamma(p, q) \langle c_r, t_r \rangle$ must hold for the instance $J$. By assumption, $\bigcap \{U^i \mid m \in B\} \neq \emptyset$; hence, $d_s(a'_m, c_r) = 0$ for all $m \in B$ (since $c_m < a'_m$). We conclude that (3) of the definition of $\Gamma(p, q) \langle a, t \rangle \in \text{En}(q)$ is false. Therefore, $(\exists i)(\forall \langle a, b \rangle \in J \cup \{\langle c_r, b'_r \rangle\}) \mathcal{A}_2(i) \in U^i_q$, i.e. $(\exists i)(\forall m \in B)(\forall i \in U^i_q)$ (remember that $p(n_r) = 1 + \langle a'_r, b'_r \rangle$); hence, $\bigcap \{U^i_q \mid d \in \text{pr}_2 B\} \neq \emptyset$. This shows that $\text{En} \Gamma(p, q)$ is finitely consistent.

Note that in Theorems 3.3 and 3.4 we have not proved reducibility, i.e. $\delta_5 = \delta_6 \Gamma$, but only $(\forall p)\delta_5(p) \subseteq \delta_6 \Gamma(p)$. For each of the computable metric spaces in Example 2.3(1)-(5) and 2.3(8), the set $D_\succ$ is indeed recursively enumerable. Furthermore, the property “$D_\succ$ is r.e.” is transmitted under Cartesian product (Example 2.3(6)). As a corollary, each of the naming systems $\delta_5$ to $\delta_8$ supplies names for exactly all the continuous functions $f: \subseteq M_1 \to M_2$.

Corollary 3.8. (1) For any $f: \subseteq M_1 \to M_2$ and $i = 5, \ldots, 8$,

$f$ is continuous $\iff (\exists p \in B) f \in \delta_i(p)$.

(2) For any $f: \subseteq M_1 \to M_2$,

$f$ is $\delta_5$-computable $\iff f$ is $\delta_6$-computable,

$\iff f$ is $\delta_7$-computable.

(3) If $D_\succ = \{\langle i, j, k \rangle \mid d(x_k(i), x_k(j)) > v_0(k)\}$ is r.e. then, for any $f: \subseteq M_1 \to M_2$,

$f$ is $\delta_5$-computable $\iff f$ is $\delta_6$-computable.

For any $i = 5, \ldots, 8$, if $f \in \delta_i(p)$ then $f$ is not determined uniquely by $p$, but “uniquely up to its domain”. Note that $f \in \delta_i(p)$ if $\text{dom}(f) = \emptyset$.

Lemma 3.9. For any $i \in \{5, \ldots, 8\}$ and $p \in \text{dom}(\delta_i)$ there is a function $f \in \delta_i(p)$ such that, for all $f \in \delta_i(p)$, $f$ extends $f$.

The proof is left to the reader. By Lemma 3.9, from $\delta_i$ a representation $\delta_i$ of certain continuous functions can be defined by $\delta_i(p) :=$ the maximally defined function $f \in \delta_i(p)$. Simple characterizations of the classes of the represented functions are not known.
4. Extension and \( G_\delta \)-characterization of domains

It is known from topology [9, Section 35.1] that every partial continuous function \( f: \subseteq M_1 \rightarrow M_2 \) from a metric space \( M_1 \) to a complete metric space \( M_2 \) can be extended to a continuous function \( \tilde{f}: \subseteq M_1 \rightarrow M_2 \) such that \( \text{dom}(\tilde{f}) \) is a \( G_\delta \)-subset of \( M_1 \). (A \( G_\delta \)-set is a denumerable intersection of open sets.) Thus, \( G_\delta \)-sets are the "natural" domains of continuous functions \( f: \subseteq M_1 \rightarrow M_2 \) if \( M_2 \) is complete. An extension \( \tilde{f} \) can be defined as follows. For \( n \in \mathbb{N} \), let \( G_n := \bigcup \{ \{ \forall \langle i, n \rangle \in \text{En}(p) \text{ and } \text{rad}(j) \leq 2^{-n} \} \mid G_n \mid n \in \mathbb{N} \} \), \( \text{dom}(\tilde{f}) := G \cap \text{cls}(\text{dom}(f)) \), \( \{ \tilde{f}(x) \} := \bigcap \{ \text{cls}(f(H)) \mid H \text{ is a neighbourhood of } x \} \). This proof, however, is not effective. The naming system \( \delta_n \) is sufficiently strong to admit a computationally effective \( G_\delta \)-extension theorem. According to the representation \( \xi \) of the \( G_\delta \)-subsets of \( \mathbb{B} \) [18, Definition 3.2.10], we define a representation \( \eta: \mathbb{B} \rightarrow \{ A \subseteq M_1 \mid A \text{ is a } G_\delta \text{-set} \} \) by

\[
\eta(p) := \bigcap_n G_n, \quad \text{where } G_n := \bigcup \{ U_{\frac{1}{n}} \mid \langle i, n \rangle \in \text{En}(p) \}
\]

for all \( p \in \mathbb{B} \). As usual, we define \( \bigcup \emptyset := \emptyset \).

**Theorem 4.1.** For \( i = 1, 2 \), let \( M_i = (M_i, d, A_i, x_i) \) be a computable metric space with numbering \( U_i \) of basic neighbourhoods as defined above. Let \( M_2 \) be complete. Then there are computable functions \( \Sigma, \Gamma: \mathbb{B} \rightarrow \mathbb{B} \) such that for any \( p \in \text{dom}(\delta_0) \) there is a function \( \tilde{f}: \subseteq M_1 \rightarrow M_2 \) such that

1. \( \tilde{f} \) extends \( f \) for every \( f \in \delta_0(p) \),
2. \( \tilde{f} \in \delta_0 \Sigma(p) \),
3. \( \text{dom}(\tilde{f}) = \eta \Gamma(p) \).

**Proof.** There are computable functions \( \Sigma, \Gamma: \mathbb{B} \rightarrow \mathbb{B} \), with

\[
\begin{align*}
\text{En} \Sigma(p) & = \{ \langle i, n \rangle \mid (\exists j)(\langle i, j \rangle \in \text{En}(p) \text{ and } \text{rad}(j) \leq 2^{-n}) \}, \\
\text{En} \Gamma(p) & = \{ \langle k, m \rangle \mid (\exists \langle i, j \rangle \in \text{En}(p)) (k <_1 i \text{ and } j <_2 m) \}.
\end{align*}
\]

Assume \( p \in \text{dom}(\delta_0) \). For \( n \in \mathbb{N} \), define \( G_n := \bigcup \{ U_{\frac{1}{n}} \mid \langle i, n \rangle \in \text{En}(\Gamma(p)) \} \) and define \( \tilde{f}: \subseteq M_1 \rightarrow M_2 \) by \( \text{dom}(\tilde{f}) := G := \bigcap \{ G_n \mid n \in \mathbb{N} \} = \eta \Gamma(p) \) and

\[
\{ \tilde{f}(x) \} := \bigcap \{ U_{\frac{1}{n}} \mid (\exists i)(\langle i, j \rangle \in \text{En}(p) \text{ and } x \in U_{\frac{1}{n}}) \}
\]

for all \( x \in G \).

**Proposition 4.2.** \( \tilde{f}(x) \) is well-defined for \( x \in G \).

**Proof.** For \( m \in \mathbb{N} \), define

\[
A_m := \bigcap \{ U_{\frac{1}{n}} \mid (\exists i)(\langle i, j \rangle \leq m \text{ and } \langle i, j \rangle \in \text{En}(p) \text{ and } x \in U_{\frac{1}{n}}) \},
\]

where \( \bigcap \emptyset := M_2 \). Obviously, \( A_m \neq \emptyset \) by the consistency property of \( p \in \text{dom}(\delta_0) \), \( A_{m+1} \subseteq A_m \), and \( A_m \) is closed for any \( m \in \mathbb{N} \). Since \( x \in G \), \( x \in G_n \) for all \( n \), i.e. for all \( n \), there
are \( i, j \), with \( x \in U_i \), \( \langle i, j \rangle \in \text{En}(p) \) and \( \text{rad}(j) \leq 2^{-n} \). This shows that diameter \((A_m)\) converges to 0 with increasing \( m \). For any \( m \in \mathbb{N} \), let \( y_m \in A_m \) be arbitrary. The sequence \((y_m)_{m \in \mathbb{N}}\) is a Cauchy sequence. Since \( M_2 \) is complete, it has a limit \( y \). Since each \( A_m \) is closed, \( y \in A_m \) for all \( m \); hence,

\[
\{ y \} = \bigcap \{ A_m \mid m \in \mathbb{N} \} = \bigcap \{ \bar{U}_j^2 \mid (3i)(\langle i, j \rangle \in \text{En}(p) \text{ and } x \in U_i) \}.
\]

Therefore, \( f(x) = y \) is well-defined. \( \square \)

**Proposition 4.3.** If \( f \in \delta_8(p) \) then \( \bar{f} \) extends \( f \).

**Proof.** Assume \( f \in \delta_8(p) \) and \( x \in \text{dom}(f) \). Then, for each \( n \), there are \( i, j \), with \( \langle i, j \rangle \in \text{En}(p) \), \( x \in U_i \) and \( \text{rad}(j) \leq 2^{-n} \); hence, \( x \in G_n \) for each \( n \), i.e., \( x \in G \). Since \( f \in \delta_7(p) \),

\[
\{ f(x) \} \cap \{ \bar{U}_j^2 \mid (3i)(\langle i, j \rangle \in \text{En}(p) \text{ and } x \in U_i) \} = \{ f(x) \}.
\]

\( \square \)

**Proposition 4.4.** \( \bar{f} \in \delta_8 \Sigma(p) \).

**Proof.** Assume \( \langle i, j \rangle \in \text{En}(p) \) and \( x \in U_i \cap \text{dom}(\bar{f}) \). Then \( \bar{f}(x) \in U_j^2 \) by the definition of \( \bar{f} \). Therefore, \( f(U_i^1) \subseteq U_j^2 \) if \( \langle i, j \rangle \in \text{En}(p) \). Assume \( \langle k, m \rangle \in \text{En} \Sigma(p) \). Then for some \( \langle i, j \rangle \in \text{En}(p) \), \( f(U_k^1) \subseteq f(U_i^1) \subseteq U_j^2 \subseteq U_m^2 \). This proves the first condition for \( \bar{f} \in \delta_8 \Sigma(p) \).

Assume \( x \in \text{dom}(f) \) and \( \varepsilon > 0 \). Choose \( n \) with \( 2^{-n} < \frac{1}{2} \varepsilon \). We obtain

\[
x \in G_n \Rightarrow (3i, j)(\langle i, j \rangle \in \text{En}(p) \land x \in U_i \land \text{rad}(j) \leq 2^{-n})
\]

\[
\Rightarrow (3k, m, i, j)(\langle i, j \rangle \in \text{En}(p) \land k < i \land x \in U_k \land j < m \land \text{rad}(m) \leq 2^{-2n})
\]

\[
\Rightarrow (3k, m)(\langle k, m \rangle \in \text{En} \Sigma(p) \land x \in U_k \land \text{rad}(m) \leq \varepsilon).
\]

This proves the second condition for \( \bar{f} \in \delta_8 \Sigma(p) \). The consistency condition for \( \Sigma(p) \) follows easily from that for \( p \). \( \square \)

By means of Theorems 3.3 and 3.4, Theorem 4.1 can be transferred from \( \delta_8 \) to \( \delta_5 \), \( \delta_5 \) and \( \delta_7 \) with continuous functions \( \Sigma \) and \( \Gamma \), which are even computable if \( D_{\text{ar}} \) is r.e.

Theorem 4.1 implies another effective extension theorem, which we shall prove below. Let \( \delta_i : \subseteq \mathbb{B} \rightarrow M_i (i = 1, 2) \) be representations and let \( f_i : M_1 \rightarrow M_2 \) be a function. According to the definitions in [18], \( f \) is called

1. \( (\delta_1, \delta_2) \)-continuous \(( \text{-computable}) \) iff there is a continuous \(( \text{computable}) \) function \( \Gamma : \subseteq \mathbb{B} \rightarrow \mathbb{B} \), with \( f \delta_1 (p) = \delta_2 \Gamma (p) \) for all \( p \in \text{dom}(f \delta_1) \);

2. strongly \((\delta_1, \delta_2) \)-continuous \(( \text{-computable}) \) iff, in addition to (1), \( (\forall p \in \text{dom}(\delta_1), \Gamma \setminus \text{dom}(\delta_1)) \) \( p \in \text{dom}(\Gamma) \).

With Definition 3.1, \( f \in \delta_8(p) \) for some \( p \in \mathbb{B} \) \(( \text{computable} \ p \in \mathbb{B}) \) iff \( f \) is \((\delta_1, \delta_2) \)-continuous \((\text{-computable}) \).

For the strongly continuous functions there is a canonical representation; see [18, Definition 3.3.11].
Definition 4.5. Let $\delta_i : \mathbb{B} \rightarrow M_i$ $(i=1, 2)$ be representations. A representation $[\delta_1 \rightarrow \delta_2]$ of the strongly $(\delta_1, \delta_2)$-continuous functions is defined as follows:

$$p \in \text{dom} [\delta_1 \rightarrow \delta_2] : \iff \begin{cases} \{ \psi_p(\text{dom}(\delta_1)) \subseteq \text{dom}(\delta_2) \text{ and } (\forall q, q' \in \text{dom}(\delta_1)), \delta_1(q) = \delta_1(q') \Rightarrow \delta_2 \psi_p(q) = \delta_2 \psi_p(q') \} \\
\end{cases}$$

for all $p \in \mathbb{B}$, and

$$[\delta_1 \rightarrow \delta_2](\delta_1(q)) := \delta_2 \psi_p(q)$$

for all $p \in \text{dom} [\delta_1 \rightarrow \delta_2]$ and $q \in \text{dom}(\delta_1)$.

First we show that the restriction of a continuous function to a $G_\delta$-subset of its domain yields a strongly $(\delta_1, \delta_2)$-continuous function. We immediately prove an effective version.

Lemma 4.6. For $i=1, 2$, let $\bar{M}_i$ be a computable metric space with $c$-admissible representation $\delta_i$ and standard basis numbering $U_i$. There is a computable function $\Sigma : \mathbb{B} \rightarrow \mathbb{B}$ such that

$$f|_G = [\delta_1 \rightarrow \delta_2] \Sigma \langle p, r \rangle$$

whenever $f \in \delta_5(p)$ and $G := \eta(r) \leq \text{dom}(f)$.

Proof. First we consider the case $\delta_1(p) = x \iff \text{En}(p) = \{ i \mid x \in U_i \}$. For any Type 2 machine $T$, let $T(p)(n)$ be the output on position $n$ of $T$, with input $p \in \mathbb{B}$. Let $T_1$ be a Type 2 machine which computes the universal function $\Gamma_u$ of $\psi$ (i.e. $\Gamma_u \langle p, q \rangle = \psi_p(q)$). There is a Type 2 machine $T_2$ such that for all $p, q, r \in \mathbb{B}$ and $n \in \mathbb{N}$,

$$T_2 \langle p, q, r \rangle(n) = \begin{cases} T_1 \langle p, q \rangle(n) & \text{if } (n=0 \text{ or } T_2 \langle p, q \rangle(n-1) \text{ exists}) \\
\text{and } (\exists i, j)[i \in \text{En}(q) \text{ and } j, n \in \text{En}(r) \text{ and } i < j], \\
\text{div} & \text{otherwise.} \\
\end{cases}$$

Let $\Gamma : \mathbb{B} \rightarrow \mathbb{B}$ be the function computed by $T_2$. By the smn-theorem, there is a computable function $\Sigma : \mathbb{B} \rightarrow \mathbb{B}$, with $\Gamma \langle p, q, r \rangle = \psi_{\Sigma(p, r)}(q)$. Assume $f \in \delta_5(p)$, $q \in \text{dom}(\delta_1)$ and $G \subseteq \text{dom}(f)$. If $\delta_1(q) \in G$ then $f \delta_1(q) = \delta_2 \psi_p(q) = \delta_2 \psi_{\psi_{\Sigma(p, r)}(q)}$. If $\delta_1(q) \notin G$ then $\psi_{\Sigma(p, r)}(q)$ does not exist. Hence, $f|_G = [\delta_1 \rightarrow \delta_2] \Sigma \langle p, r \rangle$. If $\delta_1$ is $c$-admissible then $\delta_1 = \epsilon \delta_1$ and $[\delta_1 \rightarrow \delta_2] = \epsilon [\delta_1 \rightarrow \delta_2]$. This shows that the desired property holds for any $c$-admissible representation $\delta_1$. \qed

We can now prove the extension theorem.

Theorem 4.7. Let $\bar{M}_1 = (M_1, d, A_1, x_i)$ be a computable metric space with numbering $U_i$ of basic neighbourhoods as defined above and $c$-admissible representation $\delta_i$ $(i=1, 2)$. Let $\bar{M}_2$ be complete. Then there are computable functions $\Lambda, \Gamma : \mathbb{B} \rightarrow \mathbb{B}$ such that, for all
f: \subseteq M_1 \to M_2 and p \in \text{dom}(\delta_8),
\quad f \notin \delta_8(p) \Rightarrow [\delta_1 \to \delta_2]A(p) \text{ extends } f \text{ and } \text{dom}[\delta_1 \to \delta_2]A(p) = \eta \Gamma(p).

\textbf{Proof.} By Theorems 3.3 and 3.4, there is a computable function \( \Gamma_1 : \mathbb{R} \to \mathbb{R} \), with 
\( g \in \delta_8(p) \Rightarrow g \in \delta_5 \Gamma_1(p) \). Let \( \Sigma \) and \( \Gamma \) be the functions with the properties given in
Theorem 4.1. Let \( \Sigma' \) be the function \( \Sigma \) from Lemma 4.6. Define \( A(p) := \Sigma'(\Gamma_1 \Sigma(p), \Gamma(p)) \). Obviously, \( A \) is computable. Assume \( f \in \delta_8(p) \). Let \( \tilde{f} \) be the function from
Theorem 4.1. Then \( \tilde{f} \in \delta_8 \Sigma(p) \) and \( \eta \Gamma(p) = G = \text{dom}(\tilde{f}) \). Since \( \tilde{f} \in \delta_5 \Gamma_1 \Sigma(p) \), by
Lemma 4.6,
\[
\tilde{f} = f \mid_{\eta} = [\delta_1 \to \delta_2]A(p).
\]
Therefore, \( A \) has the desired properties. \( \square \)

If \( D_\pi \) is r.e., then Theorem 4.7 holds correspondingly for \( \delta_5, \delta_6 \) and \( \delta_7 \) instead of \( \delta_8 \).
In any case, for \( \delta_5 (\delta_6 \) and \( \delta_7 \) there are continuous functions \( A \) and \( \Gamma \). As a simple
consequence, we obtain the following corollary.

\textbf{Corollary 4.8.} Let \( M_1, \delta_i (i = 1, 2) \) be as in Theorem 4.7. For every \( f: \subseteq M_1 \to M_2, \)
(1) if \( f \) is \( (\delta_1, \delta_2) \)-continuous then \( f \) has a strongly \( (\delta_1, \delta_2) \)-continuous extension.

(2) If \( D_\pi \) is r.e. and \( f \) is \( (\delta_1, \delta_2) \)-computable then \( f \) has a strongly \( (\delta_1, \delta_2) \)-computable
extension.

In Theorem 4.7 the domain of the function \( [\delta_1 \to \delta_2]A(p) \) is a \( G_\delta \)-set. Since
\( \psi = [1_\mathbb{N} \to 1_\mathbb{N}] \), the domains of all strongly \( (1_\mathbb{R}, 1_\mathbb{R}) \)-computable functions are \( G_\delta \)-
subsets of \( \mathbb{B} \) [18]. If \( \delta \) is the standard representation of the set \( \mathbb{R} \) of real numbers, then
the strongly \( (\delta, \delta) \)-continuous functions are the continuous functions the domains of
which are \( G_\delta \)-subsets of \( \mathbb{R} \) (see [5]). In what follows, we generalize this result. As
a condition, we assume that the representation \( \delta \) is "almost injective"; more precisely,
that \( \delta^{-1}\{x\} \) is compact for every \( x \in D_1 \).

A subset \( A \subseteq \mathbb{B} \) of Baire's space is compact iff it is closed and bounded, where \( g \in \mathbb{B} \)
is a bound of \( A \) iff \( (\forall f \in A) (\forall n) f(n) \leq g(n) \). For the real line \( \mathbb{R} \), Kreitz and Weihrauch
[8] have studied different representations of the set \( K(\mathbb{R}) \) of compact sets. The
representation we shall consider here corresponds to the "weak" representation of
\( K(\mathbb{R}) \).

\textbf{Definition 4.9.} Define the weak representation \( \kappa_\omega: \subseteq \mathbb{B} \to K(\mathbb{B}) \) of the compact sub-
sets of Baire's space \( \mathbb{B} \) by
\[
\kappa_\omega(s, t) = A \iff A \text{ is bounded by } t \text{ and } \mathbb{B} \setminus A = \bigcup \{v^*(i)\mid i \in \text{En}(s)\} \text{ for all } s, t \in \mathbb{B}.
\]
(\text{Remember that } v^* \text{ is the standard numbering of } N^* \text{ and } [w] = \{p \in \mathbb{B} \mid w \subseteq p\}, \text{ where } w \subseteq p \text{ means that } w \text{ is a prefix of } p; \text{ see [18]}.)

\textbf{Theorem 4.10.} For \( i = 1, 2 \), let \( \bar{M}_i \) be a computable metric space with \( c \)-admissible
representation \( \delta_i \). Assume that there is a computable function \( I_0: \subseteq \mathbb{B} \to \mathbb{B} \), with
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\[ \delta^{-1}_1 \delta^1(q) = \kappa \Gamma_0(q) \text{ for all } q \in \text{dom}(\delta_1). \]

Then there is a computable function

\[ \Sigma: \subseteq B \rightarrow B, \]

with

\[ \text{dom}([\delta_1 \rightarrow \delta_2](p)) = \eta \Sigma(p) \]

for all \( p \in \text{dom}([\delta_1 \rightarrow \delta_2]). \)

Especially, the domain of every strongly \((\delta_1, \delta_2)\)-continuous function is a \(G_\delta\)-set.

**Proof.** Any computable function \( \Gamma: \subseteq B \rightarrow B \) can be generated by an isotone computable (more precisely, \((\nu^*, \nu^*)\)-computable) function \( \gamma: \mathbb{N}^* \rightarrow \mathbb{N}^* \), where isotone means \( x \leq y \Rightarrow \gamma(x) \leq \gamma(y) \), as follows:

\[ \Gamma(p) = q \iff q = \sup \{ \gamma(p^n) \mid n \in \mathbb{N} \}; \]

see [18, Theorem 3.1.26]. Let \( \delta_4 \) be the c-admissible representation of \( M_1 \) corresponding to \( \delta_2 \) in Definition 2.4. Note that \( \delta_4[x_i] = U_1 \) for any \( x \in \mathbb{N}^* \) and \( i \in \mathbb{N} \), and that \( \{ y \in \mathbb{N}^* \mid [y] \cap \text{dom}(\delta_4) \neq \emptyset \} \) is recursively enumerable (more precisely, \( \nu^*\)-r.e.). Since \( \delta_4 \leq \delta_1 \), there is a computable function \( \Sigma_4: \subseteq B \rightarrow B \) such that \( (\forall q \in \text{dom}(\delta_4))\delta_4(q) = \delta_1 \Sigma_4(q) \). Let \( \gamma_1, \gamma_2, \gamma_4: \mathbb{N}^* \rightarrow \mathbb{N}^* \) be isotone computable functions such that \( \gamma_1 \) generates \( \Gamma_1 \), \( \gamma_2 \) generates \( \Gamma_2 \) and \( \gamma_4 \) generates \( \Sigma_4 \), where \( \Gamma_1 \) and \( \Gamma_2 \) are defined by \( \Gamma_0(q) = \langle \Gamma_1(q), \Gamma_2(q) \rangle \). For \( q \in B \), let \( B(q) = \{ r \in B \mid (\forall i) r(i) \leq q(i) \} \) and for \( z \in \mathbb{N}^* \), let \( B(z) = \{ w \mid \text{lg}(w) = \text{lg}(z) \) and \( (\forall i < \text{lg}(z)) w(i) \leq z(i) \} \).

Let \( M \) be a Type 2 machine which computes the universal function of \( \psi \). For \( p \in \mathbb{N} \), \( w \in \mathbb{N}^* \) and \( n \in \mathbb{N} \) define \( Q(p, w, n) \Rightarrow M \) with inputs \( p \) and \( w \) writes, after some time, some number on the output position \( n \). Finally, for \( v \in \mathbb{N}^* \), define \( \text{En}(v) = \{ m \mid (\exists i < \text{lg}(v)) v(i) = 1 + m \} \). There is a computable function \( \Sigma: \mathbb{N} \rightarrow \mathbb{N} \) such that, for all \( p \in \mathbb{N} \) and \( i \in \mathbb{N} \), \( \langle i, n \rangle \in \text{En}(\Sigma(p)) \) if there is some \( y \in \mathbb{N}^* \), satisfying

1. \( i \) is the last symbol of \( y \),
2. \( \nu \gamma_1(\nu \gamma_2(\nu))(y) \) is \( \nu^* \gamma^*(m) \subseteq w \) \( \lor \) \( Q(p, w, n) \).
3. \( (\forall w \in B \gamma_2 \gamma_4(v))(y)[(\exists m \in \text{En} \gamma_1 \gamma_4(v))(y)) \nu^*(m) \subseteq w \lor Q(p, w, n)] \)

We prove that \( \text{dom}([\delta_1 \rightarrow \delta_2](p)) = \eta \Sigma(p) \) (remember that \( \eta \Sigma(p) = \bigcap G^p_\Gamma \), where \( G^p_\Gamma = \bigcup \{ U \mid \langle i, n \rangle \in \text{En}(\Sigma(p)) \} \)). Assume \( p \in \text{dom}([\delta_1 \rightarrow \delta_2]), x \in \text{dom}([\delta_1 \rightarrow \delta_2](p)) \) and \( n \in \mathbb{N} \). We shall prove \( x \in G^p_\Gamma \). There is some \( r \in \text{dom}(\delta_4) \), with \( \delta_4(r) = x \). Define \( q := \Sigma_4(r) \), then \( q \in \text{dom}(\delta_1) \) and \( x = \delta_1(q) \).

By the assumption on \( \Gamma_1 \) and \( \Gamma_2 \), for all \( q' \in B \Gamma_2(q) \),

\[ (\exists m \in \text{En} \Gamma_1(q)) v^*(m) \subseteq q' \text{ or } \nu \psi(q')(n) \text{ exists.} \]

Hence, for all \( q' \in B \Gamma_2(q) \), there is some \( w \neq q' \), with

\[ (\exists m \in \text{En} \Gamma_1(q)) v^*(m) \subseteq w \lor Q(p, w, n). \]

Since \( B \Gamma_2(q) \) is compact, the words \( w \) can be chosen from a finite set \( X_0 \) of words. Furthermore, there is only a finite set \( A_0 \) of values \( m \in \text{En} \Gamma_1(q) \) which have to be considered.

Let \( k_0 = \max \{ \text{lg}(w) \mid w \in X_0 \} \) and let \( z \) be a prefix of \( q \) such that \( k := \text{lg} \gamma_2(z) \geq k_0 \) and \( A_0 \subseteq \text{En} \gamma_1(z) \). The finite set \( X_0 \) can be replaced by the set \( B \gamma_2(z) \) of words with
length $k$; hence,

$$(\forall y \in B \gamma_2(z))[(\exists m \in \Sigma_1 \gamma_1(z))v^*(m) \subseteq w \land Q(p, w, n)].$$

Every $z'$ with $z \subseteq z'$ has the same property. There is some $y$, with $y \subseteq r$, such that $z \subseteq \gamma_4(y)$. Let $i$ be the last symbol of $y$. Then, obviously, properties (1)–(3) hold; hence, $\langle i, n \rangle \in \Sigma(p)$. Since $x \in U_i$, we obtain $x \in G^p_n$. On the other hand, assume $p \in \text{dom}(\delta_1 \rightarrow \delta_2)$, $x = \delta_1(q)$. Assume $n \in \mathbb{N}$ and $x \in G^p_n$. There are some $y \in \Sigma^*$ and $i \in \mathbb{N}$ such that $x \in U_i$ and (1)–(3) hold. From (1) and (2) we conclude $x \in \delta_4[y]$; hence, $x \in \delta_1[\gamma_4(y)]$. We obtain $\delta_1^{-1}\{x\} \subseteq \bigcup\{[w] \mid w \in B \gamma_2 \gamma_4(y)\}$ and $\delta_1^{-1}\{x\} \cap [v^*(m)] = \emptyset$ for all $m \in \Sigma \gamma_1 \gamma_4(y)$. Let $w$ be the prefix of $q$ with $\lg(w) = \lg(\gamma_2 \gamma_4(y))$. Then $w \in B(\gamma_2 \gamma_4(y))$ since $q \in [v^*(m)]$. Thus, (3) holds for this word $w$. $v^*(m) \subseteq w$ for some $m \in \Sigma \gamma_1 \gamma_4(y)$ would imply $q \in [v^*(m)]$ and $\delta_1^{-1}\{x\} \cap [v^*(m)] \neq \emptyset$. Therefore, $Q(p, w, n)$ must hold. Consequently, $M$ with inputs $p$ and $q$ writes on the output position $n$. Thus, if $x \in \bigcap G^p_n$, then $\psi_p(q)$ exists, i.e. $x \in \text{dom}([\delta_1 \rightarrow \delta_2](p))$. □

Since $[\delta_1 \rightarrow \delta_2] = [\delta_1 \rightarrow \delta_2]$ if $\delta_1 \equiv \varepsilon \delta_1$ and $\delta_2 \equiv \varepsilon \delta_2$, it suffices to assume in Theorem 4.10 that there are some $\delta_1 \equiv \varepsilon \delta_1$ and a computable function $\Gamma_0$ with $Q(q \in \text{dom}([\delta_1 \rightarrow \delta_2]^{-1}\{\delta(q)\})) = \kappa \mu \Gamma_0(q)$. By Lemma 4.6 and Theorem 4.10, under the compactness condition of Theorem 4.10, the strongly $(\delta_1, \delta_2)$-computable (-continuous) functions $f : M_1 \rightarrow M_2$ are exactly the $\delta_2$-computable (-continuous) functions the domains of which are computable (arbitrary) $G_\delta$-sets. Consider the computable Euclidean $n$-space $\mathbb{E}_n := (\mathbb{R}^n, d, A_n, \tau_n)$ as an example. Clearly, $\mathbb{M}$ is computable, and $\mathbb{D}_\mathbb{M}$ is r.e. Hence, $\delta_\mathbb{M} \equiv \varepsilon \delta_5$ for $M$. Therefore, Theorems 4.1 and 4.7 hold immediately for the case $\mathbb{M}_1 = \mathbb{E}_m, \mathbb{M}_2 = \mathbb{E}_n$.

For applying Theorem 4.10 to the real line we need a representation $\delta$ which is $\varepsilon$-equivalent to the standard representation and has the property

$$(\forall q \in \text{dom}(\delta))\delta^{-1}\{\delta(q)\} = \kappa \mu \Gamma_0(q)$$

for some computable $\Gamma_0 : \mathbb{N} \rightarrow \mathbb{R}$ as follows (see [14]):

$p \in \text{dom}(\delta) \iff (\forall i \in \mathbb{N})p(i) \leq 3, p(i) = 3$

for exactly one number $i$, and no word from

$\{0, 02, 20\}$

is a prefix of $p$.

$\delta(a_k a_{k-1} \ldots a_0 3 a_{-1} a_{-2} \ldots) = \Sigma \{a_i - 2^i \mid i \in \mathbb{Z}, \ i \leq k\}$, where $a_k, a_{k-1}, \ldots \in \{0, 1, 2\}$.

Thus, $\delta$ is the binary representation with digits $-1, 0, 1$ (represented by 0, 1, 2) and normed integer part. Note that, for norming, a leading 0 can be omitted, $-11$ can be replaced by $-1$ and $1-1$ can be replaced by 1 in front of the binary point. This representation $\delta$ is indeed $\varepsilon$-admissible and a $\kappa \mu$-name of $\delta^{-1}\{\delta(q)\}$ can be determined computably from $q$. Thus, Theorem 4.10 can be applied to the real line [5]. If $\mathbb{M}_1$ and $\mathbb{M}_2$ have $\varepsilon$-admissible representations $\delta_1$ and $\delta_2$, respectively, satisfying the compactness condition from Theorem 4.10, then $[\delta_1, \delta_2]$ (see [18]) is a $\varepsilon$-admissible representation of $\mathbb{M}_1 \times \mathbb{M}_2$, again satisfying this condition. Hence, Theorem 4.10 can also be applied to the Euclidean $n$-spaces. For example, if $\delta$ is a standard
representation of \( \mathbb{R} \), and \( \delta^n \) a standard representation of \( \mathbb{R}^n \), then a function \( f: \subseteq \mathbb{R}^n \rightarrow \mathbb{R} \) is strongly \((\delta^n, \delta)\)-computable iff \( f \) is \((\delta^n, \delta)\)-computable and \( \text{dom}(f) \) is a computable \( G_\alpha \)-subset of \( \mathbb{R}^n \).

5. Conclusions

The general results obtained here can be immediately applied to analysis and functional analysis.

From computability theory on \( \mathbb{R} \), it is known that only very special representations admit a reasonable theory of computational complexity, e.g. the modified binary representations defined above \([4, 14]\). As a general requirement for representations which admit a reasonable complexity theory, we suggest that \( \delta^{-1} K \) be compact for each compact subset \( K \) of the metric space. Detailed investigations have to be performed. In \([18, \text{Chapter 3.7}]\) a general effectivity theory for the class of all domains (i.e. certain complete partial orders) is outlined. Similarly, a general effectivity theory of essentially all separable metric spaces, not only the computable ones, can be developed. Let \( M = (M, d, A, x) \) be a computable metric space. If \( (M, d) \) is complete, then it is determined uniquely up to an isomorphism by the recursively enumerable set

\[
D_\prec := \{ \langle i, j, k \rangle | d(x(i), x(j)) < v_q(k) \},
\]

i.e. \((M, d)\) is essentially the completion of the pseudometric space \((\mathbb{N}, d')\), where \( d'(i, j) := d(x(i), x(j)) \). Omitting the requirement of recursive enumerability, we first define a representation \( \delta_0 \) of the pseudometric spaces with carrier \( \mathbb{N} \) by

\[
\delta_0(p) = (\mathbb{N}, d') \iff \text{En}(p) = \{ \langle i, j, k \rangle | d(i, j) < v_q(k) \}.
\]

Define a representation \( \delta \) of metric spaces by

\[
\delta(p) = \text{the metric completion of } \delta_0(p).
\]

Then a complete metric space \((M, d)\) is separable iff it is isomorphic to some space in \( \text{range}(\delta) \). If incomplete separable metric spaces have to be considered, define a representation \( \delta \) of sets of metric spaces by \( \delta(p) = \{(M, d) | (M, d) \text{ is a restriction of } \delta(p)\} \). Each of the theorems presented in this paper has a version \( \delta \)-computationally uniform on all separable metric spaces, which is obtained by omitting the condition that \( D_\prec \) is recursively enumerable and adding a parameter \( p \) with \( \text{En}(p) = D_\prec \).

References