Constrained Stochastic Controllability of Infinite-Dimensional Linear Systems

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The main purpose of this article is to investigate the problem of \((\varepsilon, \delta)\)-stochastic controllability for linear systems of evolution type in infinite-dimensional spaces, wherein the controls are subjected to norm-bounded constrained sets. Some basic prerequisites of infinite-dimensional measures, in particular, Gaussian distributed type, are discussed. Corresponding to this measure, various properties of \((\varepsilon, \delta)\)-stochastic attainable sets in Hilbert spaces are studied. Necessary and sufficient conditions for \((\varepsilon, \delta)\)-stochastic controllability with respect to Hilbert space valued linear systems are obtained. Relationships with the deterministic counterpoint are noted. Pursuit game problems are also considered. Examples on systems governed by stochastic linear partial differential equations and stochastic differential delay equations are given for illustration.

1. INTRODUCTION

Recently, there are many reports on the study of controllability for infinite-dimensional systems, including those presented by Balakrishnan [1], Curtain and Pritchard [3] and the comprehensive survey given by Russell [2]. However, the study of stochastic controllability in this area, as far as the authors are aware, seems not to have been taken up by many researchers. The present article is devoted to an investigation of the results on \((\varepsilon, \delta)\)-stochastic controllability of linear systems in infinite-dimensional spaces. Here, the control inputs under consideration are also restricted to some simple constrained (or, more explicitly, norm-bounded) sets. In the first part, by taking up the approach as presented by Boyarsky [4] in \(\mathbb{R}^n\), we give the definition of deterministic and \((\varepsilon, \delta)\)-stochastic attainable sets for infinite-dimensional spaces. In the second part, conditions for \((\varepsilon, \delta)\)-stochastic controllability w.r.t. a single control and stochastic non-cooperative game problems are considered. For the former case, it is interesting to note that when the system is being reduced to the deterministic situation, \((\varepsilon, \delta)\)-stochastic controllability will simply imply approximate controllability. Further, the results we obtained in the theorems are illustrated by some
examples dealing with stochastic linear partial differential equations and stochastic differential delay equations which usually arise in stochastic control problems for distributed systems.

2. Preliminaries and Basic Problems

Let $X$ be a real separable Hilbert space and $\mathcal{B}$ be the $\sigma$-algebra of Borel sets in $X$ (because of the separability of the space $X$, it is sufficient that $\mathcal{B}$ contain all spheres). Let $\mu$ be a normalized Gaussian measure defined on the measurable space $(X, \mathcal{B})$.

**Definition 2.1.** By a Gaussian measure $\mu$ in Hilbert space $(X, \mathcal{B})$ we mean that $\mu$ has the characteristic functional in the form

$$\Theta(x) = \exp\{i\langle a, z \rangle_x - \frac{1}{2}\langle Az, z \rangle_x\}, \quad (2.1)$$

where $a \in X$ and $A$ is a symmetric bounded non-negative nuclear operator on $X$ and is completely continuous. If $A$ is strictly positive, the measure $\mu$ is called non-degenerate.

**Definition 2.2.** Let $L$ be a finite-dimensional subspace of $X$, $\mathcal{P}_L$ be the orthogonal projection operator on $L$ and $B$ a Borel set from $L$. A set of the form

$$\{x: \pi_L x \in B\} \quad (2.2)$$

will be defined as a cylinder set with set $B$ as its base. The set of cylinder sets with bases in $L$ is a $\sigma$-algebra written as $\mathcal{B}^L$, $\mathcal{B}^L \subset \mathcal{B}$.

For each finite-dimensional subspace $L$ of $X$, we can consider the restriction of this measure to $\mathcal{B}^L$ and define the measure $\mu_L$ on the $\sigma$-algebra $\mathcal{B}_L$ of Borel sets of $L$ as follows:

$$\mu_L(B) = \mu(\{x: P_L x \in B\}). \quad (2.3)$$

Hence, corresponding to each measure $\mu$, we can associate the set of its projections $\{\mu_L\}$ on finite-dimensional subspaces of $X$. In turn, knowing $\mu_L$, we can also determine $\mu$ on $\mathcal{B}^L$. With these facts, we come to the following observation which will be useful in the sequel.

It follows from (2.1) that all finite-dimensional projections $\mu_L$ of the Gaussian measure $\mu$ are also Gaussian measures in the respective subspaces. Hence, we can define a set of subspaces $L$, where projections upon which completely determine $\mu$ and, on $L$ the structure of $\mu$ is relatively simple.
Let \( \{f_i\} \) be a complete orthonormalized system of eigenvectors of operator \( A \) and \( \lambda_i \) the corresponding eigenvalues. Let \( L_n = \langle \{f_1, \ldots, f_n\} \rangle \) and \( L^{(i)} = \langle \{f_i\} \rangle \). Then for \( z = \sum_{k=1}^n \xi_k f_k \in L_n, \xi_k \in \mathbb{R}, \)

\[
\Theta(z) = \text{the respective } n\text{-dimensional characteristic functional}
\]

\[
= \exp \left\{ i \left( a, \sum_{k=1}^n \xi_k f_k \right)_X - \frac{1}{2} \left( A \sum_{k=1}^n \xi_k f_k, \sum_{j=1}^n \xi_j f_j \right)_X \right\}
\]

\[
= \prod_{k=1}^n \frac{1}{\sqrt{2\pi\lambda_k}} \int_{-\infty}^{\infty} e^{it_k x_k} - \frac{(x_k - \beta_k)^2}{2\lambda_k} \, dx_k
\]

\[
= \sum_{k=1}^n (2\pi\lambda_k)^{-1/2} \int_{-\infty}^{\infty} e^{-i(x,x)-(1/2)(A^{-1}(x-\alpha_n)\cdot x-\alpha_n)m_{L_n}(dx)}.
\]

where \( \beta_k = \langle a, f_k \rangle_X, x = \sum_{k=1}^n x_k f_k, \alpha_n = \sum_{k=1}^n \beta_k f_k, m_{L_n}(dx) = \prod_{k=1}^n dx_k \)
equivalently, a Lebesgue measure on \( L_n \).

Hence

\[
\mu_{L_n}(B) = \prod_{k=1}^n \frac{1}{\sqrt{2\pi\lambda_k}} \int_B \exp \left\{ -\frac{1}{2} \left( A^{-1}(x-\alpha_n), (x-\alpha_n) \right) \right\} m_{L_n}(dx) \quad (2.4)
\]

for any Borel set \( B \in \mathcal{B}_{L_n} \). Together with the k-projection \( \mathcal{R}_{(k)} \), defined as

\[
\mathcal{R}_{(k)} L_n = L^{(k)}
\]
we obtain

\[
\mu_{L^{(k)}}(B) = \frac{1}{\sqrt{2\pi\lambda_k}} \int_B \exp \left\{ -\frac{1}{2} \left( A^{-1}(x-\beta_k f_k), (x-\beta_k f_k) \right) \right\} m_{L^{(k)}}(dx). \quad (2.5)
\]

Since \( L_n = \prod_{i=1}^n L^{(i)} \), by (2.4) and (2.5) we have

\[
\mu_{L_n} = \prod_{n=1}^n \mu_{L^{(n)}}. \quad (2.6)
\]

Remark 2.1. \( X \) may be viewed as \( L_\infty = \prod_{k=1}^\infty L^{(k)} \) and \( X \) is measurable w.r.t. \( \mathcal{B}_{L_\infty} \). In fact \( X = \bigcup_{k=1}^\infty \bigcap_{n=1}^\infty C_{n,k}, C_{n,k} \) is a cylinder set from \( L_\infty \) and \( C_{n,k} = \{(x_1, x_2, \ldots): \sum_{j=1}^n \langle x_j, f_j \rangle^2 \leq k, x_j \in L^{(j)}\} \).

With the definition of an infinite product measure, \( \prod_{k=1}^\infty \mu_{L^{(k)}}(k) \) is defined on cylinder sets \( Z \subset L_\infty \) where \( Z = \{(z_1, z_2, \ldots): z_k \in B_k, B_k \in \mathcal{B}_{L^{(k)}}, k = 1, \ldots, n\} \). Since \( (\prod_{k=1}^\infty \mu_{L^{(k)}}(k)) (Z) \) (by definition) = \( \prod_{k=1}^n \mu_{L^{(k)}}(B_k) = \mu(\{x: P_L^{(k)}(k)x \in B_k, k = 1, 2, \ldots, n\}) \), the extension of measure \( \prod_{k=1}^\infty \mu_{L^{(k)}}(k) \) to \( \mathcal{B}_{L_\infty} \), the minimal \( \sigma \)-algebra containing all cylinder sets in \( L_\infty \), is possible. Besides, extension of \( \mu \) to \( L_\infty \) done by setting \( \mu_{L_\infty}(E) = \mu(E \cap X), E \in \mathcal{B}_{L_\infty} \) is
also possible. Further $\prod_{k=1}^{\infty} \mu_L(k)$ and $\mu_{L^X}$ coincide on all cylinder sets from $L_X$, hence coincide on $\mathcal{B}_{L^X}$, so this leads to

$$\mu_{L^X} = \prod_{k=1}^{\infty} \mu_L(k).$$

Further, $\mu_{L^X}$, defined on the measurable space $(L^X, \mathcal{B}_{L^X})$, has support $X$. Then, by Remark 2.1, $\mu_{L^X} = \mu$ and so

$$\mu = \prod_{k=1}^{\infty} \mu_L(k). \quad (2.7)$$

Thus, the non-degenerate Gaussian measure $\mu$, given in Definition 2.1 can be represented as an infinite product of measures defined on an orthogonal system of linear one-dimensional subspaces. For the case of partially degenerate $\mu$, a product expression like (2.7) also exists, but it consists of two different product measures, one defined on $M$ (the closure of the range of operator $A$) and the other on $M'$'s orthogonal complement.

Let $X$ and $U$ be real separable Hilbert spaces and $(X, \mathcal{B}, \mu)$ be a measure space, $\mu$ being a Gaussian measure and $(\Omega, \mathcal{F}, P)$ be a given complete probability space, $x(t, \omega)$ is an $X$-valued stochastic process from $\mathbb{R}^+ \times \Omega \rightarrow X$, satisfying the stochastic evolution equation

$$dx + A(t)x = B(t)u(t) \, dt + C(t) \, dw(t)$$

with initial condition

$$x(0) = x_0, \quad (2.8)$$

where $x_0 \in L_2(\Omega, \mathbb{P}; X)$ is a second-order Gaussian $X$-valued random variable with mean zero and covariance operator $P_0 \in L(X)$; $w(t) \in L^2(\Omega, \mathbb{P}; X)$ is an $X$-valued Wiener process and $w(t) = \sum_{i=1}^{\infty} \beta_i(t) e_i \, w.p. \, 1$, $\{e_i\}_{i=0}^{\infty}$ a complete orthonormal basis for $X$, where $\beta_i(t)$ are mutually independent real Wiener process with incremental covariance $\lambda_i$ and $\sum_{i=0}^{\infty} \lambda_i < \infty$, i.e., trace $W = \sum_{i=0}^{\infty} \lambda_i < \infty$, where $W$, nuclear in its nature, is the covariance operator of $w(t)$, besides $w(t)$ is independent with $x_0$.

Further, $B(t)$ and $C(t)$ are respective norm-continuous bounded linear operators from $U$ to $X$ and $X$ to $X$. $A(t)$, being a non-constant operator, is linearly defined on $X$.

To have a solution for (2.8), we require the following assumptions on the operator $A(t)$ (cf. [6]):

($A_1$) The domain $D_A$ of $A(t)$ $(t \in \mathbb{R}^+)$ is dense in $X$ and is independent of $t$, and $A$ is a closed operator.
(A₂) For each \( t \in \mathbb{R}^+ \), the resolvent set \( R[\lambda; A(t)] = [\lambda I - A(t)]^{-1} \) of \( A(t) \) exists for all \( \lambda \) and
\[
\|R[\lambda; A(t)]\| \leq \frac{\beta}{|\lambda| + 1} \quad \text{with} \quad \Re \lambda \leq 0. \tag{2.9}
\]

(A₃) For any \( t, s, \tau \in \mathbb{R}^+ \)
\[
\|[A(t) - A(\tau)] A^{-1}(s)\| \leq \beta |t - \tau|^\alpha \quad (0 < \alpha < 1), \tag{2.10}
\]

where the constants \( \beta \) and \( \alpha \) are independent of \( t, \tau, s \), i.e., by (2.10), the bounded operator \( A(t) A^{-1}(s) \) is Hölder continuous in \( t \) in the uniform operator topology for each fixed \( s \).

Then there exists a unique fundamental solution \( S(t, \tau) \) of (2.8), belonging to \( B(X) \) (the space of bounded continuous operators from \( X \) to itself) and it is norm continuous in \( t, \tau \in [0, \infty) \). Further it satisfies the following properties:

(a) \( \partial S(t, \tau)/\partial t \in B(X) \) for \( 0 \leq \tau < t < \infty \) and is norm-continuous in \( t \), for \( t \in [\tau, \infty) \), \( \tau \geq 0 \);
(b) \( S(t, \tau): D(A) \to D(A) \);
(c) \( \partial S(t, \tau) z/\partial t = -A(t) S(t, \tau) z, z \in D(A), 0 \leq \tau < t < \infty, S(t, \tau) = 1 \).

Hence, \( S(t, \tau) \) is an almost strong evolution operator. With the operator \( S(t, \tau) \) defined as above, we obtain the mild solution of (2.8):
\[
x(t) = S(t, 0) x_0 + \int_0^t S(t, \tau) B(\tau) u(\tau) \, d\tau + \int_0^t S(t, \tau) C(\tau) \, dw(\tau), \tag{2.11}
\]

where \( u \), the admissible control, belongs to certain abstract Lebesgue integrable space \( L_p([0, T]; U) \) defined later. Further, all integrations appearing in (2.11) and all expressions in the following are in Bochner sense.

In the autonomous situation, the fundamental solution of (2.8) has the form \( T_{t - \tau} \). We note that assumption (A₂) implies
\[
(A₂') \quad \lambda \in \rho(A) = R[\lambda, A] \quad \text{for} \quad \Re \lambda \leq 0
\]

and
\[
\|(R[\lambda; A])^n\| \leq \frac{\beta}{|\lambda|^n}, \quad n = 1, 2, 3, \ldots.
\]

If (A₁) and (A₂') hold, then \(-A\) will generate a strongly continuous semigroup \( \{T_t\}: t \in \mathbb{R}^+ \) with the following additional properties (cf. [3]):
(a) $T_t$ can be continued analytically into the sector
$$S_\omega: |\arg t| < \omega; \omega \in (0, \pi/2)t \neq 0.$$

(b) For each $t \in S_\omega$, and all $x \in X$
$$\frac{d}{dt} T_t x = -AT_t x \quad \text{and} \quad AT_x \in L(X).$$

(c) For any $0 < \epsilon < \omega$, $\exists k > 0$ such that
$$\|T_t\| \leq k, \quad \|AT_t\| \leq \frac{k}{|t|}, \quad t \in S_{\omega-\epsilon},$$
where we can represent $T_t$ by $\exp\{tA\}$.

Although in the sections which follow, we only deal with time-dependent systems, the same results will still hold for the autonomous situation only with the replacement of $S(t, \tau)$ by $T_{t-\tau}$ and all time-varying matrices replaced by constant matrices.

Let
$$L_p([0, t_1]; U) = \left\{ u: [0, t_1] \to U \text{ be abstract Lebesgue measurable} \right\},$$
and
$$\int_0^{t_1} \|u(t)\|_U^p \, dt < \infty, \quad (2.12)$$
and denote the equivalent classes by $L_p([0, t_1]; U)$ with respect to the norm
$$\|u\|_p = \left( \int_0^{t_1} \|u(t)\|_U^p \, dt \right)^{1/p}, \quad 1 \leq p < \infty, \quad (2.12a)$$

Then $L_p([0, t_1]; U)$ is a Hilbert space.

Define the admissible constrained control set by
$$U_{ad}^p = \bigcup_{t_1 \geq 0} L_p^0([0, t_1]; U)$$
$$= \bigcup_{t_1 \geq 0} \{ u \in L_p([0, t_1]; U): \|u\|_p^p \leq \rho^{1/2} < \infty, \rho > 0 \}. \quad (2.13)$$

In the sequel we need the following lemmas.

**Lemma 2.1.** The sets $L_p^0([0, t_1]; U)$ are convex and weakly compact in $L_p([0, t_1]; U)$. 

Lemma 2.2. For each \( t \in \mathbb{R}^+ \), the mapping \( A_t: L_\rho([0, t]; U) \to X \) defined by
\[
A_t(u) = \int_0^t S(t, \tau) B(\tau) u(\tau) \, d\tau
\]
is a linear continuous operator and
\[
R_t = A_t(L_\rho^0([0, t]; U))
\]
is convex and weakly compact.

Lemma 2.3. If \( M \) and \( N \) are closed convex sets in \( X \) with \( M \) being weakly compact, then \( M + N \) is closed convex in \( X \). In addition, if \( N \) is weakly compact, then \( M + N \) is weakly compact in \( X \).

3. Attainable Sets

Corresponding to system (2.8) we have the following deterministic version
\[
dx(t) + A(t) x(t) \, dt = B(t) u(t) \, dt,
\]
with \( x(0) = x_0 \).

Definition 3.1. The deterministic attainable set of (3.1) at time \( t \) is defined by
\[
K(t) = \left\{ z \in X : z = S(t, 0) x_0 + \int_0^t S(t, \tau) B(\tau) u(\tau) \, d\tau : u \in L_\rho^0([0, t]; U) \right\}.
\]
In other words,
\[
K(t) = S(t, 0) x_0 + R_t,
\]
so by Lemmas 2.2 and 2.3, \( K(t) \) is closed, convex and weakly compact in \( X \).

Definition 3.2. For every fixed \( \varepsilon \geq 0, \delta > 0 \), we say that a point \( z \) is \((\varepsilon, \delta)\) stochastically attained at time \( t \) by the solution process \( x(t) \) (2.11) if
\[
\exists u \in U_{ad} \exists P_{x_0} | x(t) \in S_\varepsilon(z) | \geq \delta,
\]
where \( P_{x_0} \) is the probability measure \( \mathbb{P} \)-conditioned on \( \{x(0) = x_0\} \). Hence,
the stochastic attainable set $A^\delta(t)$ for system (2.8) at time $t$ is the collection of all points $z$ defined in the above, i.e.,

$$A^\delta(t) = \{z \in X : \exists u \in U_{ad}^\infty, P_{x_0}\{x(t) \in S_z(x)\} \geq \delta\}. \quad (3.5)$$

It can be shown that the solution process $x(t)$ of (2.8) (or expressed explicitly in (2.11)) is Gaussian with measure $\mu$ and mean $m^\mu_t = S(t, 0)x_0 + \int_0^t S(t, \tau)B(\tau)u(\tau)\,d\tau$ and covariance

$$G(t) = \int_0^t S(t, \tau)C(\tau)S^*(\tau)\,d\tau. \quad (3.6)$$

From now on $*$ denotes adjoint operators.

Here the Gaussian measure $\mu$ has the characteristic functional

$$\exp\{i\langle m^\mu_t, z \rangle - \frac{1}{2}\langle G(t)z, z \rangle\}, z \in X.$$ Suitable assumptions can be imposed to $A(t), B(t)$ and $C(t)$ such that $G(t)$ is positive definite for each $t$, i.e., we only consider the non-degenerate distribution.

Since

$$\mu = \prod_{k=1}^{\infty} \mu_t(k), \quad (2.7)$$

where $x_k$ is a point in $L^{(k)}$, which is a one-dimensional subspace of $X$ spanned by $\langle e_k \rangle$, and $\mu_t(k)$ is the projection $\mathcal{S}_t(k)$ of $\mu$ on $L^{(k)}$. Note that $\mu_t(k)$ is also a Gaussian measure with mean $a_k = \langle m^\mu_t, e_k \rangle_x$ and covariance $\lambda_k$, where $\{e_k\}_{k=1,2,...}$ is the set of eigenvectors of $G(t)$ and $\lambda_k$ is the eigenvalue corresponding to $e_k$.

Hence in terms of the distribution (3.6) of $x(t)$, (3.4) can be written as

$$\int_{S_z(x)} \mu(dx) = \prod_{k=1}^{\infty} \frac{1}{(2\pi \lambda_k)^{1/2}} \times \int_{S_z(x)} \exp \left\{ - \frac{1}{2} \langle G^{-1}(t)(x - a_k e_k), (x - a_k e_k) \rangle_x \right\} \mu_t(k)(dx) \geq \delta, \quad (3.8)$$

where $m_t(k)$, a Lebesgue measure on $L^{(k)}$, is defined by $m_t(k)(dx) = dx_k$, $x_k \in L^{(k)}$.

From the definition, we see that if $C(t) = 0, \varepsilon = 0$, the form of $A^\delta(t)$ will reduce to $K(t)$; so $A^\delta(t)$ seems to be the natural extension of $K(t)$. Further, in view of (3.2) and (3.6), we see that elements in $K(t)$ are precisely the means of the random solution process $x(t)$ of system (2.8).
Now, assume that $\varepsilon$ and $\delta$ are chosen such that

$$\int_{S_{\varepsilon}(a)} \frac{1}{\sqrt{2\pi\lambda_k}} \exp \left\{ -\frac{1}{2} \langle G^{-1}(t)z, z \rangle \right\} m_{L(t)}(dz) > \delta \quad \forall k,$$  

(3.9)

otherwise $A_{\varepsilon}(t) = \emptyset$. For $A_{\varepsilon}(t) \neq \emptyset$, we can show that the interior of $A_{\varepsilon}(t)$ is non-empty and

$$A_{\varepsilon}(t) \supset K(t)$$

(3.10)

(because every point in $K(t)$ is the mean centre of some Gaussian distribution (3.6)).

Now fix $t \in [0, t_1]$, and let

$$Q^{(k)}(t) = \left\{ x \in L^{(k)}: \int_{S_{\varepsilon}(x)} \frac{1}{\sqrt{2\pi\lambda_k}} \exp \left\{ -\frac{1}{2} \langle G^{-1}(t)z, z \rangle \right\} m_{L(t)}(dz) \geq \delta \right\},$$

(3.11)

where $z = \sum_{k=1}^{\infty} z_k e_k$, $z_k = \langle z, e_k \rangle$.

Assume that $\varepsilon$, $\delta$ are such that $Q^{(k)}(t) \neq \emptyset$ and is not a singleton, $\forall k$.

**Lemma 3.1.** $Q(t) = \prod_{k=1}^{\infty} Q^{(k)}(t)$ is compact and convex.

**Proof:** Following [4], it can be shown that $Q^{(k)}(t)$, $k = 1, 2, \ldots$ are all compact. Hence by Tychonoff's theorem, the compactness of $Q(t)$ follows. Moreover, $Q(t)$ is also closed since $Q(t)$ is the Cartesian product of orthogonal closed sets $Q^{(k)}(t)$, $k = 1, 2, \ldots$. To prove convexity, it is sufficient to show that $x_1 = \lambda x_1 + (1 - \lambda) x_2 \in Q(t)$ for $\lambda \in [0, 1]$, $x_1$, $x_2 \in Q(t)$, or equivalently,

$$x_1^{(k)} = \lambda x_1^{(k)} + (1 - \lambda) x_2^{(k)} \in Q^{(k)}(t) \quad \forall k.$$  

(3.12)

(The equivalence follows from the orthogonal properties between $L^{(k)}$.) But $Q^{(k)}(t)$ is convex; hence (3.12) holds and thus the convexity of $Q(t)$ follows.

Now, we consider the Gaussian distribution with mean $m_t^u$ and covariance $G(t)$ in (3.6), where $u \in U_{ad}^u$. Then each point $z$ in the set $m_t^u + Q(t)$ will satisfy (3.8), and therefore $\bigcup \{ m_t^u + Q(t): u \in U_{ad}^u \}$ is the set of points in $X$ such that the $\varepsilon$-balls centered at these points have probability $\geq \delta$. Further, in view of the definitions of $K(t)$ and $A_{\varepsilon}(t)$, we see that

$$A_{\varepsilon}(t) = \bigcup \{ m_t^u + Q(t): u \in U_{ad}^u \} = K(t) + Q(t).$$

(3.13)

Hence, by Lemma 2.3, $A_{\varepsilon}(t)$ is closed, convex and weakly compact in $X$. 

4. \((\varepsilon, \delta)\)-Stochastic Controllability

In this section, we are interested in investigating the conditions of \((\varepsilon, \delta)\)-stochastic controllability for system (2.8) with input constraints (2.13). We first need the following lemma:

**Lemma 4.1.** If \(M\) and \(N\) are non-empty closed convex sets in \(X\) with one of them being weakly compact, then

(i) \(M \cap N \neq \emptyset\) iff for all \(g \in X\)

\[
\inf_{m \in M} \langle g, m \rangle \leq \sup_{n \in N} \langle g, n \rangle, \quad (4.1)
\]

(ii) \(M \subseteq N\) iff for all \(g \in X\)

\[
\sup_{m \in M} \langle g, m \rangle \leq \sup_{n \in N} \langle g, n \rangle. \quad (4.2)
\]

**Proof.** (i) Necessity: Obvious.

Sufficiency: If \(M \cap N = \emptyset\), and since they are non-empty, closed and convex and either one of them is compact in the locally convex space \(X_w\) with weak topology, then, by the strict Separation Theorem, \(0 \neq \hat{g} \in X_w^* = X^*\) exists such that

\[
\sup_{n \in N} \langle \hat{g}, n \rangle < \inf_{m \in M} \langle \hat{g}, m \rangle.
\]

Thus, (4.1) cannot hold \(\forall g \in X\).

(ii) Necessity: Obvious.

Sufficiency: If \(h \in M \setminus N\), then \(\{h\}\) is compact and convex in \(X\). By the strict Separation Theorem, \(0 \neq \hat{g} \in X\) exists such that

\[
\sup_{n \in N} \langle \hat{g}, n \rangle < \langle \hat{g}, h \rangle \leq \sup_{n \in M} \langle \hat{g}, m \rangle.
\]

Hence, (4.2) does not hold for all \(g \in X\).

**Definition 4.1.** System (2.8) is said to be \((\varepsilon, \delta)\)-stochastically null-controllable at \(x_0\) on \([0, t_1]\) if \(\exists u \in U^\text{ad}\) such that

\[
A^\delta_{\varepsilon}(t_1) \cap \{0\} = \emptyset. \quad (4.3)
\]

**Definition 4.2.** System (2.8) is said to be \((\varepsilon, \delta)\)-stochastically locally null-controllable at \(x_0\) on \([0, t_1]\) if there exists a neighborhood \(N(x_0)\) of \(x_0\) in \(X\) such that for each \(y \in N(x_0)\), in stochastic sense defined by (4.3), system (2.8) is \((\varepsilon, \delta)\)-stochastically null-controllable at \(y_0\) on \([0, t_1]\).
Remark 4.1. The neighborhood \( N(x_0) \) may be taken as an \( r \)-ball, \( B_r(x_0) \). Then \( y \in B_r(x_0) \) in the stochastic sense is simply defined as 
\[
E \| y - x_0 \|_2^2 < r^2;
\]
on the other hand we see that 
\[
E \langle g, (y - x_0) \rangle_x^2 = r^2 \| g \|_x^2 \forall g \in X.
\]

DEFINITION 4.3. System (2.8) is said to be \((\varepsilon, \delta)\)-stochastically \( \Omega \)-controllable at \( x_0 \) on \([0, t_1]\) if \( \exists u \in U_{ad}^\varepsilon \) which steers \( x_0 \) to \( \Omega \) at \( t_1 \) in a stochastic sense, i.e.,
\[
A^\varepsilon_{t_1}(x_0) \cap \Omega \neq \emptyset.
\]

The main results on \((\varepsilon, \delta)\)-stochastic controllability are presented in the following theorems.

THEOREM 4.1. System (2.8) is \((\varepsilon, \delta)\)-stochastically null-controllable at \( x_0 \) iff \( \exists t_1 \geq 0 \) and for all \( g \in X \)
\[
\delta E_{x_0} \| \langle g, S(t_1, 0) x_0 \rangle_x \|_2^2 
\leq \text{trace } W \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) g \|_x^2 d\tau 
+ \left( \varepsilon \| g \|_x + \rho^{1/2} \left[ \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_{x_0}^q d\tau \right]^{1/q} \right)^2. \tag{4.5}
\]

Proof. In view of Definition 4.1, system (2.8) is \((\varepsilon, \delta)\)-stochastically null-controllable at \( x_0 \) iff \( \exists t_1 \geq 0 \) such that
\[
A^\varepsilon_{t_1}(x_0) \cap \{0\} \neq \emptyset \tag{4.3}
\]
or, iff \( \exists u \in U_{ad}^\varepsilon \), \( P_{x_0}(x(t_1) \in S_\varepsilon(0)) \geq \delta \) (where \( x(t_1) = S(t_1, 0) x_0 + \int_0^{t_1} S(t_1, \tau) B(\tau) u(\tau) d\tau + \int_0^{t_1} S(t_1, \tau) C(\tau) dw(\tau) \)) or, iff \( \exists u \in U_{ad}^\varepsilon \) such that
\[
P_{x_0} \left( \{ -S(t_1, 0) x_0 \} \subset B_\varepsilon(0) \right)
+ \left\{ \int_0^{t_1} S(t_1, \tau) B(\tau) u(\tau) d\tau + \int_0^{t_1} S(t_1, \tau) C(\tau) dw(\tau) \right\} \geq \delta,
\]
or, iff
\[
P_{x_0} \left( \{ -S(t_1, 0) x_0 \} \subset B_\varepsilon(0) \right) \tag{4.6}
+ \left\{ \int_0^{t_1} S(t_1, \tau) B(\tau) u(\tau) d\tau + \int_0^{t_1} S(t_1, \tau) C(\tau) dw(\tau): u \in U_{ad}^\varepsilon \right\} \geq \delta.
By Definition 3.1 and Lemma 3.2, we see that $B_\varepsilon(0) + K(t)S(t_1,0) x_0 + \int_0^{t_1} \Phi(t_1, \tau) C(\tau) \, dw(\tau)$ is closed, convex and weakly compact in $X$.

Therefore (4.6) holds iff $\forall g \in X$

$$P_{x_0} \left( \langle g, -S(t_1, 0) x_0 \rangle \right)$$

$$+ \sup_{u \in U_{ad}^g} \left( \left| g, \int_0^{t_1} S(t_1, \tau) \left[ B(\tau) u(\tau) \, d\tau + C(\tau) \, dw(\tau) \right] \right| \right) \geq \delta$$

(by Lemmas 4.1(i) and (4.2)) (4.6a)

In view of the symmetry of $B_\varepsilon(0)$ and of $U_{ad}^g$ about the origin, (4.6a) is equivalent to

$$P_{x_0} \left( \langle g, S(t_1, 0) x_0 \rangle \right) \leq \varepsilon \| g \|_X + \rho^{1/2} \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_X^q \, d\tau \right)^{1/q}$$

$$+ \left| \left\langle g, \int_0^{t_1} S(t_1, \tau) C(\tau) \, dw(\tau) \right\rangle \right| \geq \delta, \quad (4.6b)$$

where $1/p + 1/q = 1, 1 \leq p \leq \infty$. By Chebyshev's inequality, (4.6b) leads to

$$E_{x_0} \left[ \varepsilon \| g \|_X + \rho^{1/2} \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_X^q \, d\tau \right)^{1/q} \right]$$

$$+ \left| \left\langle g, \int_0^{t_1} S(t_1, \tau) C(\tau) \, dw(\tau) \right\rangle \right| \geq \delta E_{x_0} |\langle g, S(t_1, 0) x_0 \rangle|^2,$$

or

$$\varepsilon \| g \|_X + \rho \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_X^q \, d\tau \right)$$

$$+ \text{trace} \left( W \int_0^{t_1} \| C(\tau) S^*(t_1, \tau) g \|_X^2 \, d\tau \right) \geq \delta E_{x_0} |\langle g, S(t_1, 0) x_0 \rangle|^2.$$
Remark 4.2b. By Theorem 4.1, we have shown that system (2.8) is 
\((\varepsilon, \delta)\)-stochastically null-controllable iff (4.5) holds. From the definition of
\(U_{ad}^p\), we have \(\rho > 0\). Hence, by (4.5), the following expression follows:

\[
\frac{\delta}{\rho} E |\langle g, S(t_1, 0)x_0 \rangle_x|^2
\leq \frac{1}{\rho} \text{trace} W \int_0^{t_1} \|C^*(\tau) S^*(t_1, \tau) g\|_x^2 d\tau
\]
\[
+ \left( \frac{\varepsilon \|g\|_x}{\rho^{1/2}} + \left[ \int_0^{t_1} \|B^*(\tau) S^*(t_1, \tau) g\|_0^q d\tau \right]^{1/q} \right)^2 \forall g \in X. 
\]

(4.5a)

Now suppose that system (2.8) is reduced to a deterministic version. Then
(4.5a) leads to

\[
\frac{1}{\rho} |\langle g, S(t_1, 0)x_0 \rangle_x|^2
\leq \left( \frac{\varepsilon \|g\|_x}{\rho^{1/2}} + \left[ \int_0^{t_1} \|B^*(\tau) S^*(t_1, \tau) g\|_0^q d\tau \right]^{1/q} \right)^2. 
\]

(4.5b)

Let \(\varepsilon \downarrow 0\); (4.5b) will asymptotically tend to

\[
\frac{1}{\rho} |\langle g, S(t_1, 0)x_0 \rangle_x|^2
\leq \left[ \int_0^{t_1} \|B^*(\tau) S^*(t_1, \tau) g\|_0^q d\tau \right]^{2/q} \forall g \in X. 
\]

(4.5c)

In view of (4.5c), it is seen that

\[ B^*(\tau) S^*(t_1, \tau) g = 0 \]

implies \(g = 0\) on \([0, t_1]\) and this result agrees with the sufficient condition for
deterministic version of (2.8) to be approximately controllable on \([0, t_1]\). Consequently, we can say that when system (2.8) is being reduced to the
deterministic situation, condition (4.5) implies the deterministic approximate controllability condition (4.5d).

**Theorem 4.2.** System (2.8) is \((\varepsilon, \delta)\)-stochastically locally null-
controllable at \(x_0\) on \([0, t_1]\) iff \(\exists r > 0\) and for each \(g \in X\),
\[ \begin{align*}
\delta(E |\langle g, S(t_1, 0) x_0 \rangle|^2 + r^2 \| S^*(t_1, 0) g \|_X^2) \\
\leq \text{trace } W \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) g \|_X^2 \, d\tau \\
+ \left[ \epsilon \| g \|_X + \rho^{1/2} \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_U^q \, d\tau \right)^{1/q} \right]^2. \\
\end{align*} \] (4.7)

**Proof.** By Definition 4.2, (2.8) is \((\epsilon, \delta)\)-stochastically locally null-controllable at \(x_0\) on \([0, t_1]\) iff for each \(y \in N(x_0)\), (4.5) holds for this \(y\), i.e.,

\[ \begin{align*}
\delta E |\langle g, S(t_1, 0) y \rangle|^2 \\
\leq \text{trace } W \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) g \|_X^2 \, d\tau \\
+ \left[ \epsilon \| g \|_X + \rho^{1/2} \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_U^q \, d\tau \right)^{1/q} \right]^2. \\
\end{align*} \] (4.8)

Observe that

\[ \begin{align*}
\langle g, S(t_1, 0) x_0 \rangle + \sup_{(y-x_0) \in B_r(0)} \langle g, S(t_1, 0)(y-x_0) \rangle \\
= \sup_{y \in B_r(x_0)} \langle g, S(t_1, 0) y \rangle \leq \sup_{y \in B_r(x_0)} |\langle g, S(t_1, 0) y \rangle| \\
\end{align*} \] (4.9)

holds w.p. 1, \(\forall g \in X\).

Since \(B_r(0)\) is symmetric about the origin and (4.9) holds not only for \(g\) but also \(-g \in X\), (4.9) becomes

\[ \begin{align*}
|\langle g, S(t_1, 0) x_0 \rangle| + \sup_{(y-x_0) \in B_r(0)} |\langle g, S(t_1, 0)(y-x_0) \rangle| \\
\leq \sup_{y \in B_r(x_0)} |\langle g, S(t_1, 0) y \rangle| \\
\end{align*} \] (4.9a)

w.p. 1, \(\forall g \in X\).

In turn (4.9a) leads to

\[ \begin{align*}
E \langle g, S(t_1, 0) x_0 \rangle^2 + E \left\{ \sup_{(y-x_0) \in B_r(0)} |\langle g, S(t_1, 0)(y-x_0) \rangle| \right\}^2 \\
\leq E \left\{ \sup_{y \in B_r(x_0)} |\langle g, S(t_1, 0) y \rangle| \right\}^2,
\end{align*} \]

or equivalently.

\[ \begin{align*}
E |\langle g, S(t_1, 0) x_0 \rangle|^2 + r^2 \| S^*(t_1, 0) g \|_X^2 \leq \sup_{y \in B_r(x_0)} E |\langle g, S(t_1, 0) y \rangle|^2. \quad (4.9b)
\end{align*} \]

Hence substituting this into (4.8), (4.7) follows.
Theorem 4.3. If $\Omega$ is a closed convex set in $X$, then system (2.8) is $(\varepsilon, \delta)$-stochastically $\Omega$-controllable at $x_0$ on $[0, t_1]$ iff for all $g \in X$,

$$\delta E \left| \langle g, S(t_1, 0) x_0 \rangle \right|^2 \leq \text{trace} \left( W \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) g \|_X^2 \, d\tau \right)$$

$$+ \left[ \sup_{z \in \Omega} \langle g, z \rangle + \varepsilon \left\| g \right\|_X + \rho^{1/2} \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_U^2 \, d\tau \right) \right]^{1/\rho} \leq 1.$$  

Proof. From Definition 4.3, system (2.8) is $(\varepsilon, \delta)$-stochastically $\Omega$-controllable iff

$$A^\delta(t_1) \cap \Omega \neq \emptyset,$$  

and by definition of $A^\delta(t_1)$, (4.4) holds iff $\exists u \in U^0_{ad}$ such that $P \{ \Omega \cap S_\varepsilon(x(t_1)) \neq \emptyset \} \geq \sigma$, or iff

$$P \{ (\Omega \backslash S(t_1, 0) x_0) \cap \{ S_\varepsilon(x(t_1)) \backslash S(t_1, 0) x_0 : u \in U^0_{ad} \} \} \geq \delta.$$  

Since

$$\{ S_\varepsilon(x(t_1)) \backslash S(t_1, 0) x_0 : u \in U^0_{ad} \}$$

$$= \left\{ \int_0^{t_1} S(t_1, \tau) C(\tau) \, dw(\tau) + \left\{ B^\varepsilon \left( \int_0^{t_1} S(t_1, \tau) B(\tau) u(\tau) \, d\tau \right) : u \in U^0_{ad} \right\} \right\},$$

and

$$R_{t_1} = \left\{ \int_0^{t_1} S(t_1, \tau) B(\tau) u(\tau) \, d\tau : u \in U^0_{ad} \right\}$$

is closed, convex and weakly compact in $X$ (by Lemma 2.2), it follows that

$$\{ B^\varepsilon \left( \int_0^{t_1} S(t_1, \tau) B(\tau) u(\tau) \, d\tau \right) : u \in U^0_{ad} \}$$

inherits all properties of $R_{t_1}$, and so does the same for $\{ S_\varepsilon(x(t_1)) \backslash S(t_1, 0) x_0 \}$ (by Lemma 2.3).

Therefore with Lemma 4.1(i) and (4.1), (4.11) holds iff

$$P \left( \inf_{u \in U^0_{ad}} \left\langle g, \int_0^{t_1} S(t_1, \tau) B(\tau) u(\tau) \, d\tau + \int_0^{t_1} S(t_1, \tau) C(\tau) \, dw(\tau) \right\rangle \right.$$ 

$$+ \inf_{a \in B^\varepsilon(0)} \left\langle g, a \right\rangle \leq \sup_{z \in \Omega} \left( \langle g, z \rangle - \langle g, S(t_1, 0) x_0 \rangle \right) \geq \delta.$$  

When symmetries of $U^0_{ad}$ and $B^\varepsilon(0)$ about the origin are being considered, (4.12) holds iff $\forall g \in X$. 


\[ P \left( |\langle g, \delta(t_1, 0) x_0 \rangle| \leq \sup_{z \in \Omega} |\langle g, z \rangle| + \sup_{a \in B(0)} |\langle g, a \rangle| - \left( \langle g, \int_0^{t_1} S(t_1, \tau) C(\tau) \, dw(\tau) \right) \right. \]
\[ \left. + \sup_{u \in U_{reg}} \left| \left( \langle g, \int_0^{t_1} S(t_1, \tau) B(\tau) u(\tau) \, d\tau \right) \right| \right) \geq \delta. \quad (4.12a) \]

By Chebyshev's inequality, (4.12a) leads to
\[
\delta E |\langle g, S(t_1, 0) x_0 \rangle|^2 
\leq \sup_{z \in \Omega} |\langle g, z \rangle|^2 + \varepsilon^2 \| g \|_x^2 + \text{trace } W \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) g \|_x^2 \, d\tau 
+ 2 \sup_{z \in \Omega} |\langle g, z \rangle| \varepsilon \| g \|_x + \rho \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_y^2 \, d\tau \right)^{2/q} 
+ 2 \left( \sup_{z \in \Omega} |\langle g, z \rangle| + \varepsilon \| g \|_x \right) \left[ \rho \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_y^2 \, d\tau \right)^{2/q} \right]^{1/2}. 
\]

Hence, (4.10) follows.

**Remark 4.3.** Suppose that \( \Omega \) is assumed to be an \( r^{1/2} \)-ball centered at \( y_0 \), i.e., \( B(y_0, r) = \{ y : \| y - y_0 \|_x \leq r \} \). Then
\[
\delta E |\langle g, S(t_1, 0) x_0 \rangle|^2 
\leq |\langle g, y_0 \rangle|^2 + 2 |\langle g, y_0 \rangle| r^{1/2} \| g \|_x + (r^{1/2} + \varepsilon)^2 \| g \|_x^2 
+ 2\varepsilon \| g \|_x |\langle g, y_0 \rangle| + \rho \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_y^2 \, d\tau \right)^{2/q} 
+ 2 \left( |\langle g, y_0 \rangle| + (r^{1/2} + \varepsilon) \| g \|_x \right) \left[ \rho \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) g \|_y^2 \, d\tau \right)^{2/q} \right]^{1/2} 
+ \text{trace } W \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) g \|_x^2 \, d\tau, \quad (4.13) 
\]

\[
(\sup_{z \in \Omega} |\langle g, z \rangle|)^2 = (\sup_{z \in \Omega} |\langle g, z \rangle|)^2 = (|\langle g, y_0 \rangle| + r^{1/2} \| g \|_x)^2). 
\]

Now, suppose \( \Omega \) is a null subspace of \( \mathcal{P} \), where \( \mathcal{P} \) is the continuous projection: \( X \to X \). With \( \mathcal{P} \) defined, \( X = R(\mathcal{P}) \oplus N(\mathcal{P}) \) where the range \( R(\mathcal{P}) \) and the kernel \( N(\mathcal{P}) \) are closed subspaces of \( X \).
THEOREM 4.4. If $\Omega = N(\mathcal{F})$, then system (2.8) is $(\varepsilon, \delta)$-stochastically $\Omega$-controllable at $x_0$ on $[0, t_1]$ iff for all $g \in X$

$$\mathcal{E}\langle g, \mathcal{F}S(t_1, 0)x_0 \rangle^2$$

$$\leq \left[ \varepsilon \| \mathcal{F}g \|_X + \rho^{1/2} \left( \int_0^{t_1} \| B^*(\tau) S^*(t_1, \tau) \mathcal{F}g \|_{L^q}^q d\tau \right)^{1/q} \right]^2$$

$$+ \text{trace} \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) \mathcal{F}g \|_X^2 d\tau.$$

(4.14)

Remark 4.4. If $\Omega = \{0\}$, then $\mathcal{F} = 1$ (identity operator) and (4.14) reduces to (4.5).

5. STOCHASTIC GAME PROBLEM

In this section, we are interested in the stochastic game problem defined on a real separable Hilbert space $X$ with two non-cooperative controllers, a pursuer and an evader, respectively.

The game system is governed by the stochastic evolution equation:

$$dx(t) + A(t)x(t) dt = [B_p(t)u(t) + B_e(t)v(t)] dt + C(t)dw(t),$$

$$x(0) = x_0,$$

(5.1)

where $B_p(t)$ and $B_e(t)$ are bounded operators and norm-continuous for $t > 0$. Assumptions on $A(t)$ as prescribed in Section 2 still hold in the present section. $u(t)$, the pursuer's control, lies in the Hilbert space $L_{p_1}([0, t_1], U)$; while $v(t)$, the evader's control, lies in the Hilbert space $L_{p_2}([0, t_1], V)$. Here $U, V$ are also separable Hilbert spaces. Further, the admissible controls are restricted in the constraint sets

$$U_{ad}^\rho = \bigcup_{t_i > 0} L_{p_1}([0, t_1], U)$$

$$\bigcup_{t_i > 0} \{ u \in L_{p_1}([0, t_1]; U): \| u \|_{p_1} \leq \rho^{1/2} < \infty, \rho > 0 \},$$

(5.2)

$$V_{ad}^\sigma = \bigcup_{t_i > 0} L_{p_2}([0, t_1]; V)$$

$$\bigcup_{t_i > 0} \{ v \in L_{p_2}([0, t_1]; V): \| v \|_{p_2} \leq \sigma^{1/2} < \infty, \sigma > 0 \}.$$
Define
\begin{align*}
R_\tau^p &= A_i(L^p_{\text{ad}}([0, t_1]; U)), \\
R_\tau^c &= A_i(L^c_{\text{ad}}([0, t_1]; V)),
\end{align*}
(5.4)

\begin{align*}
A_i^p(u) &= \int_0^t S(t, \tau) B_\rho(\tau) u(\tau) \, d\tau, \\
A_i^c(v) &= \int_0^t S(t, \tau) B_\nu(\tau) v(\tau) \, d\tau,
\end{align*}
(5.5)

with \( t \in [0, t_1] \).

Then, we admit the mild solution of (5.1) to be
\begin{align*}
x(t) &= S(t, 0) x_0 + \int_0^t S(t, \tau) B_\rho(\tau) u(\tau) \, d\tau
\end{align*}
(5.6)

where \( S(t, \tau) \) is the almost strong continuous evolution operator generated by \(-A(t)\).

Analogous to Lemmas 2.1 and 2.2, we have \( U^p_{\text{ad}} \) and \( V^c_{\text{ad}} \) are closed, convex and weakly compact in \( X \); \( A_i^p \) and \( A_i^c \) are linear continuous mappings and \( R_\tau^p \) and \( R_\tau^c \) are closed, convex and weakly compact sets in \( X \).

Consider the deterministic version of (5.1)
\begin{align*}
\frac{d x(t)}{d t} + A(t) x(t) &= B_\rho(t) u(t) + B_\nu(t) v(t) \\
\quad x(0) &= x_0.
\end{align*}
(5.7)

DEFINITION 5.1. For \( t \geq 0 \), the deterministic attainable set \( K(t) \) for system (5.7) at time \( t \) is the set
\begin{align*}
K(t) &= \left\{ z \in X : z = S(t, \tau) x_0 \\
& \quad + \int_0^t S(t, \tau) [B_\rho(\tau) u(\tau) + B_\nu(\tau) v(\tau)] \, d\tau : u \in U^p_{\text{ad}}, v \in V^c_{\text{ad}} \right\} \\
&= S(t, 0) x_0 + R_\tau^p + R_\tau^c \quad \text{(by (5.4) and (5.5)).}
\end{align*}
(5.8)

From the properties of \( R_\tau^p \) and \( R_\tau^c \), we see that \( K(t) \) is weakly compact, closed and convex in \( X \) (by Lemma 2.3).
DEFINITION 5.2. For every fixed $\epsilon > 0$ and $1 \geq \delta > 0$, the stochastic attainable set at time $t$ for system (5.1) is defined as

$$A^\delta_\epsilon(t) = \{ z \in X : \exists u \in U^\omega_{ad}, v \in V^\omega_{ad} : P_{x_0}(x(t) \in S_\epsilon(x)) \geq \delta \}. \quad (5.9)$$

Here $x(t)$ is the mild solution (5.6). Parallel to the one-control case (cf. Section 3) we adopt similar procedures to prove the following assertions:

$$K(t) \subset A^\delta_\epsilon(t), \ A^\delta_\epsilon(t) \text{ is convex, weakly compact and closed in } X.$$  

DEFINITION 5.3. System (5.1) is $(\epsilon, \delta)$-stochastically max–min null-controllable at $x_0$ on $[0, t_1]$ if for each announced $v \in V^\omega_{ad}$, $\exists u \in U^\omega_{ad}$ such that

$$A^\delta_\epsilon(t_1) \cap \{0\} \neq \emptyset$$

or

$$A^\delta_\epsilon(t_1) \ni \{0\}. \quad (5.10)$$

DEFINITION 5.4. The stochastic game system (5.1) is said to be $(\epsilon, \delta)$-stochastically max–min $\Omega$-controllable at $x_0$ on $[0, t_1]$ if for each announced $v \in V^\omega_{ad}$, $\exists u \in U^\omega_{ad}$ such that

$$A^\delta_\epsilon(t_1) \cap \Omega \neq \emptyset. \quad (5.11)$$

THEOREM 5.1. System (5.1) is $(\epsilon, \delta)$-stochastically max–min null-controllable on $[0, t_1]$ iff for each $g \in X$, we have

$$\delta E |\langle g, S(t_1, 0) x_0 \rangle|^2 \leq \left[ \epsilon \| g \|_X + \rho^{1/2} \left( \int_0^{t_1} \| B^*_\rho(\tau) S^*(t_1, \tau) g \|_{Q_1} d\tau \right)^{1/q_1} \right]^2$$

$$+ \text{trace } W \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) g \|_{F_2}^2 d\tau$$

$$- \delta \sigma \left( \int_0^{t_1} \| B^*_\sigma(\tau) S^*(t_1, \tau) g \|_{P_2}^2 d\tau \right)^{2/q_2} \quad (5.12)$$

where

$$\frac{1}{p_1} + \frac{1}{q_1} = 1, \quad \frac{1}{p_2} + \frac{1}{q_2} = 1.$$  

Proof. By Definition 5.3, system (5.1) is $(\epsilon, \delta)$-stochastically max–min null-controllable on $[0, t_1]$ iff

$$A^\delta_\epsilon(t_1) \cap \{0\} \neq \emptyset. \quad (5.10)$$
or iff \( \exists u \in U_{\text{ad}}^0 \),

\[
v \in V_{\text{ad}}^\alpha \ni P_{x_0} \{ x(t_1) \in S_x(0) \} \geq \delta.
\]

(here \( x(t_1) \) is the mild solution process of (5.1), described by (5.6)) or iff

\[
P_{x_0} \left\{ \{-S(t_1, 0)x_0\} \subseteq B_\epsilon(0) + \left\{ \int_0^{t_1} S(t_1, \tau) [B_\rho(\tau)u(\tau) + B_\epsilon(\tau)v(\tau)] \, d\tau \right\} \right. \\
+ \left. \int_0^{t_1} S(t_1, \tau) C(\tau) \, dw(\tau) : u \in U_{\text{ad}}^0, v \in V_{\text{ad}}^\alpha \right\} \geq \delta, \quad (5.13)
\]
or, iff

\[
P_{x_0} \left\{ \{-S(t_1, 0)x_0\} \subseteq B_\epsilon(0) + K(t_1) \backslash S(t_1, 0)x_0 \right. \\
+ \left. \int_0^{t_1} S(t_1, \tau) C(\tau) \, dw(\tau) \right\} \geq \delta. \quad (5.13a)
\]

Then by the weakly compactness of \( K(t) \) and \( B_\epsilon(0) \) in \( X \), \( \forall t \in [0, t_1] \), and together with Lemmas 2.3 and 4.1(ii) being applied, we see that (5.13a) holds iff \( \forall g \in X \)

\[
P_{x_0} \left\{ \langle g, -S(t_1, 0)x_0 \rangle \leq \sup_{a \in K(t_0)} \left\langle g, a \right\rangle \\
+ \sup_{u \in U_{\text{ad}}^0} \left\{ g, \int_0^{t_1} S(t_1, \tau) B_\rho(\tau)u(\tau) \, d\tau + \int_0^{t_1} S(t_1, \tau) B_\epsilon(\tau)v(\tau) \, d\tau \right\} \\
+ \left\langle g, \int_0^{t_1} S(t_1, \tau) C(\tau) \, dw(\tau) \right\rangle \right\} \geq \delta. \quad (5.14)
\]

Since \( U_{\text{ad}}^0, V_{\text{ad}}^\alpha \) and \( B_\epsilon(0) \) are symmetric about the origin, it follows that (5.14) is equivalent to

\[
P_{x_0} \left\{ |\langle g, S(t_1, 0)x_0 \rangle| - \sup_{v \in V_{\text{ad}}^\alpha} \left| \left\langle g, \int_0^{t_1} S(t_1, \tau) B_\rho(\tau)u(\tau) \, d\tau \right\rangle \right| \\
- \sup_{a \in K(t_0)} |\langle g, a \rangle| + \sup_{u \in U_{\text{ad}}^0} \left| \left\langle g, \int_0^{t_1} S(t_1, \tau) B_\rho(\tau)u(\tau) \, d\tau \right\rangle \right| \\
+ \left| \left\langle g, \int_0^{t_1} S(t_1, \tau) C(\tau) \, dw(\tau) \right\rangle \right| \right\} \geq \delta \quad \forall g \in X. \quad (5.14a)
\]
By Chebyshev's inequality, \((5.14a)\) leads to

\[
\delta E \left| \langle g, S(t_1, 0) x_0 \rangle \right|^2 \leq \varepsilon^2 \| g \|_X^2 + \rho \left( \int_0^{t_1} \| B_p^*(\tau) S^*(t_1, \tau) g \|_U^2 d\tau \right)^{2/q_1} + 2\varepsilon \| g \|_X \left( \rho^{1/2} \left[ \int_0^{t_1} \| B_p^*(\tau) S^*(t_1, \tau) g \|_U^2 d\tau \right]^{1/q_1} \right) + \text{trace } W \left( \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) g \|_X^2 d\tau \right) - \delta \sigma \left( \int_0^{t_1} \| B_e^*(\tau) S^*(t_1, \tau) g \|_Y^2 d\tau \right)^{2/q_2}.
\]

Hence \((5.12)\) follows.

**Theorem 5.2.** If the target set \(\Omega\) is closed and convex, system \((5.1)\) is \((\varepsilon, \delta)\)-stochastically max-min \(\Omega\)-controllable on \([0, t_1]\) iff for all \(g \in X\), we have

\[
\delta E \left| \langle g, S(t_1, 0) x_0 \rangle \right|^2 \leq \left( \varepsilon \| g \|_X + \rho^{1/2} \left[ \int_0^{t_1} \| B_p^*(\tau) S^*(t_1, \tau) g \|_U^2 d\tau \right]^{1/q_1} \right)^2 + \sup_{a \in \Omega} |\langle g, a \rangle|^2 + 2\varepsilon \| g \|_X \sup_{a \in \Omega} |\langle g, a \rangle| + 2 \sup_{a \in \Omega} |\langle g, a \rangle| \rho^{1/2} \left[ \int_0^{t_1} \| B_p^*(\tau) S^*(t_1, \tau) g \|_U^2 d\tau \right]^{1/q_1} + \text{trace } W \left( \int_0^{t_1} \| C^*(\tau) S^*(t_1, \tau) g \|_X^2 d\tau \right) - \delta \sigma \left( \int_0^{t_1} \| B_e^*(\tau) S^*(t_1, \tau) g \|_Y^2 d\tau \right)^{2/q_2}.
\]

**Proof.** In view of Definition 5.4, system \((5.1)\) is \((\varepsilon, \delta)\)-stochastically max-min \(\Omega\)-controllable on \([0, t_1]\) iff

\[
A^*_\varepsilon(t_1) \cap \Omega \neq \emptyset,
\]

or, iff

\[
P_{x_0}((\Omega \setminus S(t_1, 0) x_0)) \cap \{ S_e(x(t_1)) \setminus S(t_1, 0) x_0 \} \neq \emptyset: u \in U^o_{ad}, v \in V^*_{ad} \geq \delta
\]

\((5.16)\)
STOCHASTIC CONTROLLABILITY

\[ \{ S_e(x(t_1)) \backslash S(t_1, 0) x_0 : u \in U_{sa}^0, v \in V_{sa}^0 \} \]

\[ = R_{x_1}^e + R_{e_1}^e + \int_0^{t_1} S(t_1, \tau) C(\tau) dw(\tau). \]

and is hence a closed, convex and weakly compact set in \( X \) (by Lemma 2.3)).

Therefore, by Lemma 4.1(i) and (4.1), (5.16) holds iff \( \forall g \in X \)

\[ P \left( - \sup_{u \in U_{sa}^0} \left( g, \int_0^{t_1} S(t_1, \tau) [B_p(\tau) u(\tau) + B_e(\tau) v(\tau)] d\tau \right) \right. \]

\[ - \sup_{z \in \Gamma_0(\tau)} - \langle g, z \rangle + \left. \left( g, \int_0^{t_1} S(t_1, \tau) C(\tau) dw(\tau) \right) \right. \]

\[ \leq \sup_{a \in \Omega} \langle g, a \rangle - \langle g, S(t_1, 0) x_0 \rangle \geq \delta. \]  

(5.16a)

Since \( U_{sa}^0, V_{sa}^0 \) and \( B_e(0) \) are symmetric about the origin, (5.16a) holds iff \( \forall g \in X \)

\[ P \left( |\langle g, S(t_1, 0) x_0 \rangle| - \sup_{v \in V_{sa}^0} \left( g, \int_0^{t_1} S(t_1, \tau) B_p(\tau) v(\tau) d\tau \right) \right. \]

\[ \leq \sup_{a \in \Omega} |\langle g, a \rangle| + \sup_{z \in \Gamma_0(\tau)} |\langle g, z \rangle| + \sup_{u \in U_{sa}^0} \left( g, \int_0^{t_1} S(t_1, \tau) B_p(\tau) u(\tau) d\tau \right) \]

\[ - \left( g, \int_0^{t_1} S(t_1, \tau) C(\tau) d\tau(\tau) \right) \right) \geq \delta. \]  

(5.16b)

By Chebyshev’s inequality, (5.16b) leads to (5.15).

Remark 5.1. If \( \Omega = \{0\} \), then (5.15) reduces to (5.12).

If \( \Omega \) is a null space corresponding to a continuous projection \( \mathcal{P} : X \rightarrow X \) such that \( X = N(\mathcal{P}) \oplus \mathbb{R}(\mathcal{P}) \) where the null space \( N(\mathcal{P}) \) and the range \( R(\mathcal{P}) \) are closed subspaces of \( X \). Then we will have the following theorem.

Theorem 5.3. If \( \Omega = N(\mathcal{P}) \), system (5.1) is \((\varepsilon, \delta)\)-stochastically \( \Omega \)-controllable at \( x_0 \) on \([0, t_1]\) iff \( \forall g \in X \)

\[ \delta E |\langle g, \mathcal{P} S(t_1, 0) x_0 \rangle|^2 \]

\[ \leq \left( \varepsilon \|\mathcal{P} g\|_X + \rho^{1/2} \left[ \int_0^{t_1} \| B_p^*(\tau) S^*(t_1, \tau) \mathcal{P}^* g \|_{\mathcal{P}^*}^2 d\tau \right]^{1/2} \right)^2 \]
In realistic applications, many systems, governed by partial differential equations or by delay differential equations, have their appropriate state space being infinite-dimensional function space. Now, we will present some examples to illustrate the applicability of our results derived in the previous sections.

**Example 6.1.** Heat Equation (cf. [3]). Suppose we have the following stochastic equation

\[
\begin{align*}
\frac{dz(\xi, t)}{dt} &= A z(\xi, t) dt + u(t) dt + v(t) dt + dw(t), \\
z(\xi, 0) &= z_0(\xi), \quad 0 < \xi < 1, \quad t > 0,
\end{align*}
\]

(6.1.1)

where \(A h = \frac{\partial^2 h}{\partial \xi^2}\) for \(h \in D(A)\),

\[
D(A) = \{x \in X = L^2[0, 1]; \ x_t, x_{tt} \in X, \ x_t(0) = x_t(1) = 0\},
\]

(6.1.2)

and \(w(t)\) is an \(X\)-valued Wiener process

\[
w(t)(\xi) = \sum_{n=1}^{\infty} \beta_n(t) \sqrt{2} \cos n \pi \xi,
\]

(6.1.3)

where \(\beta_n(t)\) are real Wiener processes, independent, with incremental covariance \(\lambda_n\) and such that \(\sum_{n=1}^{\infty} \lambda_n < \infty\), i.e., \(\sum_{n=1}^{\infty} \lambda_n = K < \infty\). \(z_0(\xi)\) is a second-order \(X\)-valued Gaussian random variable with expectation zero, covariance operator \(P_0 \in L(X)\) such that \(P_0 \sqrt{2} \cos n \pi \xi = \alpha_n \sqrt{2} \cos n \pi \xi\), \(\sum_{n=1}^{\infty} \alpha_n = M < \infty\) and \(z_0(\xi)\) is also independent of \(w(t)\), \(\forall t > 0\).

\[\{\sqrt{2} \cos n \pi \xi\}_{n=1,2,...} \quad \text{generates a complete orthonormal basis for} \ X = L^2[0, 1]. \ A \text{ generates an analytic semigroup } S_t, \text{ given by}
\]

\[
(S_t h)(\xi) = \sum_{n=1}^{\infty} 2e^{-\alpha_n^2 t} \cos n \pi \xi \int_0^1 \cos n \pi y h(y) dy.
\]

(6.1.4)
So, $S_t: X \to D(A)$ and

$$z(\xi, t) = S_t z(\xi, 0) + \int_0^t S_{t-\tau} u(\tau) \, d\tau$$

$$+ \int_0^t S_{t-\tau} v(\tau) \, d\tau \cdot \int_0^t S_{t-\tau} \, dw(\tau). \quad (6.1.5)$$

In order to examine the $(\varepsilon, \delta)$-stochastic null-controllability of system (6.1), we need the following computation. For any $x \in L_2[0, 1] = X$, we have the following representation

$$x(\xi) = \sum_{n=1}^{\infty} \hat{x}(n) \sqrt{2} \cos n\pi \xi \quad (6.1.6)$$

where $\hat{x}(n) = 2 \int_0^1 x(y) \cos n\pi y \, dy$.

\[
\langle x, S_T z_0 \rangle_x = \left\langle \sum_{n=1}^{\infty} \hat{x}(n) \sqrt{2} \cos n\pi \xi, \sum_{n=1}^{\infty} 2e^{-k^2\pi^2 T} \cos k\pi \xi \right\rangle_x
\]

\[
= \sum_{n=1}^{\infty} \hat{x}(n) \sqrt{2} \cos n\pi \xi, \sum_{n=1}^{\infty} \hat{z}_0(k) e^{-k^2\pi^2 T} \cos k\pi \xi \rangle_x
\]

\[
= \sum_{n=1}^{\infty} \hat{x}(n) \hat{z}_0(n) e^{-n^2\pi^2 T}
\]

\[
\therefore \delta E \| \langle x, S_T z_0 \rangle_x \|^2 \leq \delta E \left\| \left( \sum_{n=1}^{\infty} \hat{x}^2(n) \right)^{1/2} \left( \sum_{n=1}^{\infty} \hat{z}_0(n) e^{-n^2\pi^2 T} \right)^{1/2} \right\|^2
\]

\[
\leq \delta \left( E \sum_{n=1}^{\infty} \hat{x}^2(n) \right) \left[ \sum_{n=1}^{\infty} \hat{x}_0^2(n) \sum_{n=1}^{\infty} e^{-2n^2\pi^2 T} \right]
\]

\[
\leq M\delta \sum_{n=1}^{\infty} \hat{x}_0^2(n) \sum_{n=1}^{\infty} \frac{1}{n^2} \left( \frac{1}{2\pi^2 T} \right)
\]

\[
= \frac{\delta M}{12T} \sum_{n=1}^{\infty} \hat{x}_0^2(n) \quad (6.1.7)
\]

\[
(\hat{z}_0(n))^2 = \left( \sqrt{2} \int_0^1 \cos n\pi y z_0(y) \, dy \right)^2 \text{ by (6.1.6). Therefore}
\]

\[
E \sum_{n=1}^{\infty} |\hat{z}_0(n)|^2 \leq \sum_{n=1}^{\infty} E \int_0^1 \sqrt{2} \cos n\pi y z_0^2(y) \sqrt{2} \cos n\pi y \, dy
\]
\[
= \sum_{n=1}^{\infty} \int_{0}^{1} \sqrt{2} \cos n\pi y \, \alpha_n \sqrt{2} \cos n\pi y \, dy \quad \text{(By Fubini's theorem)}
\]

\[
= \sum_{n=1}^{\infty} \alpha_n = M < \infty,
\]

\[
e^{-n^2\pi^2T} = \frac{1}{1 + n^2\pi^2T + (n^4\pi^4T^2/2!) + \cdots} \leq \frac{1}{n^2\pi^2T},
\]

\[
\int_{0}^{T} \|S_{T-\tau}^{x} x\|_{X}^2 \, d\tau = \int_{0}^{T} \left\| \sum_{n=1}^{\infty} \hat{x}(n) e^{-n^2\pi^2(T-\tau)} \sqrt{2} \cos n\pi \xi \right\|_{X}^2 \, d\tau
\]

\[
= \sum_{n=1}^{\infty} \hat{x}^2(n) \int_{0}^{T} e^{-2n^2\pi^2(T-\tau)} \, d\tau
\]

\[
\geq \sum_{n=1}^{\infty} \hat{x}^2(n) \int_{0}^{T} e^{-2n^2\pi^2(T-\tau)} \, d\tau
\]

\[
= \sum_{n=1}^{\infty} \frac{\hat{x}^2(n)}{2\pi^2} (1 - e^{-2\pi^2T}), \quad (6.1.8)
\]

\[
\|x\|_X = \left( \sum_{n=1}^{\infty} \hat{x}(n) \sqrt{2} \cos n\pi \xi, \sum_{k=1}^{\infty} \hat{x}(k) \sqrt{2} \cos k\pi \xi \right)_X^{1/2}
\]

\[
= \left[ \sum_{n=1}^{\infty} \hat{x}^2(n) \right]^{1/2}. \quad (6.1.9)
\]

By (6.1.8) and (6.1.9), we see that

\[
\left( \varepsilon \|x\|_X + \left[ \rho \int_{0}^{T} \|S_{T-\tau}^{x} x\|_{X}^2 \, d\tau \right]^{1/2} \right)^2
\]

\[\quad - \delta \sigma \int_{0}^{T} \|S_{T-\tau}^{x} x\|_{X}^2 \, d\tau + \text{trace } W \int_{0}^{T} \|S_{T-\tau}^{x} x\|_{X}^2 \, d\tau
\]

\[\geq \varepsilon^2 \|x\|_X^2 + (\rho - \delta \sigma + k) \int_{0}^{T} \|S_{T-\tau}^{x} x\|_{X}^2 \, d\tau
\]

\[= \left[ \varepsilon^2 + (\rho - \delta \sigma + K) \frac{1}{2\pi^2} \right] \sum_{n=1}^{\infty} \hat{x}^2(n)(1 - e^{-2\pi^2T}). \quad (6.1.10)
\]

Hence, with the comparison between (6.1.10) and (6.1.7), we observe that for all \(x \in X\), (6.1.7) \(\leq\) (6.1.10) for sufficiently large value of \(T\). So as to find the time \(T\), depending on the initial condition \(x_0\), we let \(T \geq 1/\pi^2\), then
(1 - e^{-2\pi^2 T}) \geq (1 - e^{-2}) \geq \frac{1}{4}. Hence in order that (6.1.7) \leq (6.1.10) we should have

\[ \frac{\delta M}{12T} \sum_{n=1}^{\infty} \chi^2(n) \leq \left[ \epsilon^2 + (\rho - \delta \sigma + K) \right] \frac{1}{2\pi^2} \frac{3}{4} \sum_{n=1}^{\infty} \chi^2(n). \]

Solving for \( T \),

\[ \frac{\delta M}{12T} \leq 3 \left[ \frac{2\pi^2 \epsilon^2 + (\rho - \delta \sigma + K)}{8\pi^2} \right]. \]

or

\[ T \geq \frac{2\pi^2 \delta M}{9[2\pi^2 \epsilon^2 + (\rho - \delta \sigma + K)]}. \]

Thus if we take

\[ T = \max \left\{ \frac{1}{\pi^2}, \frac{2\pi^2 \delta M}{9[2\pi^2 \epsilon^2 + (\rho - \delta \sigma + K)]} \right\}, \]

then the system (6.1.1) is \((\epsilon, \delta)\)-stochastically null-controllable at \( x_0 \in X \).

**Example 6.2.** One-dimensional wave equation (cf. [3]). Consider the formal stochastic evolution equation

\[ \begin{align*}
\frac{\partial z}{\partial t} &= z_{\xi \xi} + u(t) + \eta(t, \xi), \\
z(0, t) &= 0 = z(1, t), \\
z(\xi, 0) &= z_0(\xi), \quad z_1(\xi, 0) = z_1(\xi), \quad t \geq 0, \quad 1 \geq \xi \geq 0,
\end{align*} \tag{6.2.1} \]

where \( \eta(t, \xi) \) represents some distributed noise disturbance. \( z_0(\xi) \) and \( z_1(\xi) \) are independent second-order Gaussian \( X \)-valued random variables with expectation zero and covariance operator \( P_0, P_1 \in L(X) \), where \( P_0 \sqrt{\xi} \cos n\pi \xi = \alpha_n \sin n\pi \xi, \quad P_1 \sqrt{\xi} \cos n\pi \xi = \beta_n \sin n\pi \xi \) and \( \sum_{n=1}^{\infty} \alpha_n = M < \infty, \quad \sum_{n=1}^{\infty} \beta_n = N < \infty \); further, \( z_0(\xi) \) and \( z_1(\xi) \) are both independent with \( w(t) \) defined later.

Now \( X = L_2[0, 1]; u \in L_2^2(X), \quad v \in L_2^2(X), \quad \rho > \sigma > 0 \). In abstract form, (6.2.1) may be written as

\[ dg = [\mathcal{A}g + B_p u + B_v v] dt + C dw \tag{6.2.2} \]

where

\[ g = \begin{pmatrix} z_0(\xi) \\ z_1(\xi) \end{pmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \]
$w(t)(\xi)$ is a $X$-valued Wiener process, with the same properties as those mentioned in Example 6.1. $Az = -z_{tt}$, $z \in D(A)$ where $D(A) = H^2[0, 1] \cap H^1_0[0, 1]$, $A = A^*$, i.e., $A$ is self-adjoint. The operator

$$\tilde{A}g = \begin{pmatrix} 0 & 1 \\ -A & 0 \end{pmatrix} (z_0(\xi), z_1(\xi)),$$

$D(\tilde{A}) = D(A) \times D(A^{1/2})$

on $\tilde{H} = D(A^{1/2}) \times L_2[0, 1]$ generates a strongly continuous semigroup $S_t$. Here for $a, b \in H^1[0, 1] \times L_2[0, 1]$, the inner product on $\tilde{H}$ is given by

$$\langle a, b \rangle = \int_0^1 a_{1t}(\xi) b_{1t}(\xi) \, d\xi + \int_0^1 a_2(\xi) b_2(\xi) \, d\xi,$$

and

$$S_t \begin{pmatrix} z_0(\xi) \\ z_1(\xi) \end{pmatrix} = \begin{pmatrix} z(\xi, t) \\ z_t(\xi, t) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} 2 \left[ \langle z_0(\xi), \phi_n \rangle \cos n\pi t + \frac{1}{n\pi} \langle z_1(\xi), \phi_n \rangle \sin n\pi t \right] \phi_n \\ \sum_{n=1}^{\infty} 2 [-n\pi \langle z_0(\xi), \phi_n \rangle \sin n\pi t + \langle z_1(\xi), \phi_n \rangle \cos n\pi t] \phi_n \end{pmatrix}$$

(6.2.4)

where $\phi_n = \sin n\pi \xi$, $n = 1, 2, 3, \ldots$, $\{\sqrt{2} \sin n\pi \xi\}_{n=1,2,3,\ldots}$ generates a complete orthonormal basis for $L_2[0, 1] = X$.

$$\sqrt{2} \langle z_0(\xi), \phi_n \rangle = \sqrt{2} \int_0^1 z_0(\xi) \sin n\pi \xi \, d\xi = z_0(n),$$

$$\sqrt{2} \langle z_1(\xi), \phi_n \rangle = \sqrt{2} \int_0^1 z_1(\xi) \sin n\pi \xi \, d\xi = z_1(n).$$

It is easy to show that $S_t^* = S_{-t}$, and therefore

$$S_t^* \begin{pmatrix} z_0(\xi) \\ z_1(\xi) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} \left[ z_0(n) \cos n\pi t - \frac{1}{n\pi} z_1(n) \sin n\pi t \right] 2\phi_n \\ \sum_{n=1}^{\infty} \left[ n\pi z_0(n) \sin n\pi t + z_1(n) \cos n\pi t \right] 2\phi_n \end{pmatrix}$$

(6.2.5)
Let $P = [I, 0]$. Since we want the final state $z(\xi, T) = 0$, $\xi \in [0, 1]$ only, our target set is $\Omega = N(P)$. Thus the system (6.2.1) is $(\epsilon, \delta)$-stochastically null-controllable at $(x^{(m)}, x^{(n)})$ iff $\exists T \geq 0$ and for all $g \in X = L_2[0, 1]$

$$
\delta E \left| \left( g, PS_T \left( \frac{z_0(\xi)}{z_1(\xi)} \right) \right) \right|_X^2 \leq \left[ \rho^{1/2} \left( \int_0^T \left\| B_p^*(\tau) S_{T-\tau}^P P^* g \right\|_X^2 \, d\tau \right)^{1/2} + \epsilon \left\| g \right\|_X \right]^2
$$

$$
+ \text{trace } W \int_0^T \left\| C^*(\tau) S_{T-\tau}^P P^* g \right\|_{L(X, X)}^2 \, d\tau
$$

$$
\delta \sigma \int_0^T \left\| B_e^*(\tau) S_{T-\tau}^P P^* g \right\|_X^2 \, d\tau.
$$

(6.2.6)

Now, we first compute the following terms:

$$
\left\| g \right\|_X = \left\| \sum_{n=1}^{\infty} \hat{g}(n) \frac{\sqrt{2}}{n} \sin \pi n \xi \right\|_X
$$

$$
= \left\{ \sum_{n=1}^{\infty} \hat{g}^2(n) \right\}^{1/2}
$$

$$
\left| \left( g, PS_T \left( \frac{z_0}{z_1} \right) \right) \right|_X = \left| \left( \sum_{n=1}^{\infty} \hat{g}(n) \frac{\sqrt{2}}{n} \sin \pi n \xi, \sum_{n=1}^{\infty} \frac{\hat{z}_0(n)}{n} \cos k \pi t + \frac{1}{k \pi} \hat{z}_1(n) \sin k \pi t \right) \right|
$$

so that

$$
\delta E \left| \left( g, PS_T \left( \frac{z_0}{z_1} \right) \right) \right|_X^2 \quad \text{(with } \delta > 0) \quad \leq 2\delta E \left( \sum_{n=1}^{\infty} \frac{1}{n \pi} \hat{g}(n) \frac{\sqrt{2}}{n} \sin \pi n \xi \left( \sum_{n=1}^{\infty} \frac{\hat{z}_0(n)}{n} \cos k \pi t \right) \right)^2
$$

$$
\leq 2\delta E \left( \sum_{n=1}^{\infty} n^2 \hat{g}^2(n) \right)^{1/2} \left( \sum_{n=1}^{\infty} \frac{\hat{z}_0^2(n)}{n^2} \cos^2 \frac{n \pi T}{n \pi} \right)^{1/2} \leq 2\delta M \sum_{n=1}^{\infty} n^2 \hat{g}^2(n) \sum_{n=1}^{\infty} \frac{\cos^2 \frac{n \pi T}{n \pi}}{n^2} + 2\delta N \sum_{n=1}^{\infty} n^2 \hat{g}^2(n) \sum_{n=1}^{\infty} \frac{\sin^2 \frac{n \pi T}{n \pi}}{n^4 \pi^2}
$$
\[ \leq 2\delta \left( M \sum_{n=1}^{\infty} n^2 g^2(n) \sum_{n=1}^{\infty} \frac{1}{n^2} + N \sum_{n=1}^{\infty} n^2 g^2(n) \frac{1}{n^2} \left( \sum_{n=1}^{\infty} \frac{1}{n^2} \right)^2 \right) \]

\[ = \pi^2 \delta \left[ \frac{6M + N}{18} \right] \sum_{n=1}^{\infty} n^2 g^2(n). \quad (6.2.7) \]

Since \( B_p = B_e = C \), we only need to evaluate the term

\[ \int_0^T \| B_p S_{T-\tau}^* P^* g \|_x^2 d\tau = \int_0^T \left\| \sum_{n=1}^{\infty} n\pi \tilde{g}(n) (\sin n\pi(T-\tau)) \sqrt{2} \sin n\pi \xi \right\|_x^2 d\tau \]

\[ = \int_0^T \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \pi^2 \sin^2 n\pi(T-\tau) d\tau \]

\[ = \pi^2 \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \left( \frac{T}{2} - \frac{\sin 2n\pi T}{2n\pi} \right). \]

**Remarks.** Let \( T = t, t \in \mathbb{Z}^+ \). Then the right-hand side of (6.2.6) becomes

\[ \left[ \rho^{1/2} \left( \pi^2 \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \left( \frac{T}{2} - \frac{\sin 2n\pi T}{2n\pi} \right) + \varepsilon \sum_{n=1}^{\infty} \tilde{g}^2(n) \right)^{1/2} \right]^2 \]

\[ + (K - \delta \sigma) \left[ \pi^2 \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \left( \frac{T}{2} - \frac{\sin 2n\pi T}{2n\pi} \right) \right] \]

\[ \geq \rho \pi^2 \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \frac{T}{2} - \varepsilon^2 \sum_{n=1}^{\infty} \tilde{g}^2(n) + (K - \delta \sigma) \pi^2 \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \frac{T}{2} \]

\[ \geq (\rho + K - \delta \sigma) \pi^2 \frac{T}{2} \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) - \varepsilon^2 \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \sum_{n=1}^{\infty} \frac{1}{n^2} \]

\[ = \left( \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \right) \pi^2 \left[ \rho + K - \delta \sigma \right] + \frac{\varepsilon^2}{6} \]

\[ = \left( \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \right) \frac{\pi^2}{6} \left[ 3T(\rho + K - \delta \sigma) - \varepsilon \right]. \quad (6.2.8) \]

By comparing (6.2.7) and (6.2.8) in order to ensure the system be \((\varepsilon, \delta)\) stochastically null-controllable, we should have

\[ \frac{\pi^2}{6} \left( \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \right) \delta \left( \frac{6M + N}{3} \right) \]

\[ \leq \left( \frac{\pi^2}{6} \sum_{n=1}^{\infty} n^2 \tilde{g}^2(n) \right) \left[ 3T(\rho + K - \delta \sigma) - \varepsilon \right], \]
or
\[
\delta \left( \frac{6M + N}{3} \right) \leq 3T(\rho + K - \delta \sigma) - \varepsilon^2,
\]
or
\[
T \geq \frac{\delta(6M + N) + 3\varepsilon^2}{9(\rho + K - \delta \sigma)}.
\] (6.2.9)

Then, let \( T = l \) and \( l \) be an integer such that \( l \geq (\delta(6M + N) + 3\varepsilon^2)/9(\rho + K - \delta \sigma) \); the system (6.2.1) is \((\varepsilon, \delta)\)-stochastically null-controllable at \((\varepsilon_l(\delta))\).

Now let us test for locally null-controllability with the assumption that the neighborhood \( N(x_o) \) of \( x_o \) is a ball with radius \( r \). We first have the following computation.

\[
r^2 \| S^*(T, 0) P^*g \|_x^2 = r^2 \left\| S^*(T, 0) \begin{pmatrix} g \\ 0 \end{pmatrix} \right\|_x^2
\]
\[
= r^2 \left\| \sum_{n=1}^{\infty} \hat{g}(n) \cos n\pi t \sqrt{2} \sin n\pi \xi + \sum_{n=1}^{\infty} n\pi \hat{g}(n) \sin n\pi t \sqrt{2} \sin n\pi \xi \right\|_x^2
\]
\[
\leq 2r^2 \sum_{n=1}^{\infty} \hat{g}^2(n) \cos^2 n\pi t + 2r^2 \sum_{n=1}^{\infty} \hat{g}^2(n) n^2 \pi^2 \sin^2 n\pi t
\]
\[
\leq 2r^2 \pi^2 \sum_{n=1}^{\infty} n^2 \hat{g}^2(n) \sum_{n=1}^{\infty} \frac{\cos^2 n\pi T}{n^2} + 2r^2 \sum_{n=1}^{\infty} \hat{g}^2(n) n^2 \pi^2 \sin^2 n\pi t
\]
\[
= 2r^2 \frac{\pi^2}{6} \sum_{n=1}^{\infty} n^2 \hat{g}^2(n) \quad \text{(when } T = l \in \mathbb{Z}^+).\]

In order for the system to be \((\varepsilon, \delta)\)-stochastically locally null-controllable, we should have

\[
\pi^2 \left( \sum_{n=1}^{\infty} n^2 \hat{g}^2(n) \right) \delta \left( \frac{6M + N}{3} \right) + 2\delta r^2 \frac{\pi^2}{6} \sum_{n=1}^{\infty} n^2 \hat{g}^2(n)
\]
\[
\leq \pi^2 \left( \sum_{n=1}^{\infty} n^2 \hat{g}^2(n) \right) \left[ 3T(\rho + K - \delta \sigma) - \varepsilon^2 \right],
\]
or, equivalently,
\[
T \geq \frac{(6r^2 + 6M + N)\delta + 3\varepsilon^2}{p(\rho + K - \delta \sigma)}.
\] (6.2.10)
We observe that the value of $T$ in (6.2.10) $\geq$ the value of $T$ in (6.2.9). Hence, we can conclude that the time needed for the system to be locally null-controllable is longer than the null-controllable case.

**EXAMPLE 6.3.** Stochastic retarded functional differential equation (cf. [7]). We consider the linear system on $[0, t_1]$:

$$
d\xi(t) = A_0(t) \xi(t) dt + A_1(t) \xi(t + \Theta) dt + \int_{-b}^{0} A_2(t) \xi(t + \Theta) d\Theta \\
+ B(t) u(t) dt + C(t) dw(t),
$$

(6.3.1)

where $-b < 0$ and $b$ is a positive number; $X$ is a Hilbert space; $A_0 \in L_\infty([0, t_1]; L(X))$, $A_1 \in L_\infty(L(X))$, $A_2 \in L_\infty([0, t_1] \times (-b, 0); L(X))$, $B \in L_\infty([0, t_1]; L(U_{ad}, X))$; $w(t)$ is an $X$-valued Wiener process described same as in Section 2; $h \in L_2([-b, 0]; X)$ and control term $u(t) \in U_{ad}^p$, defined as what follows in Section 2.

Now we introduce the space $M^2(-b, 0; X)$ as follows: Let $L_2(-b, 0; X)$ be the space of measurable, square integrable $X$-valued functions on $[-b, 0)$; then $M^2(-b, 0; X)$ is the Hilbert space of equivalence classes of functions in $L_2(-b, 0; X)$ under the equivalence relation

$$\langle f, g \rangle_{M^2} = \langle f(0), g(0) \rangle_X + \int_{-b}^{0} \langle f(t), g(t) \rangle_X dt$$

and is isometrically isomorphic to the space $X \times L^2(-b, 0; X)$. Then the corresponding $M^2$-version of (6.3.1) is

$$
d_\xi(t) = \tilde{A}(t) \xi(t) dt + \tilde{B}(t) u(t) dt + \tilde{C}(t) dw(t), \\
z(0) = h_0,
$$

(6.3.2)

where $h_0 \in L_2(\Omega, M^2)$, $\tilde{B} \in B_\infty(U; M^2)$, $\tilde{C} \in B_\infty(X, M^2)$

$$
\tilde{B}u(t)(\Theta) = Bu(t), \quad \Theta = 0 \text{ for } u \in U_{ad}^p \\
= 0, \quad \Theta \in [-b, 0),
$$

$$
\tilde{C}(t) v(\Theta) = C(t)v, \quad \Theta = 0 \text{ for all } v \in X \\
= 0, \quad \Theta \in [-b, 0).
$$

$\tilde{A}(t): W^{1,2} \rightarrow M^2$ is a densely closed operator with domain $D(\tilde{A}(t)) = W^{1,2}$ and is defined by
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\[(\tilde{A}(t)h)(\Theta) = A^0(t)h, \quad \Theta = 0 \]
\[= \frac{dh}{d\Theta}, \quad \Theta \in [-b, 0), \]

\[A^0(t)h = A_0(t)h(\Theta) + A_1(t)h(\Theta) + \int_{-b}^{0} A_1(t)h(\Theta) d\Theta + B(t)u(t) + C(t)\frac{dw(t)}{dt}. \]

Here \(W^{1,2}\) is the Sobolev space

\[W^{1,2}([-b, 0]; X) = \{x \in L_2([-b, 0]; X) : Dx \text{ (the distributional derivative of } x) \in L_2([-b, 0]; X)\}\.\]

Together with the embedding \(z \rightarrow (z(0), z(\cdot))\), \(W^{1,2}\) is a subspace of \(M^2([-b, 0]; X)\).

It is readily proved that (6.3.2) has the mild solution

\[z(t) = U(t, 0)h + \int_0^t U(t, s)\tilde{B}(s)u(s)ds + \int_0^t U(t, s)\tilde{C}(s)dw(s), \quad (6.3.3)\]

where \(U(t, s)\) is the evolution operator associated with \(\tilde{A}(t)\) and is given by

\[(U(t, s)h)\Theta = \Phi(t, s)h(\Theta), \quad \Theta = 0 \]
\[\quad = \Phi(t + s, s)h(\Theta) + \Phi'(t + s, s)h(\Theta), \quad t + s \geq s, s \neq 0, \]

and \(\Phi(t, s) : X \rightarrow X\) is the unique solution of

\[\frac{\partial}{\partial t} \Phi(t, s) = A_0(t)\Phi(t, s) + A_1(t)(\Phi(t + s, s) - \Phi(t, s)) + \int_{-b}^{0} A_2(t)\Phi(t + s, s) d\Theta, \quad t + s \geq s, \]

\[\Phi(s, s) = 1, \quad \Phi'(t, s) \in L(L_2([-b, 0]; X)).\]

Now let \(u \in U^+_0\); thus the system (6.3.2) is \((\varepsilon, \delta)\)-stochastically null-controllable at \(\tilde{h}\) iff \(\exists T \geq 0\), for all \(g \in M^*_2 = M_2\), we have

\[\delta E |\langle g, U(T, 0)\tilde{h}\rangle_{M^*_2}|^2 \leq \left(\varepsilon \|g\|_{M^*_2} + \left[\rho \int_0^T \|\tilde{B}(t)U^*(T, t)g\|_X d\tau\right]^{1/2}\right)^2 \]
\[+ \text{trace} \int_0^T \|\tilde{C}^*(t)U^*(T, t)g\|_X^2dt.\]
Let
\[ \mathcal{P}_x: M^2([-b, 0]; X) \to X, \]
\[ \mathcal{P}_{L_2}: M^2([-b, 0]; X) \to L_2([-b, 0]; X) \]
be projections on \( M^2 \) such that
\[ R(\mathcal{P}_x) = N(\mathcal{P}_{L_2}) = X \quad \text{and} \quad R(\mathcal{P}_{L_2}) = N(\mathcal{P}_x) = L_2([-b, 0]; X). \]

Here system (6.3.2) is \( X - (\epsilon, \delta) \)-stochastically controllable at \( \hat{h} \) with target set \( H \) iff \( \exists T \geq 0 \) and for all \( g \in L_2([-b, 0]; H) \)
\[
\delta E \left| \left( \langle g, \mathcal{P}_{L_2} U(T, 0) \hat{h} \rangle_{L_2} \right)^2 \right| \leq \left( \epsilon \| g \|_{L_2} + \left[ \rho \int_0^T \| \mathcal{B}^*(\tau) U^*(T, \tau) \mathcal{P}_{L_2} g \|_0^2 \, d\tau \right]^{1/2} \right)^2 + \text{trace } W \int_0^T \| \mathcal{C}^*(\tau) U^*(T, \tau) \mathcal{P}_{L_2} g \|_x^2 \, d\tau;
\]
on the other hand, system (6.3.2) is \( L_2 - (\epsilon, \delta) \)-stochastically controllable at \( \hat{h} \) with target set \( L_2([-b, 0]; X) \) iff \( \exists T \geq 0 \) and for all \( g \in X \)
\[
\delta E \left| \left( \langle g, \mathcal{P}_x U(T, 0) \hat{h} \rangle_{X} \right)^2 \right| \leq \left( \epsilon \| g \|_{X} + \left[ \rho \int_0^T \| \mathcal{B}(\tau) U^*(T, \tau) \mathcal{P}_x g \|_0^2 \, d\tau \right]^{1/2} \right)^2 + \text{trace } W \int_0^T \| \mathcal{C}^*(\tau) U^*(T, \tau) \mathcal{P}_x g \|_x^2 \, d\tau.
\]

References