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Varieties of Hexagonal Quasigroups

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Hexagonal quasigroups arise from the decomposition of complete graphs into 2-perfect edge-disjoint systems of 6-cycles: given distinct vertices a, b we define the product ab to be the unique vertex such that (a, b) and (b, ab) are adjacent edges in the same 6-cycle. With one exception, the axioms for these decompositions can be expressed in terms of identities. The one exception is the requirement that 6-cycles consist of 6 distinct vertices; this requirement can be met by the implication $ab = ba \Rightarrow a = b$. Can this implication be replaced by some identities? The answer is no and yes. It is no in that the implication is not equivalent to a set of identities; hexagonal quasigroups form a quasivariety which is not a variety and so cannot be axiomatized by identities alone. But it is yes in the sense that the quasivariety of hexagonal quasigroups contains a subvariety whose spectrum (the set of cardinalities of its finite members) differs from that of hexagonal quasigroups by only a finite set.

Let G_n be the complete graph (undirected and without loops) on n vertices. Suppose that we can decompose G_n into edge-disjoint 6-cycles (a 6-cycle consists of the 6 edges (a, b), (b, c), (c, d), (d, e), (e, f),(f, a) where $\{a, b, c, d, e, f\}$ is a set of 6 distinct vertices); call such a decomposition a *6-cycle system*. Define a binary operation (denoted by $\bullet$ or merely by juxtaposition) on the vertex set by: for distinct a, b let ab be the unique vertex such that edges (a, b) and (b, ab) belong to the same 6-cycle. Complete the definition of the operation by setting $aa = a$ for every vertex a. Since we are dealing with undirected graphs and undirected 6-cycles, we must have $(ab)b = a$; note that this holds even when $a = b$ since $aa = a$. The 6 elements in the 6-cycle determined by (a, b) are $a(ba)$, ba, a, b, ab, $b(ab)$; thus, we must have $a(ba) = (ab)(b(ab))$ (which again is true when $a = b$).

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Unfortunately, our operation is not a quasigroup operation. The identity $(ab)b = a$ forces every column of the opertion's table to be a permutation, but nothing so far forces every row to be a permutation. Indeed, on a 9-element set it is easy to find a 6-cycle system, but it is known that no such system yields a quasigroup. The additional condition we need to impose on 6-cycle systems is that for distinct vertices a, b there is a unique 6-cycle in which they are distance 2 apart. That is to say, there is a unique vertex c such that (a, c) and (c, b) are consecutive edges on the same 6-cycle. Such 6-cycle systems will be called *2-perfect*. So we define a new operation (denoted $*$) so that $a * b$ is the unique vertex such that $(a, a * b)$ and $(a * b, b)$ belong to the same 6-cycle; again, we define $a * a = a$ (notice that $*$ is commutative). By existence, we have $a(a * b) = b$, while by uniqueness, we have $a * ab = b$. Thus, our new operation guarantees that our old operation is now a quasigroup. For an extensive discussion of 2-perfect 6-cycle systems, see the paper of C. C. Lindner, K. T. Phelps and C. A. Rodger, [1].

We have still to guarantee that a 6-cycle consists of 6 distinct vertices. The implication $ab = ba \Rightarrow a = b$ must hold if a 6-cycle consists of 6 distinct vertices. On the other hand, it is straightforward to show that the above identities and the implication force every 6-cycle to consist of 6 distinct vertices. Thus, we define $\mathbb{H}$, the class of *hexagonal quasigroups*, to be the class of all algebras $\langle Q; \cdot, * \rangle$ such that:

(*) $xx = x$, $(xy)y = x$, $x(yx) = (xy)(y(xy))$, $x * x = x$, $x(x * y) = y$, $x * xy = y$ and

(**) $xy = yx \Rightarrow x = y$.

It is now a simple exercise to show the equivalence of hexagonal quasigroups with 2-perfect 6-cycle systems.

While 2-perfect 6-cycle systems are of interest to combinatorialists only on finite sets (in part because on any infinite set it is easy to construct a 2-perfect 6-cycle system), the definition applies equally well to infinite sets. The *spectrum* of a class of structures is the set of cardinalities of the finite members of the class. It is easy to see that finite 6-cycle systems must have cardinality $n \equiv 1$ or 9 mod 12: since each vertex of G_n has degree $n - 1$ and each vertex of a 6-cycle has degree 2, n must be odd, and as each 6-cycle contains 6 edges, $n(n-1)$ must be divisible by 12. It is known that the spectrum of the class of 2-perfect 6-cycle systems is all $n \equiv 1$ or 9 mod 12 except for $n = 9$ (see [1]). Thus, in a sense now to be made precise, 2-perfect 6-cycle systems are a "large" subclass of the class of 6-cycle systems. Call two classes of structures *cospectral* if they have the same spectra and *eventually cospectral* if the symmetric difference of their spectra is a finite set (of course, this latter condition is equivalent to saying that for some n their spectra coincide past n). Thus, a subclass of a class is "large" precisely if its spectrum is eventually cospectral with that of the larger class.

Now we can precisely formulate our two questions, the answers to which are, respectively, no and yes. Is $\mathbb{H}$ axiomatizable by a set of identities (i.e., is $\mathbb{H}$ a variety)? If not, does $\mathbb{H}$ contain an eventually cospectral subvariety?

To show that the answer to the first equation is no, we must construct a hexagonal quasigroup with a quotient that is not a hexagonal quasigroup. For this, note that the 3-element quasigroup $\langle\{0, 1, 2\}; \bullet, *\rangle$ where $x \bullet y = x * y = -x - y \pmod 3$ satisfies (*) but not (**). The proof is then completed by showing that the 2-generated free algebra satisfying (*) also satisfies (**). But this follows readily from an inductive construction of the "free" 2-generated 2-perfect 6-cycle system: every cycle in it consists of 6 distinct elements and so (**) is satisfied.

THEOREM 1. *$\mathbb{H}$ is not a variety.*

Since combinatorialists only care about the finite members of $\mathbb{H}$, we should eliminate the possibility that the finite members of $\mathbb{H}$ might be contained in a proper subvariety of $\mathbb{H}$. This will not be possible if we can show that $\mathbb{H}$ is generated by its finite members. But this will follow if every finite partial 2-perfect 6-cycle system can be embedded in a finite 2-perfect 6-cycle system. This, in turn, follows from a generalization by R. M. Wilson of his block design techniques found in [7]. Unfortunately, this generalization has not yet appeared in print.

Showing that the answer to the second question is yes is more difficult, but interesting. Since the spectrum of $\mathbb{H}$ contains 13 and 21, we can choose $\mathbf{H}_{13}$, $\mathbf{H}_{21} \in \mathbb{H}$ with $|H_{13}| = 13$ and $|H_{21}| = 21$. Let $\mathbb{V}_2(\mathbf{H}_{13}, \mathbf{H}_{21}) = \mathbb{V}$ be the variety determined by the set of identities in at most 2 variables true in both $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$. The proof is completed by showing that $\mathbb{V}$ is contained in $\mathbb{H}$, and that the spectrum of $\mathbb{V}$ is eventually cospectral with that of $\mathbb{H}$. This is done in the following two lemmas.

LEMMA 1. *$\mathbb{V}$ is a subclass of $\mathbb{H}$.*

Proof. Both $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$ are plain; that is, each is simple and has no proper non-trivial subquasigroups. To see this, note that for idempotent algebras, congruence blocks are subalgebras, so the second property implies the first. As $\mathbb{H}$ is a quasivariety, it is closed under forming subquasigroups, and 13 is the cardinality of the smallest non-trivial member of $\mathbb{H}$. But in any quasigroup, subquasigroups have cardinality at most 1/2 that of the quasigroup; hence, $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$ are plain. Since quasigroups are congruence permutable and hexagonal quasigroups are idempotent, it now follows from a lemma of R. S. Pierce (stated as problem 15a on page 91 of [5]) that every finite quasigroup in the variety generated by $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$ is a direct product of copies of $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$. Thus, $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$ are the only finite subdirectly irreducible algebras in the variety they generate; a compactness argument (see [6]) now shows that that they are the only subdirectly irreducible algebras in the variety they generate. This means that $\mathbb{V}(\mathbf{H}_{13}, \mathbf{H}_{21})$, the variety generated by $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$, is actually the quasivariety generated by $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$, and so $\mathbb{V}(\mathbf{H}_{13}, \mathbf{H}_{21})$ is a subclass of $\mathbb{H}$. As $\mathbb{H}$ is axiomatized by identities and implications in at most 2 variables, it has the property that if $\mathbf{H}$ is any quasigroup such that all its 2-generated subquasigroups belong to $\mathbb{H}$, then so does $\mathbf{H}$ itself. This means that $\mathbb{V}$ is a subclass of $\mathbb{H}$.

LEMMA 2. *The spectrum of* $\mathbb{V}$ *is eventually cospectral with that of* $\mathbb{H}$.

Proof. Consider the class $\mathbb{B}(13, 21)$ of linear spaces with block sizes from $\{13, 21\}$; that is, the class of all decompositions of complete graphs into edge-disjoint copies of complete graphs on 13 and/or 21 vertices. A simple argument shows that the spectrum of $\mathbb{B}(13, 21)$ is contained in the set $\{n \mid n \equiv 1 \text{ or } 9 \bmod 12\}$. R. M. Wilson's Eventual Sufficiency Theorem (see [7]) states that the spectrum of $\mathbb{B}(13, 21)$ differs from $\{n \mid n \equiv 1$ or 9 mod $12\}$ by only a finite set. Let $\mathbf{B} \in \mathbb{B}(13, 21)$; if on the vertices of $\mathbf{B}$ we can construct a quasigroup $\mathbf{H}(\mathbf{B}) \in \mathbb{V}$, then we will have proved the lemma. But this is easy: on each block of size 13, define a quasigroup isomorphic to $\mathbf{H}_{13}$, and on each block of size 21, define a quasigroup isomorphic to $\mathbf{H}_{21}$. Since distinct blocks intersect in at most one vertex, this defines our quasigroup $\mathbf{H}(\mathbf{B})$; since each pair of distinct vertices of $\mathbf{B}$ belong to a unique block, each 2-generated subquasigroup of $\mathbf{H}(\mathbf{B})$ belongs to $\mathbb{V}$ and so $\mathbf{H}(\mathbf{B})$ belongs to $\mathbb{V}$. This completes the proof of the lemma.

THEOREM 2. $\mathbb{V}$ *is an eventually cospectral subvariety of* $\mathbb{H}$.

It is trivial that $\mathbb{V}$ is a finitely based variety. In fact, it is an arithmetical variety and so is 1-based by [4] (recall that a variety is arithmetical if it is both congruence permutable and congruence distributive). This follows from R. McKenzie's analysis of paraprimal varieties in [3]. A finite algebra $\mathbf{A}$ is *paraprimal* if it is congruence permutable and is subsimple (every subalgebra, including $\mathbf{A}$ itself, is simple). Thus, both $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$ are paraprimal. A variety is *paraprimal* if it is congruence permutable and generated by a finite set of paraprimal algebras. Thus, $\mathbb{V}(\mathbf{H}_{13}, \mathbf{H}_{21})$ is paraprimal. Since arithmeticality is determined by 2-variable identities, $\mathbb{V}$ will be arithmetical provided that $\mathbb{V}(\mathbf{H}_{13}, \mathbf{H}_{21})$ is. This means that we need to show that $\mathbb{V}(\mathbf{H}_{13}, \mathbf{H}_{21})$ is congruence distributive. An algebra $\mathbf{A}$ is *affine* if there is a unitary left **R**-module **M** on the set A such that all operations of **A** are module polynomials on **M**, and $x-y+z$ is a term function of **A**. The author thanks the referee for pointing out the following lemma with a different proof (using the "term condition").

LEMMA 3. *No non-trivial hexagonal quasigroup is affine.*

Proof. Let **A** be an affine hexagonal quasigroup derived from the unitary left **R**-module **M**. We write $xy=ax+by+c$; the law $xx=x$ forces $c=0$ and $b=1-a: xy=ax+(1-a)y$. Similarly, $x*y=bx+(1-b)y$. The law $(xy)y=y$ forces $a^2x=x$, while the law $x*y=y*x$ forces $2bx=x$. The laws $x(x*y)=y=x*xy$ force $abx=bax=ax+bx$. But then $2bx=x$ forces $ax=-x$ so that $xy=-x+2y$. The law $x(yx)=(xy)(y(xy))$ forces $6x=0$. But then $2bx=x$ forces $3x=0$. Finally, the implication $xy=yx \Rightarrow x=y$ forces $3x=3y \Rightarrow x=y$. Since $3x=0$, this means that $x=y$; i.e., **A** is trivial as required.

Thus, neither $\mathbf{H}_{13}$ nor $\mathbf{H}_{21}$ is affine. Now invoke Theorem 22 of [3] which says that a paraprimal algebra either has a non-trivial affine subalgebra or is quasiprimal. Recall that **A** is *quasiprimal* if the ternary discriminator $t(x, y, z)$ (defined by $t(x, x, z)=x$ and otherwise $t(x, y, z)=z$) is a term function on A. Since $\mathbf{H}_{13}$ and $\mathbf{H}_{21}$ are both plain, they must then be quasiprimal. But then $\mathbb{V}(\mathbf{H}_{13}, \mathbf{H}_{21})$ is a quasiprimal variety and so congruence distributive by Theorem 21 of [3]. In fact, $\mathbb{V}(\mathbf{H}_{13}, \mathbf{H}_{21})$ is also finitely based, and an efficient algorithm for finding an equational base is given by R. McKenzie in [2]. The most difficult part in actually applying this algorithm is likely to be writing $t(x, y, z)$ as a quasigroup word.

THEOREM 3. *The varieties* $\mathbb{V}(\mathbf{H}_{13}, \mathbf{H}_{21})$ *and* $\mathbb{V}_2(\mathbf{H}_{13}, \mathbf{H}_{21})$ *are each* 1*-based arithmetical varieties.*

In [8], D. E. Bryant has a more interesting proof of Theorem 1. He constructs a finite hexagonal quasigroup which has as a quotient the 3-element quasigroup mentioned above. In an earlier version of the present paper, I had asserted (without proof) that a quotient of a finite hexagonal quasigroup was again a hexagonal quasigroup. I thank Dr. Bryant for bringing this mistake to my attention.

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