Reverse Plane Partitions and Tableau Hook Numbers

A. P. Hillman and R. M. Grassl

Department of Mathematics, University of New Mexico, Albuquerque, New Mexico 87131

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The generating function of R. P. Stanley for reverse plane partitions on a tableau shape is obtained by a direct method that clearly shows the combinatorial significance of the hook numbers for the shape. The process generalizes the hooks into zigzag paths.

Generating functions for enumerating plane partitions generally are surprisingly simple in view of the manipulations that have been used to derive them. (For examples, see [1–3, 5, 8, 10–12].) This is also true of the theorem of Frame, Robinson, and Thrall [4, Theorem 1, p. 318] that the number of Young tableaux of a given shape with entries 1, 2,..., n is n! divided by the product of the hook numbers for the shape.

This paper obtains the generating function of Stanley for reverse plane partitions by a direct method that clearly shows the combinatorial significance of the hook numbers, while generalizing the hooks into zigzag paths. (Stanley's derivation is in [10]; also see [11, Proposition 18.3, p. 270].) As Stanley [12, (38), p. 82] has shown, this also provides a direct proof of the Frame–Robinson–Thrall result.

Tableaux were used by Young in his studies [13, 14] of irreducible representations of the symmetric groups of permutations. Nakayama introduced hook numbers [9, p. 166] in his extension of the work of Young.

1. Ordering of Tableau Nodes

Let \( p_1 \geq p_2 \geq \cdots \geq p_t \geq 1 \) with the \( p_i \) fixed integers. The tableau shape for \( (p_1, p_2, \ldots, p_t) \) is the set \( S \) of nodes

\[ \nu = (i, j) \quad [i = 1, 2, \ldots, t \text{ and } j = 1, 2, \ldots, p_i]. \]
As in a matrix, the \( i \) and \( j \) are the row and column numbers, respectively, of the node. The \( p_i \) are the lengths of the \( i \) rows; the length \( q_j \) of the \( j \)th column is the largest \( i \) with \( p_i \geq j \).

The hook \( H(\nu) \) for a node \( \nu = (e, f) \) in \( S \) is the union

\[
\{(e, j) \mid f \leq j \leq p_e\} \cup \{(i, f) \mid e < i \leq q_f\).
\]

The hook number \( h(\nu) \) is the number \( p_e - f + 1 + q_f - e \) of nodes in \( H(\nu) \). The \( s = p_1 + \cdots + p_f \) nodes in \( S \) are linearly ordered by saying that \((i', j') \) precedes \((i, j)\) if either \( j < j' \) or \( j = j' \) and \( i > i' \). Then we label the nodes as \( \nu_1, \nu_2, \ldots, \nu_s \) so that \( \nu_k \) precedes \( \nu_{k+1} \) for \( 1 \leq k < s \) and let \( h_k = h(\nu_k) \).

We also introduce an ideal node \( \nu_0 \), that is not in \( S \), and say that \( \nu_0 \) precedes each node of \( S \).

### 2. Reverse Plane Partition

Let \( S \) be a tableau shape and \( N = \{0, 1, 2, \ldots\} \). A reverse plane partition (rpp) of \( n \) on \( S \) is a mapping \( P \) from \( S \) to \( N \) such that \( n \) is the sum of all the entries \( P(i, j) \) and \( P \) is a nondecreasing function of \( i \) for \( j \) fixed and of \( j \) for \( i \) fixed. (The word “reverse” indicates a change by Stanley to “nondecreasing” from the “nonincreasing” of earlier results.) The unique rpp of zero on \( S \) is called the null rpp.

Let \( \alpha_n \) be the number of rpp of \( n \) on \( S \) and let \( G(S) \) be the generating function \( \sum_{n=0}^{\infty} \alpha_n x^n \). The following result is equivalent to Stanley’s Proposition 18.3 [11, p. 270], in which the factor \( x^n \) occurs because he takes the \( p \) entries to be positive instead of nonnegative.

**Theorem 1.** \( G(S) = \prod_{\nu \in S} [1 - x^{h(\nu)}]^{-1} \), where the \( h(\nu) \) are the hook numbers for the nodes of \( S \).

In Sections 3–6 below, a generalization (Theorem 2 of Section 6) of Theorem 1 is proved. An example of the techniques involved is given in Section 7.

### 3. Hook Number Multiplicities for \( P \)

Clearly

\[
\prod_{\nu \in S} [1 - x^{h(\nu)}]^{-1} = \prod_{k=1}^{8} [1 - x^{h_k}]^{-1} = \prod_{k=1}^{8} \sum_{m=0}^{\infty} x^{mh_k}
\]
is the generating function for the number \( \beta_n \) of ordered \( s \)-tuples 
\[ M = (m_1, \ldots, m_s), \]
of hook number multiplicities \( m_k \), such that

\[ \sum_{k=1}^{s} m_k h_k = n, \quad m_k \in N = \{0, 1, \ldots\}. \]  

We show that the \( \alpha_n \) of Section 2 equals \( \beta_n \) by exhibiting below a one-to-one correspondence \( \mu \) from the set \( A \) of all rpp of \( n \) on \( S \) onto the set \( B \) of all \( s \)-tuples \( M \) satisfying (1). Since \( \alpha_0 = 1 = \beta_0 \), we may restrict ourselves to nonnull rpp.

4. The Zigzag Path and the Derived rpp

Let \( P \) be an rpp of \( n \) on \( S \) with \( n > 0 \). Since \( P \) is nondecreasing in every column, \( P(q_j, j) > 0 \) for some \( j \). Let \( c \) be the smallest such \( j \) and let \( q_c = b \). Let \( j_{b+1} = c \) and inductively assuming that \( j_b, j_{b-1}, \ldots, j_{i+1} \) have been defined, let \( j_i \) be the smallest \( j \) with both \( j > j_{i+1} \) and \( P(i, j) = P(i - 1, j) \), if such \( j \) exist. Ultimately, one reaches a row \( r \) such that \( j_r \) has been defined in this way and either \( r = 1 \) or there is no \( j \) with both \( j > j_{r+1} \) and \( P(r, j) = P(r - 1, j) \); for this \( r \) let \( j_r = p_r \). Now let \( Z(P) \) be the zigzag path

\[ Z(P) = \{(i, j) \mid b \geq i \geq r \text{ and } j_{i+1} \leq j \leq j_i\}. \]  

We say that the ordered pair \( \pi = (r, c) \) is the pivot of \( P \). It is easily seen that \( \pi \) is a node of \( S \), that the path length \( z(P) \) of \( Z(P) \) [i.e., the number of nodes in \( Z(P) \)] is the hook number \( h(\pi) \), and that \( P(\pi) > 0 \) for \( \nu \in Z(P) \).

The properties \( P(i, j_i) = P(i - 1, j_i) \) for \( b \geq i \geq r \) and \( P(i, j) > P(i - 1, j) \) for any \( j \) with \( j_{i+1} < j < j_i \) now imply that one obtains an rpp \( P' \) of \( n - z(P) \) when all the entries of \( P \) along \( Z(P) \) are depressed by 1 and the other entries are left unchanged; we call this \( P' \) the derived rpp for \( P \).

To eliminate special cases, we say that the null rpp \( 0 \) has the ideal node \( v_0 \) as its pivot, that \( Z(0) \) is the null set, and that \( z(0) = 0 \).

We next show that the pivot \( \pi' = (r', c') \) of the derived rpp \( P' \) either is \( \pi \) or precedes \( \pi \). Since \( P'(\nu) \leq P(\nu) \) for all \( \nu \) in \( S \), it is clear that \( c' \geq c \). We assume that \( c' = c \), since \( \pi' \) precedes \( \pi \) when \( c' > c \).

For \( b \geq i \geq r \), both \( (i, j_i) \) and \( (i - 1, j_i) \) are on the path \( Z(P) \). Hence \( P'(i, j_i) = P'(i - 1, j_i) \) is implied by the same equality for \( P \). These equalities and \( j_{b+1} \leq j_b \leq \cdots \leq j_{r+1} \) guarantee that the path \( Z(P') \) reaches at least to row \( r \) and so \( r \geq r' \). [The \( j_i' \) for \( Z(P') \) in the form (2) may be smaller than the \( j_i \) for \( Z(P) \).] This shows that either \( \pi' = \pi \) or \( \pi' \) precedes \( \pi \).
5. Return Paths

Let \( \pi = (r, c) \) be a node and let \( Q \) be an rpp whose pivot \( \pi' = (r', c') \) either is \( \pi \) or precedes \( \pi \). In this section, it is shown that an rpp \( P \) exists with \( \pi \) as pivot and with the derived rpp \( P' = Q \).

The hypothesis on \( \pi \) and \( \pi' \) means that either (a) \( c < c' \), or (b) \( c = c' \) and \( r \geq r' \).

We now seek the \( j_i \) of (2) for the desired \( P \). Let \( b = q_e \) and \( j_r = p_r \).
For \( r < i < b \), we inductively assume that \( j_r, j_{r+1}, \ldots, j_{i-1} \) have been defined and let \( j_i \) be the largest \( j \) with both \( j \leq j_{i-1} \) and \( Q(i, j) = Q(i - 1, j) \). The existence of such \( j_i \) with \( j_i \geq c \) is guaranteed in case (a) by the fact that \( Q(i, c) = 0 \) for \( 1 \leq i \leq b \) and in case (b) by the properties of the \( j'_i \) that characterize the path \( Z(Q) \) in form (2) together with the relation \( r \geq r' \). Finally, let \( j_{b+1} = c \).

\[
W = W(Q, \pi) = \{(i,j) \mid r < i < b \text{ and } j_i > j > j_{i+1}\},
\]

and \( P \) be the function on \( S \) obtained by adding 1 to the entries of \( Q \) along \( W \) and leaving the other entries unaltered. The return path \( W \) has been defined so that \( P \) is an rpp. The equalities \( P(i, j_i) = P(i-1, j_i) \) follow from the same equalities for \( Q \). For any \( j \) with \( j_i > j > j_{i+1} \) and \( (i - 1, j) \) in \( S \), the inequality \( P(i, j) > P(i - 1, j) \) results from the addition of 1 to the entries of \( Q \) along \( W \). Hence \( W = Z(P) \). Then the pivot of \( P \) is \( \pi \) and \( P' = Q \), as desired.

6. The Bijection \( \mu \)

An rpp \( P \) of \( n \) on \( S \) determines a sequence \( P, P', P'', \ldots, P^{(d)} \) where \( P^{(k+1)} \) is the derived rpp for \( P^{(k)} \) and \( P^{(d)} \) is null. For \( 0 \leq k \leq d \), let \( \pi_k \) and \( \sigma_k \) be the path length and pivot, respectively, of \( P^{(k)} \). Then either \( \pi_{k+1} = \pi_k \) or \( \pi_{k+1} \) precedes \( \pi_k \). For \( 1 \leq i \leq s \), let \( m_i \) be the number of values of \( k \) in \( \{0, 1, \ldots, d - 1\} \) such that \( m_i \) is the hook number \( h_i \) for the node \( v_i \). Then the \( s \)-tuple \( M - \mu(P) = (m_1, \ldots, m_s) \) of hook number multiplicities \( m_i \) satisfies \( \sum_{i=1}^s m_i h_i = n \), since \( P^{(k+1)} \) is an rpp of \( n \cdot z_0 \cdots z_k \) and \( P^{(d)} \) is null.

It remains to show that \( \mu \) is a bijection from \( A \) onto \( B \), i.e., that any \( M = (m_1, \ldots, m_s) \) with nonnegative \( m_i \) uniquely determines an rpp \( P \) with \( \mu(P) = M \). Let \( M \) be given and \( d = m_1 + \cdots + m_s \). Let \( \pi_0, \pi_{d-1}, \pi_{d-2}, \ldots, \pi_0 \) be the sequence of nodes in which the first \( m_1 \) terms equal \( v_1 \), the next \( m_2 \) terms equal \( v_2 \), etc. Since \( v_i \) precedes \( v_{i+1} \), either \( \pi_{k+1} = \pi_k \) or \( \pi_{k+1} \) precedes \( \pi_k \). Now we can use the method of Section 5
to determine $P^{(d-1)}$ from the null rpp $P^{(d)}$ and $\pi_{d-1}$. Then $P^{(d-1)}$ and $\pi_{d-2}$ determine $P^{(d-2)}$ and this continues until $P^{(0)} = P$ is obtained. Hence $\mu$ is a bijection.

We are now in a position to generalize on Stanley's result (Theorem 1 above). Let $\gamma(k, n)$ be the number of rpp $P$ of $n$ on $S$ with the pivot of $P$ in $\{v_0, v_1, ..., v_k\}$ and let the generating function for such rpp be $G_k = \sum_{n=0}^{\infty} \gamma(k, n) x^n$.

**Theorem 2.** $G_0 = 1$ and, for $1 \leq k \leq s$,

$$G_k = \prod_{i=1}^{k} [1 - x^{h_i}]^{-1}. \quad (3)$$

**Proof.** Clearly the right side of (3) is the generating function for the $s$-tuples $M = (m_1, ..., m_s)$ with $m_i = 0$ for $i > k$. Since such $s$-tuples are the images under the bijection $\mu$ of the rpp $P$ with pivots in $\{v_0, ..., v_k\}$, the theorem follows.

The case $k = s$ of Theorem 2 is Theorem 1.

**Corollary.** For $1 \leq k \leq s$, the generating function for the rpp $P$ of $n$ on $S$ with $v_k$ as pivot is

$$G_k - G_{k-1} = x^{h_k} \prod_{i=1}^{k} [1 - x^{h_i}]^{-1}. \quad (4)$$

7. An Example

Next we illustrate the algorithms described above using the shape $S$ with row lengths $(p_1, p_2, p_3) = (3, 3, 1)$, yielding $s = p_1 + p_2 + p_3 = 7$ nodes, and the multiplicity 7-tuple $(m_1, ..., m_7) = (0, 1, 2, 1, 2, 1, 1)$.

Each of the following arrays $I$, $L$, and $M$ has seven entries in the positions of the seven nodes of $S$; the subscripts $i$ of $v_i$ (indicating the ordering of the nodes) are given in $I$, the hook lengths $h_i$ are in $L$, and their multiplicities $m_i$ are in $M$:

\[
\begin{array}{ccc}
I & L & M \\
5 & 3 & 1 & 5 & 3 & 2 & 2 & 2 & 0 \\
6 & 4 & 2 & 4 & 2 & 1 & 1 & 1 & 1 \\
7 & 1 & 1 & & & & & & \\
\end{array}
\]

The 7-tuple $M$ shows that $d = m_1 + \cdots + m_7 = 8$ and then that the pivots $\pi_7, ..., \pi_0$ are $v_2, v_3, v_3, v_4, v_5, v_5, v_6$. The null rpp $P^{(8)}$ and
\(\pi_7\) determine \(P^{(7)}\), then \(P^{(7)}\) and \(\pi_8\) determine \(P^{(8)}\), and so on, with results as follows.

\[
\begin{array}{cccccccc}
P^{(7)} & P^{(6)} & P^{(5)} & P^{(4)} & P^{(3)} & P^{(2)} & P' & P \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 2 \\
0 & 0 & 1 & 0 & 1 & 0 & 2 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \\
\end{array}
\]

The path \(Z(P^{(k)})\) consists of the \(v\) with \(P^{([k-1])}(v) = P^{(k)}(v) \neq 1\). Also, \(n = m_1h_1 + \cdots + m_7h_7 = 24 = P(v_1) + \cdots + P(v_7)\).

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**References**