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Relative π-Blocks of π-Separable Groups

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1. INTRODUCTION

In [8–10], Slattery has developed a π-block theory for π-separable groups, using the $B_{\pi'}$-characters introduced by Isaacs in [3]. These π-blocks, in addition to being equal to the usual p-blocks when $\pi = \{p\}$, enjoy many of the properties of the latter. Indeed, among other things, Slattery extended the concept of defect groups to π-blocks and defined a version of block induction, which allowed him to prove versions of Brauer's three main theorems.

Now let G be a finite π-separable group, and let μ be a π'-special character of some normal subgroup N of G. The purpose of the present paper is to show that the set $\mathrm{Irr}(G|\,\mu)$ of irreducible characters of G lying over μ decomposes into "blocks" which behave like Slattery π-blocks. Furthermore, in case $N = \langle 1 \rangle$ and $\mu = 1_{\langle 1 \rangle}$, the trivial character of $\langle 1 \rangle$, our blocks are just the π-blocks defined by Slattery.

The main result (Theorem 3.1) of Section 3 establishes a nice correspondence between our blocks and certain π-blocks of some group closely related to G.

In Section 4, we define defect groups for these blocks and show an analogue of [9, Theorem 2.11] (see Theorem 4.4), as well as a version of Brauer's height-0-conjecture.

2. SOME π-CHARACTER THEORY

The purpose of this section is to give a summary of some of the concepts and facts needed, concerning the theory of characters of π-separable groups developed by Isaacs. (See [3, 4, 6].)

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Throughout this paper, π denotes a set of rational primes, π' denotes the complementary set of primes, and G is a finite π-separable group.

For a class function χ of G, we write χ^0 to denote the restriction of χ to the set of π'-elements of G.

In [3], Isaacs has defined a character set $B_{\pi'}(G)$, such that, in case $\pi = \{p\}$, where p is prime, $B_{\pi'}(G)$ forms a set of canonical lifts for the p-modular characters of a p-solvable group. The set $\{\chi^0 : \chi \in B_{\pi'}(G)\}$ is denoted by $I_{\pi'}(G)$. If $\pi = \{p\}$, then $I_{\pi'}(G) = I\,\mathrm{Br}(G)$, the set of all irreducible p-Brauer characters of G. (See [3, Corollary 10.3].)

Let $\theta \in \mathrm{Irr}(G)$. Then, there are uniquely determined non-negative integers $d_{\theta\varphi}$ called "decomposition numbers" such that $\theta^0 = \sum_{\varphi} d_{\theta\varphi}\varphi^0$, where φ runs through the set $B_{\pi'}(G)$. (See [3, Corollary 10.1].) We may also use the notation $d_{\theta\varphi^0}$ for $d_{\theta\varphi}$. Each φ^0 such that $d_{\theta\varphi^0} \neq 0$ is called a π'-constituent of θ.

For $\theta \in \mathrm{Irr}(G)$, Isaacs constructed a pair (W, γ), where $W \subseteq G$, $\gamma \in \mathrm{Irr}(W)$ is π-factorable (i.e., γ factors into the product of a π-special character and a π'-special character), and $\gamma^G = \theta$. This pair, which is uniquely determined up to G-conjugacy by θ is called a nucleus of θ. By definition, $\theta \in B_{\pi'}(G)$ if γ is π'-special.

We now assume that $\theta \in B_{\pi'}(G)$. Let $L \subseteq G$ be a Hall π'-subgroup of G. Then a constituent $\alpha \in \mathrm{Irr}(L)$ of θ_L is called a *Fong character* associated with θ (or with θ^0), provided $\alpha(1) = \theta(1)_{\pi'}$, the π'-part of $\theta(1)$. (See [3, Definition 8.6].) By [3, Corollary 10.1(b)], if η is an irreducible character of G, then the decomposition number $d_{\eta\theta} = [\alpha^G, \eta] = [\alpha, \eta_L]$.

3. RELATIVE π-BLOCKS

Let $N \lhd G$ and let $\mu \in \mathrm{Irr}(N)$ be a π'-special character. As usual, the set of all irreducible characters of G lying over μ is denoted by $\mathrm{Irr}(G|\mu)$.

Let $\chi, \chi' \in \mathrm{Irr}(G|\mu)$. As in [8, Sect. 2], χ and χ' are said to be linked if there is $\varphi \in B_{\pi'}(G)$ such that $d_{\chi\varphi} \neq 0$ and $d_{\chi'\varphi} \neq 0$. The transitive extension of this linking decomposes $\mathrm{Irr}(G|\mu)$ into equivalence classes. Each one of these classes is called a *relative π-block of G with respect to* (N, μ) (and the set of all the relative π-blocks of G with respect to (N, μ) is denoted by $\mathrm{Bl}_{\pi}(G|\mu)$). Note that relative π-blocks of G with respect to $(\langle 1\rangle, 1_{\langle 1\rangle})$ are exactly Slattery π-blocks of G.

The purpose of this section is to show a 1–1 correspondence between $\mathrm{Bl}_{\pi}(G|\mu)$ and the set of certain π-blocks of some group closely related to G. This correspondence allows us to conclude that relative π-blocks satisfy many of the properties enjoyed by π-blocks. To state the main theorem, we introduce the following notation. If $\nu \in I_{\pi'}(K)$ for some subgroup K of G, we write $I_{\pi'}(G|\nu)$ to denote the set of $\varphi \in I_{\pi'}(G)$ such that ν is a constituent of φ_K.

(3.1) THEOREM. *Let N be a normal subgroup of a π-separable group G and let μ be a π'-special character of N with $T = I_G(\mu)$. Then, there exist a central extension T^* of $\overline{T} = T/N$ by a π'-subgroup N^* of T^*, a linear character μ^* of N^* and bijections Ψ of* $\mathrm{Irr}(G|\mu)$ *onto* $\mathrm{Irr}(T^*|\mu^*)$ *and Ψ^0 of $I_{\pi'}(G|\mu^0)$ onto $I_{\pi'}(T^*|\mu^*)$ such that the following statements hold*:

(a) *For any $\theta \in I_{\pi'}(G|\mu^0)$, if ξ is any character of* $\mathrm{Irr}(G|\mu)$ *satisfying $\xi^0 = \theta$, we have $\Psi^0(\theta) = \Psi(\xi)^0$.*

(b) *Under the correspondences Ψ and Ψ^0, the decomposition numbers are preserved. That is, for $\chi \in \mathrm{Irr}(G|\mu)$ and $\theta \in I_{\pi'}(G|\mu^0)$, we have $d_{\chi\theta} = d_{\Psi(\chi)\Psi^0(\theta)}$.*

(c) *The correspondence $\mathscr{B} \mapsto \Psi(\mathscr{B})$ is a bijection of* $\mathrm{Bl}_\pi(G|\mu)$ *onto the set of π-blocks of T^* over μ^*.*

In (c), by a π-block of T^* over μ^*, we mean a π-block, whose characters all lie over μ^*. This definition is justified by the fact that the characters of every π-block of T^* lie over a single T^*-orbit of $\mathrm{Irr}(N^*)$. (See the observation preceding Theorem 2.8 in [8].)

To prove Theorem 3.1, we need the following series of preliminary lemmas.

(3.2) LEMMA. *Let $N \lhd G$ and let $\mu \in \mathrm{Irr}(N)$ be π'-special. Then, for $\chi \in \mathrm{Irr}(G|\mu)$, if $\varphi \in B_{\pi'}(G)$ is such that $d_{\chi\varphi} \neq 0$, we have $\varphi \in \mathrm{Irr}(G|\mu)$.*

Proof. Since χ lies over μ, Lemma 3.1 in [4] implies that φ^0 lies over μ^0. It follows by [6, Theorem 6.2] and [3, Corollary 10.2] that φ lies over μ.

(3.3) LEMMA. *Let $N \lhd G$ and let μ be a π'-special character of N with $T = I_G(\mu)$. Let $\chi \in \mathrm{Irr}(G|\mu)$ and let ψ be the unique irreducible character of T lying over μ such that $\psi^G = \chi$. If $\{\beta_1, \ldots, \beta_r\}$ is the set of distinct π'-constituents of ψ, then $\{\beta_1^G, \ldots, \beta_r^G\}$ is the set of distinct π'-constituents of χ. Furthermore, the multiplicity of β_i as a constituent of ψ^0 is equal to that of β_i^G as a constituent of χ^0.*

Proof. Write $\psi^0 = \sum_{i=1}^r m_i \beta_i$. Since ψ lies over μ, each β_i lies over μ^0. Next, as μ is uniquely determined by μ^0, we have $T = I_G(\mu^0)$. It follows by [4, Lemma 3.2] that β_i^G is irreducible for each i and that $\beta_j^G \neq \beta_i^G$ if $j \neq i$. Now, $\chi^0 = (\psi^G)^0 = (\psi^0)^G = \sum_{i=1}^r m_i \beta_i^G$. This shows that $\beta_1^G, \ldots, \beta_r^G$ are precisely the irreducible π'-constituents of χ and that, for each i, the multiplicity of β_i^G as a constituent of χ^0 is equal to that of β_i as a constituent of ψ^0.

The following result is analogous to [8, Theorem 2.10].

(3.4) LEMMA. *Let $N \lhd G$ and let μ be a π'-special character of N with $T = I_G(\mu)$. Then, there is a bijection of* $\mathrm{Bl}_\pi(T|\mu)$ *onto* $\mathrm{Bl}_\pi(G|\mu)$ *given by inducing the characters. That is, a relative π-block $\mathscr{B}_0$ of T with respect to (N,μ) corresponds to the relative π-block $\{\theta^G : \theta \in \mathscr{B}_0\}$ of G with respect to (N, μ).*

Proof. Let $\mathscr{B}_0 \in \mathrm{Bl}_\pi(T|\mu)$ and let $\mathscr{A} = \{\theta^G : \theta \in \mathscr{B}_0\}$. Assume that the characters θ and σ of $\mathscr{B}_0$ are linked. So there exists $\gamma \in B_{\pi'}(T)$ such that $d_{\theta\gamma} \neq 0$ and $d_{\sigma\gamma} \neq 0$. Since any $B_{\pi'}$-constituent of γ^G links θ^G and σ^G, we conclude that $\mathscr{A}$ is a subset of some single relative π-block $\mathscr{B}$ of G with respect to (N, μ). Next, we show that $\mathscr{A} = \mathscr{B}$.

Assume, on the contrary that $\mathscr{A} \neq \mathscr{B}$. So we can find $\chi \in \mathscr{A}$ and $\chi' \in \mathscr{B} \setminus \mathscr{A}$ such that $d_{\chi\varphi} \neq 0$ and $d_{\chi'\varphi} \neq 0$ for some $\varphi \in B_{\pi'}(G)$. Let $\psi, \psi' \in \mathrm{Irr}(T|\mu)$ be such that $\psi^G = \chi$ and $\psi'^G = \chi'$. By Lemma 3.3, there exists $\phi \in B_{\pi'}(T)$ such that $(\phi^0)^G = \varphi^0$ and ϕ^0 is an irreducible π'-constituent of both ψ and ψ'. Thus ϕ links ψ and ψ'. However, $\psi^G = \chi \in \mathscr{A}$ and so $\psi \in \mathscr{B}_0$. It follows that $\psi' \in \mathscr{B}_0$. Therefore, $\chi' = (\psi')^G \in \mathscr{A}$, contradicting our choice. Hence, we must have $\mathscr{A} = \mathscr{B}$, a relative π-block of G with respect to (N, μ).

We have obtained above an injective map from $\mathrm{Bl}_\pi(T|\mu)$ into $\mathrm{Bl}_\pi(G|\mu)$, given by inducing the characters. However, if $\mathscr{B}' \in \mathrm{Bl}_\pi(G|\mu)$, we choose $\zeta \in \mathscr{B}'$. Then, there is $\xi \in \mathrm{Irr}(T|\mu)$ such that $\xi^G = \zeta$ and the relative π-block $\mathscr{B}'_0$ of T with respect to (N, μ) containing ξ gets mapped to $\mathscr{B}'$. This shows that our map is onto, thus finishing the proof of the lemma.

Let $N \lhd G$ and let $\mu \in \mathrm{Irr}(N)$ be G-invariant. In other words, (G, N, μ) is a character-triple. We say that another character-triple (Γ, M, ν) is isomorphic to (G, N, μ) if the factor groups G/N and Γ/M are isomorphic and the character theory of G "over" μ is "similar" to the character theory of Γ over ν via the given isomorphism of G/N onto Γ/M. (See [2, Definition 11.23] for the precise definition of character-triple isomorphism.)

Assume now that (τ, σ) is a character-triple isomorphism from (G, N, μ) to (Γ, M, ν). So τ is an isomorphism of G/N onto Γ/M. Let H be a subgroup of G containing N. We write H^τ to denote the subgroup $M \subseteq H^\tau \subseteq \Gamma$ such that H^τ/M is the image of H/N under τ. For every such H, there exists a certain map σ_H from $\mathrm{Ch}(H|\mu)$ (the set of possibly reducible characters χ of H such that χ_N is a multiple of μ) to $\mathrm{Ch}(H^\tau|\nu)$. By Lemma 11.24 in [2] σ_H is a bijection.

Next, if χ is any character of H, we have $\chi^g = \chi^{g'}$ for any $g, g' \in G$ such that $gg'^{-1} \in N$. Therefore, for $\bar{t} \in G/N$, we may write $\chi^{\bar{t}}$ to denote χ^g, where g is any element of G such that $gN = \bar{t}$.

For the purpose of the next section, we need character-triple isomorphisms (τ, σ): $(G, N, \mu) \to (\Gamma, M, \nu)$ that satisfy the following property:

(P) For all subgroups H of G containing N and for all $\chi \in \mathrm{Ch}(H | \mu)$, we have

$$\sigma_{H^g}(\chi^{\bar{g}}) = \sigma_H(\chi)^{\tau(\bar{g})}$$

for all $\bar{g} = gN \in G/N$.

The following fact is easy to prove.

(3.5) LEMMA. (P) *is preserved under composition of character-triple isomorphisms, each satisfying* (P).

(3.6) LEMMA. *Let* (G, N, μ) *be a character-triple and let* φ: $G \to \Gamma$ *be a surjective homomorphism such that* $\ker(\varphi) \subseteq \ker(\mu)$. *Let* $M = \varphi(N)$ *and let* $\nu \in \mathrm{Irr}(M)$ *be the character corresponding to* μ, *viewed as a character of* $N/\ker(\varphi)$. *Then, there is an isomorphism* (τ, σ) *from* (G, N, μ) *to* (Γ, M, ν) *that satisfies* (P).

Proof. The isomorphism (τ, σ) is that provided by [2, Lemma 11.26] and the fact that this isomorphism satisfies (P) is easy to check.

(3.7) LEMMA. *Let* (G, N, μ) *be a character-triple and let* $\delta \in \mathrm{Irr}(G)$ *be such that* $\delta_N \mu = \nu \in \mathrm{Irr}(N)$. *For every subgroup* H *of* G *containing* N, *define* σ_H: $\mathrm{Ch}(H | \mu) \to \mathrm{Ch}(H|\nu)$ *by* $\sigma_H(\theta) = \theta\delta_H$. *Let* I: $G/N \to G/N$ *be the identity map. Then* (i, σ) *is an isomorphism from* (G, N, μ) *to* (G, N, ν) *that satisfies* (P).

Proof. Lemma 11.27 in [2] says that (i, σ) is a character-triple isomorphism, and the fact that (i, σ) satisfies (P) is easy to verify.

Let (G, N, μ) be a character-triple. By Theorem 11.28 in [2], it is possible to find a character-triple (Γ, M, ν) isomorphic to (G, N, μ) such that $M \subseteq Z(\Gamma)$. The proof of that theorem shows that the associated isomorphism is a composition of character-triple isomorphisms of the types of Lemmas 3.6 and 3.7. It follows by Lemma 3.5 that the isomorphism of Theorem 11.28 satisfies (P). So, we obtain

(3.8) LEMMA. *Let* (G, N, μ) *be a character-triple. Then, there exists an isomorphic character-triple* (Γ, M, ν) *satisfying* $M \subseteq Z(\Gamma)$ *and such that the associated isomorphism satisfies* (P).

Let (G, N, μ) be a character-triple. The next result shows that, in case μ is π'-special, the character-triple (Γ, M, ν) of Lemma 3.8 can be chosen so that M is a π'-group. The proof is inspired by that of [5, Theorem 5.2].

(3.9) Lemma. *Let* (G, N, μ) *be a character-triple, where* μ *is* π'*-special. Then, there exists an isomorphic triple* (G^*, N^*, μ^*)*, where* N^* *is a* π'*-group contained in* $Z(G^*)$ *and such that the associated isomorphisms satisfies* (P).

Proof. By Lemma 3.8, there exists an isomorphic character-triple (Γ, M, ν), where $M \subseteq Z(\Gamma)$ and such that the associated isomorphism satisfies (P). (Note that Γ is π-separable as $\Gamma/M \cong G/N$ is π-separable and M is central.) Since ν is linear, we may uniquely write $\nu = \alpha\beta$, where the order $o(\alpha)$ of α (in the group of linear characters of M) is a π'-number and $o(\beta)$ is a π-number. Note that in this situation, ν is π-factorable with α and β as its π'-special and π-special parts, respectively.

Since μ is G-invariant, there exists a π'-special character $\varphi \in \mathrm{Irr}(G|\mu)$ by [1, Corollary 4.8]. It follows by [2, Lemma 11.24] that there is a character $\psi \in \mathrm{Irr}(\Gamma|\nu)$ where $\psi(1)$ is a π'-number. Let (W, γ) be a nucleus of ψ. As any nucleus of ψ is Γ-conjugate to (W, γ) and as $M \subseteq Z(\Gamma)$, we have $M \subseteq W$ and $\gamma \in \mathrm{Irr}(W|\nu)$ by [9, Lemma 1.2]. By definition, the character γ is π-factorable and satisfies $\gamma^G = \psi$. Thus, since $\psi(1)$ is a π'-number, $\gamma(1)$ is a π'-number and W contains a Hall π-subgroup of Γ. Therefore, for every $p \in \pi$, a Sylow p-subgroup S_p of Γ is contained in W.

Now, factor $\gamma = \sigma\omega$, where σ is π'-special and ω is π-special, and note that ω is linear, since $\gamma(1)$ is a π'-number. Then, by [3, Lemma 2.2], $\gamma_M = \sigma_M \omega_M$, where the irreducible constituents of σ_M are π'-special linear characters and ω_M is π-special. Thus, γ_M is a sum of (linear) π-factorable characters, each of which has ω_M as its π-special part.

Since ν is Γ-invariant and since γ lies over ν, it follows that those π-factorable characters are all equal to ν. Hence, $\omega_M = \beta$ as β is uniquely determined by ν. This shows that β extends to W and hence β extends to $S_p M$ for every $p \in \pi$. Now, the quotient group $S_p M/M$ of $S_p M$ by M is a Sylow p-subgroup of Γ/M. Moreover, β is Γ-invariant as M is central in Γ. Therefore, β is extendible to some linear character δ of Γ by [2, Theorem 6.26].

Now, by Lemma 3.7, multiplication of all members of $\mathrm{Irr}(L|\nu)$ by $(\delta^{-1})_L$, for all subgroups L of Γ containing M, defines a character-triple isomorphism $(\Gamma, M, \nu) \to (\Gamma, M, \alpha)$ that satisfies (P). Next, by Lemma 3.6, factoring out $\ker(\alpha)$ yields an isomorphic triple $(\overline{\Gamma}, \overline{M}, \overline{\alpha})$ with $\overline{\alpha}$ faithful and such that the associated isomorphism $(\Gamma, M, \alpha) \to (\overline{\Gamma}, \overline{M}, \overline{\alpha})$ satisfies (P). We have thus obtained a character-triple isomorphism $(G, N, \mu) \to (\overline{\Gamma}, \overline{M}, \overline{\alpha})$. This isomorphism satisfies (P) by Lemma 3.5. Furthermore, $\overline{M} \subseteq Z(\overline{\Gamma})$ and $|\overline{M}| = o(\alpha)$ is a π'-number. Therefore, $(\overline{\Gamma}, \overline{M}, \overline{\alpha})$ fulfills the desired conditions of the lemma.

Let (G, N, μ) be a character-triple, where μ is π'-special, and let L be a Hall π'-subgroup of G. Since $L \cap N$ is a Hall π'-subgroup of N, the restriction $(\mu)_{L \cap N}$ of μ to $L \cap N$ is irreducible by [1, Proposition 6.1]. So, we obtain the following result as a direct consequence of [3, Corollary 4.2].

(3.10) LEMMA. *Let (G, N, μ) be a character-triple, where μ is π'-special, and let L be a Hall π'-subgroup of G. Then, restriction defines a bijection of* $\mathrm{Irr}(LN|\mu)$ *onto* $\mathrm{Irr}(L|\mu_{L \cap N})$. *Furthermore, for any $\chi \in \mathrm{Irr}(G|\mu)$, the multiplicity of $\gamma \in \mathrm{Irr}(LN|\mu)$ as a constituent of χ_{LN} is equal to that of γ_L as a constituent of χ_L.*

Let (G, N, μ) and (G^*, N^*, μ^*) be isomorphic character-triples. So $G/N \cong G^*/N^*$, and we fix a particular isomorphism of these groups. If $N \subseteq H \subseteq G$, we write H^* to denote the subgroup $N^* \subseteq H^* \subseteq G^*$ such that H^*/N^* is the image of H/N under the fixed isomorphism. We also denote the associated bijection $\mathrm{Ch}(H|\mu) \to \mathrm{Ch}(H^*|\mu^*)$ by $*$.

(3.11) LEMMA. *Let (G, N, μ) be a character-triple, where μ is π'-special, and assume that (G^*, N^*, μ^*) is an isomorphic character-triple such that N^* is a π'-group. Let $\xi \in \mathrm{Irr}(G|\mu)$ such that $\xi^0 \in I_{\pi'}(G)$, then,*

(1) *$(\xi^*)^0 \in I_{\pi'}(G^*)$, and for any $\chi \in \mathrm{Irr}(G|\mu)$, the multiplicity of ξ^0 as a constituent of χ^0 is equal to that of $(\xi^*)^0$ as a constituent of $(\chi^*)^0$.*

(2) *the characters $\chi_1, \chi_2 \in \mathrm{Irr}(G|\mu)$ are linked if and only if the characters $\chi_1^*, \chi_2^* \in \mathrm{Irr}(G^*|\mu^*)$ are linked.*

Proof. Fix a Hall π'-subgroup L of G and recall from our discussion preceding Lemma 3.10 that the restriction ν of μ to $L \cap N$ is irreducible. Let $\xi \in \mathrm{Irr}(G|\mu)$ such that $\xi^0 \in I_{\pi'}(G)$. We begin by showing that $(\xi^*)^0$ is irreducible.

Suppose that $(\xi^*)^0$ is reducible, in other words, $(\xi^*)^0 = \zeta_1^0 + \zeta_2^0$ for characters ζ_1 and ζ_2 of G^*. Since N^* is a π'-group and since ξ^* lies over μ^*, we have $\zeta_1, \zeta_2 \in \mathrm{Ch}(G^*|\mu^*)$.

Let $\theta_1, \theta_2 \in \mathrm{Ch}(G|\mu)$ be such that $\theta_1^* = \zeta_1$ and $\theta_2^* = \zeta_2$. Now, let α be any character of $\mathrm{Irr}(L|\nu)$ and let m, m_1, and m_2 be the multiplicities of α as a constituent of ξ_L, $(\theta_1)_L$, and $(\theta_2)_L$, respectively. By Lemma 3.10, there exists a unique character $\gamma \in \mathrm{Irr}(LN|\mu)$ such that $\gamma_L = \alpha$ and the multiplicities of γ as a constituent of ξ_{LN}, $(\theta_1)_{LN}$, and $(\theta_2)_{LN}$ are m, m_1, and m_2, respectively. It follows by the definition of character-triple isomorphism that m, m_1, and m_2 are the respective multiplicities of γ^* as a constituent of the restrictions $(\xi^*)_{(LN)^*}$, $(\zeta_1)_{(LN)^*}$, and $(\zeta_2)_{(LN)^*}$ of ξ^*, ζ_1, and ζ_2 to $(LN)^*$.

Next, we note that $(LN)^*$ is a Hall π'-subgroup of G^*, as N^* is a π'-group. Now, since $(\xi^*)^0 = \zeta_1^0 + \zeta_2^0$, we conclude that $m = m_1 + m_2$.

So, we have shown that for every $\alpha \in \mathrm{Irr}(L|\nu)$, the multiplicity of α as a constituent of ξ_L is equal to the sum of the multiplicities of α as a constituent of $(\theta_1)_L$ and $(\theta_2)_L$. As the irreducible constituents of ξ_L, $(\theta_1)_L$, and $(\theta_2)_L$ all lie over ν, we conclude that $\xi_L = (\theta_1)_L + (\theta_2)_L$ and hence $\xi^0 = \theta_1^0 + \theta_2^0$, contradicting the irreducibility of ξ^0. This shows that $(\xi^*)^0 \in I_{\pi'}(G^*)$, as desired.

Next, let $\chi \in \mathrm{Irr}(G|\mu)$. If $\delta \in \mathrm{Irr}(L)$ is any Fong character associated with ξ^0, then the multiplicity n of ξ^0 as a constituent of χ^0 is equal to that of δ as a constituent of χ_L (see Section 2).

Since δ is a constituent of ξ_L, we have $\delta \in \mathrm{Irr}(L|\nu)$. Therefore, by Lemma 3.10, there exists a unique character $\eta \in \mathrm{Irr}(LN|\mu)$ such that $\eta_L = \delta$ and the multiplicity of η as a constituent of χ_{LN} is n. Now, again by the definition of character-triple isomorphism, n is the multiplicity of η^* as a constituent of $(\chi^*)_{(LN)^*}$.

By [2, Lemma 11.24], we have $\xi^*(1) = \xi(1)\mu^*(1)/\mu(1)$ and $\eta^*(1) = \eta(1)\mu^*(1)/\mu(1)$. Since $\xi(1)_{\pi'} = \delta(1) = \eta(1)$, we conclude that $\xi^*(1)_{\pi'} = \eta^*(1)$. Moreover, as δ is a constituent of ξ_L, the character η is a constituent of ξ_{LN} by Lemma 3.10, and it follows that η^* is a constituent of $(\xi^*)_{(LN)^*}$. This says that η^* is a Fong character associated with $(\xi^*)^0$, since $(LN)^*$ is a Hall π'-subgroup of G^*. Consequently, the multiplicity of $(\xi^*)^0$ as a π'-constituent of χ^* is exactly n. This proves (1).

Next, we prove (2). First, assume that χ_1 and χ_2 are linked by $\theta \in B_{\pi'}(G)$. Then $\theta \in \mathrm{Irr}(G|\mu)$ by Lemma 3.2, and it follows by (1) that $(\theta^*)^0$ is a constituent of both $(\chi_1^*)^0$ and $(\chi_2^*)^0$. Hence χ_1^* and χ_2^* are linked. Conversely, assume that χ_1^* and χ_2^* are linked by $\epsilon \in B_{\pi'}(G^*)$. Then $\epsilon \in \mathrm{Irr}(G^*|\mu^*)$ as N^* is a π'-group. Let $\sigma \in \mathrm{Irr}(G|\mu)$ be such that $\sigma^* = \epsilon$. Then, any character $\omega \in B_{\pi'}(G)$ such that $d_{\sigma\omega} \neq 0$ lies in $\mathrm{Irr}(G|\mu)$ by Lemma 3.2. It follows by (1) that $(\omega^*)^0$ is a constituent of ϵ^0 with multiplicity $d_{\sigma\omega}$. Since ϵ^0 is irreducible, we conclude that $\sigma^0 \in I_{\pi'}(G)$.

Now, again by (1), the respective multiplicities of σ^0 as a constituent of χ_1^0 and χ_2^0 are equal to the respective multiplicities of ϵ^0 as a constituent of $(\chi_1^*)^0$ and $(\chi_2^*)^0$. Since ϵ links χ_1^* and χ_2^*, we conclude that χ_1 and χ_2 are linked. This finishes the proof of (2).

(3.12) Lemma. *Let (G, N, μ) be a character-triple, where μ is π'-special and assume that (G^*, N^*, μ^*) is an isomorphic character-triple such that N^* is a π'-group. Then, the correspondence $\mathscr{B} \mapsto \mathscr{B}^*$ $(= \{\chi^* : \chi \in \mathscr{B}\})$ is a bijection of $\mathrm{Bl}_\pi(G|\mu)$ onto the set of π-blocks of G^* over μ^*.*

Proof. Let $\mathscr{B} \in \mathrm{Bl}_\pi(G|\mu)$. If $\alpha_1, \alpha_2 \in \mathscr{B}$ are linked, then α_1^*, α_2^* are linked by Lemma 3.11(2). Therefore, $\mathscr{B}^*$ is a subset of some π-block $\overline{\mathscr{B}}$ of G^* over μ^*. We claim that $\mathscr{B}^* = \overline{\mathscr{B}}$.

Suppose that $\mathscr{B}^* \neq \overline{\mathscr{B}}$. Then, we may choose $\alpha, \alpha_0 \in \mathrm{Irr}(G|\mu)$ such that $\alpha \in \mathscr{B}$, $\alpha_0^* \in \overline{\mathscr{B}} \setminus \mathscr{B}^*$, and α^*, α_0^* are linked. By Lemma 3.11(2), it follows that α and α_0 are linked and hence $\alpha_0 \in \mathscr{B}$, contradicting our choice. Therefore, $\mathscr{B}^* = \overline{\mathscr{B}}$, as claimed.

So, the correspondence $\mathscr{B} \mapsto \mathscr{B}^*$ is a well-defined map from $\mathrm{Bl}_\pi(G|\mu)$ to the set of all π-blocks of G^* over μ^*. This map is clearly 1–1 and we see next that it is onto.

Let $\tilde{\mathscr{B}}$ be a π-block of G^* over μ^*, and let β be any character of $\tilde{\mathscr{B}}$. Then, the relative π-block $\mathscr{B}'$ of G with respect to (N, μ), containing the preimage of β under $*$, satisfies $(\mathscr{B}')^* = \tilde{\mathscr{B}}$. This proves that our map is onto, thus finishing the proof of the lemma.

Finally, we are ready to prove the main result of this section.

Proof of Theorem 3.1. Let (T^*, N^*, μ^*) be a character-triple isomorphic to (T, N, μ) as in Lemma 3.9. We use throughout this proof the $*$ notation, introduced just before Lemma 3.11.

Let Λ be the bijection of $\mathrm{Irr}(T|\mu)$ onto $\mathrm{Irr}(G|\mu)$ obtained by inducing the characters. The composition Ψ of Λ^{-1} with $*$ is clearly a bijection of $\mathrm{Irr}(G|\mu)$ onto $\mathrm{Irr}(T^*|\mu^*)$. Next, let $\theta \in I_{\pi'}(G|\mu^0)$, and note that $T = I_G(\mu^0)$, since μ^0 uniquely determines μ. Assume that ξ_1 and ξ_2 are characters of $\mathrm{Irr}(G|\mu)$ such that $\xi_1^0 = \xi_2^0 = \theta$. Let $\zeta_1 = \Lambda^{-1}(\xi_1)$ and $\zeta_2 = \Lambda^{-1}(\xi_2)$. We have

$$\left(\zeta_1^0\right)^G = \left(\zeta_1^G\right)^0 = \xi_1^0 = \theta = \xi_2^0 = \left(\zeta_2^G\right)^0 = \left(\zeta_2^0\right)^G.$$

It follows by [4, Proposition 3.2(a)] that $\zeta_1^0 = \zeta_2^0 \in I_{\pi'}(T|\mu^0)$. Now, since $\zeta_1^0 = \zeta_2^0$, Lemma 3.11(1) implies that $(\zeta_1^*)^0 = (\zeta_2^*)^0 \in I_{\pi'}(T^*)$. We have thus obtained a well-defined map Ψ^0 from $I_{\pi'}(G|\mu^0)$ to $I_{\pi'}(T^*|\mu^*)$ taking an element $\theta \in I_{\pi'}(G|\mu^0)$ to the element $(\zeta^*)^0$, where $\zeta^* = \Psi(\xi)$ for any character $\xi \in \mathrm{Irr}(G|\mu)$ satisfying $\xi^0 = \theta$.

We claim that Ψ^0 is a bijection. First, note that Ψ^0 is a composition of two maps Φ and Ω. The map Φ sends $\theta \in I_{\pi'}(G|\mu^0)$ to the unique element $\varphi \in I_{\pi'}(T|\mu^0)$ satisfying $\varphi^G = \theta$, and the map Ω sends $\tau \in I_{\pi'}(T|\mu^0)$ to the unique element $\nu \in I_{\pi'}(T^*|\mu^*)$ such that $\nu = (\rho^*)^0$, for any character $\rho \in \mathrm{Irr}(T|\mu)$ satisfying $\rho^0 = \tau$.

By [4, Proposition 3.2(a)], Φ is a bijection of $I_{\pi'}(G|\mu^0)$ onto $I_{\pi'}(T|\mu^0)$, and to show that Ψ^0 is a bijection, it suffices to show that Ω is a bijection of $I_{\pi'}(T|\mu^0)$ onto $I_{\pi'}(T^*|\mu^*)$.

Let $\tau_1, \tau_2 \in I_{\pi'}(T|\mu^0)$ such that $\tau_1 \neq \tau_2$. Since τ_1 is irreducible, it is obvious that the multiplicity of τ_2 as a constituent of τ_1 is zero. It follows by Lemma 3.11(1) that the multiplicity of $\Omega(\tau_2)$ as a constituent of $\Omega(\tau_1)$ is zero. Therefore, $\Omega(\tau_1) \neq \Omega(\tau_2)$ and Ω is 1–1.

Next, let $\nu \in I_{\pi'}(T^* | \mu^*)$. Then, there exists $\eta \in B_{\pi'}(T^*) \cap \mathrm{Irr}(T^* | \mu^*)$ such that $\nu = \eta^0$. Let ω be the element of $\mathrm{Irr}(T | \mu)$ satisfying $\omega^* = \eta$. Since ν is irreducible, Lemma 3.11(1) implies that $\omega^0 \in I_{\pi'}(T | \mu^0)$. Now, $\Omega(\omega^0) = (\omega^*)^0 = \eta^0 = \nu$. This shows that Ω is onto. Therefore, Ω is a bijection, as desired.

Part (a) is trivially satisfied by the definition of Ψ^0. To show (b), let $\chi \in \mathrm{Irr}(G | \mu)$ and let $\theta \in I_{\pi'}(G | \mu^0)$. If φ is the unique element of $I_{\pi'}(T | \mu^0)$ satisfying $\varphi^G = \theta$, the multiplicity of θ as a constituent of χ^0 is equal to that of φ as a constituent of $\Lambda^{-1}(\chi)^0$ by Lemma 3.3. Next, let ζ be the character of $B_{\pi'}(T) \cap \mathrm{Irr}(T | \mu)$ such that $\zeta^0 = \varphi$. Then, by Lemma 3.11(1), the multiplicity of φ as a constituent of $\Lambda^{-1}(\chi)^0$ is equal to that of $\Psi^0(\theta) = \Omega(\varphi) = (\zeta^*)^0$ as a constituent of $\Psi(\chi)^0$. Therefore, the multiplicity of θ as a constituent of χ^0 is equal to that of $\Psi^0(\theta)$ as a constituent of $\Psi(\chi)^0$. This proves (b).

Finally, (c) follows from Lemmas 3.4 and 3.12. This finishes the proof of the theorem.

4. DEFECT GROUPS

Throughout this section, we fix a π-separable group G, a normal subgroup N of G, and a π'-special character μ of N. Let $T = I_G(\mu)$ and denote by Λ, the bijection of $\mathrm{Irr}(T | \mu)$ onto $\mathrm{Irr}(G | \mu)$ obtained by inducing the characters.

Now, let $\mathscr{B} \in \mathrm{Bl}_\pi(G | \mu)$ and let $\mathscr{B}_0$ be the relative π-block of T with respect to (N, μ) such that $\mathscr{B} = \Lambda(\mathscr{B}_0)$. To define the "defect groups" of $\mathscr{B}$, we first need to define the defect groups of $\mathscr{B}_0$.

Let K be the normal subgroup of T containing N such that $K/N = O_{\pi'}(T/N)$. If $\zeta \in \mathrm{Irr}(T | \mu)$, then by Lemma 2.3 in [3], there exists a π'-special character δ of K such that δ is a constituent of ζ_K. By Lemma 3.2, for $\omega \in B_{\pi'}(T)$ such that $d_{\zeta\omega} \neq 0$, we have $\omega \in \mathrm{Irr}(T | \delta)$. Hence, the constituents of ω_K are precisely the constituents of ζ_K by Clifford's theorem ([2, Theorem 6.2]). It follows that if $\zeta' \in \mathrm{Irr}(T | \mu)$ is linked to ζ by ω, then ζ' also lies over the T-orbit of δ. This implies that the characters of $\mathscr{B}_0$ all lie over the T-orbit of some π'-special character ν of K and so $\mathscr{B}_0$ is a subset of some relative π-block $\widehat{\mathscr{B}_0}$ of T with respect to (K, ν). Now, assume that $\zeta_1 \in \mathscr{B}_0$ and $\zeta_2 \in \widehat{\mathscr{B}_0}$ satisfy $d_{\zeta_1\sigma} \neq 0$ and $d_{\zeta_2\sigma} \neq 0$ for some $\sigma \in B_{\pi'}(T)$. Then, since ν lies over μ, the character ζ_2 lies over μ, and it follows that $\zeta_2 \in \mathscr{B}_0$. This shows that $\mathscr{B} = \widehat{\mathscr{B}_0}$. In other words, $\mathscr{B}_0$ may also be viewed as a relative π-block of T with respect to (K, ν).

Let $S = I_T(\nu)$. We inductively define the set of *defect groups* of $\mathscr{B}_0$ as follows:

If $S = T$, the defect groups of $\mathscr{B}_0$ are the Hall π-subgroups of T.

If $S < T$, then by Lemma 3.4 and via the associated bijection, there exists a unique relative π-block $\mathscr{B}_0'$ of S with respect to (K, ν) corresponding to $\mathscr{B}_0$, regarded as a relative π-block of T with respect to (K, ν). Define the defect groups of $\mathscr{B}_0$ to be the T-conjugates of any defect group of $\mathscr{B}_0'$. Note that, since $\mathscr{B}_0$ determines ν uniquely up to T-conjugacy, this definition does not depend on the choice of ν.

Finally, we define the *defect groups* of $\mathscr{B}$ to be the G-conjugates of any defect group of $\mathscr{B}_0$. Since μ is determined by $\mathscr{B}$ up to G-conjugacy, this definition does not depend on the choice of μ.

It is clear that the defect groups just defined form a single G-conjugacy class. Furthermore, this definition agrees with that of defect groups of Slattery π-blocks, when $(N, \mu) = (\langle 1 \rangle, 1_{\langle 1 \rangle})$. (See [9, Definition 2.2].)

If $\mathscr{B} \in \mathrm{Bl}_\pi(G | \mu)$, then it clearly follows from the definition of defect groups that any Hall π-subgroup L of N is contained in some defect group P of $\mathscr{B}$. Thus, $P \cap N = L$. But, any defect group D of $\mathscr{B}$ is equal to P^x for some $x \in G$. Therefore,

$$D \cap N = P^x \cap N = (P \cap N)^x = L^x \in \mathrm{Hall}_\pi(N).$$

So, we have the following fact.

(4.1) LEMMA. *Let $\mathscr{B} \in \mathrm{Bl}_\pi(G | \mu)$. Then, for any defect group D of $\mathscr{B}$, we have $D \cap N \in \mathrm{Hall}_\pi(N)$.*

Now, let (T^*, N^*, μ^*) be a character-triple isomorphic to (T, N, μ) as in Theorem 3.1, and recall from the proof of that theorem that the associated isomorphism satisfies property (P). Accordingly, denote by Ψ the bijection of $\mathrm{Irr}(G | \mu)$ onto $\mathrm{Irr}(T^* | \mu^*)$, obtained by composing Λ^{-1} with the map $*$.

The following result shows that the defect groups of $\mathscr{B}$ and of $\Psi(\mathscr{B})$ are closely related.

(4.2) THEOREM. *Let $\mathscr{B} \in \mathrm{Bl}_\pi(G | \mu)$. Then, there exists a defect group D of $\mathscr{B}$ such that $(DN)^* = \tilde{D}N^*$ for some defect group $\tilde{D}$ of $\Psi(\mathscr{B})$.*

We prove this theorem by induction. To achieve that, we first need to construct a certain character-triple isomorphism from a character-triple isomorphism that satisfies (P). It should be noted that this construction is general.

Let (τ, σ): $(A, M, \gamma) \to (B, L, \epsilon)$ be a character-triple isomorphism that satisfies (P). Let K be a normal subgroup of A containing M, $\nu \in \mathrm{Irr}(K | \gamma)$, $J = I_A(\nu)$, and $\eta = \sigma_K(\nu)$.

Since (τ, σ) satisfies (P), we have that for every $a \in A$, $\sigma_K(\nu^a) = \eta^b$, where b is any element of B such that $bL = \tau(aM)$. It follows by the definition of character-triple isomorphism that $J^\tau = I_B(\eta)$. Next, τ restricts to an isomorphism from J/M onto J^τ/L. So, we may define an isomorphism τ': $J/K \to J^\tau/K^\tau$ by associating to the element xK of J/K, the element yK^τ of J^τ/K^τ, where y is any element of $\tau(xM)$. For $K \subseteq U \subseteq J$, it is clear that the inverse image in J^τ of $\tau'(U/K)$ is U^τ. Moreover, if $\chi \in \mathrm{Ch}(U|\nu)$, the character $\sigma_U(\chi)$ lies over η. Thus, we obtain a well-defined map σ'_U: $\mathrm{Ch}(U|\nu) \to \mathrm{Ch}(U^\tau|\eta)$ by taking $\sigma'_U(\chi) = \sigma_U(\chi)$ for $\chi \in \mathrm{Ch}(U|\nu)$. Let σ' denote the union of the maps σ'_U for all $K \subseteq U \subseteq J$. Now, the following fact can be easily verified.

(4.3) LEMMA. *The pair (τ', σ') is a character-triple isomorphism from (J, K, ν) to (J^τ, K^τ, η) which satisfies* (P).

Now, we are able to prove Theorem 4.2.

Proof of Theorem 4.2. Let $\mathscr{B}_0$ be the relative π-block of T with respect to (N, μ) such that $\mathscr{B} = \Lambda(\mathscr{B}_0)$. By definition, any defect group of $\mathscr{B}_0$ is also a defect group of $\mathscr{B}$. So, it suffices to show that $\mathscr{B}_0$ has a defect group D such that $(DN)^* = \tilde{D}N^*$ for some defect group $\tilde{D}$ of $(\mathscr{B}_0)^* = \Psi(\mathscr{B})$.

Let K be the normal subgroup of T containing N such that $K/N = O_{\pi'}(T/N)$, and choose a π'-special character ν of K that lies under every character of $\mathscr{B}_0$. Further, denote by S the inertial group $I_T(\nu)$ of ν in T.

Now, since $T/N \cong T^*/N^*$, we have $K/N \cong O_{\pi'}(T^*/N^*)$. Furthermore, as N^* is a π'-group, we have $O_{\pi'}(T^*/N^*) = O_{\pi'}(T^*)/N^*$. It follows that $K^* = O_{\pi'}(T^*)$. Note that since every character of $\mathscr{B}_0$ lies over ν, the π-block $(\mathscr{B}_0)^*$ of T^* lies over $\nu^* \in \mathrm{Irr}(O_{\pi'}(T^*))$.

Next, recall that the character-triple isomorphism $(T, N, \mu) \to (T^*, N^*, \mu^*)$ satisfies (P). So $S^* = I_{T^*}(\nu^*)$ (see the discussion preceding Lemma 4.3) and by Lemma 4.3, we get an isomorphism $(S, K, \nu) \to (S^*, K^*, \nu^*)$ which satisfies (P).

First, if $S = T$, then $S^* = T^*$ and hence $I_{T^*}(\nu^*) = T^*$. By definition, any Hall π-subgroup P of T is a defect group of $\mathscr{B}_0$.

We have $(PN)^*/N^* \cong PN/N$ and $PN/N \in \mathrm{Hall}_\pi(T/N)$. Therefore, $(PN)^*/N^* \in \mathrm{Hall}_\pi(T^*/N^*)$, and it follows that $(PN)^*$ contains a Hall π-subgroup $\tilde{P}$ of T^*. Now, as $|(PN)^*|_{\pi'} = |N^*|$, we conclude that $(PN)^* = \tilde{P}N^*$. By Definition 2.2 in [9], $\tilde{P}$ is a defect group of $(\mathscr{B}_0)^*$. So, we are done in this case.

Next, assume that $S < T$. We view $\mathscr{B}_0$ as a relative π-block of T with respect to (K, ν) and we let $\mathscr{B}'_0$ be the relative π-block of S with respect to (K, ν) corresponding to $\mathscr{B}_0$ via Lemma 3.4. By the definition of defect groups, we may choose a defect group D of $\mathscr{B}_0$ that is also a defect group of $\mathscr{B}'_0$.

The isomorphism $(S, K, \nu) \to (S^*, K^*, \nu^*)$ takes any character $\xi \in \mathrm{Irr}(S|\nu)$ to the character $\xi^* \in \mathrm{Irr}(S^* \mid \nu^*)$ and so by Lemma 3.12, the character-set $(\mathscr{B}'_0)^* = \{\theta^* : \theta \in \mathscr{B}'_0\}$ is a π-block of S^* over ν^*. By induction $(DK)^* = \hat{D}K^*$ for some defect group $\hat{D}$ of $(\mathscr{B}'_0)^*$.

Now, the index $[DK : DN]$ of DN in DK is a π'-number. Furthermore, $[(DK)^* : (DN)^*] = [DK : DN]$ as $DN/N \cong (DN)^*/N^*$ and $DK/N \cong (DK)^*/N^*$. Therefore, $[(DK)^* : (DN)^*]$ is a π'-number, and hence $(DN)^*$ contains a Hall π-subgroup of $(DK)^*$. But since $(DK)^* = \hat{D}K^*$ and K^* is a π'-group, we have that $\hat{D} \in \mathrm{Hall}_\pi((DK)^*)$. It follows that $(DN)^*$ contains a $(DK)^*$-conjugate $\tilde{D}$ of $\hat{D}$. Now, $\tilde{D} \in \mathrm{Hall}_\pi((DN)^*)$ and $(DN)^*/N^*$ is a π-group. Thus $(DN)^* = \tilde{D}N^*$.

If $\chi \in \mathscr{B}_0$, then $\chi = \theta^T$ for a unique character $\theta \in \mathscr{B}'_0$ and by [2, Lemma 11.35] $\chi^* = (\theta^*)^{T^*}$. Theorem 2.10 in [8] now says that the π-block $(\mathscr{B}'_0)^*$ of S^* over ν^* corresponds to the π-block $\mathscr{B}_0^*$ of T^* over ν^*. By [9, Definition 2.2], $\hat{D}$ is a defect group of $\mathscr{B}_0^*$. Therefore $\tilde{D}$, being a T^*-conjugate of $\hat{D}$, is a defect group of $\mathscr{B}_0^*$. This ends the proof of the theorem.

The next result is analogous to [9, Theorem 2.11].

(4.4) THEOREM. *Let $\chi \in \mathscr{B} \in \mathrm{Bl}_\pi(G|\mu)$. Then, there exist a subgroup W of G and a character $\gamma \in \mathrm{Irr}(W)$ satisfying $\gamma^G = \chi$ and such that a Hall π-subgroup Q of W is contained in some defect group of $\mathscr{B}$.*

Proof. Let (T^*, N^*, μ^*) be a character-triple isomorphic to (T, N, μ) as in Theorem 3.1, and denote by τ, the associated isomorphism from T/N onto T^*/N^*.

Let $\varphi \in \mathrm{Irr}(T|\mu)$ such that $\varphi^G = \chi$. Then, the character $\varphi^* \in \mathrm{Irr}(T^*|\mu^*)$ belongs to the π-block $\Psi(\mathscr{B})$ of T^* over μ^*. Now, by [9, Theorem 2.11], if $(\hat{W}, \hat{\gamma})$ is a nucleus for φ^*, then a Hall π-subgroup $\hat{P}$ of $\hat{W}$ is contained in some defect group $\hat{D}$ of $\Psi(\mathscr{B})$. Since N^* is a π'-group and since μ^* is T^*-invariant, $N^* \subseteq \hat{W}$ and μ^* is a constituent of $\hat{\gamma}_{N^*}$ by Lemma 1.2 in [9]. Let W be the subgroup of T containing N such that $W^* = \hat{W}$, and let $\gamma \in \mathrm{Irr}(W|\mu)$ such that $\gamma^* = \hat{\gamma}$. By Lemma 11.35 in [2], we have $(\gamma^T)^* = (\hat{\gamma})^{T^*} = \varphi^*$. But then, $\gamma^T = \varphi$, as $*$ is a bijection of $\mathrm{Irr}(T|\mu)$ onto $\mathrm{Irr}(T^*|\mu^*)$. Now, since $\varphi^G = \chi$, we have $\gamma^G = \chi$, and we show below that a Hall π-subgroup of W is contained in some defect group of $\mathscr{B}$.

Since $\hat{P} \in \mathrm{Hall}_\pi(\hat{W})$, we have $\hat{P}N^*/N^* \in \mathrm{Hall}_\pi(\hat{W}/N^*)$ and therefore $\tau^{-1}(\hat{P}N^*/N^*) \in \mathrm{Hall}_\pi(W/N)$ as $\tau(W/N) = \hat{W}/N^*$. Hence, if V is the subgroup of T such that $V/N = \tau^{-1}(\hat{P}N^*/N^*)$, we have $V = QN$ for

$Q \in \mathrm{Hall}_\pi(V)$. Now, the index $[W:Q] = [W:V][V:Q]$ is a π'-number as $[W:V]$ and $[V:Q]$ are π'-numbers. It follows that $Q \in \mathrm{Hall}_\pi(W)$.

Next, $\tau(QN/N) = \tau(V/N) = \hat{P}N^*/N^*$ and as $\hat{P} \subseteq \hat{D}$, we get $\tau(QN/N) \subseteq \hat{D}N^*/N^*$. Now, by Theorem 4.2, we can find a defect group D of $\mathscr{B}$ such that $(DN)^* = (\hat{D})^b N^*$ for some $b \in T^*$. Therefore, $\{(DN)^*\}^{b^{-1}} = \hat{D}N^*$. Then, if $a \in \tau^{-1}(b^{-1}N^*)$, we have

$$\tau\{(DN)^a/N\} = \{(DN)^*/N^*\}^{\tau(aN)} = \{(DN)^*\}^{b^{-1}}/N^* = \hat{D}N^*/N^*.$$

It follows that $QN/N \subseteq (DN)^a/N$, since τ is an isomorphism, and so $Q \subseteq QN \subseteq (DN)^a = D^aN$. By definition of defect groups, D^a is also a defect group of $\mathscr{B}$ and by Lemma 4.1, $D^a \cap N \in \mathrm{Hall}_\pi(N)$. Hence, $|D^aN|_\pi = |D^a|[N:D^a \cap N]_\pi = |D^a|$. In other words, $D^a \in \mathrm{Hall}_\pi(D^aN)$. Consequently, $Q \subseteq D^{an}$ for some $n \in N$. So, we have shown that the Hall π-subgroup Q of W is contained in the defect group D^{an} of $\mathscr{B}$, thus finishing the proof of the theorem.

As a consequence of Theorem 4.4, we obtain the following analogue of [9, Theorem 2.12].

(4.5) THEOREM. *Let $\chi \in \mathscr{B} \in \mathrm{Bl}_\pi(G|\mu)$ and let D be a defect group of $\mathscr{B}$. Then, for $x \in G$ such that x_π is not conjugate to any element of D, we have $\chi(x) = 0$.*

Proof. By Theorem 4.4, there exist a subgroup W of G, a character $\gamma \in \mathrm{Irr}(W)$, and a Hall π-subgroup Q of W such that $\chi = \gamma^G$ and Q is contained in some G-conjugate of D.

Let $x \in G$ be such that x_π is not conjugate to any element of D. Then x_π is not conjugate to any element of Q, and it follows that x is not conjugate to any element of W. Since χ is induced from W, we conclude that $\chi(x) = 0$, as claimed.

In the remainder of this paper, we present a version of Brauer's height-0-conjecture. We start by defining a (relative) height function following [9].

(4.6) DEFINITION. Let $\mathscr{B} \in \mathrm{Bl}_\pi(G|\mu)$ having D as a defect group. For $\chi \in \mathscr{B}$, define

$$h_\mu(\chi) = \frac{\chi(1)_\pi |D|}{|G|_\pi}.$$

The number $h_\mu(\chi)$ is called the *relative height* of χ (with respect to (N, μ)). Note that our definition agrees with [9, Definition 2.13] when $\mu = 1_{\langle 1 \rangle}$, the trivial character of $\langle 1 \rangle$.

Now, throughout the remainder of this section, we let (T^*, N^*, μ^*) be a character-triple isomorphic to (T, N, μ) as in Theorem 3.1, and as before, we let Ψ be the corresponding bijection of $\mathrm{Irr}(G|\mu)$ onto $\mathrm{Irr}(T^*|\mu^*)$. For a character θ in a π-block of T^*, we write $h(\theta)$ to denote Slattery's height of θ.

Our version of Brauer's height-0-conjecture, as well as other results relating to relative heights, are consequences of the following key result.

(4.7) LEMMA. *Let $\chi \in \mathscr{B} \in \mathrm{Bl}_\pi(G|\mu)$. Then $h_\mu(\chi) = h(\Psi(\chi))$.*

Proof. By Theorem 4.2, we may choose a defect group D of $\mathscr{B}$ such that $(DN)^* = \tilde{D}N^*$ for some defect group $\tilde{D}$ of $\Psi(\mathscr{B})$. Let $\varphi \in \mathrm{Irr}(T|\mu)$ such that $\chi = \varphi^G$. Then,

$$h_\mu(\chi) = \frac{\chi(1)_\pi |D|}{|G|_\pi} = \frac{[G:T]_\pi \varphi(1)_\pi |D|}{|G|_\pi} = \frac{\varphi(1)_\pi |D|}{|T|_\pi}.$$

Next, by [2, Lemma 11.24], $\varphi(1)\mu^*(1) = \varphi^*(1)\mu(1)$. Therefore, $\varphi(1)_\pi = \varphi^*(1)_\pi$, as both $\mu(1)$ and $\mu^*(1)$ are π'-numbers. Furthermore, $T/N \cong T^*/N^*$ and N^* is a π'-group. Hence $|T|_\pi = |T^*|_\pi |N|_\pi$ and we get

$$h_\mu(\chi) = \frac{\varphi(1)_\pi |D|}{|T|_\pi} = \frac{\varphi^*(1)_\pi |D|}{|T^*|_\pi |N|_\pi}.$$

As $(DN)^* = \tilde{D}N^*$, we have $DN/N \cong \tilde{D}N^*/N^*$. Now, since N^* is a π'-group, it follows that $D/D \cap N \cong \tilde{D}$. So $|D| = |D \cap N|\,|\tilde{D}|$, and by Lemma 4.1 $|D| = |N|_\pi |\tilde{D}|$.

Finally, we have

$$h_\mu(\chi) = \frac{\varphi^*(1)_\pi |D|}{|T^*|_\pi |N|_\pi} = \frac{\varphi^*(1)_\pi |\tilde{D}|}{|T^*|_\pi} = h(\Psi(\chi)),$$

as $\Psi(\chi) = \varphi^* \in \Psi(\mathscr{B})$.

As a consequence of this lemma and [9, Theorem 2.5], we deduce that relative heights are positive integers, as should be expected.

To formulate the next results, we need one further definition.

(4.8) DEFINITION. Let $\mathscr{B} \in \mathrm{Bl}_\pi(G|\mu)$. We say that a character χ of $\mathscr{B}$ is of *relative height* 0 in $\mathscr{B}$, provided $h_\mu(\chi) = 1$.

Clearly, relative height 0 characters are height 0 in the sense of [9] when $\mu = 1_{\langle 1 \rangle}$.

Our next result follows from Lemma 4.7 and [9, Theorem 2.15].

(4.9) THEOREM. *Every relative π-block of G with respect to (N, μ) has a relative height 0 character.*

In [9], Slattery proved half of Brauer's height-0-conjecture for π-blocks of π-separable groups (see [9, Corollary 2.17]). The other half was established by Manz and Staszewski (see [7, Theorem 3.3]). We present here a version of that conjecture for relative π-blocks of π-separable groups.

(4.10) THEOREM. *Let $\mathscr{B} \in \mathrm{Bl}_\pi(G|\mu)$ having D as a defect group. Then $D/D \cap N$ is abelian if and only if each character in $\mathscr{B}$ is of relative height* 0.

Proof. By Theorem 4.2, we may choose a defect group D_0 of $\mathscr{B}$ such that $D_0N/N \cong \tilde{D}N^*/N^*$ for some defect group $\tilde{D}$ of the π-block $\Psi(\mathscr{B})$ of T^*. Now, since N^* is a π'-group, $D_0/D_0 \cap N \cong \tilde{D}$, and as D is G-conjugate to D_0, we have $D/D \cap N \cong \tilde{D}$ as well. Hence, $D/D \cap N$ is abelian if and only if $\tilde{D}$ is abelian. By [9, Theorem 2.18], $\tilde{D}$ is abelian if and only if each character in $\Psi(\mathscr{B})$ is height 0. On the other hand, Lemma 4.7 implies that each character in $\Psi(\mathscr{B})$ is height 0 if and only if each character in $\mathscr{B}$ is of relative height 0. Our result is now immediate.

5. SOME EXAMPLES

Let G be a π-separable group, N a normal subgroup of G, and μ a π'-special character of N. A natural question one may ask is whether each relative π-block of G with respect to (N, μ) is just the intersection of some ordinary π-block of G with the set $\mathrm{Irr}(G|\mu)$. The answer is "no" in general, as illustrated by the following example.

(5.1) EXAMPLE. Let $G = \Sigma_4$, the symmetric group on four symbols, $\pi = \{2\}$, $N = O_2(G)$, and $\mu = 1_N$, the trivial character of N. It is clear that μ is $2'$-special.

The set $\mathrm{Irr}(G|\mu)$ consists exactly of three characters. Two of these characters $\chi_1 = 1_G$ and χ_2 are linear and the third χ_3 has degree 2.

Referring to the character table of Σ_4 (see p. 287 in [2]), we see that $\chi_1^0 = \chi_2^0$. It follows that χ_1 is the only linear character in $B_{2'}(G) \cap \mathrm{Irr}(G|\mu)$ and that χ_1 and χ_2 are linked by χ_1.

By Lemma 3.2, every character $\varphi \in B_{2'}(G)$ such that $d_{\chi_3\varphi} \neq 0$ must lie in $\mathrm{Irr}(G|\mu)$. So, were χ_1 or χ_2 linked to χ_3, we would necessarily have $\chi_3^0 = 2\chi_1^0$. But this is impossible, as $\chi_3(g) = -1$ and $2\chi_1(g) = 2$ for any element $g \in G$ of order 3. Hence, we conclude that χ_3 is not linked to either χ_1 or χ_2. Therefore, $\mathscr{B}_1 = \{\chi_1, \chi_2\}$ and $\mathscr{B}_2 = \{\chi_3\}$ are precisely the relative 2-blocks of G with respect to (N, μ).

Now, since $O_{2'}(G) = \langle 1 \rangle$, the group G has a single 2-block, namely, the principal 2-block B and $B \cap \operatorname{Irr}(G|\mu) = \operatorname{Irr}(G|\mu) = \mathscr{B}_1 \cup \mathscr{B}_2$.

We next give an example, where Theorem 4.5 applies, but its ordinary π-block analogue ([9, Theorem 2.12]) does not for the same group.

(5.2) EXAMPLE. We keep the notation of Example 5.1. In that example, we have seen that G has a relative 2-block $\mathscr{B}_2$ with respect to (N, μ) consisting of the single character χ_3 of $\operatorname{Irr}(G|\mu)$ of degree 2. By Theorem 4.9, this character χ_3 is of relative height 0. It follows that if D is some defect group of $\mathscr{B}_2$, then $|D| = 4$. Now, since $N \subseteq D$ by Lemma 4.1, we conclude that $D = N$ as $|N| = 4$.

Let h be an element of G of order 4. Then, clearly h is not conjugate to any element of D, and so $\chi_3(h) = 0$ by Theorem 4.5. On the other hand, χ_3 belongs to the unique 2-block of G, namely, the principal 2-block B. Certainly, h is contained in some Sylow 2-subgroup P of G and P is a defect group of B. In this situation, Theorem 2.12 of [9] does not apply for χ_3 and h.

The last example of this section shows that Theorem 4.10 is not just a consequence of the ordinary height-zero result ([9, Theorem 2.18]) applied to the same group. Here again, we keep the notation of Example 5.1.

(5.3) EXAMPLE. The relative 2-block $\mathscr{B}_1$ of G contains exactly 2 linear characters. These characters are of relative height 0 by Theorem 4.9. Moreover, by the definition of relative heights, $\mathscr{B}_1$ has a Sylow 2-subgroup P of G as a defect group. Clearly $P \cap N = N$ and since $[P : N] = 2$, the quotient group P/N is abelian. However, the single ordinary 2-block B of G has the Sylow 2-subgroup P as a defect group and P is not abelian.

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