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LOCALIZATION FOR NONABELIAN GROUP ACTIONS

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Suppose X is a compact symplectic manifold acted on by a compact Lie group K (which may be nonabelian) in a Hamiltonian fashion, with moment map $\mu: X \to \mathrm{Lie}(K)^*$ and Marsden–Weinstein reduction $\mathcal{M}_X = \mu^{-1}(0)/K$. There is then a natural surjective map κ_0 from the equivariant cohomology $H_K^*(X)$ of X to the cohomology $H^*(\mathcal{M}_X)$. In this paper we prove a formula (Theorem 8.1, the residue formula) for the evaluation on the fundamental class of $\mathcal{M}_X$ of any $\eta_0 \in H^*(\mathcal{M}_X)$ whose degree is the dimension of $\mathcal{M}_X$, provided that 0 is a regular value of the moment map μ on X. This formula is given in terms of any class $\eta \in H_K^*(X)$ for which $\kappa_0(\eta) = \eta_0$, and involves the restriction of η to K-orbits KF of components $F \subset X$ of the fixed point set of a chosen maximal torus $T \subset K$. Since κ_0 is surjective, in principle the residue formula enables one to determine generators and relations for the cohomology ring $H^*(\mathcal{M}_X)$, in terms of generators and relations for $H_K^*(X)$. There are two main ingredients in the proof of our formula: one is the localization theorem [3, 7] for equivariant cohomology of manifolds acted on by compact abelian groups, while the other is the equivariant normal form for the symplectic form near the zero locus of the moment map.

We also make use of the techniques appearing in our proof of the residue formula to give a new proof of the nonabelian localization formula of Witten ([35, Section 2]) for Hamiltonian actions of compact groups K on symplectic manifolds X; this theorem expresses $\eta_0[\mathcal{M}_X]$ in terms of certain integrals over X.

1. INTRODUCTION

SUPPOSE X is a compact oriented manifold acted on by a compact connected Lie group K of dimension s; one may then define the equivariant cohomology $H_K^*(X)$. Throughout this paper we shall consider only cohomology with complex coefficients. If X is a symplectic manifold with symplectic form ω and the action of K is Hamiltonian (in other words, there is a moment map $\mu: X \to \mathbf{k}^*$), then we may form the symplectic quotient $\mathcal{M}_X = \mu^{-1}(0)/K$. The inclusion map $i_0: \mu^{-1}(0) \to X$ gives a ring homomorphism $i_0^*: H_K^*(X) \to H_K^*(\mu^{-1}(0))$. Using Morse theory and the gradient flow of the function $|\mu|^2: X \to \mathbb{R}$, it is proved in [29] that the map i_0^* is surjective.

Suppose, in addition, that 0 is a *regular value* of the moment map μ. This assumption is equivalent to the assumption that the stabilizer K_x of x under the action of K on X is *finite* for every $x \in \mu^{-1}(0)$, and it implies that $\mathcal{M}_X$ is an orbifold, or V-manifold, which inherits a symplectic form ω_0 from the symplectic form ω on X. In this situation there is a canonical isomorphism $\pi_0^*: H^*(\mu^{-1}(0)/K) \to H_K^*(\mu^{-1}(0))$.‡ Hence, we have a surjective ring homomorphism

$$\kappa_0 = (\pi_0^*)^{-1} \circ i_0^*: H_K^*(X) \to H^*(\mathcal{M}_X). \tag{1.1}$$

Henceforth, if $\eta \in H_K^*(X)$ we shall denote $\kappa_0(\eta)$ by η_0. Previous work [11, 30] on determining the ring structure of $H_K^*(X)$ has presented methods which in some situations

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‡ This isomorphism is induced by the map $\pi_0: \mu^{-1}(0) \times_K EK \to \mu^{-1}(0)/K$. Recall that we are only considering cohomology with complex coefficients.

permit the direct determination of the kernel of the map κ_0, and hence of generators and relations in $H^*(\mathscr{M}_X)$ in terms of generators and relations in $H_K^*(X)$. (Note that the generators of $H_K^*(X)$ give generators of $H^*(\mathscr{M}_X)$ via the surjective map κ_0, and also that generators of $H^*(BK)$ together with extensions to $H_K^*(X)$ of generators of $H^*(X)$ give generators of $H_K^*(X)$ because the spectral sequence of the fibration $X \times_K EK \to BK$ degenerates [29].) Here we present an alternative approach to determining the ring structure of $H^*(\mathscr{M}_X)$ when 0 is a regular value of μ, which complements the results obtained by directly studying the kernel of κ_0. Our approach is based on the observation that since $H^*(\mathscr{M}_X)$ satisfies Poincaré duality, a class $\eta \in H_K^*(X)$ is in the kernel of κ_0 if and only if for all $\zeta \in H_K^*(X)$ we have

$$\eta_0 \zeta_0 [\mathscr{M}_X] = (\eta\zeta)_0 [\mathscr{M}_X] = 0. \tag{1.2}$$

Hence, to determine the kernel of κ_0 (in other words, the relations in the ring $H^*(\mathscr{M}_X)$) given the ring structure of $H_K^*(X)$, it suffices to know the *intersection pairings*; equivalently, the evaluations on the fundamental class $[\mathscr{M}_X]$ of all possible classes $\xi_0 = \kappa_0(\xi)$. In principle, the intersection pairings thus determine generators and relations for the cohomology ring $H^*(\mathscr{M}_X)$, given generators and relations for $H_K^*(X)$.

There is a natural pushforward map $\Pi_*: H_K^*(X) \to H_K^* = H_K^*(\mathrm{pt}) \cong S(k^*)^K$, where we have identified H_K^* with the space of K-invariant polynomials on the Lie algebra $\boldsymbol{k}$. This map can be thought of as integration over X and will sometimes be denoted by $\int_X$. If T is a compact *abelian* group (i.e. a torus) and $\zeta \in H_T^*(X)$, there is a formula† (the *abelian localization theorem*) for $\Pi_*\zeta$ in terms of the restriction of ζ to the components of the fixed point set for the action of T. In particular, for a general compact Lie group K with maximal torus T there is a canonical map $\tau_X: H_K^*(X) \to H_T^*(X)$, and we may apply the abelian localization theorem to $\tau_X(\zeta)$ where $\zeta \in H_K^*(X)$.

In terms of the components F of the fixed point set of T on X, we obtain a formula (the residue formula, Theorem 8.1) for the evaluation of a class $\eta_0 \in H^*(\mathscr{M}_X)$ on the fundamental class $[\mathscr{M}_X]$, when η_0 comes from a class $\eta \in H_K^*(X)$. There are two main ingredients in the proof of Theorem 8.1. One is the abelian localization theorem [3, 7], while the other is an equivariant normal form for ω in a neighbourhood of $\mu^{-1}(0)$, given in [22] as a consequence of the coisotropic embedding theorem. The result is the following Theorem.

THEOREM 8.1. *Let $\eta \in H_K^*(X)$ induce $\eta_0 \in H^*(\mathscr{M}_X)$, and assume K acts effectively on $\mu^{-1}(0)$. Then we have*

$$\eta_0 e^{i\omega_0}[\mathscr{M}_X] = \frac{(-1)^{n_+}}{(2\pi)^{s-l}|W|\mathrm{vol}(T)} \mathrm{Res}\left(\varpi^2(\psi) \sum_{F \in \mathscr{F}} e^{i\mu_T(F)(\psi)} \int_F \frac{i_F^*(\eta(\psi)e^{i\omega})}{e_F(\psi)} [d\psi] \right).$$

In this formula, n_+ is the number of positive roots of K, and $\varpi(\psi) = \Pi_{\gamma>0}\gamma(\psi)$ is the product of the positive roots, while $\mathscr{F}$ is the set of components of the fixed point set of the maximal torus T on X. If $F \in \mathscr{F}$ then i_F is the inclusion of F in X and e_F is the equivariant Euler class of the normal bundle to F in X. Finally s and l are the dimensions of K and T.

Here, via the *Cartan model*, the class $\tau_X(\eta) \in H_T^*(X)$ has been identified with a family of differential forms $\eta(\psi)$ on X parametrized by $\psi \in \mathfrak{t}$. The definition of the residue map Res (whose domain is a suitable class of meromorphic differential forms on $\mathfrak{t} \otimes \mathbb{C}$) will be given in Section 8 (Definition 8.5). It is a linear map, but in order to apply it to the individual terms

† Atiyah and Bott [3] give a cohomological proof of this formula, which was first proved by Berline and Vergne [7].

in the statement of Theorem 8.1 some choices must be made. The choices do not affect the residue of the whole sum. When $K = SU(2)$ the formula becomes

$$\eta_0 e^{i\omega_0}[\mathscr{M}_X] = -2\,\mathrm{Res}_0\left(\psi^2 \sum_{F\in\mathscr{F}_+} e^{i\mu_T(F)(\psi)} \int_F \frac{i_F^*(\eta(\psi)e^{i\omega})}{e_F(\psi)}\right),$$

where Res_0 denotes the coefficient of $1/\psi$, and $\mathscr{F}_+$ is the subset of the set of components of the fixed point set of $T = U(1)$ consisting of those components F of the T fixed point set for which $\mu_T(F) > 0$.

We note that if $\dim \eta_0 = \dim \mathscr{M}_X$ then the left-hand side of the equation in Theorem 8.1 is just $\eta_0[\mathscr{M}_X]$. More generally, one may obtain a formula for $\eta_0[\mathscr{M}_X]$ by replacing the symplectic form ω by $\delta\omega$ (where $\delta > 0$ is a small parameter), and taking the limit as $\delta \to 0$. This has the effect of replacing the moment map μ by $\delta\mu$. In the limit $\delta \to 0$, the Residue Formula (Theorem 8.1) becomes a sum of terms corresponding to the components F of the fixed point set, where the term corresponding to F is (up to a constant) the residue (in the sense of Section 8) of $\varpi^2(\psi)\int_F i_F^*(\eta(\psi))/e_F(\psi)$, and the only role played by the symplectic form and the moment map is in determining which F give a nonzero contribution to the residue of the sum and the signs with which individual terms enter.

Results for the case when $K = S^1$, which are related to our Theorem 8.1, may be found in the papers of Kalkman [27] and Wu [36].

Witten in Section 2 of [35] gives a related result, the *nonabelian localization theorem*, which also interprets evaluations $\eta_0[\mathscr{M}_X]$ of classes on the fundamental class $[\mathscr{M}_X]$ in terms of appropriate data on X. For $\varepsilon > 0$ and $\zeta \in H_K^*(X)$, he defines†

$$\mathscr{I}^\varepsilon(\zeta) = \frac{1}{(2\pi i)^S \operatorname{vol} K}\int_{\phi\in\mathbf{k}} [d\phi] e^{-\varepsilon\langle\phi,\phi\rangle/2}\Pi_*\zeta(\phi) \tag{1.3}$$

(where $\langle\cdot,\cdot\rangle$ is a fixed invariant inner product on $\mathbf{k}$, which we shall use throughout to identify $\mathbf{k}^*$ with $\mathbf{k}$) and expresses it as a sum of local contributions.

Witten's theorem tells us that just as $\Pi_*\zeta$ would have contributions from the components of the fixed point set of K if K were abelian, the quantity $\mathscr{I}^\varepsilon(\zeta)$ (if K is not necessarily abelian) reduces to a sum of integrals localized around the critical set of the function $\rho = |\mu|^2$, i.e. the set of points x where $(d|\mu|^2)_x = 0$. (Of course $d|\mu|^2 = 2\langle\mu, d\mu\rangle$, so the fixed point set of the K action, where $d\mu = 0$, is a subset of the critical set of $d|\mu|^2$.) More precisely, the critical set of $\rho = |\mu|^2$ can be expressed as a disjoint union of closed subsets C_β of X indexed by a finite subset $\mathscr{B}$ of the Lie algebra $\mathbf{t}$ of the maximal torus T of K which is explicitly known in terms of the moment map μ_T for the action of T on X [29]. If $\beta \in \mathscr{B}$ then the critical subset C_β is of the form $C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$, where Z_β is a union of connected components of the fixed point set of the subtorus of T generated by β. The subset $\mu^{-1}(0)$ on which $\rho = |\mu|^2$ takes its minimum value is C_0. There is a natural map‡ $\mathrm{pr}^*: H_K^* \to H_K^*(\mu^{-1}(0))$ so that the distinguished class $f(\phi) = -\langle\phi,\phi\rangle/2$ in H_K^4 gives rise to a distinguished class $\Theta \in H^4(\mu^{-1}(0)/K) \cong H_K^4(\mu^{-1}(0))$. Witten's result can then be expressed in the following form.

† The normalisation of the measure in $\mathscr{I}^\varepsilon(\zeta)$ will be described at the beginning of Section 3. As above, $\Pi_*: H_K^*(X) \to H_K^* \cong S(k^*)^K$ is the natural pushforward map, where $S(k^*)^K$ is the space of K-invariant polynomials on $\mathbf{k}$.

‡ This map is induced by the projection $\mathrm{pr}: \mu^{-1}(0) \to \mathrm{pt}$.

THEOREM 1.1.

$$\mathscr{I}^{\varepsilon}(\zeta) = \zeta_0 \mathrm{e}^{\varepsilon\Theta}[\mathscr{M}_X] + \sum_{\beta \in \mathscr{B} - \{0\}} \int_{U_\beta} \zeta'_\beta .$$

Here, the U_β are open neighbourhoods in X of the nonminimal critical subsets C_β of the function ρ. The ζ'_β are certain differential forms on U_β obtained from ζ. .

In the special case $\zeta = \eta \exp \mathrm{i}\bar{\omega}$ (where $\bar{\omega}(\phi) = \omega + \mu(\phi)$ is the standard extension of the symplectic form ω to an element of $H^2_K(X)$, and η has polynomial dependence[§] on the generators of H^*_K), Witten's results give us the following estimate on the growth of the terms $\int_{U_\beta} \zeta'_\beta$ as $\varepsilon \to 0$.

THEOREM 1.2. *Suppose $\zeta = \eta \exp \mathrm{i}\bar{\omega}$ for some $\eta \in H^*_K(X)$. If $\beta \in \mathscr{B} - \{0\}$ then $\int_{U_\beta} \zeta'_\beta = \mathrm{e}^{-\rho\beta/2\varepsilon} h_\beta(\varepsilon)$, where $\rho_\beta = |\beta|^2$ is the value of $|\mu|^2$ on the critical set C_β and $|h_\beta(\varepsilon)|$ is bounded by a polynomial in ε^{-1}.*

Thus, one should think of $\varepsilon > 0$ as a small parameter, and one may use the asymptotics of the integral $\mathscr{I}^{\varepsilon}$ over X to calculate the intersection pairings $\eta_0 \mathrm{e}^{\varepsilon\Theta} \mathrm{e}^{\mathrm{i}\omega_0}[\mathscr{M}_X]$, since the terms in Theorem 1.2 corresponding to the other critical subsets of ρ vanish exponentially fast as $\varepsilon \to 0$. Notice that when $\zeta = \exp \mathrm{i}\bar{\omega}$, the vanishing of μ on $\mu^{-1}(0)$ means that $\zeta_0 = \exp \mathrm{i}\omega_0$, where ω_0 is the symplectic form induced by ω on $\mathscr{M}_X = \mu^{-1}(0)/K$.

In this paper we shall give a proof of a variant of Theorems 1.1 and 1.2, for the case $\zeta = \eta \exp \mathrm{i}\bar{\omega}$ where $\eta \in H^*_K(X)$. Before outlining our proof, it will be useful to briefly recall Witten's argument. Witten introduces a K-invariant 1-form λ on X, and shows that $\mathscr{I}^{\varepsilon}(\zeta) = \mathscr{I}^{\varepsilon}(\zeta \exp sD\lambda)$, where D is the differential in equivariant cohomology and $s \in \mathbb{R}^+$. He then does the integral over $\phi \in \mathbf{k}$ and shows that in the limit as $s \to \infty$, this integral vanishes over any region of X where $\lambda(V^j) \neq 0$ for at least one of the vector fields $V^j, j = 1, \ldots, s$ given by the infinitesimal action of a basis of $\mathbf{k}$ on X indexed by j. Thus, after integrating over $\phi \in \mathbf{k}$, the limit as $s \to \infty$ of $\mathscr{I}^{\varepsilon}(\zeta)$ reduces to a sum of contributions from sets where $\lambda(V^j) = 0$ for all the V^j.

In our case, when X is a symplectic manifold and the action of K is Hamiltonian, Witten chooses $\lambda(Y) = d|\mu|^2(JY)$, where J is a K-invariant almost complex structure on X. Thus, $\lambda(V^j)(x) = 0$ for all j if and only if $(d|\mu|^2)_x = 0$, so $\mathscr{I}^{\varepsilon}(\zeta)$ reduces to a sum of contributions from the critical sets of $\rho = |\mu|^2$. Further, he obtains the contribution from $\mu^{-1}(0)$ as $\mathrm{e}^{\varepsilon\Theta}\zeta_0[\mathscr{M}_X]$. If $\zeta = \eta \mathrm{e}^{\mathrm{i}\bar{\omega}}$ he also obtains the estimates in Theorem 1.2 on the contributions from the neighbourhoods U_β.

In general, the contributions to the localization theorem depend on the choice of λ. In the symplectic case, with $\lambda = Jd|\mu|^2$, the contribution from $\mu^{-1}(0)$ is canonical but the contributions from the other critical sets C_β depend in principle on the choice of J. Further, the properties of these other terms are difficult to study. Ideally, they should reduce to integrals over the critical sets C_β, and indeed when proving Theorem 1.2, Witten makes the assumption (before (2.52)) that the C_β are nondegenerate critical manifolds in the sense of Bott [9]. In general, the C_β are not manifolds; and even when they are manifolds, they are

[§] The equivariant cohomology $H^*_K(X)$ is defined to consist of classes which have polynomial dependence on the generators of H^*_K, but we shall also make use of formal classes such as $\exp \mathrm{i}\bar{\omega}$ which are formal power series in these generators.

not necessarily nondegenerate. They satisfy only a weaker condition called *minimal degeneracy* [29].† We shall treat the integrals over neighbourhoods of the C_β in a future paper.

In the case when X is a symplectic manifold and $\zeta = \eta \exp i\bar{\omega}$ for any $\eta \in H_K^*(X)$, we have been able to use methods to prove a variant of Theorems 1.1 and 1.2 (see Theorems 4.1, 4.7 and 7.1 below) which bypasses these analytical difficulties and reduces the result to fairly well-known results on Hamiltonian group actions on symplectic manifolds. We assume that 0 is a regular value of μ, or equivalently, that K acts on $\mu^{-1}(0)$ with finite stabilizers.‡ By treating the pushforward $\Pi_*\zeta$ as a function on $\mathbf{k}$, we may use the abelian localization formula [3, 7] for the pushforward in equivariant cohomology of torus actions to find an explicit expression for $\Pi_*\zeta$ as a function on $\mathbf{k}$. Thus, analytical problems relating to integrals over neighbourhoods of C_β are circumvented, and localization reduces to studying the image of the moment map and the pushforward of the symplectic or Liouville measure under the moment map.§ Seen in this light, the nonabelian localization theorem is a consequence of the same results that underlie the residue formula: the abelian localization formula for torus actions [3, 7] and the normal form for ω in a neighbourhood of $\mu^{-1}(0)$.

We now summarize the key steps in our proof. Having replaced integrals over X by integrals over $\mathbf{k}$, we observe that in turn these may be replaced by integrals over the Lie algebra $\mathbf{t}$ of the maximal torus. Then, applying properties of the Fourier transform, we rewrite $\mathscr{I}^\varepsilon$ as the integral over $\mathbf{t}^*$ of a Gaussian $\widetilde{g_{\varepsilon^{-1}}}(y) \sim e^{-|y|^2/(2\varepsilon)}$ multiplied by a function $Q = D_\varpi R$ where R is piecewise polynomial and D_ϖ is a differential operator on $\mathbf{t}^*$:

$$\mathscr{I}^\varepsilon = i^{-s} \int_{y \in \mathbf{t}^*} \widetilde{g_{\varepsilon^{-1}}}(y) Q(y)[dy] \tag{1.4}$$

where s is the dimension of K. The function Q is obtained by combining the abelian localization theorem (Theorem 2.1) with a result (Proposition 3.6) on Fourier transforms of a certain class of functions which arise in the formula for the pushforward.

The function Q is smooth in a neighbourhood of the origin when 0 is a regular value of μ; thus, there is a polynomial $Q_0 = D_\varpi R_0$ which is equal to Q near 0. It turns out that the cohomological expression $e^{\varepsilon\Theta} e^{i\omega_0}[\mathscr{M}_X]$ is obtained as the integral over $\mathbf{t}^*$ of a Gaussian multiplied not by Q but by the polynomial Q_0:

$$e^{\varepsilon\Theta} e^{i\omega_0}[\mathscr{M}_X] = i^{-s} \int_{y \in \mathbf{t}^*} \widetilde{g_{\varepsilon^{-1}}}(y) Q_0(y)[dy]. \tag{1.5}$$

This result follows from a normal form for ω near $\mu^{-1}(0)$.†

To obtain our analogue of Witten's estimate (Theorem 1.2) for the asymptotics of $\mathscr{I}^\varepsilon - e^{\varepsilon\Theta} e^{i\omega_0}[\mathscr{M}_X]$ as $\varepsilon \to 0$, we then write

$$\mathscr{I}^\varepsilon - e^{\varepsilon\Theta} e^{i\omega_0}[\mathscr{M}_X] = i^{-s} \int_{y \in \mathbf{t}^*} \widetilde{g_{\varepsilon^{-1}}}(y) D_\varpi(R - R_0)(y)[dy]. \tag{1.6}$$

Here, $R - R_0$ is piecewise polynomial and supported *away* from 0. By studying the minimum distances from 0 in the support of $R - R_0$, we obtain an estimate (Theorem 4.1)

† However, minimal degeneracy may be sufficient for Witten's argument.

‡ Witten assumes that K acts freely on $\mu^{-1}(0)$.

§ If K is abelian, this pushforward measure is equal (at $\phi \in \mathbf{k}$) to Lebesgue measure multiplied by a function which gives the symplectic volume of the reduced space $\mu^{-1}(\phi)/K$; this function is sometimes called the Duistermaat–Heckman polynomial [15].

† This normal form is a key tool in the original proof [15] of the Duistermaat–Heckman theorem; this theorem motivated the proof by Atiyah and Bott [3] of the abelian localization theorem.

similar to Witten's estimate (Theorem 1.2). In our estimate, the terms in the sum are indexed by the set $\mathscr{B} - \{0\}$; however, our estimate is weaker than Witten's estimate since some of the subsets C_β indexed by $\beta \in \mathscr{B} - \{0\}$ (which *a priori* contribute to our sum†) may be empty in which case $\rho_\beta = |\beta|^2$ may not be a critical value of $|\mu|^2$.

To summarize, the following related quantities appear in this paper:

1. The cohomological quantity $\eta_0 e^{i\omega_0}[\mathscr{M}_X]$.
2. The integral $\mathscr{I}^\varepsilon$ (1.3) coming from the pushforward of an equivariant cohomology class $\eta \in H_K^*(X)$ to H_K^*.
3. Sums of terms of the form

$$\int_F \frac{i_F^* \eta}{e_F}$$

where F is a connected component of the fixed point set of the maximal torus T acting on X. Such sums appear after mapping $\eta \in H_K^*(X)$ into $H_T^*(X)$ and then applying the abelian localization theorem.

Witten's work relates (1) and (2), while our Theorem 8.1 relates (1) and (3).

This paper is organized as follows. Section 2 contains background material on equivariant cohomology and the abelian localization formula. In Section 3 we collect a number of preliminary results which we use in Section 4 to reduce our integral $\mathscr{I}^\varepsilon$ to an integral over $\mathbf{t}^*$ of a piecewise polynomial function multiplied by a Gaussian. Section 4 also contains the statement of two of our main results, Theorems 4.1 and 4.7; Theorem 4.7 is proved in Section 5, and Theorem 4.1 in Section 6. In Section 7, Theorems 4.1 and 4.7 (which are for the case $\zeta = \exp i\bar{\omega}$) are extended to the case $\zeta = \eta \exp i\bar{\omega}$ for $\eta \in H_K^*(X)$: the result is Theorem 7.1. Finally, in Section 8 we prove the residue formula (Theorem 8.1) for the evaluation of cohomology classes from $H_K^*(X)$ on the fundamental class of $\mathscr{M}_X$, and in Section 9 we apply it when $K = SU(2)$ to specific examples. This formula may be related to an unpublished formula due to Donaldson.

In future papers we shall treat the case when $\mathscr{M}_X$ is singular using intersection homology; we shall also apply the nonabelian localization formula to moduli spaces of bundles over Riemann surfaces regarded as finite-dimensional symplectic quotients, in singular as well as nonsingular cases.

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2. EQUIVARIANT COHOMOLOGY AND PUSHFORWARDS

In this section we recall the localization formula for torus actions (Theorem 2.1) and express it in a form convenient for our later use (Lemma 2.2).

Let X be a compact manifold equipped with the action of a compact Lie group K of dimension s with maximal torus T of dimension l. We denote the Lie algebras of K and T by $\mathbf{k}$ and $\mathbf{t}$, respectively, and the Weyl group by W. We assume an invariant inner product $\langle \cdot, \cdot \rangle$ on $\mathbf{k}$ has been chosen (for example, the Killing form): we shall use this to identify $\mathbf{k}$ with its dual. The orthocomplement of $\mathbf{t}$ in $\mathbf{k}$ will be denoted $\mathbf{t}^\perp$.

† In a future paper we hope to prove that the nonzero contributions to our estimate (Theorem 4.1) come only from those $|\beta|^2$ which are nonzero critical values of $|\mu|^2$.

Throughout this paper all cohomology groups are assumed to have coefficients in the field $\mathbb{C}$. The K-equivariant cohomology of a point is $H_K^* = H^*(BK)$, and similarly the T-equivariant cohomology is $H_T^* = H^*(BT)$. We identify H_K^* with $S(\mathbf{k}^*)^K$, the K-invariant polynomial functions on $\mathbf{k}$, and H_T^* with $S(\mathbf{t}^*)$. Hence, we have a bijective map (obtained from the restriction from $\mathbf{k}^*$ to $\mathbf{t}^*$) which identifies H_K^* with the subset of H_T^* fixed by the action of the Weyl group W:

$$H_K^* \cong S(\mathbf{k}^*)^K \cong S(\mathbf{t}^*)^W \subseteq S(\mathbf{t}^*) \cong H_T^*. \tag{2.1}$$

This natural map $H_K^* \to H_T^*$ will be denoted τ or τ_X. We shall use the symbol ϕ to denote a point in $\mathbf{k}$, and ψ to denote a point in $\mathbf{t}$. For $f \in H_K^*$ we shall write $f = f(\phi)$ as a function of ϕ.

The K-equivariant cohomology of X is the cohomology of a certain chain complex (see, for instance, [6, Chapter 7 or 32, Section 5]; the construction is due to Cartan [12]) which can be expressed as

$$\Omega_K^*(X) = (S(\mathbf{k}^*) \otimes \Omega^*(X))^K \tag{2.2}$$

(where $\Omega^*(X)$ denotes differential forms on X). An element in $\Omega_K^*(X)$ may be thought of as a K-equivariant polynomial function from $\mathbf{k}$ to $\Omega^*(X)$. For $\alpha \in \Omega^*(X)$ and $f \in S(\mathbf{k}^*)$, we write $(\alpha \otimes f)(\phi) = f(\phi)\alpha$. In this notation, the differential D on the complex $\Omega_K^*(X)$ is then defined by†

$$D(\alpha \otimes f)(\phi) = f(\phi)(d\alpha - \iota_{\tilde{\phi}}\alpha) = f(\phi)\, d\alpha - \sum_{j=1}^{s} \phi_j f(\phi)\, \iota_{V^j}\alpha. \tag{2.3}$$

Here, $\tilde{\phi}$ is the vector field on X given by the action of $\phi \in \mathbf{k}$, and $\iota_{\tilde{\phi}}$ is the interior product with the vector field $\tilde{\phi}$. We have introduced an orthonormal basis $\{\hat{e}^j : j = 1, \ldots, s\}$ for $\mathbf{k}$, and the $\phi_j \in \mathbf{k}^*$ are simply the coordinate functions $\phi_j = \langle \hat{e}^j, \phi \rangle$, while the V^j are the vector fields on X generated by the action of $\hat{e}^j$. The ϕ_j are assigned degree 2, so that the differential D increases degrees by 1.

One may define the pushforward $\Pi_*^K : H_K^*(X) \to H_K^*$, which corresponds to integration over the fibre of the map $X \times_K EK \to BK$ (see [3, Section 2]). The pushforward satisfies $\Pi_*^T = \tau \circ \Pi_*^K$. Because of this identification, we shall usually simply write Π_* for Π_*^K or Π_*^T. A localization formula for Π_* was given by Berline and Vergne [7]; a more topological proof of this formula is given in [3, Section 3].

THEOREM 2.1 (Berline and Vergne [7]). *If $\sigma \in H_T^*(X)$ and $\psi \in \mathbf{t}$ then*

$$(\Pi_* \sigma)(\psi) = \sum_{F \in \mathscr{F}} \int_F \frac{i_F^* \sigma(\psi)}{e_F(\psi)}.$$

Here we sum over the set $\mathscr{F}$ of components F of the fixed point set of T, and e_F is the T-equivariant Euler class of the normal bundle of F; this Euler class is an element of $H_T^*(F) \cong H^*(F) \otimes H_T^*$, as is $i_F^* \sigma$. The map $i_F : F \to X$ is the inclusion map. The right-hand side of the above expression is to be interpreted as a rational function of ψ.

We shall now prove a lemma about the image of the pushforward, which will be applied in Section 4.

† This definition and the definition (2.8) of the extension $\bar{\omega}(\phi) = \omega + \mu(\phi)$ of the symplectic form ω to an equivariant cohomology class are different from the conventions used by Witten [35]: a factor ϕ appears in the our definitions where $\mathrm{i}\phi$ appears in Witten's definition. In other words Witten's definition is $D(\alpha \otimes f)(\phi) = f(\phi)(d\alpha - \mathrm{i}\iota_{\tilde{\phi}}\alpha)$ and $\bar{\omega}(\phi) = \omega + \mathrm{i}\mu(\phi)$. Witten makes this substitution so that the oscillatory integral $\int_X \exp(\omega + \mathrm{i}\mu(\phi))$ will appear as the integral of an equivariant cohomology class.

LEMMA 2.2. *If $\sigma \in H_T^*(X)$ then $(\Pi_*\sigma)(\psi)$ is a sum of terms*

$$(\Pi_*\sigma)(\psi) = \sum_{F \in \mathcal{F},\, \alpha \in A_F} \tau_{F,\alpha} \tag{2.4}$$

such that each term $\tau_{F,\alpha}$ is of the form

$$\tau_{F,\alpha} = \frac{\int_F c_{F,\alpha}(\psi)}{e_F^{(0)}(\psi)\prod_j \beta_{F,j}(\psi)^{n_{F,j}(\alpha)}} \tag{2.5}$$

for some component F of the fixed point set of the T action. Here, the $\beta_{F,j}$ are the weights of the T action on the normal bundle ν_F, and $e_F^{(0)}(\psi) = \Pi_j \beta_{F,j}(\psi)$ is the product of all the weights, while $n_{F,j}(\alpha)$ are some nonnegative integers. The class $C_{F,\alpha}$ is in $H^*(F) \otimes H_T^*$, and is equal to $i_F^*\sigma \in H^*(F) \otimes H_T^*$ times some characteristic classes of subbundles of ν_F.

Proof. The normal bundle ν_F to F decomposes as a direct sum of weight spaces $\nu_F = \bigoplus_{j=1}^r \nu_F^{(j)}$, on each of which T acts with weight $\beta_{F,j}$. All these weights must be nonzero. By passing to a split manifold if necessary (see [10, Section 21]), we may assume, without loss of generality, that the subbundle on which T acts with a given weight decomposes into a direct sum of T-invariant real subbundles of rank 2. In other words, we may assume that the $\nu_F^{(j)}$ are rank 2 real bundles, and the T action enables one to identify them in a standard way with complex line bundles.

Then the equivariant Euler class $e_F(\psi)$ is given for $\psi \in \mathbf{t}$ by

$$e_F(\psi) = \prod_{j=1}^{r} (c_1(\nu_F^{(j)}) + \beta_{F,j}(\psi)). \tag{2.6}$$

Thus, we have

$$\begin{aligned} \frac{1}{e_F(\psi)} &= \frac{1}{e_F^{(0)}(\psi)} \prod_j \left(1 + \frac{c_1(\nu_F^{(j)})}{\beta_{F,j}(\psi)}\right)^{-1} \\ &= \frac{1}{e_F^{(0)}(\psi)} \prod_j \sum_{r_j \geq 0} (-1)^{r_j} \left(\frac{c_1(\nu_F^{(j)})}{\beta_{F,j}(\psi)}\right)^{r_j}. \end{aligned} \tag{2.7}$$

Here, $c_1(\nu_F^{(j)}) \in H^2(F)$, so that $c_1(\nu_F^{(j)})/\beta_{F,j}(\psi)$ is *nilpotent* and the inverse makes sense in $H^*(F) \otimes \mathbb{C}(\psi_1, \ldots, \psi_l)$, where $\mathbb{C}(\psi_1, \ldots, \psi_l)$ denotes the complex valued-rational functions on $\mathbf{t}$. □

Let us now assume that X is a symplectic manifold and the action of K is Hamiltonian with moment map $\mu: X \to \mathbf{k}^*$. Denote by μ_T the moment map for the action of T given by the composition of μ with the restriction map $\mathbf{k}^* \to \mathbf{t}^*$. We shall be interested in one particular (formal) equivariant cohomology class σ, defined by

$$\sigma(\phi) = \exp i\bar{\omega}(\phi), \quad \bar{\omega}(\phi) = \omega + \mu(\phi). \tag{2.8}$$

For this class the localization formula gives

$$(\Pi_*\sigma)(\psi) = \sum_F r_F(\psi), \quad r_F(\psi) = \int_F \frac{e^{i\mu_T(F)(\psi)} e^{\omega}}{e_F(\psi)}. \tag{2.9}$$

(This formula does not require the fixed point set of T to consist of isolated fixed points.)

Remark: For any $\eta \in H_K^*(X)$ the function $\Pi_*\eta \in H_K^*$ is a polynomial on $\mathbf{k}$, and in particular is smooth. However, $\sigma = e^{i\bar{\omega}}$ does not have polynomial dependence on ϕ. Although it is not immediately obvious from the formula (2.9), the function $\Pi_*(\eta e^{i\bar{\omega}})$ is still

a smooth function on $\mathbf{k}$ (for any $\eta \in H_K^*(X)$ represented by an element $\tilde{\eta} \in \Omega_K^*(X)$): this follows from its description as

$$\Pi_*(\eta e^{i\tilde{\omega}})(\phi) = \int_{x \in X} e^{i\omega} \tilde{\eta}(\phi) e^{i\mu(x)(\phi)}.$$

3. PRELIMINARIES

This section contains results which will be applied in the next section to reduce the integral $\mathscr{I}^\varepsilon$ to an integral over $\mathbf{t}^*$ of a Gaussian multiplied by a piecewise polynomial function. The first, Lemma 3.1, reduces integrals over $\mathbf{k}$ to integrals over $\mathbf{t}$. Lemma 3.2 enables us to replace the L^2 inner product of two functions by the L^2 inner product of their Fourier transforms. Lemma 3.4 relates Fourier transforms on $\mathbf{k}$ to Fourier transforms on $\mathbf{t}$. Finally, Proposition 3.6 describes certain functions whose Fourier transforms are the terms appearing in the localization formula (2.7).

We would like to study a certain integral that arises out of equivariant cohomology:

$$\mathscr{I}^\varepsilon = \frac{1}{(2\pi i)^s \operatorname{vol} K} \int_{\phi \in \mathbf{k}} [d\phi]\, e^{-\varepsilon\langle\phi,\phi\rangle/2} \int_X \sigma(\phi). \tag{3.1}$$

Here, $\sigma \in \Omega_K^*(X)$ (see (2.2)); we are mainly interested in the class σ defined by (2.8). Also, $\varepsilon > 0$ and we shall consider the behaviour of $\mathscr{I}^\varepsilon$ as $\varepsilon \to 0^+$. The measure $[d\phi]$ is a measure on $\mathbf{k}$ which corresponds to a choice of invariant metric on $\mathbf{k}$ (for instance, the metric given by the Killing form): such a metric induces a volume form on K, and vol K is the integral of this volume form over K. Thus, $[d\phi]/\operatorname{vol} K$ is independent of the choice of metric on $\mathbf{k}$. The metric also gives a measure $[d\psi]$ on $\mathbf{t}$ and a volume form on T: it is implicit in our notation that the measures on T and $\mathbf{t}$ come from the same invariant metric as those on K and $\mathbf{k}$.

It will be convenient to recast integrals over $\mathbf{k}$ in terms of integrals over $\mathbf{t}$. For this we use a function $\varpi : \mathbf{t} \to \mathbb{R}$, satisfying $\varpi(w\psi) = (\det w)\varpi(\psi)$ for all elements w of the Weyl group W, and defined by

$$\varpi(\psi) = \prod_{\gamma > 0} \gamma(\psi), \tag{3.2}$$

where γ runs over the positive roots. Using the inner product to identify $\mathbf{t}$ with $\mathbf{t}^*$, ϖ also defines a function $\mathbf{t}^* \to \mathbb{R}$. We have the following lemma.

LEMMA 3.1 (Weyl integration formula) *If $f : \mathbf{k} \to \mathbb{R}$ is K-invariant, then*

$$\int_{\phi \in \mathbf{k}} f(\phi)[d\phi] = C_K^{-1} \int_{\psi \in \mathbf{t}} f(\psi)\varpi(\psi)^2 [d\psi]$$

where s and l are the dimensions of K and T, and $C_K = |W| \operatorname{vol} T / \operatorname{vol} K$.

Proof. There is an orthonormal basis $\{X_\gamma, Y_\gamma | \gamma$ *a* positive root$\}$ for $\mathbf{t}^\perp$ such that

$$[X_\gamma, \psi] = \gamma(\psi) Y_\gamma, \qquad [Y_\gamma, \psi] = -\gamma(\psi) X_\gamma$$

for all $\psi \in \mathbf{t}$. The Riemannian volume form of the coadjoint orbit through $\psi \in \mathbf{t} \cong \mathbf{t}^*$ (with the metric on the orbit pulled back from the metric on $\mathbf{k}^*$ induced by the inner product $\langle\cdot,\cdot\rangle$) evaluated on the tangent vectors $[X_\gamma, \psi]$ and $[Y_\gamma, \psi]$ is thus $\prod_{\gamma>0} \gamma(\psi)^2$, while the volume form of the homogeneous space K/T (induced by the chosen metric on $\mathbf{k}$) evaluated on the

tangent vectors corresponding to X_γ, $Y_\gamma \in \mathbf{k}$ is 1. Hence, the Riemannian volume of the orbit through $\psi \in \mathbf{t}$ is $\varpi(\psi)^2$ times the volume of the homogeneous space K/T.

□

It will be convenient also to work with the Fourier transform. Given $f: \mathbf{k} \to \mathbb{R}$ we define $F_K f: \mathbf{k}^* \to \mathbb{R}$, $F_T f: \mathbf{t}^* \to \mathbb{R}$ by

$$(F_K f)(z) = \frac{1}{(2\pi)^{s/2}} \int_{\phi \in \mathbf{k}} f(\phi) e^{-iz(\phi)} [d\phi] \tag{3.3}$$

$$(F_T f)(y) = \frac{1}{(2\pi)^{l/2}} \int_{\psi \in \mathbf{t}} f(\psi) e^{-iy(\psi)} [d\psi]. \tag{3.4}$$

More invariantly, the Fourier transform is defined on a vector space V of dimension n with dual space V^* as a map $F: \Omega^{\max}(V) \to \Omega^{\max}(V^*)$ where $\Omega^{\max}(V) = \Lambda^{\max}(V^*) \otimes \mathscr{D}'(V)$, $\Lambda^{\max}(V^*)$ is the top exterior power V^* and $\mathscr{D}'(V)$ are the tempered distributions on V (see [25]). Indeed for $z \in V^*$, $f \in \mathscr{D}'(V)$ and $u \in \Lambda^{\max}(V^*)$ we define

$$(F(u \otimes f))(z) = \frac{v}{(2\pi)^{n/2}} \int_{\phi \in V} f(\phi) e^{-iz(\phi)} u \tag{3.5}$$

where the element $v \in \Lambda^{\max}(V)$ satisfies $u(v) = 1$ under the natural pairing $\Lambda^{\max}(V^*) \cong (\Lambda^{\max}(V))^*$. (The normalization has been chosen so that $\tilde{f}(\phi) = F(F\tilde{f})(-\phi)$ for any $\tilde{f} \in \Omega^{\max}(V)$.) For notational convenience we shall often ignore this subtlety and identify $\mathbf{k}$, $\mathbf{t}$ with $\mathbf{k}^*$, $\mathbf{t}^*$ under the invariant inner product $\langle \cdot, \cdot \rangle$: we shall also suppress the exterior powers of $\mathbf{k}$ and $\mathbf{t}$ and write $F_K: \mathscr{D}'(\mathbf{k}) \to \mathscr{D}'(\mathbf{k}^*)$, $F_T: \mathscr{D}'(\mathbf{t}) \to \mathscr{D}'(\mathbf{t}^*)$. Further, although we shall work with functions whose definition depends on the choice of the element $[d\phi] \in \Lambda^{\max}(\mathbf{k}^*)$ (associated to the inner product), some of our end results do not depend on this choice† and the use of such functions is just a notational convenience.

A fundamental property of the Fourier transform is that it preserves the L^2 inner product.

LEMMA 3.2. (Parseval's theorem [25, Section 7.1]). *If $f: \mathbf{k} \to \mathbb{C}$ is a tempered distribution and $g: \mathbf{k} \to \mathbb{C}$ is a Schwartz function then $F_K f: \mathbf{k} \to \mathbb{C}$ is also a tempered distribution and $F_K g: \mathbf{k} \to \mathbb{C}$ a Schwartz function, and we have*

$$\int_{\phi \in \mathbf{k}} \overline{g(\phi)} f(\phi) [d\phi] = \int_{x \in \mathbf{k}^*} \overline{(F_K g)(z)} (F_K f)(z) [dz]$$

$$\int_{\phi \in \mathbf{t}} \overline{g(\psi)} f(\psi) [d\psi] = \int_{y \in \mathbf{t}^*} \overline{(F_T g)(y)} (F_T f)(y) [dy]$$

We note also that if $g_\varepsilon: \mathbf{k} \to \mathbb{R}$ is the Gaussian defined by $g_\varepsilon(\phi) = e^{-\varepsilon\langle \phi, \phi \rangle/2}$, then

$$(F_K g_\varepsilon)(z) = \frac{1}{\varepsilon^{s/2}} e^{-\langle z, z \rangle/2\varepsilon} = \frac{1}{\varepsilon^{s/2}} g_{\varepsilon^{-1}}(z), \qquad (F_T g_\varepsilon)(y) = \frac{1}{\varepsilon^{l/2}} e^{-\langle y, y \rangle/2\varepsilon} = \frac{1}{\varepsilon^{l/2}} g_{\varepsilon^{-1}}(y). \tag{3.6}$$

We also have the following lemma.

LEMMA 3.3. *The symplectic volume form $d\Omega_\phi^S$ at a point ϕ in the orbit $K \cdot \psi$ through $\psi \in \mathbf{k}$ is related to the Riemannian volume form $d\Omega_\phi^R$ (induced by the metric on $\mathbf{k}$) by*

$$d\Omega_\phi^R = \varpi(\psi)\, d\Omega_\phi^S.$$

† The statement of Theorem 8.1, for instance, does not depend on the invariant inner product $\langle \cdot, \cdot \rangle$.

Proof. The symplectic form is K-invariant and is given at the point $\psi \in \mathbf{t}_+$ in the orbit $K \cdot \psi$ (for $\xi, \eta \in \mathbf{t}^\perp$ giving rise to tangent vectors $[\xi, \psi]$, $[\eta, \psi]$ to the orbit) by

$$\omega([\xi, \psi], [\eta, \psi]) = \langle [\xi, \psi], \eta \rangle = \langle \psi, [\eta, \xi] \rangle. \tag{3.7}$$

In the notation of Lemma 3.1, the symplectic volume form evaluated on the tangent vectors $[X_\gamma, \psi]$, $[Y_\gamma, \psi]$ is given by $\prod_{\gamma > 0} \omega([X_\gamma, \psi], [Y_\gamma, \psi])$. But $\omega([X_\gamma, \psi], [Y_\gamma, \psi])$, $= \langle [X_\gamma, \psi], Y_\gamma \rangle = \gamma(\psi)$, from which (comparing with the proof of Lemma 3.1) the Lemma follows. □

We shall use this Lemma to prove the following Lemma relating Fourier transforms on $\mathbf{k}$ to those on $\mathbf{t}$.

LEMMA 3.4. *Let $f \in \mathscr{D}'(\mathbf{k})$ be K-invariant, and let ϖ be defined by* (3.2). *Then*

$$F_T(\varpi f) = \varpi F_K(f)$$

as distributions on $\mathbf{t}$.

Proof.

$$\begin{aligned}(F_K f)(z) &= \frac{1}{(2\pi)^{s/2}} \int_{\phi \in \mathbf{k}} e^{-iz(\phi)} f(\phi)[d\phi] \\ &= \frac{1}{(2\pi)^{s/2}} \int_{\psi \in \mathbf{t}_+} [d\psi] f(\psi) \int_{\phi \in K.\psi} e^{-iz(\phi)} d\Omega_\phi^R \end{aligned} \tag{3.8}$$

(where $\mathbf{t}_+$ denotes the fundamental Weyl chamber).

We have from Lemma 3.3 that $d\Omega_\phi^R = \varpi(\psi)\, d\Omega_\phi^S$. Thus, we have

$$(F_K f)(z) = (2\pi)^{-s/2} \int_{\psi \in \mathbf{t}_+} [d\psi] f(\psi) \varpi(\psi) \int_{\phi \in K \cdot \psi} e^{-iz(\phi)}\, d\Omega_\phi^S. \tag{3.9}$$

Now the integral over the coadjoint orbit may be computed by the Duistermaat–Heckman theorem [15] applied to the action of T on the orbit $K \cdot \psi$ (or equivalently as a consequence of the abelian localization theorem, Theorem 2.1): we have (see e.g. [6 Theorem 7.24])

$$\int_{\phi \in K \cdot \psi} e^{-iz(\phi)}\, d\Omega_\phi^S = \frac{(2\pi)^{(s-l)/2}}{\varpi(z)} \sum_{w \in W} e^{-iz(w\psi)} (\det w). \tag{3.10}$$

This is a well-known formula due originally to Harish-Chandra ([23, Lemma 15]). (Notice that the z in (3.10) plays the role of the ψ in Theorem 2.1, while the ψ in (3.10) specifies the orbit.) Now since $\varpi(w\psi) = (\det w)\varpi(\psi)$, we may replace the integral over $\mathbf{t}_+$ by an integral over $\mathbf{t}$; in other words, we have

$$\varpi(z)(F_K f)(z) = (2\pi)^{-l/2} \int_{\psi \in \mathbf{t}} [d\psi] f(\psi) \varpi(\psi) e^{-iz(\psi)} = F_T(\varpi f)(z). \quad \square \tag{3.11}$$

Applying this to the Gaussian $g_\varepsilon(\psi) = e^{-\varepsilon\langle \psi, \psi \rangle/2}$ we have the following corollary.

COROLLARY 3.5.

$$F_T(g_\varepsilon \varpi)(y) = \frac{1}{\varepsilon^{s/2}} \varpi(y) e^{-\langle y, y \rangle/2\varepsilon} = \frac{1}{\varepsilon^{s/2}} \varpi(y) g_{\varepsilon^{-1}}(y).$$

We shall also need the following result which occurs in the work of Guillemin, Lerman, Prato and Sternberg concerning Fourier transforms of a class of functions on $\mathbf{t}^*$. This result will be applied to functions appearing in the abelian localization formula (2.7).

PROPOSITION 3.6. (a) (*See* [18], [19, *Section* 3.2], *and* [20]) *Define* $h(y) = H_{\bar\beta}(y+\tau)$ *for some* $\tau \in \mathbf{t}$, *where* $H_{\bar\beta}(y) = \mathrm{vol}\{(s_1, \dots, s_N): s_i \geq 0,\ y = \Sigma_j s_j \beta_j\}$ *for some* N*-tuple* $\bar\beta = \{\beta_1, \dots, \beta_N\}$, $\beta_j \in \mathbf{t}^*$, *such that* β_j *all lie in the interior of some half-space of* $\mathbf{t}^*$. *Thus,* $H_{\bar\beta}$ *is a piecewise polynomial function supported on the cone* $C_{\bar\beta} = \{\Sigma_j s_j \beta_j : s_j \geq 0\}$. *Then the Fourier transform of* h *is given for* ψ *in the complement of the hyperplanes* $\{\psi \in \mathbf{t}: \beta_j(\psi) = 0\}$ *by the formula*

$$F_T h(\psi) = \frac{e^{i\tau(\psi)}}{i^N \prod_{j=1}^N \beta_j(\psi)}. \tag{3.12}$$

(b) (*See* [18, *Section* 2] *and* [20, (2.15)]) *The function* $H_{\bar\beta}(y)$ *is also given as*

$$H_{\bar\beta}(y) = H_{\beta_1} * H_{\beta_2} * \cdots * H_{\beta_\tau},$$

where for $\beta \in Hom(\mathbb{R}^l, \mathbb{R})$ *we have* $H_\beta = (i_\beta)_* \, dt$, *i.e.* H_β *is the pushforward of the Euclidean measure* dt *on* $\mathbb{R}^+$ *under the map* $i_\beta: \mathbb{R}^+ \to \mathbb{R}^l$ *given by* $i_\beta(t) = \beta t$. *Here,* $*$ *denotes convolution.*

(c) (*See* [18, *Section* 2]) *The function* $H_{\bar\beta}$ *satisfies the differential equation*

$$\prod_{j=1}^N \beta_j(\partial/\partial y) H_{\bar\beta}(y) = \delta_0(y)$$

where δ_0 *is the Dirac delta distribution.*

(d) (*See* [18, *Proposition* 2.6]) *The function* $H_{\bar\beta}$ *is smooth at any* $y \in U_{\bar\beta}$, *where* $U_{\bar\beta}$ *are the points in* $\mathbf{t}^*$ *which are not in any cone spanned by a subset of* $\{\beta_1, \dots, \beta_N\}$ *containing fewer than* l *elements.*

4. REDUCTION TO A PIECEWISE POLYNOMIAL FUNCTION

In this section we apply the results stated in Section 3 to reduce $\mathscr{I}^\varepsilon$ to the form given in Proposition 4.4, as the integral over $y \in \mathbf{t}^*$ of a Gaussian $\widetilde{g_{\varepsilon^{-1}}}(y) = g_{\varepsilon^{-1}}(y)/((2\pi)^s |W| \mathrm{vol}(T) \varepsilon^{s/2})$ times a function $Q(y)$ which is $D_\varpi R(y)$, where D_ϖ is a differential operator and R is a piecewise polynomial function. As a by-product we obtain also a generalization of Theorem 2.16 of [20]: we may relax the hypothesis of [20] that the action of the maximal torus T have isolated fixed points.

Two of our main results related to Witten's work [35] are Theorems 4.1 and 4.7: these are stated in this section. The proof of Theorem 4.7 will be given in Section 5. It tells us that the cohomological contribution $e^{\varepsilon\Theta} e^{i\omega_0}[\mathscr{M}_X]$ given in Witten's theorem 1.2 for the zero locus of the moment map is given by the integral over $y \in \mathbf{t}^*$ of $\widetilde{g_{\varepsilon^{-1}}}(y)$ times a polynomial $Q_0(y)$ which is equal to Q near $y = 0$. Theorem 4.1 is our version of the asymptotic estimates given in Witten's theorem 1.1, and will be proved in Section 6 below. Theorem 7.1 in Section 7 extends Theorems 4.1 and 4.7 to more general equivariant cohomology classes.

We assume throughout the rest of the paper that K acts on X in a Hamiltonian fashion, and that 0 is a regular value of the moment map μ for the K action. This is equivalent to the assumption that K acts on $\mu^{-1}(0)$ with finite stabilizers, and it implies that $\mu^{-1}(0)$ is a smooth manifold. Under these hypotheses, the space $\mathscr{M}_X = \mu^{-1}(0)/K$ is a V-manifold or orbifold (see [28]) and $P = \mu^{-1}(0) \to \mu^{-1}(0)/K$ is a V-bundle: we have a class $\Theta \in H^4(\mathscr{M}_X)$ which represents the class $-\langle \phi, \phi \rangle/2 \in H_K^4(\mu^{-1}(0)) \cong H^4(\mu^{-1}(0)/K)$, and which is a four-dimensional characteristic class of the bundle $P \to \mathscr{M}_X$.

In [29] it is proved that the set of critical points of the function $\rho = |\mu|^2 : X \to \mathbb{R}$ is a disjoint union of closed subsets C_β in X indexed by a finite subset $\mathscr{B}$ of $\mathbf{t}$. In fact, if $\mathbf{t}_+$ is a fixed positive Weyl chamber for K in $\mathbf{t}$ then $\beta \in \mathscr{B}$ if and only if $\beta \in \mathbf{t}_+$ and β is the closest point to 0 of the convex hull in $\mathbf{t}$ of some nonempty subset of the finite set $\{\mu_T(F) : F \in \mathscr{F}\}$, i.e. the image under μ_T of the set of fixed points of T in X. Moreover, if $\beta \in \mathscr{B}$ then

$$C_\beta = K(Z_\beta \cap \mu^{-1}(\beta))$$

where Z_β is the union of those connected components of the set of critical points of the function μ_β defined by $\mu_\beta(x) = \mu(x)(\beta)$ on which μ_β takes the value $|\beta|^2$. Note that $C_0 = \mu^{-1}(0)$ and in general the value taken by the function $\rho = |\mu|^2$ on the critical set C_β is just $|\beta|^2$.

We shall prove the following version of Witten's nonabelian localization theorem, for the integral $\mathscr{I}^\varepsilon$ defined in (3.1) with the class $\sigma = e^{i\varpi}$ defined by (2.8).

THEOREM 4.1. *For each $\beta \in \mathscr{B}$, let $\rho_\beta = |\beta|^2$ (this is the critical value of the function $\rho = |\mu|^2 : X \to \mathbb{R}$ on the critical set C_β when this set is nonempty). Then there exist functions $h_\beta : \mathbb{R}^+ \to \mathbb{R}$ such that for some $N_\beta \geq 0$, $\varepsilon^{N_\beta} h_\beta(\varepsilon)$ remains bounded as $\varepsilon \to 0^+$, and for which*

$$|\mathscr{I}^\varepsilon - e^{\varepsilon\Theta} e^{\omega}[\mathscr{M}_X]| \leq \sum_{\beta \in \mathscr{B} - \{0\}} e^{-\rho_\beta/2\varepsilon} h_\beta(\varepsilon).$$

Remark. The estimate given in Theorem 4.1 is *weaker* than Witten's estimate (Theorem 1.2) since $|\beta|^2$ is not in fact a critical value of $|\mu|^2$ when C_β is empty. Nevertheless, in many interesting cases all the C_β are nonempty, and our estimate then coincides with Witten's.

To prove this result, we shall rewrite $\mathscr{I}^\varepsilon$ using Lemma 3.1:

$$\mathscr{I}^\varepsilon = \frac{1}{(2\pi i)^s |W| \operatorname{vol}(T)} \int_{\psi \in \mathfrak{t}} [d\psi]\, (g_\varepsilon(\psi)\varpi(\psi))(\varpi(\psi)(\Pi_* \sigma)(\psi)). \tag{4.1}$$

Now we apply Lemma 3.2 to get

$$\mathscr{I}^\varepsilon = \frac{1}{(2\pi i)^s |W| \operatorname{vol}(T)} \int_{y \in \mathfrak{t}^*} [dy]\, (F_T(g_\varepsilon \varpi)(y))(F_T(\varpi \Pi_* \sigma)(y)). \tag{4.2}$$

Applying Corollary 3.5 we have

$$\mathscr{I}^\varepsilon = \frac{1}{(2\pi i)^s |W| \operatorname{vol}(T) \varepsilon^{s/2}} \int_{\psi \in \mathfrak{t}^*} [dy]\, \varpi(y)\, e^{-\langle y, y\rangle/2\varepsilon} (F_T(\varpi \Pi_* \sigma)(y)). \tag{4.3}$$

Following Guillemin et al. [18], we may use the abelian localization formula (Theorem 2.1) to give a formula for $F_T(\Pi_* \sigma)$, where $\sigma = e^{i\varpi}$ and from it obtain a formula for $F_T(\varpi \Pi_* \sigma)$. In terms of the notation of Lemma 2.2, we choose a component Λ of the set $\cap_{F,j}\{\psi \in \mathbf{t} : \beta_{F,j}(\psi) \neq 0\}$, where $\beta_{F,j}$ are the weights of the action of T on the normal bundle to a component F of the fixed point set. Thus, Λ is a cone in $\mathbf{t}$. If we denote by $C_F = C_{F,\Lambda}$ the component of $\cap_j\{\psi \in \mathbf{t} : \beta_{F,j}(\psi) \neq 0\}$ containing Λ, then $\Lambda = \cap_F C_{F,\Lambda}$. Also, $\beta_{F,j} \in \mathbf{t}^*$ lies in the dual cone $\check{C}_{F,\Lambda}$ of $C_{F,\Lambda}$; indeed, this dual cone is simply the cone $\check{C}_{F,\Lambda} = \{\sum_j s_j \beta_{F,j} : s_j \geq 0\}$. We then define $\sigma_{F,j} = \operatorname{sign} \beta_{F,j}(\xi)$ for any $\xi \in \Lambda$, and $\hat{\beta}_{F,j} = \sigma_{F,j} \beta_{F,j}$. Then we set $k_F(\alpha) = \sum_{j, \sigma_{F,j} = -1} n_{F,j}(\alpha)$.

We define a function $H : \mathbf{t}^* \to \mathbb{C}$ by

$$H(y) = \sum_{F \in \mathscr{F}} \sum_{\alpha \in \mathscr{A}_F} (-1)^{k_F(\alpha)} H_{\overline{\gamma_F(\alpha)}}(-y + \mu_T(F)) \int_F (e^{i\omega} \tilde{c}_{F,\alpha}). \tag{4.4}$$

Here as before $\mathscr{F}$ is the set of components F of the fixed point set of T and the $\mathscr{A}_F$ are the indexing sets which appeared in Lemma 2.2. If $\alpha \in \mathscr{A}_F$ then $\bar{\gamma}_F(\alpha)$ consists of the elements $\beta^{\Lambda}_{F,j}$ where each $\beta^{\Lambda}_{F,j}$ appears with multiplicity $n_{F,j}(\alpha)$. Then $H_{\gamma F(\alpha)}$ is as defined in Proposition 3.6. The $\tilde{c}_{F,\alpha} \in H^*(F)$ are related to the $c_{F,\alpha}$ in (2.5), in that

$$c_{F,\alpha}(\psi) = e^{i\omega} e^{i\mu_T(F)(\psi)} \tilde{c}_{F,\alpha}. \tag{4.5}$$

We then have the following theorem, which in the case when the action of T has isolated fixed points is the main theorem of Section 3 of [18]. For the most part, our proof is a direct extension of the proof given in that paper; the major difference is in the use of the abelian localization theorem rather than stationary phase.

THEOREM 4.2. *The (piecewise polynomial) function H given in* (4.4) *is identical to the distribution $G = F_T(\Pi_* e^{i\bar{\omega}})$.*

Proof. We first apply the abelian localization formula (2.9) to $\Pi_*\sigma$. Then the formula obtained from Lemma 2.2 for $\Pi_*\sigma$ is

$$\Pi_*\sigma = \sum_{F \in \mathscr{F}} r_F, \quad r_F = \sum_{\alpha \in \mathscr{A}_F} \tau_{F,\alpha} \tag{4.6}$$

$$\tau_{F,\alpha} = (-1)^{k_F(\alpha)} \frac{e^{i\mu_T(F)(\psi)} \int_F (e^{i\omega} \tilde{c}_{F,\alpha})}{\Pi_j (\beta^{\Lambda}_{F,j}(\psi))^{n_{F,j}(\alpha)}} \tag{4.7}$$

where μ_T, the moment map for the T action, is simply the projection of μ onto $\mathbf{t}$ and the $\beta_{F,j}$ are the weights of the T action. Recall that the quantity $\beta^{\Lambda}_{F,j}$ is $\beta_{F,j}$ if $\beta_{F,j}(\xi) > 0$ and $-\beta_{F,j}$ if $\beta_{F,j}(\xi) < 0$. The class $\tilde{c}_{F,\alpha} \in H^*(F)$ is equal to some characteristic classes of subbundles of the normal bundle ν_F; it is independent of ψ.

Notice that each $\beta^{\Lambda}_{F,j} \in \mathbf{t}^*$ lies in the half-space $\{y \in \mathbf{t}^* : y(\xi) > 0\}$, for any $\xi \in \Lambda$. Equation (4.7) is hence of the form appearing on the right-hand side of (3.12), up to multiplication by a factor independent of ψ. The conclusion of the proof goes, as in Section 3 of [18], by applying a lemma about distributions (see [20, Appendix A]):

LEMMA 4.3. *Suppose G and H are two tempered distributions on $\mathbb{R}^l$ such that*

1. *$F_T G - F_T H$ is supported on a finite union of hyperplanes.*
2. *There is a half-space $\{y : \langle y, \zeta \rangle > k_0\}$ containing the support of $H - G$.*

Then $G = H$.

Here we apply the lemma to H as given in (4.4) and $G = F_T(\Pi_* e^{i\bar{\omega}})$. The first hypothesis is satisfied because we know from Proposition 3.6 that $F_T H$ is given by the formula (4.6) on the complement of the hyperplanes $\{\psi \mid \beta^{\Lambda}_{F,j}(\psi) = 0\}$; but this is just the formula for $F_T G = \Pi_* e^{i\bar{\omega}}$. Further, H is supported in a half-space since all the weights $\beta^{\Lambda}_{F,j}$ satisfy $\beta^{\Lambda}_{F,j}(\xi) > 0$ for any $\xi \in \Lambda$, while the support of G is contained in the compact set $\mu_T(X)$ (see Section 5 below). Therefore, the support of $H - G$ is contained in a half-space of the form $\{y : \langle y, \xi \rangle > k_0\}$ for some k_0. This completes the proof of Theorem 4.2. □

Define

$$R(y) = F_T(\Pi_*\sigma)(y) \tag{4.8}$$

where $\sigma = \exp i\bar{\omega}$. Then (4.3) and (4.4) give us the following proposition.

PROPOSITION 4.4. *The function R is a piecewise polynomial function supported on cones each of which has apex at $\mu_T(F)$ for some component F of the fixed point set of T. Let Q be the distribution defined by*

$$Q(y) = \varpi(y) D_\varpi \pi(y)$$

where the differential operator D_ϖ is given by

$$D_\varpi = \prod_{\gamma > 0} (\mathrm{i}\gamma(\partial/\partial y))$$

and γ runs over the positive roots (cf. (3.2)). Then

$$\mathscr{I}^\varepsilon = \frac{1}{(2\pi\mathrm{i})^s |W| \operatorname{vol}(T) \varepsilon^{s/2}} \int_{y \in \mathfrak{t}^*} [dy]\, \mathrm{e}^{-\langle y, y\rangle/2\varepsilon}\, Q(y). \tag{4.9}$$

Proof. It only remains to note that $F_T(\varpi \Pi_* \sigma) = D_\varpi F_T(\Pi_* \sigma)$, where D_ϖ is defined above. □

Remark. The formulas for $Q(y)$ obtained from (4.6) will in general be different for different choices of Λ.

In addition, certain formulas simplify if we impose the additional assumption that at any point x in a component F of the fixed point set of the T action, the orthocomplement $\mathbf{t}^\perp$ of $\mathbf{t}$ in $\mathbf{k}$ injects into the tangent space $T_x X$ under the infinitesimal action of K; in other words, that the stabilizer $\operatorname{Stab}(x)$ of x is such that $\operatorname{Stab}(x)/T$ is a finite group. Under this additional hypothesis, we may indeed prove a somewhat stronger result. Notice that by Lemma 2.2, each term (corresponding to a component F in the fixed point set of T) in the localization formula for $\Pi_* \sigma$ has a factor $e_F^{(0)}(\psi)$ in the denominator. Now for $x \in F$, the fibre $(\nu_F)_x$ over x of the normal bundle to F will contain $\mathbf{t}^\perp \cdot x$, the image of $\mathbf{t}^\perp$ under the infinitesimal action of K. Under the additional assumption that $\mathbf{t}^\perp$ injects into $T_x X$, the set of weights for ν_F contains for each root $\gamma > 0$ either the root γ or the root $-\gamma$; in other words, $e_F^{(0)}(\psi)$ is divisible by $\varpi(\psi)$. Thus, from Lemma 2.2 we obtain the formula for $\varpi(\psi)\Pi_* \sigma$:

$$\varpi(\psi)\Pi_* \sigma = \sum_{F \in \mathscr{F},\, \alpha \in \mathscr{A}_F} \tilde{\tau}_{F,\alpha}, \tag{4.10}$$

$$\tilde{\tau}_{F,\alpha} = (-1)^{k_F(\alpha)} \frac{\mathrm{e}^{\mathrm{i}\mu_T(F)(\psi)}}{\Pi_j (\beta_{F,j}^\wedge(\psi))^{\tilde{n}_{F,j}(\alpha)}} \int_F (\mathrm{e}^{\mathrm{i}\omega} \tilde{c}_{F,\alpha}). \tag{4.11}$$

Here, the notation is as in (4.6) and (4.7) except that $\tilde{n}_{F,j}(\alpha) = n_{F,j}(\alpha)$ if $\beta_{F,j}$ is not a root, while $\tilde{n}_{F,j}(\alpha) = n_{F,j}(\alpha) - 1$ if $\beta_{F,j}$ is a root.

We may then use the abelian localization formula to give a formula for $F_T(\varpi \Pi_* \sigma)$ where $\sigma = \mathrm{e}^{\mathrm{i}\varpi}$. As in (4.4), we define a function $H^\varpi(y)$ for $y \in \mathbf{t}^*$ by

$$H^\varpi(y) = \sum_{F \in \mathscr{F}} \sum_{\alpha \in \mathscr{A}_F} (-1)^{k_F(\alpha)} H\hat{\gamma}_F(\alpha)(-y + \mu_T(F)) \int_F \mathrm{e}^{\omega} \tilde{c}_{F,\alpha}. \tag{4.12}$$

The notation is as in (4.4) except that each $\beta_{F,j}^\wedge$ appears in $\hat{\gamma}_F(\alpha)$ with multiplicity $n_{F,j}(\alpha)$ if it is not a root and $n_{F,j}(\alpha) - 1$ if it is a root. The function $H\hat{\gamma}_F(\alpha)$ is then as defined in Proposition 3.6.

In the case when the action of T has isolated fixed points, Theorem 4.5 below is the main result (Theorem 2.16) of Part I of Guillemin and Prato [20]. For the most part, our proof translates directly from the proof given in that paper, except that we use the abelian localization theorem in place of stationary phase.

THEOREM 4.5. *Suppose that* $\mathfrak{t}^{\perp}$ *injects into* T_xX *for all fixed points* x *of the action of* T. *Then the distribution* H^{ϖ} *given in* (4.12) *is identical to the distribution* $G = F_T(\varpi\Pi_* e^{i\tilde{\omega}})$.

Proof. This theorem is proved in exactly the same way as Theorem 4.2. The conclusion of the proof goes, as in the proof of Theorem 2.16 of [20], by applying Lemma 4.3 directly to H^{ϖ} as given in (4.12) and $G = F_T(\varpi\Pi_* e^{i\tilde{\omega}})$. □

In particular we have the following proposition.

PROPOSITION 4.6. *Suppose that* $\mathfrak{t}^{\perp}$ *injects into* T_xX *for all fixed points* x *of the action of* T. *Define*

$$Q(y) = \varpi(y)F_T(\varpi\Pi_*\sigma)(y) \tag{4.13}$$

so that

$$\mathscr{I}^{\varepsilon} = \frac{1}{(2\pi i)^s|W|\operatorname{vol}(T)\varepsilon^{s/2}}\int_{y\in\mathfrak{t}^*}[dy]\,e^{-\langle y,y\rangle/2\varepsilon}Q(y). \tag{4.14}$$

Then Q *is a piecewise polynomial function supported on cones each of which has apex at* $\mu_T(F)$ *for some component* F *of the fixed point set of* T.

We now drop the hypothesis that $\mathfrak{t}^{\perp}$ injects into T_xX at the fixed points x of the action of T, and return to the general situation described in Proposition 4.4. It will follow from (5.6) that Q is smooth near $y = 0$; thus, in particular there is a polynomial Q_0 which is equal to Q near $y = 0$. Of course, $Q_0 = D_{\varpi}R_0$ where R_0 is the polynomial which is equal to $F_T(\Pi_*\sigma)$ near $y = 0$. In the next section we shall provide an alternative description of Q_0 and prove the following theorem.

THEOREM 4.7. *Assume* K *acts effectively on* $\mu^{-1}(0)$. *Then we have*

$$\mathscr{I}_0^{\varepsilon} \overset{\text{def}}{=} \frac{1}{(2\pi i)^s|W|\operatorname{vol}(T)\varepsilon^{s/2}}\int_{y\in\mathfrak{t}^*}[dy]\,e^{-(y,y)/2\varepsilon}Q_0(y) = e^{\varepsilon\Theta}e^{i\omega_0}[\mathscr{M}_X].$$

This tells us that the contribution to $\mathscr{I}^{\varepsilon}$ from $\mu^{-1}(0)/K$ is obtained by integrating Q_0 rather than Q (weighted by the Gaussian $\widetilde{g_{\varepsilon^{-1}}}$ defined in the first paragraph of this section) over $y\in\mathfrak{t}^*$.

5. THE PROOF OF THEOREM 4.7

This section gives the proof of Theorem 4.7, which identifies $e^{\varepsilon\Theta}e^{i\omega_0}[\mathscr{M}_X]$ with the integral of a Gaussian $\widetilde{g_{\varepsilon^{-1}}}$ times a polynomial Q_0. The key step in the proof is the well-known result Proposition 5.2 below, which gives a normal form for the symplectic form, the K action and the moment map in a neighbourhood $\mathcal{O}$ of $\mu^{-1}(0)$. We first recast the distribution Q in terms of an integral over X (Proposition 5.1), so that $Q(y)$ is given by the integral over X of a distribution supported where μ takes the value y. (This step occurs also in the proof [15] of the Duistermaat–Heckman theorem.) Hence, for sufficiently small y, this distribution is supported in $\mathcal{O}$ and we may do the integral over X to obtain the value of the polynomial Q_0 which is equal to Q near 0. This turns out to be given by an integral over $\mathscr{M}_X$ involving the symplectic form and the curvature of a bundle over $\mathscr{M}_X$ (see (5.6)). Finally, we multiply Q_0 by the Gaussian $\widetilde{g_{\varepsilon^{-1}}}$ and integrate over $\mathfrak{t}$ to see that the result is $e^{\varepsilon\Theta}e^{i\omega_0}[\mathscr{M}_X]$.

PROPOSITION 5.1. *$Q(y) = \varpi^2(y)(2\pi)^{s/2}\int_{x\in X} e^{i\omega}\delta(y-\mu(x))$, where δ denotes the (Dirac) delta distribution.*

Proof. We have by Lemma 3.4 that

$$Q = \varpi F_T(\varpi\Pi_*\sigma) = \varpi^2 F_K(\Pi_*\sigma)$$

so that

$$Q(y) = \frac{\varpi^2(y)}{(2\pi)^{s/2}}\int_{\phi\in\mathbf{k}}[d\phi]\,e^{-iy(\phi)}\int_{x\in X} e^{i\omega}\,e^{i\mu(x)(\phi)}$$

$$= \varpi^2(y)(2\pi)^{s/2}\int_X e^{i\omega}\delta(\mu - y). \qquad \square \quad (5.1)$$

We would like to study this for $|y| < h$ for sufficiently small $h > 0$. Now there is a neighbourhood of $\mu^{-1}(0)$ on which the symplectic form is given in a standard way related to the symplectic form ω_0 on $\mathcal{M}_X$; this follows from the coisotropic embedding theorem (see [22, Sections 39–41]).

PROPOSITION 5.2 (Gotay [16], Guillemin and Sternberg [22] and Marle [31]). *Assume 0 is a regular value of μ (so that $\mu^{-1}(0)$ is a smooth manifold and K acts on $\mu^{-1}(0)$ with finite stabilizers). Then there is a neighbourhood $\mathcal{O} \cong \mu^{-1}(0)\times\{z\in\mathbf{k}^*, |z|\le h\} \subseteq \mu^{-1}(0)\times\mathbf{k}^*$ of $\mu^{-1}(0)$ on which the symplectic form is given as follows. Let $P \overset{\text{def}}{=} \mu^{-1}(0) \xrightarrow{q} \mathcal{M}_X$ be the orbifold principal K-bundle given by the projection map $q:\mu^{-1}(0)\to\mu^{-1}(0)/K$,* and let *$\theta\in\Omega^1(P)\otimes\mathbf{k}$ be a connection for it. Let ω_0 denote the induced symplectic form on $\mathcal{M}_X$; in other words, $q^*\omega_0 = i_0^*\omega$. Then if we define a 1-form τ on $\mathcal{O}\subset P\times\mathbf{k}^*$ by $\tau_{p,z} = z(\theta)$ (for $p\in P$ and $z\in\mathbf{k}^*$), the symplectic form on $\mathcal{O}$ is given by*

$$\omega = q^*\omega_0 + d\tau. \qquad (5.2)$$

Further, the moment map on $\mathcal{O}$ is given by $\mu(p, z) = z$.

Proof of Theorem 4.7. We assume, for simplicity of notation, that K acts freely on $\mu^{-1}(0)$, but all of the following may be transferred to the case when K acts with finite stabilizers by introducing V-manifolds or orbifolds (see [28]). In other words, we work locally on finite covers of subsets of $\mu^{-1}(0)$ and $\mu^{-1}(0)/K$, where the covering group is the stabilizer of the K action at a point $x\in\mu^{-1}(0)$.

When $|y| < h$ and h is sufficiently small, the distribution $\delta(\mu(x) - y)$ is supported in $\mathcal{O}$, so we may compute $Q_0(y)$ from (5.1) by restricting to $\mathcal{O}$. We have

$$Q_0(y) = (2\pi)^{s/2}\varpi^2(y)\int_{(p,z')\in P\times\mathbf{k}^*} e^{i\omega}\delta(y - z') \qquad (5.3)$$

$$= (2\pi)^{s/2}\varpi^2(y)\int_{(p,z')\in P\times\mathbf{k}^*}\exp i(q^*\omega_0 + z'(d\theta))\exp i\,dz'(\theta)\delta(y - z'). \qquad (5.4)$$

Now the term in $\exp i\,dz'(\theta)$ which contributes to the integral (5.4) is $i^s\Omega[dz']$ where $[dz']$ is the volume form on $\mathbf{k}$ (since all factors $dz'_1 \dots dz'_l$ must appear in order to get a contribution to the integral). Here, $\Omega = \Pi_{j=1}^s\,\theta^j$ (for j indexing an orthonormal basis of $\mathbf{k}$ and θ^j the corresponding components of the connection θ) is a form integrating to $\mathrm{vol}(K)$ over each fibre of $P\to\mathcal{M}_X$.

Doing the integral over $z' \in \mathbf{k}^*$, we get

$$Q_0(y) = \mathrm{i}^s \varpi^2(y)(2\pi)^{s/2} \int_P \exp \mathrm{i}(q^*\omega_0 + y(d\theta + [\theta, \theta]/2))\Omega \tag{5.5}$$

$$= \mathrm{i}^s \varpi^2(y)(2\pi)^{s/2} \int_{\mu^{-1}(0)} \exp \mathrm{i}(q^*\omega_0 + y(F_\theta))\Omega. \tag{5.6}$$

Here, $F_\theta = d\theta + \frac{1}{2}[\theta, \theta]$ is the curvature associated with the connection θ; we may introduce the term $[\theta, \theta]$ into the exponential in (5.5) since the additional factors θ will give zero under the wedge product with Ω. Formula (5.6) shows that Q_0 is a polynomial in y.

Now we were interested in

$$\mathscr{I}_0^\varepsilon = \frac{1}{(2\pi\mathrm{i})^s |W| \operatorname{vol}(T)\, \varepsilon^{s/2}} \int_{y \in \mathfrak{t}^*} [dy] \mathrm{e}^{-\langle y, y\rangle/2\varepsilon} Q_0(y)$$

$$= \frac{1}{(2\pi)^{s/2} |W| \operatorname{vol}(T)\, \varepsilon^{s/2}} \int_{y \in \mathfrak{t}^*} [dy] \varpi^2(y)\, \mathrm{e}^{-\langle y, y\rangle/2\varepsilon} \int_P \exp(q^*\omega_0 + y(F_\theta))\Omega \tag{5.7}$$

$$= \frac{1}{(2\pi\varepsilon)^{s/2} \operatorname{vol}(K)} \int_{z \in \mathbf{k}^*} [dz]\, \mathrm{e}^{-\langle z, z\rangle/2\varepsilon} \int_P \exp(q^*\omega_0 + z(F_\theta))\Omega \tag{5.8}$$

where the last step uses Lemma 3.1 and the fact that $\int_P \exp(q^*\omega_0 + z(F_\theta))\Omega$ is an invariant function of z.

We now regard F_θ as a formal parameter and complete the square to do the integral over z; we have (identifying $z(F_\theta)$ with $\langle F_\theta, z\rangle$ using the invariant inner product $\langle \cdot, \cdot \rangle$)

$$\int_{z \in \mathbf{k}^*} [dz]\, \mathrm{e}^{-\langle z, z\rangle/2\varepsilon} \exp\langle F_\theta, z\rangle = (2\pi\varepsilon)^{s/2} \exp \varepsilon\langle F_\theta, F_\theta\rangle/2. \tag{5.9}$$

But $\langle F_\theta, F_\theta\rangle/2$ is just the class $\pi^*\Theta$ on P, for $\Theta \in H^4(\mathscr{M}_X)$. Hence, we obtain (integrating over the fibre of $P \to \mathscr{M}_X$ and using the fact that the integral of Ω over the fibre is $\operatorname{vol}(K)$)

$$\mathscr{I}_0^\varepsilon = \int_{\mathscr{M}_X} \exp \mathrm{i}\omega_0 \exp \varepsilon\Theta, \tag{5.10}$$

completing the proof of Theorem 4.7. □

6. THE PROOF OF THEOREM 4.1

In this section we complete the proof of Theorem 4.1. This is done by observing that $\mathscr{I}^\varepsilon - \mathrm{e}^{\varepsilon\Theta}\mathrm{e}^{\mathrm{i}\omega_0}[\mathscr{M}_X]$ is of the form $\mathrm{i}^{-s} \int_{\mathfrak{t}} \widetilde{g_{\varepsilon^{-1}}}(Q - Q_0) = \int_{\mathfrak{t}} \widetilde{g_{\varepsilon^{-1}}} D_\varpi(R - R_0) = \int_{\mathfrak{t}} (D_\varpi^* \widetilde{g_{\varepsilon^{-1}}})(R - R_0)$, where $R - R_0$ is piecewise polynomial and supported away from 0. (Here, $D_\varpi^* = (-1)^{(s-l)/2} D_\varpi$.) The results of [29] establish that the distance of any point of $\operatorname{Supp}(Q - Q_0)$ from 0 is at least $|\beta|$ for some nonzero β in the indexing set $\mathscr{B}$ defined in Section 4. Hence, we obtain the estimates in Theorem 4.1.

In fact, the function $R - R_0$ is known explicitly in terms of the values of $\mu_T(F)$ (where F are the components of the fixed point set), the integrals over F of characteristic classes of subbundles of the normal bundle ν_F, and the weights of the action of T on ν_F (see (2.7) and Proposition 3.6). The function $R - R_0$ is polynomial on polyhedral regions of $\mathfrak{t}$, so that the quantity $\mathscr{I}^\varepsilon - \mathrm{e}^{\varepsilon\Theta}\mathrm{e}^{\mathrm{i}\omega_0}[\mathscr{M}_X]$ can, in principle, be computed from the integral of a polynomial times a Gaussian over these polyhedral regions. We shall study these integrals in another paper and relate them to the cohomology of the higher strata in the stratification of X according to the gradient flow of $|\mu|^2$ given in [29].

We now examine $\mathscr{I}^{\varepsilon} - \mathscr{I}_0^{\varepsilon}$ and prove Theorem 4.1. Recall from Section 4 that the indexing set $\mathscr{B}$ of the critical sets C_β for the function $\rho = |\mu|^2$ is $\mathscr{B} = \mathbf{t}_+ \cap W\mathscr{B}$ where $W\mathscr{B} = \{w\mathscr{B} : \mathscr{B} \in \mathscr{B},\ w \in W\}$ is the set of closest points to 0 of convex hulls of nonempty subsets of the set $\{\mu_T(F) : F \in \mathscr{F}\}$ of images under μ_T of the connected components of the fixed point set of T in X. We shall refer to $\{\mu_T(F) : F \in \mathscr{F}\}$ as the set of *weights*† associated to X equipped with the action of T.

Let $\mathscr{J}$ denote the locus

$$\mathscr{J} = \{y \in \mathbf{t}^* : Q \text{ is not smooth at } y\}.$$

Then we have the following proposition.

PROPOSITION 6.1. *$\mathscr{J} \subset \mathscr{J}^{ab}$, where $\mathscr{J}^{ab} = \{y \in \mathbf{t}^* : F_T(\Pi_* e^{i\varpi})$ is not smooth at $y\}$.*

Proof. $Q = \varpi F_T(\varpi \Pi_* e^{i\varpi}) = \varpi D_\varpi F_T(\Pi_* e^{i\varpi})$. Hence, if $F_T(\Pi_* e^{i\varpi})$ is smooth at y then so is Q. □

Now it follows from [18, Section 5] that $\mathscr{J}^{ab} = \cup_{\gamma \in \Gamma} \mu_T(V_\gamma)$ where V_γ is a component of the fixed point set of a one parameter subgroup T_γ of T and Γ indexes all such one parameter subgroups and components of their fixed point sets. Let

$$D = \cap \{D_\beta : \beta \in W\mathscr{B} - \{0\}\}$$

where D_β denotes the open half-space

$$D_\beta = \{y \in \mathbf{t}^* : y(\beta) < |\beta|^2\}.$$

Note that if $\beta \in \mathscr{B} - \{0\}$ then D_β contains 0 and its boundary is the hyperplane

$$H_\beta = \{y \in \mathbf{t}^* : y(\beta) = |\beta|^2\}.$$

LEMMA 6.2. *The support of $Q - Q_0$ is contained in the complement of D (or equivalently $Q = Q_0$ on D).*

Proof. Suppose V_γ is a component of the fixed point set of a one parameter subgroup T_γ. By the Atiyah–Guillemin–Sternberg convexity theorem [1, 21], $\mu_T(V_\gamma)$ is the convex hull of some subset of the weights. Hence, the closest point to 0 in $\mu_T(V_\gamma)$ is in $W\mathscr{B}$. Now either this closest point is 0, or else the closest point is a point $\beta \in W\mathscr{B} - \{0\}$ (in which case $\mu_T(V_\gamma) \subset \mathbf{t} - D_\beta \subset \mathbf{t} - D$).

Now if x is a point in Supp $(Q - Q_0)$ then the ray from 0 to x must pass through at least one point in $\mathscr{J}$; hence, it suffices to prove $\mathscr{J} \subset \mathbf{t} - D$ since D is the intersection of a number of open half-spaces all of which contain 0.

But $\mathscr{J} \subset \cup_\gamma \mu_T(V_\gamma)$ and every point in $\mu_T(V_\gamma)$ is in $\mathbf{t} - D$ unless 0 is in $\mu_T(V_\gamma)$. Moreover, if $x \in \mu_T(V_\gamma) \cap \mathscr{J}$ and $0 \in \mu_T(V_\gamma)$, we may consider the function $Q - Q_0$ restricted to a small neighbourhood of the ray from 0 through x. This ray lies in the hyperplane $\tilde{H}_\gamma$ which is the orthocomplement in $\mathbf{t}$ to the Lie algebra $\mathbf{t}_\gamma$ of T_γ. (Since the component of μ_T in the direction of $\mathbf{t}_\gamma$ is constant along V_γ and since $0 \in \mu_T(V_\gamma)$, it follows that $\mu_T(V_\gamma)$ is contained in $\tilde{H}_\gamma$.) Since $Q - Q_0$ is identically zero near 0 but not near x, the ray from 0 to x must contain a point x' in $\mathscr{J}$ which is contained in some $\mu_T(V_{\gamma'})$ with $t_\gamma \neq t_{\gamma'}$. If $0 \notin \mu_T(V_{\gamma'})$ then $\mu_T(V_{\gamma'}) \subset \mathbf{t} - D$ and so $x \in \mathbf{t} - D$. If $0 \in \mu_T(V_{\gamma'})$ then we simply repeat the argument, considering the restriction of

† The motivation for this is that if X is a nonsingular subvariety of complex projective space $\mathbb{P}_n$ and T acts on X via a linear action on $\mathbb{C}^{n+1}$, then each $\mu_T(F)$ is a weight of this action when appropriate identifications are made.

$Q - Q_0$ to a neighbourhood of $\tilde{H}_\gamma \cap \tilde{H}_{\gamma'}$. Since $0 \notin \mathscr{I}$, after finitely many repetitions of this argument we get the required result. Hence, the lemma is proved. □

To complete the proof of Theorem 4.7, we then use Lemma 6.2 to express $\mathscr{I}^\varepsilon - \mathscr{I}_0^\varepsilon$ as

$$\mathscr{I}^\varepsilon - \mathscr{I}_0^\varepsilon = \frac{1}{(2\pi i)^s |W| \operatorname{vol} T \varepsilon^{s/2}} \int_{\mathfrak{t}^* - D} [dy] (Q - Q_0) \, e^{-|y|^2/2\varepsilon}. \tag{6.1}$$

Denote by C the set $\{y \in \mathfrak{t}^* - D : |Q(y) - Q_0(y)| \leq 1\}$. Then

$$(2\pi)^s |W| \operatorname{vol}(T) \varepsilon^{s/2} |\mathscr{I}^\varepsilon - \mathscr{I}_0^\varepsilon| \leq \int_C [dy] \, e^{-|y|^2/2\varepsilon} + \int_{\mathfrak{t}^* - D} [dy] |Q - Q_0|^2 \, e^{-|y|^2/2\varepsilon}.$$

If b is the minimum value of $|\beta|$ over all $\beta \in \mathscr{B} - \{0\}$, then

$$\int_C [dy] \, e^{-|y|^2/2\varepsilon} \leq \int_{|y| \geq b} [dy] e^{-|y|^2/2\varepsilon} \leq e^{-b^2/2\varepsilon} q(\varepsilon)$$

where $q(\varepsilon)$ is a polynomial in $\varepsilon^{1/2}$.

Further, denote the p the function $|Q - Q_0|^2$. Then

$$\int_{\mathfrak{t}^* - D} [dy] p(y) e^{-|y|^2/2\varepsilon} \leq \sum_{\beta \in W\mathscr{B} - \{0\}} \int_{y \in D_\beta} [dy] \, e^{-|y|^2/2\varepsilon} \, p(y). \tag{6.2}$$

For each $\beta \in W\mathscr{B} - \{0\}$ one can now decompose $y \in \mathfrak{t}^*$ into $y = w_0 \hat{\beta} + w$, $w \in \beta^\perp$, $w_0 \in \mathbb{R}$ (where $\hat{\beta} = \beta/|\beta|$). Hence, each of the integrals (6.2) is of the form

$$\int_{w_0 \geq |\beta|} \int_{w \in \beta^\perp} e^{-w_0^2/2\varepsilon} \, e^{-|w|^2/2\varepsilon} \, p(w_0, w)$$

and this is clearly bounded by $e^{-|\beta|^2/2\varepsilon}$ times a polynomial in ε. This completes the proof of Theorem 4.1.

7. EXTENSION OF THEOREMS 4.1 AND 4.7 TO OTHER CLASSES

In this section we extend Theorems 4.1 and 4.7 to equivariant cohomology classes of the form $\zeta = \eta e^{i\varpi}$ where $\eta \in H_K^*(X)$. More precisely, we shall show the following theorem.

THEOREM 7.1. *Suppose $\eta \in H_K^*(X)$ and suppose that $i_0^* \eta \in H_K^*(\mu^{-1}(0))$ is represented by $\eta_0 \in H^*(\mu^{-1}(0)/K)$ (where $i_0 : \mu^{-1}(0) \to X$ is the inclusion map). Then we have the following.*

(a) *We have that*

$$\eta_0 \, e^{\varepsilon\Theta} \, e^{i\omega_0} [\mathscr{M}_X] = \frac{1}{(2\pi i)^s |W| \operatorname{vol}(T) \varepsilon^{s/2}} \int_{y \in \mathfrak{t}^*} [dy] \, e^{-|y|^2/2\varepsilon} \, Q_0^\eta(y)$$

where $Q^\eta(y) = \varpi(y) F_T(\varpi \Pi_*(\eta \exp i\bar{\omega}))$ and $Q_0^\eta(y)$ is a polynomial which is equal to $Q^\eta(y)$ near $y = 0$.

(b) *Let $\rho_\beta = |\beta|^2$ be the value of the function $|\mu|^2 : X \to \mathbb{R}$ on the critical set C_β. Then there exist functions $h_\beta : \mathbb{R}^+ \to \mathbb{R}$ such that for some $N_\beta \geq 0$, $\varepsilon^{N_\beta} h_\beta(\varepsilon)$ remains bounded as $\varepsilon \to 0^+$, and for which*

$$\left| \frac{1}{(2\pi i)^s \operatorname{vol}(K)} \int_{\phi \in \mathfrak{k}} [d\phi] \, e^{-\varepsilon|\phi|^2/2} \, \Pi_* \eta \, e^{i\bar{\varpi}} - \eta_0 \, e^{\varepsilon\Theta} \, e^{i\omega_0} [\mathscr{M}_X] \right| \leq \sum_{\beta \in \mathscr{B} - \{0\}} e^{-\rho_\beta/2\varepsilon} \, h_\beta(\varepsilon).$$

(c) *Suppose η is represented by $\tilde{\eta} = \sum_J \eta_J \phi^J \in \Omega_K^*(X)$ for $\eta_J \in \Omega^*(X)$. Then Q^η is of the form $Q^\eta(y) = \sum_J D_J R_J(y)$, where D_J are differential operators on $\mathbf{t}^*$ and R_J are piecewise polynomial functions on $\mathbf{t}^*$.*

Proof. (a) We examine the function $Q^\eta(y)$ near $y = 0$. We have from Lemma 3.4 and the paragraph before Theorem 2.1 that

$$\varpi F_T(\varpi \Pi_*(\eta e^{i\bar{\omega}}))(y) = (\varpi^2 F_K \Pi_*(\eta\, e^{i\bar{\omega}}))(y)$$
$$= \frac{\varpi^2(y)}{(2\pi)^{s/2}} \int_{\phi \in \mathbf{k}} [d\phi] \int_{x \in X} e^{-iy(\phi)} e^{i\omega} e^{i\mu(x)(\phi)} \eta(\phi).$$

Now since $\eta(\phi) = \sum_I \phi^I \eta_I$ for $\eta_I \in \Omega^*(X)$ (where the I are multi-indices), we may define for any $x \in X$ a distribution S_x with values in $\Lambda^* T_x^* X$ as follows: for any $y \in \mathbf{t}^*$ we have

$$S_x(y) = \int_{\phi \in \mathbf{k}} [d\phi]\, e^{-iy(\phi)} e^{i\omega} e^{i\mu(x)(\phi)} \sum_I \eta_I \phi^I$$
$$= \sum_I (i\partial/\partial y)^I \int_{\phi \in \mathbf{k}} [d\phi]\, e^{i(\mu(x)-y)(\phi)} e^{i\omega} \eta_I$$
$$= (2\pi)^s \sum_I (i\partial/\partial y)^I \delta(\mu(x) - y)\, e^{i\omega} \eta_I. \tag{7.1}$$

Thus, the distribution $S_x(y)$, viewed as a distribution $S(x, y)$ on $X \times \mathbf{t}^*$, is supported on $\{(x, y) \in X \times \mathbf{t}^* : \mu(x) = y\}$. Hence, for sufficiently small y, $S(x, y)$ (viewed now as a function of x) is supported on $\mathcal{O}$ (in the notation of Section 5) and we find that

$$Q^\eta(y) = \frac{\varpi^2(y)}{(2\pi)^{s/2}} \int_{x \in X} S(x, y) \tag{7.2}$$
$$= \frac{\varpi^2(y)}{(2\pi)^{s/2}} \int_{\phi \in \mathbf{k}} [d\phi] \int_{P \times \mathbf{k}^*} e^{i(\mu(x)-y)(\phi)} e^{i\omega} \eta(\phi).$$

Now consider the restriction of η to $H_K^*(P \times \mathbf{k}^*) \cong H_K^*(P)$ (where $P = \mu^{-1}(0)$ as in Section 5). Recall that there exists $\eta_0 \in \Omega^*(P/K)$ such that $\eta - \pi^* q^* \eta_0 = D\gamma$ for some γ, where D is the equivariant cohomology differential on $P \times \mathbf{k}^*$ and $\pi : P \times \mathbf{k}^* \to P \times \{0\}$ and $q : P \to \to P/K$ are the projection maps.† We then have that

$$Q^\eta(y) - \frac{\varpi^2(y)}{(2\pi)^{s/2}} \int_{\phi \in \mathbf{k}} [d\phi] \int_{x \in P \times \mathbf{k}^*} e^{i(\mu(x)-y)(\phi)} e^{i\omega} \pi^* q^* \eta_0$$
$$= \frac{\varpi^2(y)}{(2\pi)^{s/2}} \int_{\phi \in \mathbf{k}} [d\phi] \int_{x \in P \times \mathbf{k}^*} e^{i(\mu(x)-y)(\phi)} e^{i\omega} D\gamma \overset{\text{def}}{=} \frac{\varpi^2(y)}{(2\pi)^{s/2}} \Delta \tag{7.3}$$

But also

$$e^{i(\mu(x)-y)(\phi)} e^{i\omega} D\gamma = D(e^{i(\mu(x)-y)(\phi)} e^{i\omega} \gamma)$$

(since $D\phi_j = 0$ and $D\bar{\omega} = 0$). Hence $\Delta = \int_{\phi \in \mathbf{k}} [d\phi] \int_{P \times \mathbf{k}^*} d(e^{i(\mu(x)-y)(\phi)} e^{i\omega} \gamma)$ (since for any differential form Ψ, the term $\iota_{\tilde{\phi}} \Psi$ in $D\Psi$ cannot contain differential forms of top degree in x). Using Stokes' theorem and replacing $P \times \mathbf{k}^*$ by $P \times B(\mathbf{k}^*)$ where $B(\mathbf{k}^*)$ is a large ball in $\mathbf{k}^*$

† This is because the map $i \circ \pi : P \times \mathbf{k}^* \to P \times \mathbf{k}^*$ is homotopic to the identity by a homotopy through equivariant maps, where $i : P \times \{0\} \to P \times \mathbf{k}^*$ is the inclusion map. Hence, i induces an isomorphism

$$i^* : H_K^*(P \times \mathbf{k}^*) \to H_K^*(P \times \{0\}).$$

with boundary $S(\mathbf{k}^*)$, we have that

$$\Delta = \int_{\phi\in\mathbf{k}} [d\phi] \int_{P\times S(\mathbf{k}^*)} e^{i(\mu(x)-y)(\phi)} e^{i\omega} \sum_I \gamma_I \phi^I$$

for $\gamma_I \in \Omega^*(P\times\mathbf{k}^*)$. Thus, we have $\Delta = \sum_I (i\partial/\partial y)^I S_I(y)$ where

$$S_I(y) = \int_{\phi\in\mathbf{k}} [d\phi] \int_{x\in P\times S(\mathbf{k}^*)} e^{i(\mu(x)-y)(\phi)} e^{i\omega}\gamma_I.$$

Now we do the integral over ϕ to obtain

$$S_I(y) = (2\pi)^s \int_{x\in P\times S(\mathbf{k}^*)} \delta(\mu(x)-y) e^{i\omega}\gamma_I.$$

This is zero since the delta distribution is supported off $S(\mathbf{k}^*)$ (recall that $\mu(p,z)=z$ for $(p,z)\in P\times\mathbf{k}^*$). Hence, we have that $\Delta = 0$, and so

$$Q^\eta(y) = \frac{\varpi^2(y)}{(2\pi)^{s/2}} \int_{\phi\in\mathbf{k}} [d\phi] \int_{(p,z)\in P\times\mathbf{k}^*} e^{i(z-y)(\phi)} e^{i\omega}\pi^*\eta_0 \tag{7.4}$$

and the rest of the proof is exactly the same as the proof of Theorem 4.7 which was for the case $\eta_0 = 1$. In particular, the analogue of (5.6) is

$$Q_0^\eta(y) = i^s\varpi^2(y)(2\pi)^{s/2} \int_{\mu^{-1}(0)} \eta_0 \exp(q^*\omega_0 + y(F_\theta))\Omega; \tag{7.5}$$

this equation shows that Q_0^η is a polynomial (and in particular smooth) in y.

(b) This is a direct extension of the proof of Theorem 4.1, with Q and Q_0 replaced by Q^η and Q_0^η.

(c) Since $\eta = \sum_J \eta_J \phi^J$, the abelian localization formula for $\Pi_*(\eta e^{i\bar\omega})$ yields

$$\begin{aligned}\varpi(\phi)\Pi_*(\eta e^{i\bar\omega})(\phi) &= \varpi(\phi)\sum_{F\in\mathscr{F}} r_F^\eta(\phi), \quad r_F^\eta(\phi) = \int_F \frac{i_F^*\eta(\phi)e^{i\bar\omega(\phi)}}{e_F(\phi)} \\ &= \sum_J \phi^J e^{i\mu_T(F)(\phi)} \int_F \frac{i_F^*\eta_J e^{i\bar\omega}}{e_F(\phi)} \\ &= \varpi(\phi)\sum_J \phi^J \sum_{F\in\mathscr{F},\,\alpha\in\mathscr{A}_F} \tau_{F,\alpha,J}^\eta(\phi)\end{aligned} \tag{7.6}$$

where

$$\tau_{F,\alpha,J}^\eta(\phi) = (-1)^{k_F(\alpha)} \int_F i_F^*\eta_J e^{i\omega}\tilde c_{F,\alpha} \frac{e^{i\mu_T(F)(\phi)}}{\Pi_j(\beta_{F,j}^\wedge(\phi))^{n_{F,j}(\alpha)}} \tag{7.7}$$

(and the $\tilde c_{F,\alpha}$ are as in (4.5)). Here, as in Section 4, we may form a distribution

$$H(y) = D_\varpi \tilde H, \quad \tilde H = \sum_J (i\partial/\partial y)^J \sum_{F\in\mathscr{F},\,\alpha\in\mathscr{A}_F} (-1)^{k_F(\alpha)} \int_F i_F^*\eta_J e^{i\omega}\tilde c_{F,\alpha})\, H_{\tilde\gamma_F(\alpha)}(\mu_T(F)-y) \tag{7.8}$$

where the piecewise polynomial function $H_{\bar\beta}$ is as in Proposition 3.6. Thus, $\tilde H$ is of the form $\tilde H(y) = \sum_J D_J R_J(y)$ where the $R_J(y)$ are piecewise polynomial and the D_J are differential operators.

We also define the distribution

$$G(y) = F_T(\varpi\Pi_*\eta e^{i\bar\omega}). \tag{7.9}$$

Thus, $\varpi(y)G(y) = Q^\eta(y)$.

Then we may show that the distributions G and H are identical. For we apply Lemma 4.3 as before. We find by Proposition 3.6(a) that $(F_T G)(\psi)$ and $(F_T H)(\psi)$ are identical off the hyperplanes $\beta_{F,j}(\psi) = 0$, so the first hypothesis of Lemma 4.3 is satisfied. Moreover, H is supported in a half-space and

$$G(y) = \varpi(y) F_K(\Pi_* \sum_J \eta_J \phi^J e^{i\bar{\omega}})(y)$$

$$= (2\pi)^s \varpi(y) \sum_J (i\partial/\partial y)^J \int_{x \in X} \eta_J \delta(\mu(x) - y) e^{i\omega} \tag{7.10}$$

so G is supported in the compact set $\mu(X) \cap \mathbf{t}^*$ and hence $H - G$ is supported in a half-space. Thus, the second hypothesis of Lemma 4.3 is also satisfied and we may conclude that $G = H$, which completes the proof. □

8. RELATIONS IN THE COHOMOLOGY RING OF SYMPLECTIC QUOTIENTS

In this section we shall prove a formula for the evaluation of cohomology classes from $H_K^*(X)$ on the fundamental class of $\mathscr{M}_X$, and apply it to study the cohomology ring $H^*(\mathscr{M}_X)$ in two examples.

Henceforth we assume K acts effectively on $\mu^{-1}(0)$. More generally the right hand side of (8.1) should be multiplied by n_0 where n_0 is the order of the subgroup of K that acts trivially on $\mu^{-1}(0)$.

We shall prove the following theorem.

THEOREM 8.1. *Let $\eta \in H_K^*(X)$ induce $\eta_0 \in H^*(\mathscr{M}_X)$ and assume K acts effectively on $\mu^{-1}(0)$. Then we have*

$$\eta_0 e^{i\omega_0}[\mathscr{M}_X] = \frac{(-1)^{n_+}}{(2\pi)^{s-l}|W|\mathrm{vol}(T)} \mathrm{Res}(\varpi^2(\psi) \sum_{F \in \mathscr{F}} r_F^\eta(\psi)[d\psi]) \tag{8.1}$$

where

$$r_F^\eta(\psi) = e^{i\mu_T(F)(\psi)} \int_F \frac{i_F^*(\eta(\psi) e^{i\omega})}{e_F(\psi)}.$$

Here s and l are the dimensions of K and its maximal torus T, and $\varpi(\psi) = \Pi_{\gamma > 0}\gamma(\psi)$ where γ runs over the positive roots of K; the number of positive roots $(s - l)/2$ is denoted by n_+. If $F \in \mathscr{F}$ (where $\mathscr{F}$ denotes the set of components of the T fixed point set) then $i_F : F \to X$ is the inclusion of F in X and e_F is the equivariant Euler class of the normal bundle to F in X. The quantity $\mathrm{Res}(\Omega)$ will be defined below (Definition 8.5). The definition of $\mathrm{Res}(\Omega)$ will depend on the choice of a cone Λ, a test function χ, and a ray in $\mathbf{t}^*$ specified by a parameter $\rho \in \mathbf{t}^*$—but in fact if the form Ω is sufficiently well behaved (as is the case in (8.1)) then the quantity $\mathrm{Res}(\Omega)$ will turn out to be independent of these choices (see Propositions 8.6, 8.7 and 8.9).

In the case of $K = SU(2)$ the result of Theorem 8.1 is as follows.

COROLLARY 8.2. *Let $K = SU(2)$, let $\eta \in H_K^*(X)$ induce $\eta_0 \in H^*(\mathscr{M}_X)$ and assume K acts effectively on $\mu^{-1}(0)$. Then the cohomology class $\eta_0 e^{i\omega_0}$ evaluated on the fundamental class of $\mathscr{M}_X$ is given by the following formula:*

$$\eta_0 e^{i\omega_0}[\mathscr{M}_X] = -2\mathrm{Res}_0(\psi^2 \sum_{F \in \mathscr{F}_+} r_F^\eta(\psi)), \quad \text{where } r_F^\eta(\psi) = e^{i\mu_T(F)(\psi)} \int_F \frac{i_F^* \eta(\psi) e^{i\omega}}{e_F(\psi)}.$$

Here, Res_0 denotes the coefficient of $1/\psi$, and $\mathscr{F}_+$ is the subset of the fixed point set of $T = U(1)$ consisting of those components F of the T fixed point set for which $\mu_T(F) > 0$. We have identified $\psi \in \mathbb{R}$ with $\mathrm{diag}(2\pi i\psi, -2\pi i\psi) \in \mathbf{SU(2)}$

An important special case (cf. Witten [35, Section 2.4]) is as follows.

COROLLARY 8.3. *Let $\eta \in H_K^*(X)$ induce $\eta_0 \in H^*(\mathscr{M}_X)$, and let $\Theta \in H^4(\mathscr{M}_X)$ be induced by the polynomial function $-\langle\phi, \phi\rangle/2$ of ϕ, regarded as an element of H_K^4. Then if $\varepsilon > 0$ we have*

$$\eta_0 e^{\varepsilon\Theta} e^{i\omega_0}[\mathscr{M}_X] = \frac{(-1)^{n_+}}{(2\pi)^{s-l}|W|\mathrm{vol}(T)} \sum_{m\geq 0} \frac{1}{m!} \mathrm{Res}\left((-\varepsilon|\psi|^2/2)^m \varpi^2(\psi) \sum_{F\in\mathscr{F}} r_F^\eta(\psi)[d\psi] \right) \tag{8.2}$$

where

$$r_F^\eta(\psi) = e^{i\mu_T(F)(\psi)} \int_F \frac{i_F^*(\eta(\psi)e^{i\omega})}{e_F(\psi)}.$$

We now give a general definition (Definition 8.5) of $\mathrm{Res}\, h[d\psi]$ when h is a meromorphic function on an open subset of $\mathbf{t}\otimes\mathbb{C}$ satisfying certain growth conditions at infinity.† For the restricted class of meromorphic forms of the form $e^{i\lambda(\psi)}[d\psi]/\Pi_j\beta_j(\psi)$, Definition 8.5 implies the existence of an explicit procedure for computing these residues by successive contour integrations, which is outlined in Proposition 8.11.

Our definition of the residue will be based on the following well-known result ([25], Theorem 7.4.2 and Remark following Theorem 7.4.3):

PROPOSITION 8.4 (i) *Suppose u is a distribution on $\mathbf{t}^*$. Then the set $\Gamma_u = \{\xi \in \mathbf{t} : e^{(\cdot,\xi)}u$ is a tempered distribution$\}$ is convex. (Here, $(\cdot,\cdot)$ denotes the pairing between $\mathbf{t}$ and $\mathbf{t}^*$.)*

(ii) *If the interior Γ_u^0 of Γ_u is nonempty, then there is an analytic function $\hat{u}$ in $\mathbf{t} + i\Gamma_u^0$ such that the Fourier transform of $e^{(\cdot,\xi)}u$ is $\hat{u}(\cdot + i\xi)$ for all $\xi \in \Gamma_u^0$.*

(iii) *For every compact subset M of Γ_u^0 there is an estimate*

$$|\hat{u}(\zeta)| \leq C(1 + |\zeta|)^N, \quad \mathrm{Im}(\zeta) \in M. \tag{8.3}$$

(iv) *Conversely if Γ is an open convex set in $\mathbf{t}$ and h is a holomorphic function on $\mathbf{t} + i\Gamma$ with bounds of the form (8.3) for every compact $M \subset \Gamma$, then is a distribution u on $\mathbf{t}^*$ such that $e^{(\cdot,\xi)}u$ is a tempered distribution and has Fourier transform $h(\cdot + i\xi)$ for all $\xi \in \Gamma$.*

(v) *Finally, if u itself is tempered then the Fourier transform $\hat{u}$ is the limit (in the space S' of tempered distributions) of the distribution $\psi \to \hat{u}(\psi + it\theta)$ as $t \to 0^+$, for any $\theta \in \Gamma_u^0$.*

Definition 8.5. Let Λ be a (proper) cone in $\mathbf{t}$. Let h be a holomorphic function on $\mathbf{t} - i\mathrm{Int}(\Lambda) \subseteq \mathbf{t}\otimes\mathbb{C}$ such that for any compact subset M of $\mathbf{t} - i\mathrm{Int}(\Lambda)$ there is an integer $N \geq 0$ and a constant C such that $|h(\zeta)| \leq C(1 + |\zeta|)^N$ for all $\zeta \in M$. Let $\chi : \mathbf{k}^* \to \mathbb{R}$ be a smooth invariant function with compact support and strictly positive in some neighbourhood of 0, and let $\hat{\chi} = F_K\chi : \mathbf{k} \to \mathbb{C}$ be its Fourier transform. Let $\hat{\chi}_\varepsilon(\phi) = \hat{\chi}(\varepsilon\phi)$ so that $(F_K\chi_\varepsilon)(z) = \chi_\varepsilon(z) = \varepsilon^{-s}\chi(z/\varepsilon)$. Assume $\hat{\chi}(0) = 1$. Then we define

$$\mathrm{Res}^{\Lambda,\chi}(h[d\psi]) = \lim_{\varepsilon\to 0^+} \frac{1}{(2\pi i)^l} \int_{\psi\in\mathbf{t}-i\xi} \hat{\chi}(\varepsilon\psi)h(\psi)[d\psi] \tag{8.4}$$

where ξ is any element of Λ.

† Our definition is somewhat related to the definition of the residue of a meromosphic l-form at a point in $\mathbb{C}^l$ given for example in [17, Chapter 5.1] and [24, Chapter 9], though it is not the same.

By the Paley–Wiener theorem ([25, Theorem 7.3.1]), for any fixed $\xi \in \mathbf{k}$ the function $\hat{\chi}(\psi - \mathrm{i}\xi) = F_K(e^{-\langle \xi, \cdot \rangle}\chi)(\psi)$ is rapidly decreasing since $F_K\hat{\chi} = \chi$ is smooth and compactly supported. Hence, the integral (8.4) converges. Now the function $\hat{\chi}$ extends to a holomorphic function on $\mathbf{k}\otimes\mathbb{C}$ and in particular on $\mathbf{t}\otimes\mathbb{C}$ (Proposition 8.4), and by assumption h extends to a holomorphic function on $\mathbf{t} - \mathrm{i}\mathrm{Int}(\Lambda)$; hence, by Cauchy's theorem, the integral (8.4) is independent of $\xi \in \mathrm{Int}\,(\Lambda)$.

The independence of $\mathrm{Res}^{\Lambda,\chi}(\Omega)$ of the choices of Λ and χ when Ω is sufficiently well behaved is established by the next results.

PROPOSITION 8.6. *Let $h:\mathbf{k} \to \mathbb{C}$ be a K-invariant function. Assume that $F_K h$ is compactly supported; it then follows that $h:\mathbf{k} \to \mathbb{C}$ is smooth* [25, *Lemma* 7.1.3] and extends to a holomorphic function on $\mathbf{k} \otimes \mathbb{C}$ (*Proposition* 8.4). *Then $\mathrm{Res}^{\Lambda,\chi}(h[d\psi])$ is independent of the cone Λ.*

Proof. As above, define $\chi_\varepsilon(z) = \varepsilon^{-s}\chi(z/\varepsilon)$, so that $\hat{\chi}_\varepsilon(\phi) = \hat{\chi}(\varepsilon\phi)$. The function h extends in particular to a holomorphic function on $\mathbf{t}\otimes\mathbb{C}$, and by the remarks after Definition 8.5, the function $\hat{\chi}_\varepsilon$ also extends to a holomorphic function on $\mathbf{t}\otimes\mathbb{C}$. Hence, Cauchy's theorem shows that for any choice of the cone Λ,

$$\mathrm{Res}^{\Lambda,\chi}(h(\psi)[d\psi]) = \lim_{\varepsilon\to 0^+} \frac{1}{(2\pi\mathrm{i})^l} \int_{\psi\in\mathfrak{t}} \hat{\chi}(\varepsilon\psi)h(\psi)[d\psi]. \qquad \square \quad (8.5)$$

Remark. If h is as in the statement of Proposition 8.6, then $\varpi^2 h$ also satisfies the hypotheses of the proposition, so $\mathrm{Res}^{\Lambda,\chi}(\varpi^2 h[d\psi])$ is also independent of the cone Λ.

The following is a consequence of Proposition 8.4.

PROPOSITION 8.7. *Let $u:\mathbf{t}^* \to \mathbb{C}$ be a distribution, and assume the set Γ_u defined in Proposition* 8.4 *contains -Int(Λ). Thus, $h = F_T u$ is a holomorphic function on $\mathbf{t} - \mathrm{i}\mathrm{Int}\ \Lambda$, satisfying the hypotheses in Definition* 8.5. *Suppose in addition that u is smooth at* 0. *Then $\mathrm{Res}^{\Lambda,\chi}(h[d\psi])$ is independent of the test function χ, and equals $i^{-1}u(0)/(2\pi)^{l/2}$.*

We shall be dealing with functions $h:\mathbf{k} \to \mathbb{C}$ whose Fourier transforms are smooth at 0 but which are sums of other functions not all of whose Fourier transforms need be smooth at 0. For this reason we must introduce a small generic parameter $\rho \in \mathbf{t}^*$ where all the functions in this sum are smooth. More precisely, we define the following result.

Definition 8.8. Let Λ, χ and h be as in Definition 8.5. Let $\rho \in \mathbf{t}^*$ be such that the distribution $F_T h$ is smooth on the ray $t\rho$ for $t \in (0,\delta)$ for some $\delta > 0$, and suppose $(F_T h)(t\rho)$ tends to a well defined limit as $t \to 0^+$. Then we define

$$\mathrm{Res}^{\rho,\Lambda,\chi}(h[\mathrm{d}\psi]) = \lim_{t\to 0^+} \mathrm{Res}^{\Lambda,\chi}(h(\psi)\mathrm{e}^{\mathrm{i}t\rho(\psi)}[d\psi]). \quad (8.6)$$

Under these hypotheses, $\mathrm{Res}^{\rho,\Lambda,\chi}(h[d\psi])$ is independent of χ (by Proposition 8.7), but it may depend on the ray $\{t\rho : t \in \mathrm{R}^+\}$. However, we have by Proposition 8.7 the following result.

PROPOSITION 8.9. *Suppose $F_T h$ is smooth at* 0. *Then the quantity $Res^{\rho,\Lambda,\chi}(h[d\psi])$ satisfies* $\mathrm{Res}^{\rho,\Lambda,\chi}(h[d\psi]) = \mathrm{Res}^{\Lambda,\chi}(h[d\psi])$.

Remark. In the light of Propositions 8.6, 8.7 and 8.9, it makes sense to write $\mathrm{Res}(\Omega)$ for $\mathrm{Res}^{\rho,\Lambda,\chi}(\Omega)$ when $\Omega = \varpi^2 h[d\psi]$ for a K-invariant function $h:\mathbf{k}\to\mathbb{C}$ for which $F_K h$ is compactly supported and $F_T(\varpi^2 h)$ is smooth at 0. In the proof of Theorem 8.1 which we are about to give, we shall check the validity of these hypotheses for the form Ω which appears in the statement of the theorem.

We now give the proof of Theorem 8.1. We shall first show the following proposition.

PROPOSITION 8.10. (a)*The distribution* $F_K(\Pi_*\eta e^{i\varpi})(z')$ *defined for* $z'\in\mathbf{k}$ *is represented by a smooth function for* z' *in a sufficiently small neighbourhood of* 0.

(b) *We have*

$$\eta_0 e^{i\omega_0}[\mathscr{M}_X] = \frac{1}{(2\pi)^{s/2} i^s \mathrm{vol}(K)} F_K(\Pi_*\eta e^{i\varpi})(0), \tag{8.7}$$

$$= \frac{(2\pi)^{l/2}}{(2\pi)^s |W| \mathrm{vol}(T) i^s} F_T(\varpi^2\Pi_*\eta e^{i\varpi})(0). \tag{8.8}$$

Proof. (a) To evaluate $F_K(\Pi_*\eta e^{i\varpi})(0)$, we introduce a test function $\chi:\mathbf{k}^*\to\mathbf{R}^+$ which is smooth and of compact support, and for which $(F_K\chi)(0) = 1/(2\pi)^{s/2}$. We define $\chi_\varepsilon(z) = \varepsilon^{-s}\chi(z/\varepsilon)$; as $\varepsilon\to 0$, the functions χ_ε tend to the Dirac delta distribution on $\mathbf{k}^*$ (in the space $\mathscr{D}'$ of distributions on $\mathbf{k}^*$). Then we have

$$(F_K\chi_\varepsilon)(\phi) = (F_K\chi)(\varepsilon\phi). \tag{8.9}$$

Now to evaluate $F_K(\Pi_*\eta e^{i\varpi})(z')$, we integrate it against the sequence of test functions χ_ε:

$$F_K(\Pi_*\eta e^{i\varpi})(z') = \lim_{\varepsilon\to 0^+} i^s \mathscr{J}^\varepsilon(z') \tag{8.10}$$

where

$$i^s\mathscr{J}^\varepsilon(z') = \int_{z\in\mathbf{k}^*} [dz]\, F_K(\Pi_*\eta e^{i\varpi})(z)\chi_\varepsilon(z-z')$$

$$= \int_{\phi\in\mathbf{k}} [d\phi] \int_{x\in X} \eta(\phi) e^{i\omega} e^{i(\mu(x)-z')(\phi)} \hat{\chi}_\varepsilon(\phi) \tag{8.11}$$

(by Parseval's theorem). Now because χ_ε is smooth and of compact support, the Paley–Wiener theorem ([25, theorem 7.3.1]) implies that $\hat{\chi}_\varepsilon$ is rapidly decreasing. So we may use Fubini's theorem to interchange the order of integration and get

$$i^s\mathscr{J}^\varepsilon(z') = \int_{x\in X} e^{i\omega} \int_{\phi\in\mathbf{k}} [d\phi]\eta(\phi)\hat{\chi}_\varepsilon(\phi) e^{i(\mu(x)-z')(\phi)} \tag{8.12}$$

$$= (2\pi)^{s/2} \int_{x\in X} e^{i\omega}\eta(-i\partial/\partial z)\chi_\varepsilon(z)|_{z=\mu(x)-z'}. \tag{8.13}$$

As $\varepsilon\to 0$, χ_ε is supported on an arbitrarily small neighbourhood of $\mu^{-1}(0)$. Thus, because of Proposition 5.2, the integral may be replaced by an integral over $P\times\mathbf{k}^*$:

$$i^s\mathscr{J}^\varepsilon(z') = (2\pi)^{s/2} \int_{(p,z)\in P\times\mathbf{k}^*} e^{i\omega}\eta\left(-i\frac{\partial}{\partial z}\right)\chi_\varepsilon(z-z'). \tag{8.14}$$

Now by the same argument as given in the proof of Theorem 7.1(a), there is $\eta_0\in\Omega^*(P/K)$ such that

$$\eta - \pi^*\eta_0 = D\gamma \quad \text{for some } \gamma \tag{8.15}$$

where D is the equivariant cohomology differential on $P \times \mathbf{k}^*$ and $\pi: P \times \mathbf{k}^* \to P \to P/K$ is the projection map. By the argument given after (7.3), we have

$$\begin{aligned} & i^s \mathscr{J}^\varepsilon(z') - (2\pi)^{s/2} \int_{(p,z)\in P\times \mathbf{k}^*} e^{i\omega}(\pi^*\eta_0)\chi_\varepsilon(z-z') \qquad (8.16)\\ &= \int_{(p,z)\in P\times \mathbf{k}^*} e^{i\omega} \int_{\phi\in\mathbf{k}} [d\phi] D\gamma(\phi)\hat{\chi}_\varepsilon(\phi)\, e^{i(z-z')(\phi)} \\ &= \int_{\phi\in\mathbf{k}} [d\phi]\hat{\chi}_\varepsilon(\phi) \int_{(p,z)\in P\times \mathbf{k}^*} e^{i\omega}\, D\gamma(\phi)\, e^{i(z-z')(\phi)} \overset{\text{def}}{=} \Delta_\varepsilon. \end{aligned}$$

But

$$e^{i\omega} e^{i(z-z')(\phi)} D\gamma = D(e^{i\omega} e^{i(z-z')(\phi)}\gamma(\phi)). \qquad (8.17)$$

Hence,

$$\Delta_\varepsilon = \int_{\phi\in\mathbf{k}} [d\phi]\hat{\chi}_\varepsilon(\phi) \int_{(p,z)\in P\times \mathbf{k}^*} d(e^{i\omega} e^{i(z-z')(\phi)}\gamma(\phi)) \qquad (8.18)$$

(since for any differential form Ψ, the term $i_{\tilde{\phi}}\Psi$ in $D\Psi$ cannot contain differential forms of top degree in x). Replacing $P \times \mathbf{k}^*$ by $P \times B_R(\mathbf{k}^*)$ where $B_R(\mathbf{k}^*)$ is a large ball in $\mathbf{k}^*$ with boundary $S_R(\mathbf{k}^*)$, we have that

$$\Delta_\varepsilon = \lim_{R\to\infty} \int_{\phi\in\mathbf{k}} [d\phi]\hat{\chi}_\varepsilon(\phi) \int_{(p,z)\in P\times B_R(\mathbf{k}^*)} d(e^{i\omega} e^{i(z-z')(\phi)}\gamma(\phi)). \qquad (8.19)$$

We then interchange the order of integration to get

$$\begin{aligned} \Delta_\varepsilon &= \lim_{R\to\infty} \int_{(p,z)\in P\times B_R(\mathbf{k}^*)} d\Big(e^{i\omega} \int_{\phi\in\mathbf{k}} [d\phi]\hat{\chi}_\varepsilon(\phi)\, e^{i(z-z')(\phi)}\gamma(\phi)\Big) \\ &= \lim_{R\to\infty} \int_{(p,z)\in P\times B_R(\mathbf{k}^*)} d\left(e^{i\omega}\gamma\left(\frac{\partial}{\partial z}\right)\chi_\varepsilon(z-z')\right). \\ &= \lim_{R\to\infty} \int_{(p,z)\in P\times S_R(\mathbf{k}^*)} \left(e^{i\omega}\gamma\left(\frac{\partial}{\partial z}\right)\chi_\varepsilon(z-z')\right) \quad \text{by Stokes' theorem.} \qquad (8.20) \end{aligned}$$

This limit equals 0 for sufficiently small z' since χ_ε is compactly supported on $\mathbf{k}^*$. Hence, $\Delta_\varepsilon = 0$.

Finally, we obtain, using the expression for ω given in Proposition 5.2,

$$\begin{aligned} i^s \mathscr{J}^\varepsilon(z') &= (2\pi)^{s/2} \int_{(p,z)\in P\times \mathbf{k}^*} e^{i\omega}(\pi^*\eta_0)\,\chi_\varepsilon(z-z') \qquad (8.21)\\ &= (2\pi)^{s/2} \int_{(p,z)\in P\times \mathbf{k}^*} e^{i\pi^*\omega_0}(\pi^*\eta_0)\, e^{i\theta(dz)}\, e^{id\theta(z)}\,\chi_\varepsilon(z-z'). \qquad (8.22) \end{aligned}$$

Thus, we have, as in Section 5,

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathscr{J}^\varepsilon(z') &= (2\pi)^{s/2} \int_{(p,z)\in P\times \mathbf{k}^*} e^{i\pi^*\omega_0}(\pi^*\eta_0)\,\Omega[dz] e^{iF_\theta(z)}\delta(z-z')) \\ &= (2\pi)^{s/2} \int_P e^{i\pi^*\omega_0}(\pi^*\eta_0)\, e^{iF_\theta(z')}\Omega \qquad (8.23) \end{aligned}$$

where Ω is the differential form introduced after (5.4). This shows in particular that $(F_K r^\eta)(z')$ (where $r^\eta(\phi) = \Pi_*(\eta e^{i\bar{\omega}})(\phi)$) is a polynomial in z' for small z', and hence is smooth in z' for z' sufficiently close to 0. This completes the proof of (a).

When $z' = 0$, eq. (8.23) becomes

$$\lim_{\epsilon \to 0} \mathscr{J}^\epsilon(0) = i^{-s} F_K(\Pi_* \eta e^{i\bar{\omega}})(0) = (2\pi)^{s/2} \operatorname{vol}(K) \eta_0 e^{i\omega_0}[\mathscr{M}_X]$$

which proves (8.7). Using Lemma 3.1 again, we have for any K-invariant function f on $\mathbf{k}$ that

$$\frac{(2\pi)^{s/2}}{\operatorname{vol}(K)} (F_K f)(0) = \frac{(2\pi)^{l/2}}{|W| \operatorname{vol}(T)} F_T(f\varpi^2)(0). \tag{8.24}$$

Combining this with (8.7) we obtain

$$\eta_0 e^{i\omega_0}[\mathscr{M}_X] = \frac{(2\pi)^{l/2}}{(2\pi)^s |W| \operatorname{vol}(T) i^s} F_T(\varpi^2 \Pi_* \eta e^{i\bar{\omega}})(0)$$

which is (8.8). □

Theorem 8.1 follows from Propositions 8.10 and 8.7 by applying Theorem 2.1 to decompose $r^\eta = \Pi_*(\eta e^{i\bar{\omega}})$ as a sum of meromorphic functions r_F^η on $\mathbf{t} \otimes \mathbb{C}$ corresponding to the components F of the fixed point set of T. We now complete the proof of this theorem.

Proof of Theorem 8.1: As in (7.6), the abelian localization formula yields $r^\eta(\psi) = \sum_{F \in \mathscr{F}} r_F^\eta(\psi)$, *where*

$$r_F^\eta(\psi) = e^{i\mu_T(F)(\psi)} \int_F \frac{i_F^* \eta(\psi) e^{i\omega}}{e_F(\psi)}.$$

Now the distribution $(F_K r^\eta)(\phi)$ is represented by a smooth function near 0 (Proposition 8.10(a)); also, the distribution $F_T(\varpi^2 r^\eta) = D_\varpi F_T(\varpi r^\eta)$ is represented by a smooth function near 0, since $F_T(\varpi r^\eta) = \varpi F_K r^\eta$ (Lemma 3.4) and $F_K r^\eta$ is smooth near 0 (Proposition 8.10(a)). We choose $\rho \in \mathbf{t}^*$ so that the distribution $F_T r_F^\eta$ is smooth along the ray t_ρ, $t \in (0, \delta)$, for all F and sufficiently small $\delta > 0$, and that this distribution tends to a well-defined limit as $t \to 0^+$: this is possible because the $r_F^\eta(\psi)$ are sums of terms of the form $e^{i\mu_T(F)(\psi)}/\Pi_j \beta_{F,j}(\psi)^{n_j}$ so their Fourier transforms $F_T r_F^\eta$ are piecewise polynomial functions of the form $H_{\bar{\beta}}(y)$ (see Proposition 3.6). These functions are smooth on the set $U_{\bar{\beta}}$ consisting of all points y where y is not in the cone spanned by any subset of the $\beta_{Fj}^\wedge$ containing less than l elements. Thus,

$$\lim_{t \to 0^+} (2\pi)^{-l/2} i^{-l} F_T r_F^\eta(t\rho) = \operatorname{Res}^{\rho, \Lambda, \chi}(r_F^\eta[d\psi])$$

by Definition 8.5 and Proposition 8.7. It follows that

$$(2\pi)^{-l/2} i^{-l} F_T(\varpi^2 r^\eta)(0) = \operatorname{Res}^{\rho, \Lambda, \chi}(\varpi^2 r^\eta[d\psi]) \tag{8.25}$$

$$= \sum_{F \in \mathscr{F}} \operatorname{Res}^{\rho, \Lambda, \chi}(\varpi^2 r_F^\eta). \tag{8.26}$$

The residue in (8.25) is independent of χ, Λ and ρ by Propositions 8.6, 8.7 and 8.9. The residues in (8.26) are independent of χ by Proposition 8.7, but they may depend on ρ and Λ.

To conclude the proof of Theorem 8.1 we note that (8.8) gives

$$\eta_0 e^{i\omega}[\mathscr{M}_X] = \frac{(2\pi)^{l/2}}{(2\pi)^s |W| \mathrm{vol}(T) i^s} F_T(\varpi^2 \Pi_* \eta e^{i\tilde{\omega}})(0)$$

$$= \frac{i^l}{(2\pi)^{s-l} |W| \mathrm{vol}(T) i^s} \mathrm{Res}^{\Lambda,\chi}(r^\eta(\psi)\varpi^2(\psi)[d\psi]) \text{ (by Proposition 8.7)}$$

$$= \frac{(-1)^{n+}}{(2\pi)^{s-l} |W| \mathrm{vol}(T)} \mathrm{Res}^{\Lambda,\chi}\left(\sum_{F\in\mathscr{F}} r_F^\eta(\psi)\varpi^2(\psi)[d\psi]\right) \tag{8.27}$$

as claimed. □

Proof of Corollary 8.2. In a normalization where $\mathrm{vol}(T) = 1$, the factor $\varpi(\psi)$ becomes $4\pi\psi$. This gives

$$\frac{1}{(2\pi)^{s-l}} \mathrm{Res}\left(\varpi^2(\psi) \sum_{F\in\mathscr{F}} r_F^\eta(\psi)[d\psi]\right) = 4\mathrm{Res}^{\Lambda,\chi}(\psi^2 \sum_{F\in\mathscr{F}} r_F^\eta(\psi)[d\psi]).$$

Each term $r_F^\eta(\psi)$ is a sum of terms of the form $\tau_\alpha(\psi) = c_\alpha \psi^{-n_\alpha} e^{i\mu(F)\psi}$ for some constants c_α and integers n_α. By (8.4), the residue is given by

$$\mathrm{Res}^{\Lambda,\chi}(h[d\psi]) = \lim_{\epsilon\to 0^+} \frac{1}{2\pi i} \int_{\psi\in R - i\xi} \hat{\chi}(\varepsilon\psi) h(\psi)[d\psi] \tag{8.28}$$

where we choose ξ to be in the cone $\Lambda = \mathbb{R}^+$. Proposition 8.4 (i) and (ii) show that the function $\hat{\chi}: \mathbb{R} \to \mathbb{C}$ extends to an entire function on $\mathbb{C}$.

We may now decompose the integral in (8.28) into terms corresponding to the τ_α. If $n_\alpha > 0$, we complete each such integral over $\mathbb{R} - i\xi$ to a contour integral by adding a semicircular curve at infinity, which is in the upper half plane if $\mu(F) > 0$ and in the lower half plane if $\mu(F) < 0$. This choice of contour is made so that the function $\tau_\alpha(\psi)$ is bounded on the added contours, so the added semicircular curves do not contribute to the integral. Since only the contours corresponding to values of F for which $\mu(F) > 0$ enclose the pole at 0, application of Cauchy's residue formula now gives the result. A similar argument establishes that the terms τ_α for which $n_\alpha \leq 0$ contribute 0 to the sum.† □

Remarks. (a) The quantity $\mathrm{Res}^{\rho,\Lambda,\chi}(\varpi^2 r_F^\eta[d\psi])$ depends on the cone Λ for each F; however, it follows from Proposition 8.6 that the sum $\sum_{F\in\mathscr{F}} \mathrm{Res}^{\rho,\Lambda,\chi}(\varpi^2 r_F^\eta[d\psi])$ is independent of Λ.

(b) Let $\mathscr{F}_\Lambda$ be the set of those $F \in \mathscr{F}$ for which $\mu_T(F)$ lies in the cone $\check{C}_{F,\Lambda} = \{\sum_j s_j \beta_{F,j}^\Lambda : s_j \geq 0\}$ (defined in Section 4) spanned by the $\beta_{F,j}^\Lambda$. Then by Proposition 8.11 (iii), $\mathrm{Res}^{\rho,\Lambda,\chi}(\varpi^2 r_F^\eta) = 0$ if $F \notin \mathscr{F}_\Lambda$, so in fact

$$\mathrm{Res}^{\rho,\Lambda,\chi}(\varpi^2 r^\eta[d\psi]) = \mathrm{Res}^{\rho,\Lambda,\chi} \sum_{F\in\mathscr{F}_\Lambda} (\varpi^2 r_F^\eta[d\psi]).$$

(c) Finally, it follows from Proposition 8.11 that if we replace the symplectic form ω and the moment map μ by $\delta\omega$ and $\delta\mu$ (where $\delta > 0$) and then let δ tend to 0, we obtain an expression where μ and ω appear only in determining the set $\mathscr{F}_\Lambda$ indexing terms which yield a nonzero contribution.

We now restrict ourselves to the special case when Ω is of the form

$$\Omega_\lambda(\psi) = e^{i\lambda(\psi)}[d\psi] / \prod_{j=1}^N \beta_j(\psi).$$

† The formula obtained by choosing $\Lambda = \mathbb{R}^-$ is in fact equivalent to the formula we have obtained using the choice $\Lambda = \mathbb{R}^+$. This can be seen directly from the Weyl invariance of the function r^η, where the action of the Weyl group takes ψ to $-\psi$ and so converts terms with $\mu(F) > 0$ to terms with $\mu(F) < 0$.

If $\beta_j \in \Lambda$ then the distribution $\mathrm{Res}^{\rho,\Lambda,\chi}(\Omega_\lambda)$ is just (up to multiplication by a constant) the piecewise polynomial function $H_{\bar\beta}(\lambda)$ from Proposition 3.6. We shall now give a proposition which gives a list of properties satisfied by the residues $\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda)$: these properties in fact characterize the residues uniquely and enable one to compute them.

PROPOSITION **8.11.** *Let $\xi \in \mathbf{t}$ and suppose $\beta_1, \ldots, \beta_N \in \mathbf{t}^*$ are all in the dual cone of a cone $\Lambda \in \mathbf{t}$. Denote by $P : \mathbf{t} \to \mathbb{R}$ the function $P(\psi) = \prod_j \beta_j(\psi)$. Suppose $\lambda \in U_{\bar\beta} \subset \mathbf{t}^*$ (see Proposition 3.6), and define $\Omega_\lambda(\psi) = e^{i\lambda(\psi)}[d\psi]/P(\psi)$.*

Then we have

(i) $\mathrm{Res}^{\Lambda,\chi}(\psi_k \psi^J \Omega_\lambda) = (-i\partial/\partial\lambda_k)\mathrm{Res}^{\Lambda,\chi}(\psi^J \Omega_\lambda)$.

(ii) $(2\pi i)^l \mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda) = i^N H_{\bar\beta}(\lambda)$, *where $H_{\bar\beta}$ is the distribution given in Proposition* 3.6. (*Recall we have assumed that β_j is in the dual cone of Λ for all j.*)

(iii) $\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda) = 0$ *unless λ is in the cone $C_{\bar\beta}$ spanned by the β_j.*

(iv)

$$\lim_{s\to 0^+} \mathrm{Res}^{\Lambda,\chi}(\Omega_{s\lambda}\psi^J) = 0$$

unless $N - |J| = l$.

(v)

$$\lim_{s\to 0^+} \mathrm{Res}^{\Lambda,\chi}(\Omega_{s\lambda}\psi^J) = 0$$

if the monomials β_j do not span $\mathbf{t}^*$.

(vi) If $\beta_1, \ldots \beta_l$ *span $\mathbf{t}^*$ and $\lambda = \sum_j \lambda^j \beta_j$ with all $\lambda^j > 0$, then*

$$\lim_{s\to 0^+} \mathrm{Res}^{\Lambda,\chi}\left(\frac{e^{is\lambda(\psi)}[d\psi]}{\beta_1(\psi)\ldots\beta_l(\psi)}\right) = \frac{1}{\det\bar\beta}$$

where $\det\bar\beta$ is the determinant of the l by l matrix whose columns are the coordinates of $\beta_1, \ldots, \beta_l$ written in terms of any orthonormal basis of $\mathbf{t}$.

(vii)

$$\mathrm{Res}^{\Lambda,\chi}\left(\frac{e^{i\lambda(\psi)}\psi^J}{P(\psi)}[d\psi]\right) = \sum_{m\geq 0} \lim_{s\to 0^+} \mathrm{Res}^{\Lambda,\chi}\left(\frac{(i\lambda(\psi))^m e^{is(\psi)}\psi^J}{m!\,P(\psi)}[d\psi]\right).$$

Remark. The limits in Proposition 8.11 are not part of the definition of the residue map: rather these limits are described in order to specify a procedure for computing the piecewise polynomial function $H_{\bar\beta}(\lambda) = (2\pi i)^l i^{-N}\,\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda)$ (see the example below). Proposition 8.11 (ii) identifies $H_{\bar\beta}(\lambda)$ with an integral over $\mathbf{t}$, which may be completed to an appropriate contour integral: the choice of contour is determined by the value of λ, and requires λ to be a nonzero point in $U_{\bar\beta}$. Proposition 8.11(vii) says that one may compute $H_{\bar\beta}$ by expanding the numerator $e^{i\lambda(\psi)}$ in a power series, but only provided one keeps a factor $e^{is\lambda(\psi)}$ in the integrand (for small $s > 0$) in order to specify the contour. The limits in Proposition 8.11 (iv)–(vii) exist because according to Proposition 8.11(i) and (ii), they specify limits of derivatives of polynomials on subdomains of $U_{\bar\beta}$, as one approaches the point 0 in the boundary of $U_{\bar\beta}$ along the fixed direction $s\lambda$ as $\to 0$ in $\mathbb{R}^+$.

Proof. (i) This follows directly from Definition 8.5.

(ii) This follows because $(2\pi i)^l i^{-N}\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda)$ is the fundamental solution $E(\lambda)$ of the differential equation $P(\partial/\partial\lambda)E(\lambda) = \delta_0$ with support in a half-space containing the β_j. (see [4,

Theorem 4.1] or [26, Theorem 12.5.1.]) But this fundamental solution is given by $H_{\bar\beta}$ (see Proposition 3.6(c)).

(iii) This is an immediate consequence of (i) and (ii), in view of Proposition 3.6(a)†.

(iv) By (i),

$$\lim_{s\to 0^+} \mathrm{Res}^{\Lambda,\chi}(\Omega_{s\lambda_0}\psi^J) = \lim_{s\to 0^+} (-\mathrm{i}\partial/\partial\lambda)^J H_{\bar\beta}(\lambda)|_{\lambda=s\lambda_0}$$

but $H_{\bar\beta}$ is a homogeneous piecewise polynomial function of degree $N-l$, so the conclusion holds for $|J| > N-l$. If $|J| < N-l$, we find that $(\partial/\partial\lambda)^J H_{\bar\beta}(\lambda)$ is homogeneous of order $N-l-|J|$ (on any open subset of $\mathbf{t}^*$ where $H_{\bar\beta}$ is smooth). Hence, it is of order $s^{N-l-|J|}$ at $\lambda = s\lambda_0$ as $s\to 0^+$, and the conclusion also holds in this case.

(v) By (ii), we know that $\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda) = 0$ for λ in a neighbourhood of $s\lambda_0$ (since $s\lambda_0$ is not in the support of $H_{\bar\beta}$). Applying (i), $\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda\psi^J)$ must also be zero.

(vi) If $\beta = \{\beta_1, \ldots, \beta_l\}$ span $\mathbf{t}^*$, and $\lambda = \sum_j \lambda^j\beta_j$ with all $\lambda^j > 0$, we have

$$\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda) = \frac{1}{(2\pi\mathrm{i})^l}\int_{\psi\in\mathfrak{t}} \frac{[d\psi]\,\mathrm{e}^{\mathrm{i}\sum_j \lambda^j(\psi_j - \mathrm{i}\xi_j)}}{\prod_{k=1}^l(\psi_k - \mathrm{i}\xi_k)}$$

where the $\xi_k = \beta_k(\xi) > 0$. But $[d\psi] = d\psi_1 \ldots d\psi_l/(\det\bar\beta)$, where $\psi_j = \beta_j(\psi)$. Since the integrals over $\psi_1, \ldots, \psi_l$ may be completed to integrals over semicircular contours $C_+(R)$ in the upper half plane, for each of which the contour integral may be evaluated by the Residue Theorem to give the contribution $2\pi\mathrm{i}$, we obtain the result.

(vii) We have

$$\int_{\psi+\mathrm{i}\xi\in\mathbf{R}^l} \frac{\mathrm{e}^{\mathrm{i}\lambda(\psi)}}{P(\mathrm{i}\psi)}[d\psi] = (2\pi\mathrm{i})^{l_i-N}\,\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda) = H_{\bar\beta}(\lambda).$$

Also, by (i),

$$\begin{aligned}
&\sum_{m_1,\ldots,m_l\geq 0} \frac{(2\pi\mathrm{i})^{l_i-N}}{m_1!\ldots m_l!} \lim_{s\to 0^+} \mathrm{Res}^{\Lambda,\chi}((\mathrm{i}\lambda^1\psi_1)^{m_1}\cdots(\mathrm{i}\lambda^l\psi_l)^{m_l}\Omega_s\lambda_0)\\
&= \sum_{m_1,\ldots,m_l\geq 0} \frac{(2\pi\mathrm{i})^{l_i-N}(\lambda^1)^{m_1}\ldots(\lambda^l)^{m_l}}{m_1!\ldots m_l!} \lim_{s\to 0^+} (\partial/\partial\lambda^1)^{m_1}\ldots(\partial/\partial\lambda^l)^{m_l}\,\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda)|_{\lambda=s\lambda_0}\\
&= \sum_{m_1,\ldots,m_l\geq 0} \frac{(\lambda^1)^{m_1}\ldots(\lambda^l)^{m_l}}{m_1!\ldots m_l!} \lim_{s\to 0^+} (\partial/\partial\lambda^1)^{m_1}\ldots(\partial/\partial\lambda^l)^{m_l}\,H_{\bar\beta}(\lambda)|_{\lambda=s\lambda_0}
\end{aligned}$$

† Alternatively, there is the following direct argument, which was pointed out to us by J. J. Duistermaat. We recall that

$$\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda) = \frac{1}{(2\pi\mathrm{i})^l}\int_{\psi\in\mathfrak{t}} \frac{\mathrm{e}^{\mathrm{i}\lambda(\psi-\mathrm{i}\xi)}}{P(\psi-\mathrm{i}\xi)}[d\psi]$$

which is defined and independent of ξ for $\xi\in(C_{\bar\beta})^*$ (see the discussion after Definition 8.5). Hence, we may replace ξ by $t\xi$ for any $t\in\mathbb{R}^+$:

$$\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda) = \frac{1}{(2\pi\mathrm{i})^l}\int_{\psi\in\mathfrak{t}} \frac{\mathrm{e}^{\mathrm{i}\lambda(\psi-\mathrm{i}t\xi)}}{P(\psi-\mathrm{i}t\xi)}[d\psi]. \tag{8.29}$$

Taking the limit as $t\to\infty$, we see that $\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda) = 0$ if $\lambda(\xi) < 0$, because of the factor $\mathrm{e}^{t\lambda(\xi)}$ that appears in the numerator of (8.29). Since this holds for all $\xi\in(C_{\bar\beta})^*$, $\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda)$ is only nonzero when $\lambda(\xi)\geq 0$ for all $\xi\in(C_{\bar\beta})^*$; in other words, when $\lambda\in C_{\bar\beta}$.

which is equal to $H_{\bar{\beta}}(\lambda)$ since $H_{\bar{\beta}}$ is a polynomial on certain conical subregions of $\mathbf{t}^*$, and there is such a subregion containing the ray $\lambda = s\lambda_0$. □

Example. In the following simple example, the explicit formula for $H_{\bar{\beta}}$ follows immediately from the definition of $H_{\bar{\beta}}$ in Proposition 3.6(a). The example is included to show how this result may alternatively be derived by successive contour integrations.

Suppose $l = 2$ and $N = 3$, and $\beta_1(\psi) = \psi_1$, $\beta_2(\psi) = \psi_2$, $\beta_3(\psi) = \psi_1 + \psi_2$. We compute

$$\mathscr{R} = \mathrm{Res}^{\Lambda,\chi}\left(\frac{e^{i\lambda(\psi)}}{\psi_1\psi_2(\psi_1+\psi_2)}\right) \tag{8.30}$$

where $\lambda(\psi) = \lambda^1\psi_1 + \lambda^2\psi_2$. We assume $\lambda^1, \lambda^2 > 0$ and $\xi_1, \xi_2 > 0$. The quantity (8.30) is given by

$$\mathscr{R} = \frac{1}{(2\pi i)^2}\int_{\psi_2+i\xi_2\in\mathbb{R}} d\psi_2\,\frac{e^{i\lambda^2\psi_2}}{\psi_2}\int_{\psi_1+i\xi_1\in\mathbb{R}}\frac{e^{i\lambda^1\psi_1}}{(\psi_1+\psi_2)\psi_1}. \tag{8.31}$$

(This integral in fact gives the Duistermaat–Heckman polynomial $F_T(\Pi_* e^{i\bar{\omega}})(\lambda)$ near $\mu_T(F)$ where F is a fixed point of the action of T on X, when X is a coadjoint orbit of $SU(3)$ and $T \cong (S^1)^2$ is the maximal torus; see [18].) We compute this by first integrating over ψ_1: since $\lambda^1 > 0$, the integral may be completed to a contour integral over a semicircular contour in the upper half plane. We obtain contributions from the two residues $\psi_1 = 0$ and $\psi_1 = -\psi_2$. Hence, we have

$$\mathscr{R} = \frac{1}{(2\pi i)}\int_{\psi_2+i\xi_2\in\mathbb{R}} d\psi_2\,\frac{e^{i\lambda^2\psi_2}}{\psi_2^2} - \frac{1}{(2\pi i)}\int_{\psi_2+i\xi_2\in\mathbb{R}} d\psi_2\,\frac{e^{i(\lambda^2-\lambda^1)\psi_2}}{\psi_2^2}. \tag{8.32}$$

Since $\lambda^2 > 0$, the first of these integrals may be completed to a contour integral over a semicircular contour in the upper half plane, and the residue at 0 yields the value $i\lambda^2$. If $\lambda^2 - \lambda^1 > 0$, the second integral likewise yields $-i(\lambda^2 - \lambda^1)$. However, if $\lambda^2 - \lambda^1 < 0$ the second integral is instead equal to a contour integral over a semicircular contour in the *lower* half plane, which does not enclose the pole at 0, and hence the second integral gives 0. Thus, we have

$$(2\pi i)^2\mathrm{Res}^{\Lambda,\chi}(\Omega_\lambda) = \begin{cases} i\lambda^2, & \lambda^1 > \lambda^2, \\ i\lambda^1, & \lambda^2 > \lambda^1. \end{cases} \tag{8.33}$$

According to (iv) and (vii), the quantity $\mathscr{R}$ is also given by

$$\begin{aligned}\mathscr{R} &= \lim_{s\to 0+}\int_{\psi+i\xi\in\mathbb{R}^2}\frac{e^{is(\lambda^1\psi_1+\lambda^2\psi_2)}(i\lambda^1\psi_1+i\lambda^2\psi_2)[d\psi]}{\psi_1\psi_2(\psi_1+\psi_2)} \\ &= \lim_{s\to 0+}(i\lambda^1)\int_{\psi+i\xi\in\mathbb{R}^2}\frac{e^{is[\lambda^1(\psi_1+\psi_2)+(\lambda^2-\lambda^1)\psi_2]}[d\psi]}{(\psi_1+\psi_2)\psi_2} \\ &\quad + \lim_{s\to 0+}(i\lambda^2)\int_{\psi+i\xi\in\mathbb{R}^2}\frac{e^{is[(\lambda^1-\lambda^2)\psi_1+\lambda^2(\psi_1+\psi_2)]}[d\psi]}{\psi_1(\psi_1+\psi_2)}\end{aligned} \tag{8.34}$$

This clearly gives the result (8.33).

9. EXAMPLES

In this section we shall show in the case $K = SU(2)$ how Corollary 8.2 may be used to prove relations in the cohomology ring $H^*(\mathscr{M}_X)$ for two specific X. These X are the

examples treated at the end of Section 6 of [30]. There, all the relations in the cohomology ring are determined, which is equivalent to exhibiting all the vanishing intersection pairings. We shall show how the results of the present paper may be used to show these are indeed vanishing intersection pairings, although we shall not be able to rederive the result that there are no others.

Example 1. $X = (\mathbb{P}_1)^N$, N *odd*. Consider the action of $K = SU(2)$ on the space $X = (\mathbb{P}_1)^N$ of ordered N-tuples of points on the complex projective line $\mathbb{P}_1$, defined by the Nth tensor power of the standard representation of K on $\mathbb{C}^2$. Equivalently, when $\mathbb{P}_1$ is identified with the unit sphere S^2 in $\mathbb{R}^3$ then K acts on $X = (S^2)^N$ by rotations of the sphere. When the dual of the Lie algebra of K is identified suitably with $\mathbb{R}^3$ then the moment map μ is given (up to a constant scalar factor depending on the conventions used) by

$$\mu(x_1, \ldots, x_N) = x_1 + \cdots + x_N$$

for $x_1, \ldots, x_N \in S^2$. We assume that 0 is a regular value for μ; this happens if and only if there is no N-tuple in $\mu^{-1}(0)$ containing a pair of antipodal points in S^2 each with multiplicity $N/2$, and so 0 is a regular value if and only if N is odd.

In order to apply Corollary 8.2, we note that the fixed points of the action of the standard maximal torus T of K are the N-tuples $(x_1, \ldots, x_N)$ of points in $\mathbb{P}_1$ such that each x_j is either 0 or ∞. We shall index these by sequences $n = (n_1, \ldots, n_N)$ where $n_j = +1$ if $x_j = 0$ and $n_j = -1$ if $x_j = \infty$. Denote by e_n the fixed point indexed by n. Then

$$\mu_T(e_n) = \sum_{j=1}^{N} n_j$$

and the weights of the action of T at e_n are just $\{n_1, \ldots, n_N\}$. Hence, the sign of the product of weights at e_n is $\Pi_j n_j$ and its absolute value is 1.

The cohomology ring $H^*(X)$ has N generators $\xi_1, \ldots, \xi_N$ say, of degree two, satisfying $\xi_j^2 = 0$ for $1 \le j \le N$. The equivariant cohomology ring $H_T^*(X)$ with respect to the torus T has generators $\xi_1, \ldots, \xi_N$ and α of degree two subject to the relations

$$(\xi_j)^2 = \alpha^2$$

for $1 \le j \le N$. The Weyl group action sends α to $-\alpha$ so $H_K^*(X)$ has generators $\xi_1, \ldots, \xi_n, \alpha^2$ subject to the same relations.

According to the last example of [30, Section 6], the kernel of the map $H_K^*(X) \to H^*(\mathcal{M}_X)$ is spanned by elements of the form

$$(1/\alpha)\left(q(\xi_1, \ldots, \xi_N, \alpha) \prod_{i \in Q} (\xi_i + \alpha) - q(\xi_1, \ldots, \xi_N, -\alpha) \prod_{i \in Q} (\xi_i - \alpha)\right) \tag{9.1}$$

for some $Q \subset \{1, \ldots, N\}$ containing at least $(N+1)/2$ elements and some polynomial q in $N+1$ variables with complex coefficients. We can use Corollary 8.2 to give an alternative proof that the evaluation against the fundamental class $[\mathcal{M}_X]$ of the image in $H^*(\mathcal{M}_X)$ of any element of this form of degree $N-3$ is zero. This amounts to showing that

$$\operatorname{Res}_0 \frac{1}{\psi^{N-1}} \sum_{\substack{n_j = \pm 1 \\ \sum_j n_j > 0}} \left(\prod_j n_j\right) \{q(\psi n_1, \ldots, \psi n_N, \psi) \prod_{i \in Q} \psi(n_i + 1) - q(\psi n_1, \ldots, \psi n_N, -\psi) \prod_{i \in Q} \psi(n_i - 1)\} \tag{9.2}$$

is zero when q is homogeneous of degree $N-2-|Q|$. In other words, it amounts to showing that $\Delta = 0$ for any Δ of the form

$$\Delta = \sum_{\substack{n_j = \pm 1 \\ \sum_j n_j > 0}} (\prod_j n_j)\{q(n_1, \ldots, n_N, 1) \prod_{i \in Q}(n_i + 1) - q(n_1, \ldots, n_N, -1) \prod_{i \in Q}(n_i - 1)\} \tag{9.3}$$

where q is homogeneous of degree $N-2-|Q|$. Let us assume, without loss of generality, that $q(\xi_1, \ldots, \xi_N, \alpha) = \Pi_i \xi_i^{r_i} \alpha^p$ where $p + \Sigma_i r_i + |Q| = N-2$. Thus, $p - 1 = |Q| + \Sigma_i r_i$ (mod 2) since N is odd. Hence, we have that

$$\Delta = \sum_{\substack{n_j = \pm 1 \\ \sum_j n_j > 0}} \prod_j (n_j^{r_j+1})\{\prod_{i \in Q}(n_i + 1) + (-1)^{|Q|} \prod_k (-1)^{r_k} \prod_{i \in Q}(n_i - 1)\} \tag{9.4}$$

$$= \sum_{\substack{n_j = \pm 1 \\ \sum_j n_j > 0}} \prod_j (n_j^{r_j+1}) \prod_{k \in Q}(n_k + 1) - \sum_{\substack{m_j = \pm 1 \\ \sum_j m_j < 0}} \prod_j (m_j^{r_j+1}) \prod_{k \in Q}(m_k + 1) \tag{9.5}$$

where we have introduced $m_j = -n_j$.

Now the second sum vanishes, for if $m_j = 1$ for all $j \in Q$ then we must have $\Sigma_j m_j > 0$ since $|Q| > N/2$. Hence, we are reduced to proving the vanishing of

$$\sum_{n \in \Gamma} \prod_j n_j^{r_j+1}$$

where

$$\Gamma = \{n | \sum_j n_j > 0,\ n_j = 1 \text{ for } j \in Q\}.$$

Hence, we have to prove the vanishing of

$$\Delta = \sum_{n_j = \pm 1,\ j \notin Q} \prod_{j \in S} n_j$$

where S is the set $\{j \notin Q | r_j = 0 \text{ (mod 2)}\}$. The sum thus vanishes by cancellation in pairs provided S is nonempty. However, if S were empty then $r_j = 1$ (mod 2) for all $j \notin Q$, so $r_j \geq 1$ for all $j \notin Q$, which is impossible since $\Sigma_j r_j + |Q| \leq N-2$. This proves the desired result.

Example 2. $X = \mathbb{P}_N$, N *odd*: A closely related example is given by the action of $K = SU(2)$ on the complex projective space $\mathrm{X} = \mathbb{P}^N$ defined by the Nth symmetric power of the standard representation of K on $\mathbb{C}^2$. Equivalently, we can identify X with the space of *unordered* N-tuples of points in the complex projective line $\mathbb{P}_1$ or the sphere S^2, and then K acts by rotations as in Example 1. We take the symplectic form ω on X to be the Fubini-Study form on $\mathbb{P}_N$. The moment map is given by the composition of the restriction map $\mathbf{u}(N+1)^* \to \mathbf{k}^*$ with the map $\mu: \mathbb{P}_N \to \mathbf{u}(N+1)^*$ defined for $\alpha \in \mathbf{u}(N+1)$ by

$$\langle \mu(x), a \rangle = (2\pi \mathrm{i} |x^*|^2)^{-1} \bar{x}^{*t} a x^*$$

where $x^* = (x_0^*, \ldots, x_N^*)$ is any point in $\mathbb{C}^{N+1}$ lying over the point $x \in \mathbb{P}_N$. The restriction of this moment map to $\mathbf{t}^*$ is $\mu_T(x) = (2\pi \mathrm{i} |x^*|^2)^{-1})\sum_{j=0}^N (N-2j)|x_j^*|^2$. Again we assume that N is odd in order to ensure that 0 is a regular value of the moment map μ.

Again the fixed points of the action of T are the N-tuples of points in $\mathbb{P}_1$ consisting entirely of copies of 0 and ∞. Equivalently, they are the points $e_0 = [1,0, \ldots, 0]$, $e_1 = [0,1, \ldots, 0]$, $\ldots, e_N = [0, \ldots, 0, 1]$ of $\mathbb{P}_N$. The image of e_k under μ_T is $\mu_T(e_k) = N - 2k = \mu_k$ say. Since diag $(t, t^{-1}) \in T$ acts on $\mathbb{P}_N$ by sending $[x_0, \ldots, x_j, \ldots, x_N]$ to

$[t^{-N}x_0, \ldots, t^{2j-N}x_j, \ldots, t^N x_N]$ the weights at e_k are

$$\{2(j-k): 0 \le j \le N, j \ne k\}.$$

The number of negative weights at e_k is equal to k (modulo 2), and the absolute value of the product of weights at e_k is $v_k = \prod_{j \ne k}|j-k| = 2^N k!(N-k)!$

The cohomology ring $H^*(\mathbb{P}_N)$ is generated by ξ of degree two subject to the relation $\xi^{N+1} = 0$. The equivariant cohomology ring $H_T^*(\mathbb{P}_N)$ is generated by ξ and α of degree two subject to the relation $\prod_{0 \le j \le N}(\xi - (2j-N)\alpha) = 0$, and the equivariant cohomology ring $H_K^*(\mathbb{P}_N)$ is generated by ξ and α^2 subject to the same relation.

According to Section 6 of [30] the kernel of the natural map $H_K^*(X) \to H^*(\mathscr{M}_X)$ is generated as an ideal in $H_K^*(X)$ by $P_+(\xi, \alpha)$ and $P_-(\xi, \alpha)/\alpha$ where

$$P(\xi, \alpha) = \prod_{k > N/2} (\xi + \mu_k \alpha)$$

and

$$P_\pm(\xi, \alpha) = P(\xi, \alpha) \pm P(\xi, -\alpha).$$

(Note that $P_+(\xi, \alpha)$ and $P_-(\xi, \alpha)/\alpha$ are actually polynomials in ξ and α^2.) We would like to check that the evaluation against the fundamental class $[\mathscr{M}_X]$ of the image of $R_+(\xi, \alpha^2)P_+(\xi, \alpha)$ and $R_-(\xi, \alpha^2)P_-(\xi, \alpha)/\alpha$ in $H^*(\mathscr{M}_X)$ is zero for any $R_\pm(\xi, \alpha^2) \in H_K^*(X)$ of the appropriate degree.

Now we have from the abelian fixed point formula that for any $S(\xi, \alpha^2)$,

$$\Pi_*^+(S(\xi, \alpha^2)) = \sum_{k < N/2} (-1)^k \frac{S(\mu_k \psi, \psi^2)}{v_k} \psi^{-N}.$$

(Here, if $\zeta \in H_K^*(X)$, the notation $\Pi^+(\zeta)$ means the portion of the abelian formula (2.1) for $\Pi_*(\zeta)$ corresponding to fixed points F for which $\mu_T(F) > 0$.) To evaluate this on the fundamental class of $\mathscr{M}_X$ we must then find the term of degree -1 in $\varpi^2(\psi)\Pi_*^+(S(\xi, \alpha^2))(\psi)$, or in other words the term of degree $N-3$ in $\Sigma_{k<N/2}(-1)^k S(\mu_k\psi, \psi^2)/(v_k)$. Having found the term of degree $N-3$ in ψ, we evaluate it at $\psi = 1$ to get the residue. In the case when $S(\mu_k\psi, \psi^2) = R_+(\mu_k\psi, \psi^2)P_+(\mu_k\psi, \psi)$ or $S(\mu_k\psi, \psi^2) = R_-(\mu_k\psi, \psi^2)P_-(\mu_k\psi, \psi)/\psi$ is homogeneous of degree $N-3$ in ψ, we need to show that

$$\sum_{k=0}^{(N-1)/2} (-1)^k \frac{1}{v_k}\left(R_\pm(\mu_k, 1) \prod_{j \ge N/2} \mu_k + \mu_j) \pm R_\pm(\mu_k, -1) \prod_{j \ge N/2} (\mu_k - \mu_j)\right) = 0.$$

Since $\mu_k = -\mu_{N-k}$, we have $\Pi_{j \ge N/2}(\mu_k + \mu_j) = 0$ for all $k < N/2$, so we just have to prove the vanishing of

$$\sum_{k \le N/2} (-1)^k \frac{1}{v_k} R(\mu_k) \prod_{j \ge N/2} (\mu_k - \mu_j)$$

for every polynomial R of degree at most $(N-1)/2 - 2$ or, without loss of generality, the vanishing of

$$\sum_{k \le N/2} (\mu_k)^s \frac{(-1)^k}{\prod_{l \ne k}|l-k|} \prod_{j \ge N/2} (\mu_k - \mu_j)$$

where $s \le (N-1)/2 - 2$. Since $\prod_{l \ne k}|l-k| = k!(N-k)!$ and $\mu_k - \mu_j = 2(j-k)$, we have that

$$\prod_{j \ge N/2} |\mu_k - \mu_j| = 2^{(N+1)/2} \frac{(N-k)!}{((N-1)/2 - k)!}.$$

Define $r = (N-1)/2$. We need to show the vanishing of

$$\sum_{k=0}^{r} (N-2\bar{k})^s \frac{(-1)^k (N-k)!}{k!(N-k)!(r-k)!}$$

for $s \leq r-2$. It then suffices to show the vanishing of

$$\sum_{k=0}^{r} k^s (-1)^k \binom{r}{k}, \quad s \leq r-2.$$

This follows since one may expand $(1-e^{\lambda})^r$ as a power series in e^{λ} using the binomial theorem: we have

$$(1-e^{\lambda})^r = \sum_{j=0}^{r} (-1)^j \binom{r}{j} \exp j\lambda = \sum_{s \geq 0} \lambda^s/s! \sum_{j=0}^{r} (-1)^j j^s \binom{r}{j}$$

but the terms in this expansion corresponding to $s < r$ must vanish since $1 - e^{\lambda} = \lambda h(\lambda)$ for some function h *of* λ which is analytic at $\lambda = 0$.

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