The Continuity of Subdifferential Mapping

Jian-Hua Wang and Chao-Xun Nan

Department of Mathematics, Anhui Normal University, Wuhu, Anhui,
People’s Republic of China 241000

Submitted by William A. Kirk
Received July 29, 1996

In this paper we introduce and study the nearly uniformly norm upper semicontinuity for subdifferential mappings. Further we establish the interesting relations between uniform α upper semicontinuity and nearly uniformly norm upper semicontinuity. Moreover, we discuss the weakly (weak*) uniformly upper semicontinuity and give applications in differentiability theory. © 1997 Academic Press

1. INTRODUCTION

The study of the duality mapping and the subdifferential mapping has been developed extensively by the topics [1, 2, 4, 5]. In 1993, Giles and Moors [3] introduced and studied the properties of α upper semicontinuity and uniform α upper semicontinuity for set—valued mappings. This α upper semicontinuity and norm upper semicontinuity with compact values are equivalent for duality mappings, but both are different in the uniform case. The purpose of this paper is to establish the interesting relations between uniform α upper semicontinuity and uniformly norm upper semicontinuity. For this purpose, we define the concept of nearly uniformly norm upper semicontinuity. We will show that the uniform α upper semicontinuity is equivalent to the nearly uniformly norm upper semicontinuity for subdifferential mapping $x \to \partial p(x)$ from $X$ into subsets of $X^*$; $p$ is the gauge of a bounded closed convex set $K$ with $0 \in \text{int} K$. In [3], it has been proved that a Banach space $X$ has uniformly Fréchet differentiable norm if and only if the duality mapping $x \to D(x)$ is uniformly norm upper semicontinuous on $S(X)$. In this paper we will introduce the weakly (weak*) uniformly upper semicontinuity of the subdifferential mapping of the gauge and prove that for a closed bounded convex set $K$ with
CONTINUITY OF SUBDIFFERENTIAL MAPPING

0 \in \operatorname{int} K$, if the subdifferential mapping of the gauge function of $K$ is weakly (weak*) uniformly upper semicontinuous on $S(X)$, then it is single valued on $S(X)$. Further we give the characterizations of the above two continuity properties for subdifferential mappings. These characterizations have useful applications in the differentiability theory. Our work generalises some results from Giles and Moors’s paper [3].

2. NEARLY UNIFORMLY NORM UPPER SEMICONTINUITY

Let $X$ be a Banach space; the dual space of $X$ will be denoted by $X^*$. We denote the closed unit ball of $X$ by $U(X)$ and the unit sphere of $X$ by $S(X)$. A closed bounded convex set $K$ in a Banach space, given $x^* \in X^* \setminus \{0\}$ and $\delta > 0$, the slice of $K$ defined by $x^*$ and $\delta > 0$, is a subset $S(K, x^*, \delta) = \{x \in K : x^*(x) > \sup x^*(K) - \delta\}$. For a set $A$ in a metric space $X$, the Kuratowski index of noncompactness of $A$ is

$$\alpha(A) = \inf \{ \varepsilon > 0 : A \text{ is covered by a finite family of sets of diameter less than } \varepsilon \}.$$ 

$2^{-d}$ denotes the set of all nonempty subsets of the set $A$.

Let $X$ be a topological space and $Y$ be a metric space. The mapping $\Phi: X \to 2^Y$ is said to be $\alpha$ upper semicontinuous at $x \in X$ if, given $\varepsilon > 0$, there exists an open neighbourhood $U$ of $x$ such that $\alpha(\Phi(U)) < \varepsilon$.

Let $X$ be a Banach space and $Y$ be a metric space, $\Phi: X \to 2^Y$. We say that $\Phi$ is uniformly $\alpha$ upper semicontinuous on $S(X)$ if, given $\varepsilon > 0$, there exists a $0 < \delta(\varepsilon) < 1$ such that $\alpha(\Phi(B(x, \delta))) < \varepsilon$ for all $x \in S(X)$ where $B(x, \delta) = \{a \in S(X) : \|x - a\| < \delta\}$.

Consider a set-valued mapping $\Phi: X \to 2^Y$ where $X, Y$ are topological spaces and $Y$ with $\tau$ topology. We refer to $\Phi$ as a $\tau$ upper semicontinuous mapping at $x \in X$ if, given a $\tau$ open set $W$ containing $\Phi(x)$, there exists an open neighbourhood $U$ of $x$ such that $\Phi(U) \subset W$. Further we say that $\Phi$ is a $\tau$ usco mapping at $x \in X$ if $\Phi$ is $\tau$ upper semicontinuous at $x \in X$ and $\Phi(x)$ is nonempty, $\tau$ compact.

A norm usco mapping $\Phi$ from a metric space $(X, d)$ into subsets of a Banach space $Y$ is said to be uniformly norm upper semicontinuous on $X$ if, given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\Phi(x') \subset \Phi(x) + \varepsilon U(Y)$ for all $d(x, x') < \delta$.

Now we introduce a generalization of uniformly norm upper semicontinuity. Let $X, Y$ be Banach spaces, $\Phi: X \to 2^Y$. If $\Phi$ is a norm usco mapping and for any given $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that whenever $(x_n) \subset S(X)$ and $\operatorname{diam}(x_n) < \delta$, then $(x_n)$ has a subsequence
\{x_n\} which satisfies
$$
\Phi(x_n) < \Phi(x_{n_j}) + \varepsilon U(Y) \quad \text{for all } i, j \geq 1.
$$

We say that $\Phi$ is nearly uniformly norm upper semicontinuous on $S(X)$.

Given a closed bounded convex set $K$ with $0 \in \text{int } K$ in a Banach space $X$, the gauge $p$ of $K$ is defined by $p(x) = \inf(\lambda \geq 0: x \in \lambda K)$. It is known that $p$ is a uniformly continuous sublinear functional on $X$ [8]. Recall that $K^0$, the polar of $K$, is defined by $K^0 = \{x^* \in X^*: x^*(x) \leq 1 \text{ for all } x \in K\}$ and $K^0$ is weak* compact convex and $0 \in \text{int } K^0$. $K^{00}$ denotes the polar of $K^0$ in $X^{**}$.

Let $\varphi$ be a continuous convex function on a nonempty open convex set $A$ of a Banach space $X$. A subgradient of $\varphi$ at $x_0 \in A$ is a continuous linear functional $x^*$ on $X$ such that $x^*(x - x_0) \leq \varphi(x) - \varphi(x_0)$ for all $x \in A$. The subdifferential of $\varphi$ at $x_0$ is denoted by $\partial \varphi(x_0)$ and is the set of subgradients of $\varphi$ at $x_0$. The set-valued mapping $x \mapsto \partial p(x)$ from $X \setminus \{0\}$ into subsets of $X^*$ is called the subdifferential mapping.

We note some basic relations between polar and subdifferential. For a bounded closed convex set $K$ with $0 \in \text{int } K$, given $x \in X$, for any $x^*_0 \in \partial p(x_0)$, we have $x^*_0(x_0) = p(x_0)$ and $x^*_0(x) \leq p(x)$ for all $x \in X$, so $x^*_0(x) \leq 1$ for all $x \in K$ and $x^*_0 \in K^0$.

We use $\text{int}$ to denote natural embedding elements.

Consider a bounded closed convex $K$ in a Banach space $X$. We will say that $K$ has property $\text{U}_a$ if, given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\alpha(S(K, x^*, \delta)) < \varepsilon$ for all $x^* \in S(X^*)$. For a bounded closed convex set $A$ in a dual space $X^*$, we say that $A$ has property weak* $\text{U}_a$ if, given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that $\alpha(S(A, \delta, \delta)) < \varepsilon$ for all $x \in S(X)$.

Given a Banach space $X$, for each $x \in S(X)$ we denote by $D(x)$ the set $\{x^* \in S(X^*): x^*(x) = 1\}$. The set-valued mapping $x \mapsto D(x)$ from $S(X)$ into subsets of $S(X^*)$ is called the duality mapping on $S(X)$. It is clear that when $K = U(X)$ we have $D(x) = \partial p(x)$ for all $x \in S(X)$. For a sequence $\{x_n\}$ in a Banach space $X$, the separation constant of $\{x_n\}$ is defined by $\text{sep}_{n \geq m}(x_n) = \inf(||x_n - x_m||: n \neq m)$.

Let $A, B$ be nonempty, bounded, closed subsets of a metric space $X$. The Hausdorff distance from $A$ to $B$ is defined to be the real number
$$
H(A, B) = \sup \left\{ \sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A) \right\}.
$$

$F$ denotes the set of all nonempty, bounded, closed subsets of $X$. It is known that $H(\cdot, \cdot)$ is a metric on $F$. 
Theorem 2.1. Let $K$ be a bounded closed convex set with $0 \in \text{int } K$ in a Banach space and let $p$ be the gauge of $K$ on $X$. Then the following statements are equivalent:

1. $K^0$ has property weak* $U\alpha$.
2. The subdifferential mapping $x \mapsto \partial p(x)$ is nearly uniformly norm upper semicontinuous on $S(X)$.
3. The subdifferential mapping $x \mapsto \partial p(x)$ is uniformly $\alpha$ upper semicontinuous on $S(X)$.

Proof. (1) $\Rightarrow$ (2): Since $K^0$ has property weak* $U\alpha$, so $K^0$ has property $\alpha$ for each $\hat{x} \neq 0$. By [3, 5] we claim that the subdifferential mapping $x \mapsto \partial p(x)$ is norm usco at each $x \in X \setminus \{0\}$. For any given $\varepsilon > 0$, according to assumption (1), there is a corresponding $\eta > 0$ such that for all $x \in S(X)$ we have $\alpha(S(K^0, \hat{x}, \eta)) < \varepsilon$. Arbitrarily take a $\delta > 0$ with $\delta \cdot \sup_{x \in K^0} \|x^*\| < \eta/2$ such that $|p(x) - p(y)| < \eta/2$ when $\|y - x\| < \delta$. (Note that $p$ is uniformly continuous on $X$ when $K$ is convex and $0 \in \text{int } K$ [8].) Suppose that $(x_n) \subset S(X)$ and $\text{diam}((x_n)) < \delta$. If $x^* \in \partial p(x_n)$, we deduce

$$x^*(x_1) = x^*(x_n) + x^*(x_1) - x^*(x_n)$$

$$\geq \sup \hat{x}_n(K^0) - \|x^*\| \|x_n - x_1\|$$

$$> \sup \hat{x}_n(K^0) - \eta/2 = p(x_n) - \eta/2 > p(x_1) - \eta$$

$$= \sup \hat{x}_1(K^0) - \eta.$$

This shows that $\partial p(x_n) \subset S(K^0, \hat{x}_1, \eta)$ for all $n \geq 1$. Thus $\alpha((\partial p(x_n))) < \varepsilon$. So there exists a $\gamma \in (0, \varepsilon)$ and sets $A_i \subset X^*$ with $\text{diam}(A_i) \leq \gamma$, $i = 1, 2, \ldots, k$, such that $\bigcup_{i=1}^k A_i \supset (\partial p(x_n))$. Further for each $\partial p(x_n)$ we can define a nonempty index set as

$$\sigma_n = \{j : \partial p(x_n) \cap A_j \neq \emptyset, j = 1, 2, \ldots, k\}.$$

Since $k$ is a fixed integer, clearly $\sigma_n = 1$ is a finite set. We note that if $\sigma_n = \sigma_n$, then for any $a \in \partial p(x_n)$ and any $b \in \partial p(x_n)$ we have $d(a, \partial p(x_n)) \leq \gamma$ and $d(b, \partial p(x_n)) \leq \gamma$. Consequently,

$$H(\partial p(x_n), \partial p(x_m))$$

$$= \sup \left\{ \sup_{a \in \partial p(x_n)} d(a, \partial p(x_n)), \sup_{b \in \partial p(x_m)} d(b, \partial p(x_m)) \right\}$$

$$\leq \gamma < \varepsilon.$$
Hence \( \partial p(x_n) \) has a subsequence \( \{ \partial p(x_{n_j}) \} \) such that \( H(\partial p(x_{n_j}), \partial p(x_{n_j})) < \varepsilon \). This shows that \( \partial p(x_{n_j}) \subset \partial p(x_{n_j}) + \varepsilon U(X^*) \) for all \( n, j \geq 1 \).

(2) \( \Rightarrow \) (1): If \( K^0 \) does not have property \( \text{weak*} \) \( U \alpha \), then there exists an \( \varepsilon_0 > 0 \) and a sequence \( \{ x_n \} \subset S(X) \) such that \( \alpha(S(K^0, \hat{x}_k, 1/k^2)) \geq 2 \varepsilon_0 \) for all \( k \geq 1 \). Thus, for each slice of \( K^0 \), \( S(K^0, \hat{x}_k, 1/k^2) \), we can choose a sequence \( \{ x_n^{(k)} \} \) in \( S(K^0, \hat{x}_k, 1/k^2) \) with \( \text{sep}_{n \geq 1} \{ x_n^{(k)} \} > 2 \varepsilon_0 \). From \( x_n^{(k)}(x_n) > \hat{x}_k(K^0) - 1/k^2 \) and \( x_n^{(k)}(y) \leq p(y) \) for all \( y \in X \), it follows that \( x_n^{(k)}(y) - x_n^{(k)}(x_n) \leq p(y) - p(x_n) + 1/k^2 \) for all \( y \in X \). By the Brøndsted–Rockafellar theorem 7, there exist \( y_n^{(k)} \in S(X) \) and \( y_n^{(k)} \in \partial p(y_n^{(k)}) \) such that \( \| y_n^{(k)} - x_n - x_n^{(k)} \| < 2/k < \varepsilon_0 \) and \( \| y_n^{(k)} - x_n^{(k)} \| < 1/k^2 \) for all \( n \geq 1 \). By assumption (2), for given \( \varepsilon_0/8 \) there is a \( \delta_0 > 0 \) such that whenever \( \{ x_n \} \subset S(X) \) and \( \text{diam}(x_n) < \delta_0 \), \( x_n \) contains a subsequence \( \{ x_{n_j}^{(k)} \} \) with \( \partial p(x_{n_j}) \subset \partial p(x_{n_j}) + \varepsilon_0/8 \) \( U(X^*) \), \( i, j \geq 1 \). Now select an integer \( k \geq 1 \) with \( 4/k' < \min(\delta_0, \varepsilon_0/8) \). It follows from \( \| y_n^{(k)} - x_{n_j}^{(k)} \| < 2/k' \) for all \( n \geq 1 \) that \( \text{diam}(y_n^{(k)}) < \delta_0 \). Thus, without loss of generality,

\[
\partial p(y_n^{(k)}) \subset \partial p(y_n^{(k)}) + \varepsilon_0/8 \ U(X^*)
\]

for all \( n, m \geq 1 \). Observe \( \text{sep}_{n \geq 1} \{ y_n^{(k)} \} > 2 \varepsilon_0 \) and \( \| y_n^{(k)} - x_n^{(k)} \| < 2/k < \varepsilon_0/8 \) for all \( n \geq 1 \). Therefore \( \text{sep}_{n \geq 1} \{ y_n^{(k)} \} \geq 2/2 \). Arbitrarily fix a set \( \partial p(y_n^{(k)}) \). For each \( y_n^{(k)} \), we can find \( u_n^{(k)} \) in \( \partial p(y_n^{(k)}) \) such that \( \| y_n^{(k)} - u_n^{(k)} \| = d(y_n^{(k)}, \partial p(y_n^{(k)})) \) for all \( n \). Because \( \partial p(y_n^{(k)}) \subset \partial p(y_n^{(k)}) + \varepsilon_0/8 \ U(X^*) \) and \( \{ y_n^{(k)} \} \subset \partial p(y_n^{(k)}) \), we get that \( \| y_n^{(k)} - u_n^{(k)} \| < \varepsilon_0/8 \) for all \( n \). By virtue of \( \text{sep}_{n \geq 1} \{ y_n^{(k)} \} \geq \varepsilon_0/2 \), we obtain \( \text{sep}_{n \geq 1} \{ u_n^{(k)} \} \geq \varepsilon_0/4 \). However, \( u_n^{(k)} \in \partial p(y_n^{(k)}) \), \( n \geq 1 \), and \( \partial p(y_n^{(k)}) \) is norm compact, which is a contradiction.

(3) \( \Rightarrow \) (1): For any \( x \in S(X) \), an easy computation shows that \( p(x) \geq 1/\sup_{y \in K} \| y \| \), so

\[
\sup_{x \in X} \| y \| \geq \frac{1}{\sup_{y \in K} \| y \|} \quad \text{for all } x \in S(X).
\]

Given \( x \in S(X) \) and \( 0 < \delta^2 < 1/\sup_{x \in K} \| y \| \), following the proof of Theorem 3.2(ii) in [3], we have \( S(K^0, \hat{x}, \delta^2) \subset \partial p(B(x, 2 \delta)) + 2 \delta U(X^*) \). By the assumption of (3), for given \( \varepsilon > 0 \) there exists a \( 0 < \delta(\varepsilon) < \varepsilon/2 \) such that \( \alpha(\partial p(B(x, 2 \delta))) < \varepsilon \) for all \( x \in S(X) \). Thus \( \alpha(S(K^0, \hat{x}, \delta^2)) < 2 \varepsilon \) for all \( x \in S(X) \).

\[\text{Note that if } x_n^{(k)}(y) - x_n^{(k)}(x_n) \leq p(y) - p(x_n) + 1/k^2 \text{ for all } y \in X \text{ and } x_n \in S(X) \text{, by the Brøndsted–Rockafellar theorem 7 there exist } x \in X \text{ and } x^* \in \partial p(x) \text{ such that } \| x - x^* \| < 1/k, \| x^* - x_n^* \| < 1/k. \text{ Let } y_0 = x^*/\| x^* \|. \text{ Then } \| y_0 - x_0^* \| \leq \| x - x^* \| + \| y_0 - x^* \| < 1/k + \| x \| (1/k) - 1 = 1/k + \| x \| \leq 1/k + \| x \| < 2/k. \text{ Also } \partial p(x) = \partial p(y_0). \text{ Thus the result holds.}\]
(1) $\Rightarrow$ (3): Since $K^0$ has property weak* $U\alpha$, thus for given $\varepsilon > 0$ there exists an $\eta > 0$ satisfying $\alpha(S(K^0, \hat{x}, \eta)) < \varepsilon$ for all $x \in S(X)$. Now take

$$0 < \delta < \frac{\eta}{2 \sup_{y^* \in K^0} \|y^*\|}$$

such that $|p(x) - p(y)| < \eta/2$ when $\|x - y\| < \delta$. Consider $x^* \in \partial p(y)$ where $y \in B(x, \delta)$. Then

$$x^*(x) = x^*(y) + x^*(x) - x^*(y)$$
$$= p(y) + x^*(x - y)$$
$$= p(x) + p(y) - p(x) + x^*(x - y)$$
$$> \sup \hat{x}(K^0) - \frac{\eta}{2} - \frac{\eta}{2} = \sup \hat{x}(K^0) - \eta.$$

So $\partial p(B(x, \delta)) \subseteq S(K^0, \hat{x}, \eta)$ for all $x \in S(X)$. It follows that $\alpha(\partial p(B(x, \delta))) < \varepsilon$ for all $x \in S(X)$.

**Corollary 2.1.** For a Banach space $X$, the following statements are equivalent:

1. $X^*$ has property weak* $U\alpha$ (i.e., $U(X^*)$ has property weak* $U\alpha$).
2. The duality mapping $x \to D(x)$ is nearly uniformly norm upper semicontinuous on $S(X)$.
3. The duality mapping $x \to D(x)$ is uniformly $\alpha$ upper semicontinuous on $S(X)$.

From Lemma 3.1 of [3] we see that for a closed bounded convex set $K$ with $0 \in \text{int } K$ in a Banach space $X$, $K$ has property $U\alpha$ if and only if $K^0$ has property weak* $U\alpha$. Using Theorem 2.1, we can immediately derive the following result.

**Theorem 2.2.** Let $K$ be a closed bounded convex set with $0 \in \text{int } K$ in a Banach space $X$ and let $p$ be the gauge of $K^0$ on $X^*$. Then the following statements are equivalent:

1. $K$ has property $U\alpha$.
2. The subdifferential mapping $x^* \to \partial p(x^*)$ is nearly uniformly norm upper semicontinuous on $S(X^*)$.
3. The subdifferential mapping $x^* \to \partial p(x^*)$ is uniformly $\alpha$ upper semicontinuous on $S(X^*)$. 
Corollary 2.2. For a Banach space $X$, the following statements are equivalent:

1. $X$ has property $U\alpha$ (i.e., $U(X)$ has property $U\alpha$).
2. The duality mapping $x^* \mapsto D(x^*)$ is nearly uniformly norm upper semicontinuous on $S(X^*)$.
3. The duality mapping $x^* \mapsto D(x^*)$ is uniformly $\alpha$ upper semicontinuous on $S(X^*)$.

3. Applications of Weakly Weak* Uniformly Upper Semicontinuity in the Differentiability Theory

In this section we will introduce and study the properties of weakly weak* uniformly upper semicontinuity for subdifferential mappings of the gauge. Now we give the following definitions.

Consider a set-valued mapping $F$ from a metric space $(X, d)$ into subsets of a linear topological space $Y$ with $\tau$ topology, where $F$ is a $\tau$ usco mapping. We say that $F$ is uniformly $\tau$ upper semicontinuous on $X$ if, for any given $\tau$ open neighbourhood $W$ of 0 in $Y$, there exists a $\delta > 0$ such that $F(B_\delta(x, x')) \subseteq F(x) + W$ for all $x \in X$ where $B_\delta(x, x') = \{x' \in X : d(x, x') < \delta\}$.

A set-valued mapping $F$ from a Banach space $X$ into subsets of $X^*$ with weak weak* topology is said to be weakly weak* uniformly upper semicontinuous on $S(X)$ if $F$ is a weak weak* usco mapping and for any given weak weak* neighbourhood $W$ of 0 in $X^*$ there exists a $\delta > 0$ such that $F(x') \subseteq F(x) + W$ for all $x, x' \in S(X)$ and $\|x - x'\| < \delta$.

To establish the relations between weakly weak* uniformly upper semicontinuity and differentiability, we need the following concepts and proposition.

Let $X$ be a topological space, let $Y$ be a linear topological space with $\tau$ topology, and let $\Phi : X \to 2^Y$ be $\tau$ upper semicontinuous at $x \in X$. We say that $\Phi$ is $\tau$ cusco at $x \in X$ if $\Phi(x)$ is nonempty, convex, and $\tau$ compact.

A $\tau$ usco ($\tau$ cusco) mapping $\Phi$ from a topological space $X$ into subsets of a topological space (linear topological space) $Y$ with $\tau$ topology is said to be minimal if its graph does not contain the graph of any other $\tau$ usco ($\tau$ cusco) mapping with the same domain.

Proposition 3.1. Let $(X, d)$ be a metric space, let $(Y, \tau)$ be a Hausdorff linear topological space (Hausdorff locally convex space), and let $\Phi$ be a minimal $\tau$ usco ($\tau$ cusco) mapping from $X$ into subsets of $Y$. If $\Phi$ is uniformly $\tau$ upper semicontinuous on $X$, then $\Phi$ is single-valued on $X$. 


Proof. Suppose \( \Phi \) is not single-valued at \( x_0 \in X \). Then there exist \( y_1, y_2 \in \Phi(x_0) \) and \( y_1 \neq y_2 \). Select a \( y_1 \)-neighbourhood \( U_1 \) and a \( y_2 \)-neighbourhood \( U_2 \) such that \( U_1 \cap U_2 = \emptyset \). Further there is a \( \tau \) closed balanced (\( \tau \) closed balanced convex) 0-neighbourhood \( W \) with \( y_2 + W \subset U_2 \). Since \( \Phi \) is uniformly \( \tau \) upper semicontinuous on \( X \), there exists a \( \delta > 0 \) such that \( \Phi(B_\delta(x, \delta)) \subset \Phi(x) + W \). But \( \Phi(B_\delta(x_0, \delta)) \not\subset y_2 + W \). Because \( \Phi \) is minimal \( \tau \) usco (\( \tau \) usco) on \( X \), it follows from Lemma 2.5 of [3] that there exists an open subset \( V \) of \( B(x, \delta) \) such that \( \Phi(V) \subset (y_2 + W) \). For any \( x \in V \) we have \( y_2 \not\in \Phi(x_0) + W \); if not, let \( y_2 = u_1 + u_2 \) where \( u_1 \in \Phi(x) \), \( u_2 \in W \). Then \( y_2 - u_2 = u_1 \). As \( W \) is balanced, \( u_1 \in y_2 + W \), but this is impossible. However, \( x_0 \in B(x_1, \delta) \), which contradicts the uniform \( \tau \) upper semicontinuity of \( \Phi \) on \( X \).

Remark 3.1. Let \( K \) be a bounded closed convex set with \( 0 \in \text{int} \, K \), and let \( p \) be a gauge of \( K \). Following Lemma 3.5 by Giles and M oors [3], one can now prove that the subdifferential mapping \( x \to \partial p(x) \) on \( S(X) \) is minimal weak* cusco. (Note that the subdifferential mapping \( x \to \partial p(x) \) from an open convex subset \( A \) into subsets of \( X^* \) is minimal weak* cusco (see [7]).) By Proposition 3.1 we get that if the subdifferential mapping \( x \to \partial p(x) \) is weakly (weak*) uniformly upper semicontinuous on \( S(X) \), then it must be single-valued.

Using Remark 3.1, we can deduce the following result.

Remark 3.2. Let \( K \) be a bounded closed convex set with \( 0 \in \text{int} \, K \) in a Banach space \( X \) and let \( p \) be the gauge of \( K \). Then the subdifferential mapping \( x \to \partial p(x) \) is weakly uniformly (uniformly norm or weak* uniformly) upper semicontinuous on \( S(X) \) if and only if, whenever \( x^*_n, y^*_n \in K^0 \) and \( x_n \in S(X) \) with \( x^*_n(x_n) + y^*_n(x_n) - 2p(x_n) \to 0 \), we have \( x^*_n - y^*_n \to 0 \) \( (x^*_n - y^*_n \to 0 \text{ or } x^*_n - y^*_n \to 0) \).

It is known that a Banach space \( X \) has uniformly very smooth (uniformly Fréchet differentiable or uniformly Gâteaux differentiable) norm if and only if \( x^*_k, y^*_k \in S(X^*) \), \( \|x^*_k + y^*_k\| \to 2 \) imply \( x^*_k - y^*_k \to 0 \) \( (x^*_n - y^*_n \to 0 \text{ or } x^*_n - y^*_n \to 0) \).

Consequently, we have

Corollary 3.1. A Banach space \( X \) has uniformly very smooth (uniformly Fréchet differentiable or uniformly Gâteaux differentiable) norm if and only if the duality mapping \( x \to D(x) \) is weakly uniformly (uniformly norm or weak* uniformly) upper semicontinuous on \( S(X) \).
ACKNOWLEDGMENTS

We are most grateful to the referees for their comments and suggestions. For the uniform \( \alpha \) upper semicontinuity and nearly uniformly norm upper semicontinuity, the referee has provided a more general result: If \( \Phi \) is a set-valued mapping from a metric space \( A \) into totally bounded subsets of a metric space \( M \), then \( \Phi \) is uniformly \( \alpha \) upper semicontinuous if and only if it is nearly uniformly metric upper semicontinuous. The authors have benefited from this result.

REFERENCES