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Modular construction of complete coalgebraic logics

Corina Cîrstea[a,*], Dirk Pattinson[b]

[a] School of Electronics and Computer Science, University of Southampton, UK
[b] Department of Computing, Imperial College London, UK

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Abstract

We present a modular approach to defining logics for a wide variety of state-based systems. The systems are modelled as coalgebras, and we use modal logics to specify their observable properties. We show that the syntax, semantics and proof systems associated with such logics can all be derived in a modular fashion. Moreover, we show that the logics thus obtained inherit soundness, completeness and expressiveness properties from their building blocks. We apply these techniques to derive sound, complete and expressive logics for a wide variety of probabilistic systems, for which no complete axiomatisation has been obtained so far.

Keywords: Coalgebra; Modal logic; Completeness; Probabilistic systems

1. Introduction

Modularity has been a key concern in software engineering since the conception of the discipline [22]. This paper investigates modularity not in the context of building software systems, but in connection with specifying and reasoning about systems. Our work focuses on reactive systems, which are modelled as coalgebras over the category of sets and functions. The coalgebraic approach provides a uniform framework for modelling state-based and reactive systems [28]. In particular, coalgebras provide models for a large class of probabilistic systems, as shown by the recent survey [3], which discusses the coalgebraic modelling of eight different types of probabilistic systems.

In the coalgebraic approach, a system consists of a state space C and a function $\gamma : C \to TC$, which maps every state $c \in C$ to the observations $\gamma(c)$ that can be made of c after one transition step. Different types of systems can then be represented by varying the type T of observations. A closer look at the coalgebraic modelling of state-based and reactive systems reveals that, in nearly all cases of interest, the type T of observations arises as the composite of a small number of basic constructs.

The main goal of this paper is to lift this compositionality which exists at the level of observations to the level of specification logics and proof systems. That is, we associate a specification logic and a proof system to every basic

* Corresponding author. Tel.: +44 2380593625.
E-mail addresses: cc2@ecs.soton.ac.uk (C. Cîrstea), dirk@doc.ic.ac.uk (D. Pattinson).

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type construct, and show how to obtain specification logics and proof systems for a combination of constructs, in terms of the ingredients of the construction. Our main technical contribution is the study of the properties which are preserved by a combination of logics and proof systems. On the side of logics, we isolate a property which ensures that combined logics have the Hennessy–Milner property w.r.t. behavioural equivalence, that is, the logical equivalence of states coincides with behavioural equivalence. Since this property is present in all of the basic constructs and is preserved by each combination of constructs, we automatically obtain expressive specification logics for a large class of systems. Concerning proof systems, our main interests are the soundness and completeness of the combined logical systems. In order to guarantee both, we investigate conditions which ensure that soundness and completeness of a combination of logics are inherited from the corresponding properties of the ingredients of the construction. Again, we demonstrate that this property is present in all basic building blocks.

As an immediate application of our compositional approach, we obtain sound, complete and expressive specification logics for a large class of probabilistic system types. While the Hennessy–Milner property has already been established for a number of such logics [19,13,9], sound and complete axiomatisations for these logics have only been studied for a simple type of probabilistic systems, namely unlabelled probabilistic transition systems [10]. The modular approach presented in this paper allows us to also derive a completeness result for the probabilistic modal logic of [13] (interpreted over Segala's simple probabilistic automata [29]), as well as a logic with a sound and complete axiomatisation for Segala's general probabilistic automata [29].

Our main technical tool to establish the above results is the systematic exploitation of the fact that coalgebras model the one-step behaviour of systems, i.e. one application of the coalgebra map allows us to extract information about one transition step of the system being modelled. This one-step behaviour of systems is parallelled both on the level of specification logics and on the level of proof systems. Regarding specification logics, we introduce the notion of *syntax constructor*, which specifies a set of syntactic features allowing the formulation of assertions about the next transition step of a system. The notion of *one-step semantics* then specifies how to interpret these syntactic features over the next transition step. Finally, a *proof system constructor* specifies how one can infer judgements about the next transition step.

These notions are used to make assertions about the global system behaviour, by viewing this behaviour as the stratification of the observations which can be made after a (finite) number of transition steps. This is again parallelled both on the level of logics and on the level of proof systems. Completeness, for example, can be established by isolating the corresponding one-step notion, which we call *one-step completeness*, and proving by induction on the number of transition steps that one-step completeness entails completeness in the ordinary sense. Expressiveness and soundness are treated similarly by considering the associated notions of one-step expressiveness and one-step soundness. When combining the logics, we combine the syntax constructors, the associated one-step semantics and the proof system constructors, and show that such combinations preserve one-step soundness, completeness and expressiveness.

Our approach generalises previous work on coalgebraic modal logic, including the abstract coalgebraic logic of Moss [21], and the concrete logics for coalgebras proposed in [17,27,11]. In particular, our approach provides logics with sound and complete axiomatisations for probabilistic models. Moreover, thanks to the modular treatment of languages and their associated semantics, our logics are easily extensible to accommodate more features of state-based systems. A consequence of this wider generality is that our logics fail to be strongly complete, and accordingly our treatment is focused on weak completeness instead.

Regarding further work, we plan to extend our approach to more expressive logics, in particular to coalgebraic versions of CTL [7] and of the modal μ-calculus [16]. Also, it remains to be explored in what way our setup induces logics for programming languages with coalgebraically defined semantics [30,14,2].

2. Preliminaries and notation

We denote the category of sets and functions by Set, and pick a final object $1 = \{*\}$. Binary products (coproducts) in Set are written $X_1 \times X_2$ ($X_1 + X_2$), with canonical projections $\pi_i : X_1 \times X_2 \to X_i$ (canonical injections $\iota_i : X_i \to X_1 + X_2$) for $i = 1, 2$. If $R \subseteq X_1 \times X_2$ is a relation and $T : \mathsf{Set} \to \mathsf{Set}$ is an endofunctor, we write $TR \subseteq TX_1 \times TX_2$ for the relation defined by $t_1\,(TR)\,t_2$ if there exists $w \in TR$ such that $T\pi_1(w) = t_1$ and $T\pi_2(w) = t_2$. Finally, the set of functions from Y to X is denoted by X^Y.

We write $M : \mathsf{Set} \to \mathsf{Set}$ for the (inclusion-preserving) functor taking a set A (of atoms) to the set of propositional formulas built from atoms in A together with the propositional constant false (ff), by closing under implication ($\to$). The remaining propositional connectives can be defined on MA in terms of ff and $\to$ in the usual way.

We use endofunctors $T : \mathsf{Set} \to \mathsf{Set}$, subsequently referred to as *signature functors*, to specify particular system types. A signature functor T specifies the structure of the information which can be observed of the system states *in one step*. Systems themselves are modelled as T-coalgebras.

Definition 1 (*Coalgebras, Morphisms*). A *T-coalgebra* is a pair (C, γ) with C a set (the *carrier* or *state space* of the coalgebra) and $\gamma : C \to TC$ a function (the *coalgebra map*, or *transition structure*). A *T-coalgebra morphism* $f : (C, \gamma) \to (D, \delta)$ is a function $f : C \to D$ such that $Tf \circ \gamma = \delta \circ f$. The category of T-coalgebras and T-coalgebra morphisms is denoted by $\mathsf{CoAlg}(T)$.

For $(C, \gamma) \in \mathsf{CoAlg}(T)$, the transition structure γ determines the observations $\gamma(c) \in TC$ which can be made from a state $c \in C$ in one transition step. Morphisms between coalgebras preserve this one-step behaviour. The next example shows that coalgebras can be used to model a wide variety of state-based systems, including non-deterministic and probabilistic ones:

Example 2. We write $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ for the (covariant) powerset functor, $\mathcal{P}_\omega : \mathsf{Set} \to \mathsf{Set}$ for the finite powerset functor (i.e. $\mathcal{P}_\omega(X) = \{Y \subseteq X \mid Y \text{ finite}\}$), and $\mathcal{D} : \mathsf{Set} \to \mathsf{Set}$ for the finite probability distribution functor, defined by

$$\mathcal{D}X = \left\{ \mu : X \to [0, 1] \mid \mu(x) \neq 0 \text{ for finitely many } x \in X \text{ and } \sum_{x \in X} \mu(x) = 1 \right\}.$$

(1) For $TX = (\mathcal{P}X)^A$, T-coalgebras $\gamma : C \to (\mathcal{P}C)^A$ are in one-to-one correspondence with A-labelled transition systems (C, R), where $R \subseteq C \times A \times C$ is defined by $(c, a, c') \in R$ iff $c' \in \gamma(c)(a)$. Similarly, coalgebras for $TX = (\mathcal{P}_\omega X)^A$ are precisely the image-finite transition systems.
(2) For $TX = \mathcal{P}X \times \mathcal{P}D$, T-coalgebras $\gamma : C \to \mathcal{P}C \times \mathcal{P}D$ are in one-to-one correspondence with Kripke models (C, R, V) over the set D of propositional atoms, with the accessibility relation R being given by $(c, c') \in R$ iff $c' \in \pi_1(\gamma(c))$, and with the valuation $V : C \to \mathcal{P}D$ being given by $\pi_2 \circ \gamma$. Similarly, every $\mathcal{P}$-coalgebra determines a Kripke frame and vice-versa.
(3) Coalgebras for $TX = (1 + \mathcal{D}X)^A$ are the A-labelled probabilistic transition systems of [19] (see [8] for details). These have also been called reactive probabilistic systems in [31], and are different from the probabilistic transition systems considered in [13], which are treated next.
(4) The simple and general probabilistic automata of [29] can be modelled as coalgebras for $TX = (\mathcal{P}\mathcal{D}X)^A$ and $TX = \mathcal{P}(\mathcal{D}(A \times X))$, respectively. Replacing the unbounded powerset functor by the finite powerset functor in these definitions yields image-finite variants of these two types of systems. The image-finite simple probabilistic automata are called probabilistic transition systems in [13].

Note that all the endofunctors in the previous example arise as combinations of a small number of simple functors (constant, identity, powerset and probability distribution functor) using products, coproducts, exponentiation with constant sets and composition. A recent survey of existing probabilistic models of systems [3] identified no less than eight probabilistic system types of interest, all of which can be written as such combinations. This paper derives logics and proof systems for these probabilistic system types, using similar combinations on the logical level.

Apart from making this kind of compositionality explicit, the coalgebraic approach also allows a uniform definition of behavioural equivalence, which specialises to standard notions of equivalence in many important examples.

Definition 3 (*Behavioural Equivalence*). Given two T-coalgebras (C, γ) and (D, δ), two states $c \in C$ and $d \in D$ are called *behaviourally equivalent* (written $c \simeq d$) if there exist T-coalgebra morphisms $f : (C, \gamma) \to (E, \epsilon)$ and $g : (D, \delta) \to (E, \epsilon)$ such that $f(c) = g(d)$.

Any T-coalgebra (C, γ) induces an ω-indexed sequence of maps $\gamma_n : C \to T^n 1$, where T^n denotes the n-fold application of the signature functor T. The maps γ_n are defined by induction on n: $\gamma_0 : C \to 1$ is the unique such map, and $\gamma_{n+1} = T\gamma_n \circ \gamma$ for $n \in \omega$. Intuitively, $T^n 1$ contains all possible T-behaviours observable through n unfoldings of the coalgebra structure, while γ_n maps states of the coalgebra to their n-step observable behaviour. A notion of observational equivalence which only takes finitely observable behaviour into account can now be defined as follows:

Definition 4 (*ω-Behavioural Equivalence*). Given T-coalgebras (C, γ) and (D, δ), two states $c \in C$ and $d \in D$ are called *ω-behaviourally equivalent* (written $c \simeq_\omega d$) if $\gamma_n(c) = \delta_n(d)$ for all $n \in \omega$.

Remark 5. The notion of ω-behavioural equivalence is strictly weaker than behavioural equivalence. However, for endofunctors whose final sequence stabilises in at most $\omega + \omega$ steps, the two notions coincide (see [32,25]). The class of endofunctors with this property includes the ω-accessible ones, and is closed under arbitrary coproducts of functors, as well as under $\mathcal{I}$-indexed limits of functors, with $\mathcal{I}$ a small category [32].

As a result, ω-behavioural equivalence coincides with behavioural equivalence both in the case of image-finite transition systems, and in the case of probabilistic transition systems (with each of the functors ${\mathcal{P}_\omega}^A$ and $(1 + \mathcal{D})^A$ being obtained as a limit of ω-accessible functors). It is often possible to define finitary logics for which logical equivalence coincides with ω-behavioural equivalence. On the other hand, we cannot in general hope to characterise behavioural equivalence by a logic with finitary syntax.

It can also be shown that, for weak pullback preserving endofunctors, the notion of behavioural equivalence coincides with the span-based notion of coalgebraic bisimilarity, as defined by Aczel and Mendler [1] and studied by Rutten [28]. All functors considered in the following are weak pullback preserving.

Definition 6 (*Bisimulation, Bisimilarity*). A *T-bisimulation* between two T-coalgebras (C, γ) and (D, δ) is given by a relation $R \subseteq C \times D$ that can be equipped with a T-coalgebra structure (R, ρ) which makes the projections $\pi_1^R : R \to C$ and $\pi_2^R : R \to D$ T-coalgebra morphisms. The largest T-bisimulation between (C, γ) and (D, δ) is called *T-bisimilarity*.

Coalgebraic bisimulation (bisimilarity) instantiates to known notions of bisimulation (bisimilarity) in concrete cases.

Example 7. We consider the system types introduced in Example 2.

(1) For A-labelled transition systems, i.e. coalgebras for $TX = (\mathcal{P}X)^A$, T-bisimulation coincides with Park–Milner bisimulation [23,20].
(2) For Kripke models over the set D of propositional atoms, i.e. coalgebras for $TX = \mathcal{P}X \times \mathcal{P}D$, T-bisimulation coincides with the standard notion of bisimulation, as defined e.g. in [4].
(3) For coalgebras for $TX = (1+\mathcal{D}X)^A$, that is, A-labelled probabilistic transition systems, T-bisimulation coincides with the notion of probabilistic bisimulation considered in [19]. (This is proved in [8].)
(4) For coalgebras for $TX = (\mathcal{P}\mathcal{D}X)^A$, that is, simple probabilistic automata, T-bisimulation coincides with the stronger notion of bisimulation defined in [13]. (This is called simply *bisimulation* in [13], in order to distinguish it from a weaker notion of equivalence referred to as *probabilistic bisimulation*.)

A more detailed analysis of probabilistic systems from a coalgebraic point of view can be found in [3].

3. Modular construction of modal languages

In this section we introduce *syntax constructors* and the modal languages they define. If we consider a modal language $\mathcal{L}$ as an extension of the basic propositional language, the idea of a syntax constructor is that it describes what needs to be added to the propositional language in order to obtain $\mathcal{L}$. The important feature of syntax constructors is that they can be combined in the same way as the signature functors which define the particular shape of the systems under consideration.

After introducing the abstract concept, we give examples of some basic syntax constructors, and show how they can be combined in order to obtain more complex syntax constructors, and hence more structured modal languages.

Definition 8 (*Syntax Constructor, Induced Language*). A *syntax constructor* is an ω-accessible endofunctor $\mathsf{S} : \mathsf{Set} \to \mathsf{Set}$ which preserves inclusions, i.e. $\mathsf{S}X \subseteq \mathsf{S}Y$ whenever $X \subseteq Y$. The *language* $\mathcal{L}(\mathsf{S})$ associated with a syntax constructor S is the least set F (of formulas) such that

- $\mathsf{ff} \in F$,
- $\varphi \to \psi \in F$ whenever $\varphi, \psi \in F$,
- $\mathsf{S}F \subseteq F$.

In the above definition we silently assume that, for any set X, the elements of $\mathsf{S}X$ are sufficiently fresh, in the sense that $\mathsf{ff} \notin \mathsf{S}X$ and $\varphi \to \psi \notin \mathsf{S}X$ whenever $\varphi, \psi \in X$.

The assumptions in Definition 8 ensure that the language associated with a syntax constructor S carries the structure of an initial L-algebra, where $LX = 1 + (X \times X) + \mathsf{S}X$. Of course, one could have taken $\mathcal{L}(\mathsf{S})$ as (the carrier of) the initial L-algebra directly, but the present treatment avoids abstract syntax and allows us to construct languages as least fixpoints for monotone operators on sets in the usual way.

Recall that the image $\mathsf{S}X$ of a set X under an inclusion-preserving and ω-accessible endofunctor S can always be reconstructed from the sets $\mathsf{S}Y$ for finite subsets $Y \subseteq X$. More formally, if X is a set and $x \in \mathsf{S}X$, we can always find a finite subset $Y \subseteq X$ such that $x \in \mathsf{S}Y$. The requirement that $\mathsf{S}F \subseteq F$ in Definition 8 is therefore equivalent to $\mathsf{S}\Phi \subseteq F$ whenever Φ is a finite subset of F. Thus, the ω-accessibility of S ensures that the construction of $\mathcal{L}(\mathsf{S})$ terminates after ω steps, that is, we are dealing with finitary languages. Technically, we often use induction on the rank (nesting depth of modal operators) of a formula as a proof principle, and ω-accessibility guarantees that the induction terminates at ω.

Example 9. (1) If D is a set (of atomic propositions), then the constant functor $\mathsf{S}_D L = D$ is a syntax constructor. The associated language $\mathcal{L}(\mathsf{S}_D)$ is the set of propositional formulas over the set D of atoms.

(2) If $\mathcal{I}d : \mathsf{Set} \to \mathsf{Set}$ is the identity functor, then the functor $\mathsf{S}_{\mathcal{I}d} : \mathsf{Set} \to \mathsf{Set}$ which maps a set L to the set $\mathsf{S}_{\mathcal{I}d} L = \{ \circ\varphi \mid \varphi \in L \}$ is a syntax constructor. The associated language is similar to the standard modal language over the empty set of atomic propositions. However, this language will be interpreted over $\mathcal{I}d$-coalgebras, which provide a trivial model of deterministic systems.

(3) If M is a (possibly infinite) set of modal operators with associated finite arities, then S_M is a syntax constructor, where S_M maps a set L (of formulas) to the set $\mathsf{S}_\mathsf{M}(L)$ of formal expressions, given by

$$\mathsf{S}_\mathsf{M}(L) = \{m(\varphi_1, \ldots, \varphi_n) \mid m \in \mathsf{M} \text{ is } n\text{-ary and } \varphi_1, \ldots, \varphi_n \in L\}.$$

Viewing M as an algebraic signature, $\mathsf{S}_\mathsf{M}(L)$ is the set of terms with exactly one function symbol applied to variables in L. In the literature on modal logic, M is called a modal similarity type [4]. The language associated with S_M is the set of modal formulas with modalities in M over the empty set of propositional variables. When writing such formulas, we shall assume that the modal operators bind more tightly than any of the boolean operators. For later reference, we let $\mathsf{S}_\mathcal{P} = \mathsf{S}_{\mathcal{P}_\omega} = \mathsf{S}_{\{\Box\}}$ where $\Box$ has arity one, and $\mathsf{S}_\mathcal{D} = \mathsf{S}_\mathsf{M}$ where $\mathsf{M} = \{L_p \mid p \in \mathbb{Q} \cap [0, 1]\}$, $\mathbb{Q}$ denotes the set of rational numbers, and each L_p has arity one. The language associated with $\mathsf{S}_\mathcal{P}$ is the standard modal language over the empty set of atomic propositions. The language associated with $\mathsf{S}_\mathcal{D}$ has a countable number of unary modalities, and has been used to describe properties of probabilistic transition systems [19,13,10]. The intended reading of a formula $L_p\varphi$ is "the probability of φ holding in the next state is at least p".

(4) If T is an inclusion-preserving, ω-accessible endofunctor, then $\mathsf{S} = T$ qualifies as a syntax constructor, and the associated language $\mathcal{L}(\mathsf{S})$ is a variant of Moss's coalgebraic logic for the functor T. In the original treatment [21], the language is infinitary and only has modal operators (obtained using functor application) and infinitary conjunctions. In contrast, the language $\mathcal{L}(\mathsf{S})$ is finitary and comes with all standard propositional connectives.

We are now ready for the first modularity issue of the present paper: the combination of syntax constructors to build more powerful languages from simple ingredients.

Definition 10 (*Combinations of Syntax Constructors*). Consider the following operations on sets L_1, L_2 (of formulas):

$$L_1 \otimes L_2 = \{[\pi_i]\varphi \mid \varphi \in L_i,\ i = 1, 2\}$$
$$L_1 \oplus L_2 = \{\langle\kappa_i\rangle\varphi \mid \varphi \in L_i,\ i = 1, 2\}$$
$$L_1 \odot E = \{[e]\varphi \mid \varphi \in L_1,\ e \in E\}$$

where E is an arbitrary set. For syntax constructors $\mathsf{S}_1, \mathsf{S}_2$ we let

$$(\mathsf{S}_1 \otimes \mathsf{S}_2)L = M\mathsf{S}_1 L \otimes M\mathsf{S}_2 L \qquad (\mathsf{S}_1 \oplus \mathsf{S}_2)L = M\mathsf{S}_1 L \oplus M\mathsf{S}_2 L$$
$$(\mathsf{S}_1 \odot E)L = M\mathsf{S}_1 L \odot E \qquad (\mathsf{S}_1 \circledcirc \mathsf{S}_2)L = \mathsf{S}_1 M\mathsf{S}_2 L.$$

Recall that M takes a set to the set of propositional formulas over that set. Note that the above operations are of a purely syntactical nature. The addition of the symbols $[\pi_i]$, $\langle\kappa_i\rangle$ and $[e]$ will later ensure that the languages associated with $\mathsf{S}_1 \otimes \mathsf{S}_2$, $\mathsf{S}_1 \oplus \mathsf{S}_2$ and $\mathsf{S}_1 \odot E$ can be given a well-defined semantics.

When combining syntax constructors, we add another layer of modal operators to the syntax already defined. Closure under propositional connectives (through the application of M) is needed to express propositional judgements also at the level on which the construction operates, e.g. to have formulas of the form $\langle\kappa_i\rangle(\neg\Box\varphi \wedge \Box\psi)$ in $\mathcal{L}(\mathsf{S}_{\mathcal{P}} \oplus \mathsf{S}_{\mathcal{P}})$.

The above definition is modelled after the definition of signature functors. Languages of the form $\mathcal{L}(\mathsf{S}_1 \otimes \mathsf{S}_2)$ and $\mathcal{L}(\mathsf{S}_1 \oplus \mathsf{S}_2)$ will be used to formalise properties of systems whose signature functors are of the form $T_1 \times T_2$ and $T_1 + T_2$, respectively, while the language $\mathcal{L}(\mathsf{S}_1 \odot E)$ provides a means to reason about systems with signature functor ${T_1}^E$. The clause dealing with the composition of syntax constructors gives rise to S_1-modal operators which are indexed by S_2-formulas. Alternatively, the composition of two syntax constructors can be thought of as introducing an additional sort for formulas, as illustrated in the next example.

Example 11. Suppose $\mathsf{S}_i L = \{\Box_i \varphi \mid \varphi \in L\}$ for $i = 1, 2$. Then the language $\mathcal{L} = \mathcal{L}(\mathsf{S}_1 \circledcirc \mathsf{S}_2)$ can be described by the following grammar:

$$\mathcal{L} \ni \varphi, \psi ::= \mathsf{ff} \mid \varphi \to \psi \mid \Box_1 \rho \qquad (\rho \in \mathcal{L}')$$
$$\mathcal{L}' \ni \rho, \sigma ::= \mathsf{ff} \mid \rho \to \sigma \mid \Box_2 \varphi \qquad (\varphi \in \mathcal{L}).$$

Languages of this kind have a two-layer structure, corresponding to systems that exhibit two different types of behaviour (modelled by $\Box_1$ and $\Box_2$, respectively) in an alternating fashion. They are used to specify properties of systems whose signature functor T is the composition of two functors: $T = T_1 \circ T_2$. In order to capture all possible behaviours described by T, we first have to describe the T_2-behaviours, and then use these descriptions to specify the observations which can be made according to T_1. Since propositional connectives will in general be necessary to capture all possible T_2-behaviours, the definition of the syntax constructor $\mathsf{S}_1 \circledcirc \mathsf{S}_2$ involves the closure under propositional connectives before applying S_1. Thus, the introduction of new sorts in the approach of [27,11] can be explained as the construction of logics for the composition of two signature functors.

The next proposition shows that the constructions in Definition 10 indeed give rise to syntax constructors:

Proposition 12. *$\mathsf{S}_1 \otimes \mathsf{S}_2$, $\mathsf{S}_1 \oplus \mathsf{S}_2$, $\mathsf{S}_1 \odot E$ and $\mathsf{S}_1 \circledcirc \mathsf{S}_2$ are syntax constructors.*

Proof. It is clear that all four constructions are inclusion-preserving and functorial. The fact that they are ω-accessible is immediate from the fact that accessibility is preserved under arbitrary coproducts and functor composition in Set, and from the accessibility of M. $\Box$

In ordinary modal logic, the modal language $\mathcal{L} = \mathcal{L}(\mathsf{S}_{\{\Box\}})$ can be viewed as the stratification $\mathcal{L} = \bigcup_{n \in \omega} \mathcal{L}^n$, where $\mathcal{L}^n$ contains all modal formulas of rank $\leq n$. A similar characterisation holds for the language associated with an arbitrary syntax constructor. This, in particular, will allow us to use induction on the rank of formulas as a proof principle.

Definition 13. Suppose S is a syntax constructor. Let $\mathcal{L}^0(\mathsf{S}) = M\emptyset$, and $\mathcal{L}^{n+1}(\mathsf{S}) = (M \circ \mathsf{S})(\mathcal{L}^n(\mathsf{S}))$ for $n \in \omega$. If $\varphi \in \mathcal{L}^n(\mathsf{S})$, we say that φ *has rank at most* n.

If $\mathsf{S} = \mathsf{S}_{\mathsf{M}}$ for a set M of modal operators, then $\mathcal{L}^n(\mathsf{S})$ contains all modal formulas with modal operators in M whose nesting depth of modal operators is at most n.

The fact that $\mathcal{L}^n(\mathsf{S})$ for $n \in \omega$ constitutes a stratification of $\mathcal{L}(\mathsf{S})$ is the content of the next result.

Proposition 14. *The following hold:*

(1) $\mathcal{L}^n(\mathsf{S}) \subseteq \mathcal{L}^{n+1}(\mathsf{S})$ *for all* $n \in \omega$;

(2) $\mathcal{L}(\mathsf{S}) = \bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S})$.

Proof. While the inclusion in the first of the above statements might initially appear slightly surprising, it should be noted that the application of M introduces the constant truth values ff and tt (as ff → ff), and as a result, $\mathcal{L}^n(\mathsf{S})$ also contains formulas of rank strictly smaller than n.

Induction on n and the preservation of inclusions by S and M are used to prove both the first statement and the inclusion $\bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S}) \subseteq \mathcal{L}(\mathsf{S})$. The inclusion $\mathcal{L}(\mathsf{S}) \subseteq \bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S})$ follows by induction on the structure of formulas: First, $\mathrm{ff} \in \mathcal{L}^0(\mathsf{S}) \subseteq \bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S})$. Second, if $\varphi, \psi \in \bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S})$, and hence $\varphi, \psi \in \mathcal{L}^n(\mathsf{S})$ for some $n \in \omega$, then closure of $\mathcal{L}^n(\mathsf{S})$ under boolean connectives gives $\varphi \to \psi \in \mathcal{L}^n(\mathsf{S}) \subseteq \bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S})$. Finally, if $\varphi \in \mathsf{S}(\bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S}))$, then by ω-accessibility of S, $\varphi \in \mathsf{S}(\Phi)$ for some finite set $\Phi \subseteq \bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S})$. Now Φ finite together with $\mathcal{L}^n(\mathsf{S}) \subseteq \mathcal{L}^{n+1}(\mathsf{S})$ give $\Phi \subseteq \mathcal{L}^n(\mathsf{S})$ for some $n \in \omega$. Then, $\varphi \in \mathsf{S}\Phi$ implies $\varphi \in \mathsf{S}\mathcal{L}^n(\mathsf{S})$ (as S preserves inclusions), and therefore $\varphi \in M\mathsf{S}\mathcal{L}^n(\mathsf{S}) = \mathcal{L}^{n+1}(\mathsf{S}) \subseteq \bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S})$. This concludes the proof of $\mathcal{L}(\mathsf{S}) \subseteq \bigcup_{n\in\omega} \mathcal{L}^n(\mathsf{S})$. $\square$

Corollary 15. $(M \circ \mathsf{S})(\mathcal{L}(\mathsf{S})) = \mathcal{L}(\mathsf{S})$.

Proof. The definition of $\mathcal{L}(\mathsf{S})$ gives $(M \circ \mathsf{S})(\mathcal{L}(\mathsf{S})) \subseteq \mathcal{L}(\mathsf{S})$, while the second statement of Proposition 14 combined with induction on n prove the reverse inclusion. $\square$

For the subsequent development, it will be useful to regard the languages $\mathcal{L}^n(\mathsf{S})$ as closures under boolean connectives of certain sets (of atoms).

Definition 16. The sets $\mathcal{A}^n(\mathsf{S})$ of *atoms of rank* n, with $n \in \omega$, are defined by:

- $\mathcal{A}^0(\mathsf{S}) = \emptyset$;
- $\mathcal{A}^{n+1}(\mathsf{S}) = (\mathsf{S} \circ M)(\mathcal{A}^n(\mathsf{S}))$ for $n \in \omega$.

Some of the properties of the sets $\mathcal{A}^n(\mathsf{S})$ are given next.

Proposition 17. *The following hold:*

(1) $M\mathcal{A}^n(\mathsf{S}) = \mathcal{L}^n(\mathsf{S})$ *for all* $n \in \omega$*;*
(2) $\mathcal{A}^n(\mathsf{S}) \subseteq \mathcal{A}^{n+1}(\mathsf{S})$ *for all* $n \in \omega$*;*
(3) *Let* $\mathcal{A}(\mathsf{S}) = \bigcup_{n\in\omega} \mathcal{A}^n(\mathsf{S})$. *Then* $\mathsf{S}(\mathcal{L}(\mathsf{S})) = \mathcal{A}(\mathsf{S})$ *and* $\mathcal{L}(\mathsf{S}) = M(\mathcal{A}(S))$.

Proof. The first two statements follow by induction on n. For the third statement, the ω-accessibility of S together with Proposition 14 and the definition of $\mathcal{A}(\mathsf{S})$ are used to prove $\mathsf{S}(\mathcal{L}(\mathsf{S})) \subseteq \mathcal{A}(\mathsf{S})$, while induction on n, the preservation of inclusions by S and M, and Corollary 15 prove the reverse inclusion. A subsequent application of M and use of Corollary 15 yields $\mathcal{L}(\mathsf{S}) = M(\mathcal{A}(S))$. $\square$

4. Modular construction of coalgebraic semantics

In the previous section, we have argued that a syntax constructor with associated language $\mathcal{L}$ specifies those features which have to be added to the propositional language in order to obtain $\mathcal{L}$. In standard modal logic, this boils down to adding the modal operator $\Box$, which can be used to describe the observable behaviour after one transition step. Abstracting from this example, we now introduce the *one-step semantics* for a syntax constructor, which relates the additional modal structure (specified by a syntax constructor) to the observations (specified by a signature functor) which can be made of a system in one transition step. Throughout this section, S denotes a syntax constructor and T is an endofunctor. Also, we write $\hat{\mathcal{P}} : \mathsf{Set}^{\mathrm{op}} \to \mathsf{Set}$ for the contravariant powerset functor. To simplify the notation, we make no distinction between (elements of) the set $\hat{\mathcal{P}}X$ and (elements of) the set $\mathcal{P}X$.

Definition 18 (*One-Step Semantics*). If L is a set (of formulas) and X is a set (of points), then an *interpretation of* L *over* X is a function $d : L \to \mathcal{P}X$. A *morphism between interpretations* $d : L \to \mathcal{P}X$ and $d' : L' \to \mathcal{P}X'$ is a pair (t, f) with $t : L \to L'$ and $f : X' \to X$, such that $d' \circ t = \hat{\mathcal{P}}f \circ d$:

$$\begin{array}{ccc} L & \xrightarrow{t} & L' \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d'} \\ \mathcal{P}X & \xrightarrow{\hat{\mathcal{P}}f} & \mathcal{P}X' \end{array}$$

A *one-step semantics* $[\![\mathsf{S}]\!]^T$ *for a syntax constructor* S *w.r.t. an endofunctor* T maps interpretations of L over X to interpretations of $\mathsf{S}L$ over TX, in such a way that whenever $(t, f) : d \to d'$ is a morphism of interpretations, so is $(\mathsf{S}t, Tf) : [\![\mathsf{S}]\!]^T(d) \to [\![\mathsf{S}]\!]^T(d')$. We omit the superscript on the one-step semantics if the endofunctor T is clear from the context.

A one-step semantics provides the glue between a language constructor S and a signature functor T. The requirement that $[\![\mathsf{S}]\!]^T$ preserves morphisms of interpretations ensures that $[\![\mathsf{S}]\!]^T$ is defined uniformly on interpretations. This will subsequently guarantee that the coalgebraic semantics of the induced language $\mathcal{L}(\mathsf{S})$ is adequate w.r.t. behavioural equivalence, i.e. behaviourally equivalent states of coalgebras cannot be distinguished using formulas of the language.

Remark 19. Let Int be the category whose objects are interpretations and whose arrows are morphisms between interpretations. (That is, Int is the comma category $\mathcal{I}d \downarrow 2^-$.) Also, let $V : \mathsf{Int} \to \mathsf{Set}$ ($W : \mathsf{Int} \to \mathsf{Set}^{\mathrm{op}}$) take $d : L \to \mathcal{P}X$ to L (respectively X), and (t, f) to t (respectively f). A one-step semantics for S w.r.t. T can alternatively be defined as a functor $[\![\mathsf{S}]\!]^T : \mathsf{Int} \to \mathsf{Int}$ such that $V \circ [\![\mathsf{S}]\!]^T = \mathsf{S} \circ V$ and $W \circ [\![\mathsf{S}]\!]^T = T^{\mathrm{op}} \circ W$:

$$\begin{tikzcd}
\mathsf{Set} \arrow[r, "\mathsf{S}"] & \mathsf{Set} \\
\mathsf{Int} \arrow[u, "V"] \arrow[r, "{[\![\mathsf{S}]\!]^T}"] \arrow[d, "W"'] & \mathsf{Int} \arrow[u, "V"'] \arrow[d, "W"] \\
\mathsf{Set}^{\mathrm{op}} \arrow[r, "T^{\mathrm{op}}"'] & \mathsf{Set}^{\mathrm{op}}
\end{tikzcd}$$

Equivalently, if one regards the category Int as a *fibred span* [12, Section 9.1] over the categories Set and $\mathsf{Set}^{\mathrm{op}}$, a one-step semantics for S w.r.t. T can be defined as a lifting $[\![\mathsf{S}]\!]^T$ of $\mathsf{S} \times T^{\mathrm{op}}$ to such spans:

$$\begin{tikzcd}
\mathsf{Int} \arrow[r, "{[\![\mathsf{S}]\!]^T}"] \arrow[d, "{\langle V,W\rangle}"'] & \mathsf{Int} \arrow[d, "{\langle V,W\rangle}"] \\
\mathsf{Set} \times \mathsf{Set}^{\mathrm{op}} \arrow[r, "\mathsf{S}\times T^{\mathrm{op}}"'] & \mathsf{Set} \times \mathsf{Set}^{\mathrm{op}}
\end{tikzcd}$$

Lemma 20. *There is a one-to-one correspondence between one-step semantics* $[\![\mathsf{S}]\!]^T$ *for* S *w.r.t.* T *and natural transformations* $\delta : \mathsf{S}\hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}}T$.

Proof. Given a one-step semantics $[\![\mathsf{S}]\!]^T$ for S w.r.t. T, one can define a natural transformation $\delta : \mathsf{S}\hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}}T$ by letting $\delta_C = [\![\mathsf{S}]\!]^T(1_{\mathcal{P}C})$ for each set C. The naturality of δ follows from the functoriality of $[\![\mathsf{S}]\!]^T$. Conversely, given $\delta : \mathsf{S}\hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}}T$, one can define $[\![\mathsf{S}]\!]^T : \mathsf{Int} \to \mathsf{Int}$ by $[\![\mathsf{S}]\!]^T(d) = \delta_X \circ \mathsf{S}d$ for $d : L \to \mathcal{P}X$. The functoriality of $[\![\mathsf{S}]\!]^T$ follows from the functoriality of S and the naturality of δ. Moreover, the two mappings defined above are inverse to each other. $\square$

The previous correspondence establishes a connection with the work on dualities between categories of algebras and coalgebras, see e.g. [18], where natural transformations of the same kind are used to provide coalgebraic semantics for modal languages.

Now recall that, for a set A, MA gives the closure of A under propositional connectives. Then, interpretations $d : A \to \mathcal{P}X$ extend naturally to interpretations $d^\sharp : MA \to \mathcal{P}X$ (by mapping ff to $\emptyset$ and $\varphi \to \psi$ to $(X \setminus d^\sharp(\varphi)) \cup d^\sharp(\psi)$). Also, a one-step semantics $[\![\mathsf{S}]\!]^T$ for S w.r.t. T extends to a one-step semantics $[\![\mathsf{S} \circ M]\!]^T$ for $\mathsf{S} \circ M$ w.r.t. T, which maps an interpretation $d : A \to \mathcal{P}X$ to the interpretation $[\![\mathsf{S}]\!]^T(d^\sharp) : \mathsf{S}MA \to \mathcal{P}TX$. This corresponds to an extension of the natural transformation $\delta : \mathsf{S}\hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}}T$ from Lemma 20 to a natural transformation $\delta^\sharp : \mathsf{S}M\hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}}T$, with $\delta^\sharp$ being given by $\delta \circ \mathsf{S}\nu$; here, the natural transformation $\nu : M\hat{\mathcal{P}} \to \hat{\mathcal{P}}$ encodes the standard interpretation of boolean operators on subsets.

A variant of the notion of one-step semantics was introduced in [5,6]. For languages with unary modalities, a one-step semantics essentially corresponds to a choice of a predicate lifting for each modal operator; languages and proof systems arising in this way were previously studied in [24,25].

The key feature of a one-step semantics for a syntax constructor is that it gives rise to a semantics of $\mathcal{L}(\mathsf{S})$ w.r.t. T-coalgebras, that is, it induces a satisfaction relation between T-coalgebras and formulas of $\mathcal{L}(\mathsf{S})$. Furthermore, we can define a one-step semantics for a combination of syntax constructors in terms of some given one-step semantics for the ingredients. Before describing these constructions, we provide one-step semantics for some simple syntax constructors.

Example 21. We consider the syntax constructors introduced in Example 9.

(1) Suppose D is a set. The function which maps an arbitrary interpretation to the interpretation $d_D : D \to \mathcal{P}D$, $x \mapsto \{x\}$ defines a one-step semantics for S_D w.r.t. the constant functor $TX = D$.

(2) A one-step semantics for $\mathsf{S}_{\mathcal{I}d}$ w.r.t. $\mathcal{I}d$ is given by

$$[\![\mathsf{S}_{\mathcal{I}d}]\!](d) : \mathsf{S}_{\mathcal{I}d}L \to \mathcal{P}X \qquad [\![\mathsf{S}_{\mathcal{I}d}]\!](d)(\bigcirc\varphi) = \{x \in X \mid x \in d(\varphi)\}$$

for $d : L \to \mathcal{P}X$ and $\varphi \in L$.

(3) A one-step semantics for $\mathsf{S}_{\mathcal{P}}$ w.r.t. $\mathcal{P}$ is defined by

$$[\![\mathsf{S}_{\mathcal{P}}]\!](d) : \mathsf{S}_{\mathcal{P}}L \to \mathcal{P}\mathcal{P}X \qquad [\![\mathsf{S}_{\mathcal{P}}]\!](d)(\Box\varphi) = \{x \subseteq X \mid x \subseteq d(\varphi)\}$$

for $d : L \to \mathcal{P}X$ and $\varphi \in L$. Similarly, a one-step semantics for $\mathsf{S}_{\mathcal{P}_\omega}$ w.r.t. $\mathcal{P}_\omega$ is given by the same formula, with slightly different types:

$$[\![\mathsf{S}_{\mathcal{P}_\omega}]\!](d) : \mathsf{S}_{\mathcal{P}_\omega}L \to \mathcal{P}\mathcal{P}_\omega X \quad [\![\mathsf{S}_{\mathcal{P}_\omega}]\!](d)(\Box\varphi) = \{x \subseteq X \mid x \text{ finite}, x \subseteq d(\varphi)\}$$

where again $d : L \to \mathcal{P}X$ and $\varphi \in L$.

(4) For the syntax constructor $\mathsf{S}_{\mathcal{D}}$ associated with the probability distribution functor $\mathcal{D}$, we define a one-step semantics by

$$[\![\mathsf{S}_{\mathcal{D}}]\!](d) : \mathsf{S}_{\mathcal{D}}L \to \mathcal{P}\mathcal{D}X \qquad [\![\mathsf{S}_{\mathcal{D}}]\!](d)(L_p\varphi) = \left\{\mu \in \mathcal{D}X \mid \sum_{x \in d(\varphi)} \mu(x) \geq p\right\}$$

for $d : L \to \mathcal{P}X$ and $\varphi \in L$.

(5) If T is ω-accessible and preserves inclusions and weak pullbacks, a one-step semantics for the syntax constructor $\mathsf{S} = T$ associated with Moss's coalgebraic logic is defined by

$$[\![\mathsf{S}]\!](d) : \mathsf{S}L \to \mathcal{P}TX \qquad [\![\mathsf{S}]\!](d)(\Phi) = \{x \in TX \mid x\,(T \models_d)\,\Phi\}$$

for $d : L \to \mathcal{P}X$ and $\Phi \in TL$, where the relation $\models_d \subseteq X \times L$ is given by $x \models_d \varphi$ iff $x \in d(\varphi)$, and $x\,(T \models_d)\,\Phi$ iff there exists $r \in (T \models_d)$ such that $T\pi_1(r) = x$ and $T\pi_2(r) = \Phi$ as described in Section 2.

We now return to the claim made at the beginning of this section, and show that a one-step semantics for a syntax constructor S w.r.t. a signature functor T gives rise to an interpretation of the associated language $\mathcal{L}(\mathsf{S})$ over T-coalgebras.

Definition 22 (*Coalgebraic Semantics*). Suppose S is a syntax constructor with one-step semantics $[\![\mathsf{S}]\!]^T$ and $(C, \gamma) \in \mathsf{CoAlg}(T)$. The *coalgebraic semantics* $[\![\varphi]\!] = [\![\varphi]\!]_C \subseteq C$ of a formula $\varphi \in \mathcal{L}(\mathsf{S})$ w.r.t. a T-coalgebra (C, γ) is defined inductively on the structure of formulas by

$$[\![\mathsf{ff}]\!] = \emptyset \qquad [\![\varphi \to \psi]\!] = (C \setminus [\![\varphi]\!]) \cup [\![\psi]\!]$$

$$[\![\sigma]\!] = \hat{\mathcal{P}}\gamma([\![\mathsf{S}]\!]^T(d_\Phi)(\sigma)) \quad \text{for } \sigma \in \mathsf{S}\Phi$$

where we inductively assume that $[\![\varphi]\!]$ is already defined for $\varphi \in \Phi$ via the map $d_\Phi : \Phi \to \mathcal{P}C$. Given $c \in C$, we write $(C, \gamma, c) \models \varphi$ for $c \in [\![\varphi]\!]_C$, and $\mathrm{Th}(c) = \{\varphi \in \mathcal{L}(\mathsf{S}) \mid (C, \gamma, c) \models \varphi\}$. Finally, we write $(C, \gamma) \models \varphi$ if $(C, \gamma, c) \models \varphi$ for all $c \in C$, and $\models_T \varphi$ if $(C, \gamma) \models \varphi$ for all $(C, \gamma) \in \mathsf{CoAlg}(T)$.

Before showing that this definition captures the standard interpretation of some known modal logics, we need to show that the coalgebraic semantics is well defined, as we can have $\sigma \in \mathsf{S}\Phi$ and $\sigma \in \mathsf{S}\Psi$ for two different sets Φ, Ψ.

Lemma 23. *The coalgebraic semantics of $\mathcal{L}(\mathsf{S})$ is well defined, that is, for $(C, \gamma) \in \mathsf{CoAlg}(T)$ and $\Phi, \Psi \subseteq \mathcal{L}(\mathsf{S})$, we have $[\![\mathsf{S}]\!]^T(d_\Phi)(\sigma) = [\![\mathsf{S}]\!]^T(d_\Psi)(\sigma)$ for all $\sigma \in \mathsf{S}\Phi \cap \mathsf{S}\Psi$.*

Proof. The claim follows from the definition of a one-step semantics by considering the diagram

$$\begin{tikzcd}
\mathsf{S}\Phi \arrow[r, "\mathsf{S}i"] \arrow[d, "{[\![\mathsf{S}]\!]^T(d_\Phi)}"] & \mathsf{S}(\Phi \cup \Psi) \arrow[d, "{[\![\mathsf{S}]\!]^T(d_{\Phi\cup\Psi})}"] & \mathsf{S}\Psi \arrow[l, "\mathsf{S}j"'] \arrow[d, "{[\![\mathsf{S}]\!]^T(d_\Psi)}"] \\
\mathcal{P}(TC) \arrow[r, equal] & \mathcal{P}(TC) \arrow[r, equal] & \mathcal{P}(TC)
\end{tikzcd}$$

where $i : \Phi \to \Phi \cup \Psi$ and $j : \Psi \to \Phi \cup \Psi$ are the inclusions. $\square$

Our definition of the coalgebraic semantics generalises the semantics of modal formulas, as well as the semantics of the formulas considered in [10] and [21]:

Example 24. (1) Consider the syntax constructor $\mathsf{S}_\mathcal{P}$ defined in Example 9, and the associated semantics $[\![\mathsf{S}_\mathcal{P}]\!]$ as in Example 21. The induced coalgebraic semantics w.r.t. (C, γ) is defined inductively by

$$(C, \gamma, c) \models \Box\varphi \text{ iff } (C, \gamma, c') \models \varphi \text{ for all } c' \in \gamma(c)$$

with $c \in C$ and $\varphi \in \mathcal{L}(\mathsf{S}_\mathcal{P})$. This is the standard textbook semantics of modal logic [4].

(2) Consider the syntax constructor $\mathsf{S}_\mathcal{D}$ defined in Example 9, and the associated semantics $[\![\mathsf{S}_\mathcal{D}]\!]$ as in Example 21. The induced coalgebraic semantics w.r.t. (C, γ) is defined inductively by

$$(C, \gamma, c) \models L_p\varphi \text{ iff } \sum_{(C,\gamma,c')\models\varphi} \gamma(c)(c') \geq p$$

with $c \in C$, $\varphi \in \mathcal{L}(\mathsf{S}_\mathcal{D})$ and $p \in \mathbb{Q} \cap [0, 1]$. Note that this agrees with the semantics of the probabilistic modal logic of [10].

(3) Consider the syntax constructor $\mathsf{S} = T$ associated with Moss's coalgebraic logic, and the corresponding semantics $[\![\mathsf{S}]\!]$ as defined in Example 21. The induced coalgebraic semantics w.r.t. (C, γ) is defined inductively by

$$(C, \gamma, c) \models \varphi \text{ iff } \gamma(c)\,(T \models_C)\,\varphi$$

with $c \in C$ and $\varphi \in T(\mathcal{L}(\mathsf{S}))$ where the relation $\models_C \subseteq C \times \mathcal{L}(\mathsf{S})$ is given by $c \models_C \varphi$ iff $(C, \gamma, c) \models \varphi$. This agrees with the standard semantics of Moss's coalgebraic logic [21].

The above example shows that the coalgebraic semantics specialises to known semantics in concrete cases. We now turn to the issue of combining one-step semantics, and show that we can derive a one-step semantics for a combination of syntax constructors (see Definition 10) by combining one-step semantics for the ingredients. To make the notation bearable, we disregard the dependency on the signature functor.

Definition 25 (*Combinations of One-Step Semantics*). Let $d_1 : L_1 \to \mathcal{P}X_1$ and $d_2 : L_2 \to \mathcal{P}X_2$ be interpretations of L_1 and L_2 over X_1 and X_2, respectively, let E be an arbitrary set, and consider the functions

$$d_1 \otimes d_2 : L_1 \otimes L_2 \to \mathcal{P}(X_1 \times X_2),\ [\pi_i]\varphi \mapsto \{(x_1, x_2) \mid x_i \in d_i(\varphi)\}$$
$$d_1 \oplus d_2 : L_1 \oplus L_2 \to \mathcal{P}(X_1 + X_2),\ \langle\kappa_i\rangle\varphi \mapsto \{\iota_i(x_i) \mid x_i \in d_i(\varphi)\}$$
$$d_1 \odot E : L_1 \odot E \to \mathcal{P}(X^E),\ [e]\varphi \mapsto \{f : E \to X \mid f(e) \in d_1(\varphi)\}.$$

If $[\![\mathsf{S}_i]\!]$ is a one-step semantics for a syntax constructor S_i w.r.t. an endofunctor T_i, for $i = 1, 2$, the one-step semantics of various combinations of S_1 and S_2 are defined as follows:

$$\llbracket \mathsf{S}_1 \otimes \mathsf{S}_2 \rrbracket(d) = \llbracket \mathsf{S}_1 \rrbracket(d)^\sharp \otimes \llbracket \mathsf{S}_2 \rrbracket(d)^\sharp \ : \ M\mathsf{S}_1 L \otimes M\mathsf{S}_2 L \to \mathcal{P}(T_1 \times T_2)X$$
$$\llbracket \mathsf{S}_1 \oplus \mathsf{S}_2 \rrbracket(d) = \llbracket \mathsf{S}_1 \rrbracket(d)^\sharp \oplus \llbracket \mathsf{S}_2 \rrbracket(d)^\sharp \ : \ M\mathsf{S}_1 L \oplus M\mathsf{S}_2 L \to \mathcal{P}(T_1 + T_2)X$$
$$\llbracket \mathsf{S}_1 \odot E \rrbracket(d) = \llbracket \mathsf{S}_1 \rrbracket(d)^\sharp \odot E \ : \ (M\mathsf{S}_1 L) \odot E \to \mathcal{P}({T_1}^E)X$$
$$\llbracket \mathsf{S}_1 \circledcirc \mathsf{S}_2 \rrbracket(d) = \llbracket \mathsf{S}_1 \rrbracket(\llbracket \mathsf{S}_2 \rrbracket(d)^\sharp) \ : \ \mathsf{S}_1 M \mathsf{S}_2 L \to \mathcal{P}T_1 T_2 X$$

where $d : L \to \mathcal{P}X$.

The intuitions behind the definitions of $d_1 \otimes d_2$, $d_1 \oplus d_2$ and $d_1 \odot E$ are as follows. Assuming that formulas in L_1 and L_2 are interpreted over X_1 and X_2, respectively, we can interpret formulas in $L_1 \otimes L_2$ ($L_1 \oplus L_2$) over $X_1 \times X_2$ ($X_1 + X_2$). In the first case, a formula $[\pi_i]\varphi$ holds in $x = (x_1, x_2)$ iff φ holds in x_i. Also, $\langle \kappa_i \rangle \varphi$ holds in $x \in X_1 + X_2$ iff $x = \iota_i(x_i)$ and φ holds in x_i. Finally, $d_1 \odot E$ interprets $L_1 \odot E$ in the structure $\mathcal{P}(X^E)$, where, for a function $f : E \to X$, we have that f satisfies $[e]\varphi$ iff $f(e)$ satisfies φ.

Note that the presence of the $(\_)^\sharp$ operator in the definitions of $\llbracket \mathsf{S}_1 \otimes \mathsf{S}_2 \rrbracket$, $\llbracket \mathsf{S}_1 \oplus \mathsf{S}_2 \rrbracket$, $\llbracket \mathsf{S}_1 \odot E \rrbracket$ and $\llbracket \mathsf{S}_1 \circledcirc \mathsf{S}_2 \rrbracket$ ensures that the domains of $\llbracket \mathsf{S}_1 \otimes \mathsf{S}_2 \rrbracket(d)$, $\llbracket \mathsf{S}_1 \oplus \mathsf{S}_2 \rrbracket(d)$, $\llbracket \mathsf{S}_1 \odot E \rrbracket(d)$ and $\llbracket \mathsf{S}_1 \circledcirc \mathsf{S}_2 \rrbracket(d)$ are as required.

We now show that the combination of one-step semantics is well defined.

Proposition 26. *Suppose $\llbracket \mathsf{S}_i \rrbracket$ is a one-step semantics for S_i w.r.t. T_i, for $i = 1, 2$. Then $\llbracket \mathsf{S}_1 \otimes \mathsf{S}_2 \rrbracket$, $\llbracket \mathsf{S}_1 \oplus \mathsf{S}_2 \rrbracket$, $\llbracket \mathsf{S}_1 \odot E \rrbracket$ and $\llbracket \mathsf{S}_1 \circledcirc \mathsf{S}_2 \rrbracket$ are one-step semantics for $\mathsf{S}_1 \otimes \mathsf{S}_2$, $\mathsf{S}_1 \oplus \mathsf{S}_2$, $\mathsf{S}_1 \odot E$ and $\mathsf{S}_1 \circledcirc \mathsf{S}_2$ w.r.t. $T_1 \times T_2$, $T_1 + T_2$, ${T_1}^E$ and $T_1 \circ T_2$, respectively.*

Proof. Straightforward unfolding of the respective definitions. □

We have now seen how we can combine syntax constructors and their associated one-step semantics. This gives rise to a modular way of constructing logics for coalgebras. The following two sections present applications of this modular approach. In the next section, we show that a combination of logics has the Hennessy–Milner property if all the ingredients of the construction satisfy an expressiveness property. In the subsequent section, we show how to obtain sound and complete proof systems for a combination of logics, by suitably combining sound and complete proof systems for the building blocks.

5. Behavioural versus logical equivalence

In this section, we investigate the Hennessy–Milner property, stating that any two behaviourally equivalent points have the same logical theory, on logics arising from syntax constructors and associated one-step semantics. We introduce the notion of expressiveness for an interpretation, and show that if a one-step semantics for a syntax constructor preserves expressiveness, then the induced logic has the Hennessy–Milner property. To treat logics which arise from a combination of syntax constructors and associated one-step semantics, we show that a combination of one-step semantics preserves expressiveness if all the ingredients do. This allows us to establish the Hennessy–Milner property for combined logics in a modular fashion.

We begin with the easy part, and show that behaviourally equivalent states cannot be distinguished by formulas of a logic induced by a syntax constructor and associated one-step semantics.

Proposition 27. *Suppose S is a syntax constructor, $\llbracket \mathsf{S} \rrbracket^T$ is a one-step semantics for S w.r.t. an endofunctor T, and $(C, \gamma), (D, \delta) \in \mathsf{CoAlg}(T)$. Then, $\mathrm{Th}(c) = \mathrm{Th}(d)$ whenever $c \simeq d$, with $c \in C$ and $d \in D$.*

Proof. If $c \simeq d$, then there exist T-coalgebra morphisms $f : (C, \gamma) \to (E, \epsilon)$ and $g : (D, \delta) \to (E, \epsilon)$ such that $f(c) = g(d)$. Thus, to show that $\mathrm{Th}(c) = \mathrm{Th}(d)$, it suffices to show that $\mathrm{Th}(c) = \mathrm{Th}(f(c))$ for any $f : (C, \gamma) \to (E, \epsilon)$, or equivalently,

$$(C, \gamma, c) \models \varphi \ \text{ iff } \ (E, \epsilon, f(c)) \models \varphi$$

for any $\varphi \in \mathcal{L}(\mathsf{S})$ and f as above. The last statement follows by induction on the structure of formulas, using Definition 22 and the functoriality of $\llbracket \mathsf{S} \rrbracket^T$. □

The remainder of this section is concerned with proving the converse of Proposition 27. To this end, we introduce the notion of one-step expressiveness, which will allow us to derive the Hennessy–Milner property for a logic induced by a syntax constructor and associated one-step semantics. Moreover, we show that this condition automatically holds for a combination of syntax constructors and associated one-step semantics, if it holds for the ingredients of the construction.

The formal definition of one-step expressiveness is as follows:

Definition 28 (*One-Step Expressiveness*). (1) An interpretation $d : L \to \mathcal{P}X$ is *expressive* if the map $d^\dagger : X \to \mathcal{P}L$ given by $x \mapsto \{\varphi \in L \mid x \in d(\varphi)\}$ is injective.

(2) A one-step semantics $[\![\mathsf{S}]\!]^T$ is *one-step expressive* if $[\![\mathsf{S}]\!]^T(d^\sharp) : \mathsf{S}MA \to \mathcal{P}TX$ is expressive whenever $d : A \to \mathcal{P}X$ is.

The idea behind the notion of expressive interpretation is the following: if $d : L \to \mathcal{P}X$ is expressive, then the set $\{d(\varphi) \mid \varphi \in L\}$ contains enough predicates to distinguish individual elements of X. Thinking of the set $\mathrm{Th}(x) = \{\varphi \in L \mid x \in d(\varphi)\}$ as the theory of the point $x \in X$, then d is expressive if $\mathrm{Th}(x) = \mathrm{Th}(x')$ implies $x = x'$ for all $x, x' \in X$. Now recall that the language induced by a syntax constructor S is generated by the iterated application of the functor $\mathsf{S} \circ M$, to an empty set of atoms to begin with (Definitions 13 and 16, and Propositions 14 and 17). Thus, our concern is the preservation of expressiveness at each step. The notion of one-step expressiveness of a one-step semantics ensures that this is the case. (Recall that $d^\sharp : MA \to \mathcal{P}X$ denotes the natural extension of an interpretation $d : A \to \mathcal{P}X$ to propositional formulas over A.)

Given the correspondence between one-step semantics $[\![\mathsf{S}]\!]^T$ for S w.r.t. T and natural transformations $\delta : \mathsf{S}\hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}}T$, one can give an alternative characterisation of the one-step expressiveness of $[\![\mathsf{S}]\!]^T$ in terms of δ. To see this, let $\eta : \mathcal{I}d \Rightarrow \hat{\mathcal{P}} \circ \hat{\mathcal{P}}$ denote the unit of the adjunction $\hat{\mathcal{P}} \vdash \hat{\mathcal{P}}$, mapping $x \in X$ to $\{Y \subseteq X \mid x \in Y\}$, and let $\rho ::= \hat{\mathcal{P}}\mathsf{S}M\eta \circ \hat{\mathcal{P}}\delta^\sharp_{\hat{\mathcal{P}}} \circ \eta_{T\hat{\mathcal{P}}} : T\hat{\mathcal{P}} \Rightarrow \hat{\mathcal{P}}\mathsf{S}M$. Also, note that for an interpretation $d : L \to \mathcal{P}X$, the map $d^\dagger : X \to \mathcal{P}L$ from Definition 28 is the unique such map with the property that $d = \hat{\mathcal{P}}d^\dagger \circ \eta_L$ and $d^\dagger = \hat{\mathcal{P}}d \circ \eta_X$.

The natural transformation ρ embodies the interpretation of the language given by a syntax constructor, and has been used by Klin [15] in a category-theoretic setting, where the functor M is absorbed into the structure of the category on top of which the logic is defined.

Proposition 29. *Under the assumption that T preserves monomorphims, a one-step semantics $[\![\mathsf{S}]\!]^T$ for S w.r.t. T is one-step expressive if and only if all the components of the natural transformation ρ are monomorphisms.*

Proof. We use the correspondence between one-step semantics $[\![\mathsf{S}]\!]^T$ for T and natural transformations $\delta : \mathsf{S}\hat{\mathcal{P}} \to \hat{\mathcal{P}}T$ given in Lemma 20, and denote the inductively-defined interpretation of boolean operators over a set X by $\nu_X : M\hat{\mathcal{P}}X \to \hat{\mathcal{P}}(X)$; note that ν is natural.

Assume first that all the components of ρ are monomorphisms. To show one-step expressiveness of $[\![\mathsf{S}]\!]^T$, let $d : A \to \mathcal{P}X$ be an expressive interpretation. We have: $([\![\mathsf{S}]\!]^T(d^\sharp))^\dagger = \hat{\mathcal{P}}[\![\mathsf{S}]\!]^T(d^\sharp) \circ \eta_{TX} = \hat{\mathcal{P}}\mathsf{S}d^\sharp \circ \hat{\mathcal{P}}\delta_X \circ \eta_{TX} = \hat{\mathcal{P}}\mathsf{S}Md \circ \hat{\mathcal{P}}\mathsf{S}\nu_X \circ \hat{\mathcal{P}}\delta_X \circ \eta_{TX} = \hat{\mathcal{P}}\mathsf{S}Md \circ \hat{\mathcal{P}}\delta^\sharp_X \circ \eta_{TX} = \hat{\mathcal{P}}\mathsf{S}M\eta_A \circ \hat{\mathcal{P}}\mathsf{S}M\hat{\mathcal{P}}d^\dagger \circ \hat{\mathcal{P}}\delta^\sharp_X \circ \eta_{TX} = \hat{\mathcal{P}}\mathsf{S}M\eta_A \circ \hat{\mathcal{P}}\delta^\sharp_{\hat{\mathcal{P}}A} \circ \hat{\mathcal{P}}^2Td^\dagger \circ \eta_{TX} = \hat{\mathcal{P}}\mathsf{S}M\eta_A \circ \hat{\mathcal{P}}\delta^\sharp_{\hat{\mathcal{P}}A} \circ \eta_{T\hat{\mathcal{P}}A} \circ Td^\dagger = \rho_A \circ Td^\dagger$ using the naturality of $\delta^\sharp$ and η. Now expressiveness of d together with preservation of monomorphisms by T result in $Td^\dagger$ being a monomorphism, which, together with ρ_A being a monomorphism, result in $([\![\mathsf{S}]\!]^T(d^\sharp))^\dagger$ also being mono. Hence, $[\![\mathsf{S}]\!]^T$ is one-step expressive.

Assume now that $[\![\mathsf{S}]\!]^T$ is one-step expressive. To show that $\rho_A : T\hat{\mathcal{P}}A \Rightarrow \hat{\mathcal{P}}\mathsf{S}MA$ is a monomorphism, consider the interpretation $\eta_A : A \to \mathcal{P}\mathcal{P}A$. Since $\eta_A^\dagger = 1_{\mathcal{P}A}$ is a monomorphism, one-step expressiveness of $[\![\mathsf{S}]\!]^T$ results in $[\![\mathsf{S}]\!]^T(\eta_A^\sharp)^\dagger$ also being a monomorphism. But $[\![\mathsf{S}]\!]^T(\eta_A^\sharp)^\dagger = ([\![\mathsf{S}]\!]^T(1_{\mathcal{P}^2A}) \circ \mathsf{S}\eta_A^\sharp)^\dagger = \hat{\mathcal{P}}([\![\mathsf{S}]\!]^T(1_{\mathcal{P}^2A}) \circ \mathsf{S}\eta_A^\sharp) \circ \eta_{T\hat{\mathcal{P}}A} = \hat{\mathcal{P}}\mathsf{S}\eta_A^\sharp \circ \hat{\mathcal{P}}[\![\mathsf{S}]\!]^T(1_{\mathcal{P}^2A}) \circ \eta_{T\hat{\mathcal{P}}A} = \hat{\mathcal{P}}\mathsf{S}M\eta_A \circ \hat{\mathcal{P}}\mathsf{S}\nu_{\hat{\mathcal{P}}A} \circ \hat{\mathcal{P}}\delta_{\hat{\mathcal{P}}A} \circ \eta_{T\hat{\mathcal{P}}A} = \hat{\mathcal{P}}\mathsf{S}M\eta_A \circ \hat{\mathcal{P}}\delta^\sharp_{\hat{\mathcal{P}}A} \circ \eta_{T\hat{\mathcal{P}}A} = \rho_A$. Hence, ρ_A is a monomorphism. This concludes the proof. □

In what follows, we will refer to a one-step expressive one-step semantics simply as an *expressive one-step semantics*. Using this terminology, our first main result can be stated as follows:

Theorem 30. *If $[\![\mathsf{S}]\!]^T$ is an expressive one-step semantics, then $\mathcal{L}(\mathsf{S})$ is expressive w.r.t. $\simeq_\omega$, that is,* $\mathrm{Th}(c) = \mathrm{Th}(d)$ *iff $c \simeq_\omega d$, for all $(C, \gamma), (D, \delta) \in \mathsf{CoAlg}(T)$ and $(c, d) \in C \times D$.*

In other words, the induced logic is rich enough to distinguish any two states which exhibit different behaviours in finitely many steps.

The proof of this theorem uses induction on the rank of formulas (see Definition 13). We begin by showing that a formula of rank at most n can be semantically represented by a subset of $T^n 1$. This representation is computed by the functions d_n, which we now introduce.

Definition 31. For $n \in \omega$, the functions $d_n : \mathcal{A}^n(\mathsf{S}) \to \mathcal{P}T^n 1$ with $n \in \omega$ are defined inductively by

- $d_0 : \emptyset \to \mathcal{P}1$ is the unique such map, and
- $d_{n+1} = [\![S]\!](d_n^\sharp)$ for $n \in \omega$.

The relationship between the coalgebraic semantics of a formula $\varphi \in \mathcal{L}^n(\mathsf{S})$ and the semantical representation $d_n^\sharp(\varphi)$ is as follows:

Proposition 32 *([25, Lemma 4.10]). Let (C, γ) be a T-coalgebra and $\varphi \in \mathcal{L}^n(\mathsf{S})$. Then*

$$c \in [\![\varphi]\!]_C \;\; \textit{iff} \;\; \gamma_n(c) \in d_n^\sharp(\varphi)$$

where the functions $\gamma_n : C \to T^n 1$ with $n \in \omega$ are as in Definition 4.

Proof. Induction on n. □

Using this terminology, the proof of Theorem 30 can be given as follows:

Proof. Assume that $(C, \gamma), (D, \delta) \in \mathsf{CoAlg}(T)$, and $c \in C$ and $d \in D$ have the same logical theory, that is, $(C, \gamma, c) \models \varphi$ iff $(D, \delta, d) \models \varphi$ for all $\varphi \in \mathcal{L}(\mathsf{S})$. We have to show that $\gamma_n(c) = \delta_n(d)$ for all $n \in \omega$. This will follow from Proposition 32, if we show that ${d_n}^\dagger : T^n 1 \to \mathcal{P}(\mathcal{L}^n(\mathsf{S}))$ is injective for all $n \in \omega$. For $n = 0$, this is immediate. For $n > 0$, this follows from $d_n = [\![\mathsf{S}]\!](d_{n-1}^\sharp)$ using the one-step expressiveness of $[\![\mathsf{S}]\!]$. □

Using the fact that ω-behavioural equivalence coincides with behavioural equivalence for coalgebras of a functor whose final sequence stabilises in at most $\omega + \omega$ steps (see Remark 5), we have the following corollary:

Corollary 33. *Suppose the final sequence of T stabilises in at most $\omega + \omega$ steps and $[\![\mathsf{S}]\!]^T$ is one-step expressive. Then $\mathcal{L}(\mathsf{S})$ is expressive, that is,* $\mathrm{Th}(c) = \mathrm{Th}(d)$ *iff $c \simeq d$, for all $(C, \gamma), (D, \delta) \in \mathsf{CoAlg}(T)$ and $(c, d) \in C \times D$.*

Note that the accessibility degree of T basically limits the branching degree of T-coalgebras [25], so the above corollary is a coalgebraic Hennessy–Milner result.

It is easy to see that the one-step semantics for all the basic syntax constructors are one-step expressive:

Example 34. (1) Suppose D is a set. Then $[\![\mathsf{S}_D]\!]$ is one-step expressive, since the interpretation $d_D : D \to \mathcal{P}D$, $x \mapsto \{x\}$ is expressive.

(2) $[\![\mathsf{S}_{\mathcal{I}d}]\!]$ is one-step expressive. For, if $d : A \to \mathcal{P}X$ is an expressive interpretation, and if the formula $\varphi \in A \subseteq MA$ distinguishes two points $x \neq y \in X$ (under the interpretation provided by d), then the formula $\circ\varphi \in \mathsf{S}_{\mathcal{I}d}MA$ also distinguishes these two points (this time under the interpretation provided by $[\![\mathsf{S}_{\mathcal{I}d}]\!](d^\sharp)$).

(3) $[\![\mathsf{S}_{\mathcal{P}_\omega}]\!]$ is one-step expressive. To see this, let $d : A \to \mathcal{P}X$ be an expressive interpretation, and let $Y, Z \in \mathcal{P}_\omega X$ be such that $Y \neq Z$. Say $\emptyset \neq (Y \setminus Z) \ni y$. The expressiveness of d together with MA being closed under negation yields, for each $z \in Z$, a formula $\varphi_z \in MA$ such that $z \models_{d^\sharp} \varphi_z$ and $y \not\models_{d^\sharp} \varphi_z$. Then, the formula $\Box \bigvee_{z \in Z} \varphi_z \in \mathsf{S}_{\mathcal{P}_\omega} MA$ holds in Z but not in Y. Hence, $[\![\mathsf{S}_{\mathcal{P}_\omega}]\!](d^\sharp) : \mathsf{S}_{\mathcal{P}_\omega} MA \to \mathcal{P}(\mathcal{P}_\omega X)$ is expressive. We also note that one-step expressiveness does not hold for the unbounded powerset functor. This observation is consistent with the fact that Hennessy–Milner logic only characterises bisimulation on image-finite transition systems.

(4) $[\![\mathsf{S}_{\mathcal{D}}]\!]$ is one-step expressive. To see this, let $d : A \to \mathcal{P}X$ be an expressive interpretation, let $\mu, \nu \in \mathcal{D}X$ be such that $\mu \neq \nu$, and let $dom(\mu) \cup dom(\nu) = \{x_1, \ldots, x_n\}$. Since $\mu \neq \nu$, there exists $x \in dom(\mu)$ such that $\mu(x) \neq \nu(x)$. We assume without loss of generality that $x = x_1$ and $\mu(x_1) > \nu(x_1)$. The expressiveness of d together with MA being closed under negation yields, for each $i \in \{2, \ldots, n\}$, a formula $\varphi_i \in MA$ such that $x_1 \models_{d^\sharp} \varphi_i$ and $x_i \not\models_{d^\sharp} \varphi_i$. Now if $q \in \mathbb{Q}$ with $\nu(x_1) < q < \mu(x_1)$, the formula $L_q \bigwedge_{i=2,\ldots,n} \varphi_i \in \mathsf{S}_{\mathcal{D}} MA$ holds in μ but not in ν. Hence, $[\![\mathsf{S}_{\mathcal{D}}]\!](d^\sharp) : \mathsf{S}_{\mathcal{D}} MA \to \mathcal{P}(\mathcal{D}X)$ is expressive.

Alternatively, Proposition 29 can be used to derive the expressiveness of the coalgebraic semantics above, using essentially the same arguments. As the one-step semantics for all the basic syntax constructors are one-step expressive, our next goal is to show that one-step expressiveness is preserved by all the combinations of syntax constructors and associated one-step semantics. Again suppressing the dependency on the signature functor, we obtain:

Proposition 35. *Suppose* $[\![\mathsf{S}_1]\!]$ *and* $[\![\mathsf{S}_2]\!]$ *are expressive one-step semantics w.r.t.* T_1 *and* T_2*, respectively. Then so are the one-step semantics* $[\![\mathsf{S}_1 \otimes \mathsf{S}_2]\!]$, $[\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!]$, $[\![\mathsf{S}_1 \odot E]\!]$ *and* $[\![\mathsf{S}_1 \circledcirc \mathsf{S}_2]\!]$ *w.r.t.* $T_1 \times T_2$, $T_1 + T_2$, ${T_1}^E$ *and* $T_1 \circ T_2$*, respectively.*

Proof. In the case of $[\![\mathsf{S}_1 \circledcirc \mathsf{S}_2]\!]$, the claim follows immediately from the definition of $[\![\mathsf{S}_1 \circledcirc \mathsf{S}_2]\!]$ and from the one-step expressiveness of $[\![\mathsf{S}_1]\!]$ and $[\![\mathsf{S}_2]\!]$. Now suppose that $d : A \to \mathcal{P}X$ is expressive; by our assumption, $[\![\mathsf{S}_i]\!](d^\sharp) : \mathsf{S}_i MA \to \mathcal{P}(T_i X)$ are also expressive for $i = 1, 2$. To see that $[\![\mathsf{S}_1 \otimes \mathsf{S}_2]\!]$ is expressive, assume $(x_1, x_2), (y_1, y_2) \in T_1 X \times T_2 X$ with $(x_1, x_2) \neq (y_1, y_2)$. Without loss of generality, assume that $x_1 \neq y_1$. By expressiveness of $[\![\mathsf{S}_1]\!](d^\sharp)$, we find $\varphi \in \mathsf{S}_1 MA$ that distinguishes x_1 and y_1, i.e. $x_1 \in [\![\mathsf{S}_1]\!](d^\sharp)(\varphi)$ and $y_1 \notin [\![\mathsf{S}_1]\!](d^\sharp)(\varphi)$, or vice-versa. By construction of $[\![\mathsf{S}_1 \otimes \mathsf{S}_2]\!]$, the formula $[\pi_1]\varphi \in (\mathsf{S}_1 \otimes \mathsf{S}_2)MA$ then distinguishes (x_1, x_2) from (y_1, y_2). We now turn to the expressiveness of $[\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!](d^\sharp)$. Clearly the formula $\langle \kappa_1 \rangle \mathsf{tt} \in (\mathsf{S}_1 \oplus \mathsf{S}_2)MA$ distinguishes any $x \in \iota_1(T_1 X)$ from any $y \in \iota_2(T_2 X)$, whereas the formula $\langle \kappa_i \rangle \varphi \in (\mathsf{S}_1 \oplus \mathsf{S}_2)MA$ distinguishes $x = \iota_i(x_i)$ from $y = \iota_i(y_i)$ whenever $\varphi \in \mathsf{S}_i MA$ distinguishes x_i from y_i. Finally we show that $[\![\mathsf{S}_1 \odot E]\!]$ is expressive. Suppose that $f, g \in ({T_1}^E)(X)$ with $f \neq g$, that is, $f, g : E \to T_1 X$ with $f(e) \neq g(e)$ for some $e \in E$. As $[\![\mathsf{S}_1]\!]$ is one-step expressive, we find $\varphi \in \mathsf{S}_1 MA$ that distinguishes $f(e)$ from $g(e)$, hence $[e]\varphi \in (\mathsf{S}_1 \odot E)MA$ distinguishes f from g. $\square$

Thus, Theorem 30 applies to any combination of expressive one-step semantics. As a result, the logic induced by a combination of syntax constructors with associated expressive one-step semantics distinguishes any two states up to ω-behavioural equivalence, and in case the final sequence of T stabilises at, or before $\omega + \omega$, also up to behavioural equivalence. As an immediate application, we obtain expressive logics for all system types discussed in Example 2.

In particular, for image-finite simple probabilistic automata, we obtain a variant of the logic described in [13] which is expressive w.r.t. behavioural equivalence, and hence also w.r.t. the strong version of the notion of bisimulation considered in [13] (see also Example 7).

6. Modular construction of proof systems

This section extends the methods presented so far to also include the compositional construction of proof systems. Our main result shows that this can be done in such a way that the combined proof system inherits soundness and completeness from its building blocks. The key notion needed to modularise the construction of proof systems is that of a proof system constructor, which operates on the category of boolean theories, described next.

Definition 36. The category BTh of *boolean theories* is specified by the following data:

- objects are pairs (A, Φ_A) where A is a set (of *atoms*), and $\Phi_A \subseteq MA$ is a set (of *theorems over* A),
- morphisms $f : (A, \Phi_A) \to (B, \Phi_B)$ are functions $f : A \to B$ such that $\varphi \in \Phi_A \implies Mf(\varphi) \in \Phi_B$.

We write $\Pi_1 : \mathsf{BTh} \to \mathsf{Set}$ for the first projection functor.

It is easy to see that Π_1 is actually a fibration that arises by change of base of the fibration $\mathsf{Pred} \to \mathsf{Set}$ along the functor M [12], where $\mathsf{Pred} \to \mathsf{Set}$ is the standard fibration of subsets over sets. The fact that Π_1 is a fibration is however inconsequential for the later development.

The idea behind an object (A, Φ_A) is that Φ_A contains the set of all provable formulas over atoms in A. Note that closure under propositional reasoning and modus ponens is not required in general for boolean theories. However, there is a canonical way of transforming an arbitrary boolean theory into one which has the previously mentioned property. This is captured by the inclusion-preserving functor $\mathsf{Cl} : \mathsf{BTh} \to \mathsf{BTh}$ which takes a boolean theory (A, Φ) to the boolean theory (A, Φ'), with Φ' being obtained by adding all propositional tautologies over A to Φ, and closing the resulting set of formulas under modus ponens. An immediate consequence of this definition is that $\mathsf{Cl} \circ \mathsf{Cl} = \mathsf{Cl}$. The boolean theories of interest in the following will be obtained by applying Cl to certain sets of axioms.

Using the category BTh of boolean theories, we can now define the notion of proof system constructor as follows.

Definition 37 (*Proof System Constructor*). Suppose S is a syntax constructor. A *proof system constructor for* S is an inclusion-preserving, ω-accessible functor $\mathsf{P} : \mathsf{BTh} \to \mathsf{BTh}$ that satisfies $\Pi_1 \circ \mathsf{P} = \mathsf{S} \circ M \circ \Pi_1$.

Note that the above definition has several implications. First, it requires that P is compatible with S in the sense that the diagram

$$\begin{tikzcd} \mathsf{BTh} \arrow[rr, "\mathsf{P}"] \arrow[d, "\Pi_1"'] & & \mathsf{BTh} \arrow[d, "\Pi_1"] \\ \mathsf{Set} \arrow[r, "M"'] & \mathsf{Set} \arrow[r, "\mathsf{S}"'] & \mathsf{Set} \end{tikzcd}$$

commutes, that is, P lifts $\mathsf{S} \circ M$. The presence of M ensures that the syntax constructor is only applied to sets of formulas that are closed under boolean connectives. Second, the requirement that P is ω-accessible generalises a standard requirement in proof systems, namely that an inference rule can only contain a finite number of premises. Since P is ω-accessible, and since every BTh-object (A, Φ) is an ω-directed union of objects (C, Ψ) with C and Ψ finite, the value $\mathsf{P}(A, \Phi)$ is determined by the values $\mathsf{P}(C, \Psi)$ with C a finite subset of A and Ψ a finite subset of Φ. That is, any $\varphi \in \mathsf{P}(A, \Phi)$ is constructed using a finite number of atoms $C \subseteq A$, and is derived using a finite number of premises $\Psi \subseteq \Phi$. The ω-accessibility of a proof system constructor will later be shown to ensure that the induced derivability predicate can be given an inductive characterisation.

The intuition behind the definition of a proof system constructor is as follows. The syntax constructor S specifies a set of modalities to be added to the basic propositional language. The functor $\mathsf{S} \circ M$ takes a set A of atoms to the set A' obtained by applying the modal operators defined by S to propositional formulas over A exactly once. Now a corresponding proof system constructor takes a set of theorems over A, which contains all provable facts among propositional formulas over A, and produces a set of axioms over A'. Subsequently closing these axioms together with all propositional tautologies over A' under modus ponens yields a set of theorems over A', which contains all provable facts concerning the next transition step that can be derived from the given theorems over A. In other words, a proof system constructor specifies how theoremhood can be lifted to formulas containing an extra degree of nesting of the modal operators. Note that proof system constructors only axiomatise those judgements that are needed on top of propositional logic.

Since the axioms and rules of modal logic only involve formulas of rank at most one, and since the premises of these rules only involve formulas of rank zero, it is straightforward to encode the modal logic K into a proof system constructor. A similar encoding can be given for the probabilistic modal logic described in [10]. The next example describes these encodings, as well as other proof system constructors that correspond to syntax constructors defined in Example 9.

Example 38. (1) For the constant functor $TX = D$ and the associated syntax constructor S_D, we define a constant proof system constructor P_D by $\mathsf{P}_D(A, \Phi) = (D, \Phi')$, where $\varphi \in \Phi' \subseteq MD$ iff $\vdash' \varphi$, and $\vdash'$ is defined by the following axioms:

$$\vdash' \bigvee_{d \in D} d \quad \text{(only if } D \text{ finite)} \qquad\qquad \vdash' \neg(d \wedge d') \quad (d \neq d' \in D).$$

(2) For the identity functor $TX = X$ and the associated syntax constructor $\mathsf{S}_{\mathcal{I}d}$, we define a proof system constructor $\mathsf{P}_{\mathcal{I}d}$ by $\mathsf{P}_{\mathcal{I}d}(A, \Phi) = (\mathsf{S}_{\mathcal{I}d} MA, \Phi')$, where $\varphi \in \Phi' \subseteq M\mathsf{S}_{\mathcal{I}d} MA$ iff $\vdash' \varphi$, and $\vdash'$ is defined by the following axioms:

$$\vdash' \circ\mathsf{ff} \to \mathsf{ff} \qquad \vdash' \circ(\varphi \to \psi) \leftrightarrow (\circ\varphi \to \circ\psi) \qquad \frac{\vdash \varphi \to \psi}{\vdash' \circ\varphi \to \circ\psi}$$

with $\varphi, \psi \in MA$, where we write $\vdash \varphi$ for $\varphi \in \Phi$. Note that, while the last axiom above is written as an inference rule, it actually represents an axiom schema giving an axiom of the form $\vdash' \circ\varphi \to \circ\psi$ for any theorem $\varphi \to \psi \in \Phi$.

(3) Consider the syntax constructors $\mathsf{S}_{\mathcal{P}}$ and $\mathsf{S}_{\mathcal{P}_\omega}$ from Example 9. For a boolean theory (A, Φ), define $\mathsf{P}_{\mathcal{P}}(A, \Phi) = \mathsf{P}_{\mathcal{P}_\omega}(A, \Phi) = (\mathsf{S}_{\mathcal{P}} MA, \Phi')$ where $\varphi \in \Phi' \subseteq M\mathsf{S}_{\mathcal{P}} MA$ iff $\vdash' \varphi$, and $\vdash'$ is defined by the following axioms:

$$\vdash' \Box \mathsf{tt} \qquad\qquad \vdash' \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi) \qquad\qquad \frac{\vdash \varphi \to \psi}{\vdash' \Box\varphi \to \Box\psi}.$$

Then, $\mathsf{P}_{\mathcal{P}}$ ($\mathsf{P}_{\mathcal{P}_\omega}$) is a proof system constructor for $\mathsf{S}_{\mathcal{P}}$ (respectively $\mathsf{S}_{\mathcal{P}_\omega}$).

(4) Consider the syntax constructor $\mathsf{S}_{\mathcal{D}}$ from Example 9. For $p \in \mathbb{Q} \cap [0, 1]$ and $\varphi \in A$, let $M_p\varphi ::= L_{1-p}\neg\varphi \in M\mathsf{S}_{\mathcal{D}} MA$, and $E_p\varphi := L_p\varphi \wedge M_p\varphi \in M\mathsf{S}_{\mathcal{D}} MA$. Recall that L_p signifies "with probability at least p"; accordingly M_p stands for "probability at most p", while E_p stands for "probability exactly p". Thus, given the one-step semantics $[\![\mathsf{S}_{\mathcal{D}}]\!]$ of Example 21, we have

$$[\![\mathsf{S}_{\mathcal{D}}]\!](d^\sharp)(M_p\varphi) = \left\{ \mu \in \mathcal{D}X \mid \sum_{x \in d^\sharp(\varphi)} \mu(x) \leq p \right\}$$

$$[\![\mathsf{S}_{\mathcal{D}}]\!](d^\sharp)(E_p\varphi) = \left\{ \mu \in \mathcal{D}X \mid \sum_{x \in d^\sharp(\varphi)} \mu(x) = p \right\}$$

for $d : A \to \mathcal{P}X$ and $\varphi \in MA$. Now given a finite sequence of formulas $\varphi_1, \ldots, \varphi_m \in MA$, let $\varphi^{(k)}$ stand for $\bigvee_{1 \leq l_1 < \cdots < l_k \leq m} (\varphi_{l_1} \wedge \cdots \wedge \varphi_{l_k})$; in particular, $\varphi^{(k)} = \mathsf{ff}$ if $k > m$. Thus, the formula $\varphi^{(k)}$ states that, from among the formulas $\varphi_1, \ldots, \varphi_m$, at least k are true at any point in X.

Next, for each boolean theory (A, Φ), define $\mathsf{P}_{\mathcal{D}}(A, \Phi) = (\mathsf{S}_{\mathcal{D}} MA, \Phi')$, where $\varphi \in \Phi' \subseteq M\mathsf{S}_{\mathcal{D}} MA$ iff $\vdash' \varphi$, and $\vdash'$ is defined by the following axioms:

$$\vdash' L_0\varphi \qquad \vdash' L_p\mathsf{tt} \qquad \vdash' L_p\varphi \to \neg L_q\neg\varphi \qquad \vdash' \neg L_p\varphi \to M_p\varphi$$

$$\frac{\vdash \bigwedge_{k=1}^{\max(m,n)} \varphi^{(k)} \leftrightarrow \psi^{(k)}}{\vdash' \left(\bigwedge_{i=1}^{m} L_{p_i}\varphi_i\right) \wedge \left(\bigwedge_{j=2}^{n} M_{q_i}\psi_i\right) \to L_{p_1+\cdots+p_m-(q_2+\cdots+q_n)}\psi_1}$$

with $p + q > 1$ being required in the third axiom.

All but the last of the above axioms capture immediate properties of the one-step semantics $[\![\mathsf{S}_{\mathcal{D}}]\!]$ defined in Example 21. The last axiom exploits the well-definedness of integrals w.r.t. probability distributions. To see this, assume that the formulas $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n \in MA$ are interpreted using $d : A \to \mathcal{P}X$, and let $d^\sharp(\varphi_i) = E_i$ for $i = 1, \ldots, m$ and $d^\sharp(\psi_j) = F_j$ for $j = 1, \ldots, n$. Then, the "premise" of the last axiom holds precisely when the sum of the characteristic functions of $E_1, \ldots, E_m$ coincides with the sum of the characteristic functions of $F_1, \ldots, F_n$. Whenever this is the case, if $\mu : X \to [0, 1]$ is a probability distribution, we also have that $\sum_{i=1}^m \mu(E_i) = \sum_{j=1}^n \mu(F_j)$. This equality is further exploited in the "conclusion" of this axiom: whenever $\mu(E_i) \geq p_i$ for $i = 1, \ldots, m$ and $\mu(F_j) \leq q_j$ for $j = 2, \ldots, n$, then necessarily $\mu(F_1) \geq p_1 + \cdots + p_m - (q_2 + \cdots + q_n)$.

The functor $\mathsf{P}_{\mathcal{D}} : \mathsf{BTh} \to \mathsf{BTh}$ defined above qualifies as a proof system constructor for $\mathsf{S}_{\mathcal{D}}$.

We conclude this example by noting that a standard proof system consisting of the axioms and rules in the definition of $\mathsf{P}_{\mathcal{D}}$ (with $\vdash'$ replaced by $\vdash$) together with the rule:

$$\frac{\vdash \varphi \leftrightarrow \psi}{\vdash L_p\varphi \leftrightarrow L_p\psi}$$

has been studied in [10]. We note, however, that by taking $m = n = 1$ in the last axiom defining $\mathsf{P}_{\mathcal{D}}$, one obtains both of the following:

$$\frac{\vdash \varphi \leftrightarrow \psi}{\vdash' L_p\varphi \to L_p\psi} \qquad \frac{\vdash \varphi \leftrightarrow \psi}{\vdash' L_p\psi \to L_p\varphi}$$

and therefore the additional rule used in [10] is redundant.

The previous example explains our choice of terminology as regards proof system constructors. Each of the proof system constructors in Example 38 gives rise to a standard proof system for the language induced by the corresponding syntax constructor. For example, replacing the symbol $\vdash'$ by $\vdash$ in the axioms defining the proof system constructor $\mathsf{P}_{\mathcal{P}}$ for $\mathsf{S}_{\mathcal{P}}$ yields a set of axioms and rules for standard modal logic.

Now recall that the language $\mathcal{L}(\mathsf{S})$ induced by a syntax constructor S was defined as the smallest set of formulas which is closed under the application of S, as well as under propositional connectives. Similarly, a proof system constructor P for S induces a derivability predicate on the language $\mathcal{L}(\mathsf{S})$, defined as the set of facts that can be inferred by applying P together with propositional reasoning and modus ponens.

Definition 39 (*Theory Induced by* P). The *theory induced by* P, denoted $(\mathcal{A}(\mathsf{S}), \Phi_{\mathsf{P}})$, is the least boolean theory of the form $(\mathcal{A}(\mathsf{S}), \Phi)$ with the following properties:

- $\mathsf{Cl}(\mathcal{A}(\mathsf{S}), \Phi) \subseteq (\mathcal{A}(\mathsf{S}), \Phi)$, and
- $\mathsf{P}(\mathcal{A}(\mathsf{S}), \Phi) \subseteq (\mathcal{A}(\mathsf{S}), \Phi)$.

We write $\vdash_{\mathsf{P}} \varphi$ for $\varphi \in \Phi_{\mathsf{P}}$.

Equivalently, $(\mathcal{A}(\mathsf{S}), \Phi_{\mathsf{P}})$ can be characterised as the least boolean theory of the form $(\mathcal{A}(\mathsf{S}), \Phi)$ which satisfies

- $(\mathcal{A}(\mathsf{S}), \Phi)$ contains all instances of propositional tautologies,
- $(\mathcal{A}(\mathsf{S}), \Phi)$ is closed under modus ponens,
- $\mathsf{P}(A, \Psi) \subseteq (\mathcal{A}(\mathsf{S}), \Phi)$ for any $(A, \Psi) \subseteq (\mathcal{A}(\mathsf{S}), \Phi)$ with A and Ψ finite.

Note that, since P lifts $\mathsf{S} \circ M$, we have $(\Pi_1 \circ \mathsf{P})(\mathcal{A}(\mathsf{S}), \Phi_{\mathsf{P}}) = (\mathsf{S} \circ M \circ \Pi_1)(\mathcal{A}(\mathsf{S}), \Phi_{\mathsf{P}}) = (\mathsf{S} \circ M)(\mathcal{A}(\mathsf{S})) = \mathsf{S}(\mathcal{L}(\mathsf{S})) = \mathcal{A}(\mathsf{S})$, and therefore the theory induced by P is well-typed.

We now apply our main programme also to this definition, and show that the theory induced by P can be viewed as the stratification of a sequence of boolean theories $\Phi^n_{\mathsf{P}} \subseteq \mathcal{L}^n(\mathsf{S})$. This will open the road for the proof of soundness and completeness, where induction on the rank of formulas will be available as a proof technique.

Definition 40. The boolean theories $(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}})$ with $n \in \omega$ are defined by:

- $(\mathcal{A}^0(\mathsf{S}), \Phi^0_{\mathsf{P}}) = \mathsf{Cl}(\emptyset, \emptyset)$;
- $(\mathcal{A}^{n+1}(\mathsf{S}), \Phi^{n+1}_{\mathsf{P}}) = (\mathsf{Cl} \circ \mathsf{P})(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}})$ for $n \in \omega$.

We write $\vdash^n_{\mathsf{P}} \varphi$ for $\varphi \in \Phi^n_{\mathsf{P}}$, with $n \in \omega$.

Note that, since $\Pi_1 \circ \mathsf{Cl} = \Pi_1$ and $\Pi_1 \circ \mathsf{P} = \mathsf{S} \circ M \circ \Pi_1$, we have $(\Pi_1 \circ \mathsf{Cl} \circ \mathsf{P})(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}}) = (\Pi_1 \circ \mathsf{P})(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}}) = (\mathsf{S} \circ M \circ \Pi_1)(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}}) = (\mathsf{S} \circ M)(\mathcal{A}^n(\mathsf{S})) = \mathcal{A}^{n+1}(\mathsf{S})$ for all $n \in \omega$, and therefore the boolean theories $(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}})$ with $n \in \omega$ are well-typed. Moreover, the following holds:

Proposition 41. *For $n \in \omega$, $(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}}) \subseteq (\mathcal{A}^{n+1}(\mathsf{S}), \Phi^{n+1}_{\mathsf{P}})$.*

Proof. Induction on n, using the fact that both Cl and P preserve inclusions. □

We are now ready to give an inductive characterisation of the theory induced by P.

Proposition 42. *The boolean theories $(\mathcal{A}(\mathsf{S}), \Phi_{\mathsf{P}})$ and $\bigcup_{n \in \omega} (\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}})$ coincide.*

Proof. Induction on n proves $(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}}) \subseteq (\mathcal{A}(\mathsf{S}), \Phi_{\mathsf{P}})$ for all $n \in \omega$, which then yields $\bigcup_{n \in \omega} (\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}}) \subseteq (\mathcal{A}(\mathsf{S}), \Phi_{\mathsf{P}})$. The reverse inclusion follows by structural induction over Φ_{P}: First, any $\varphi \in \Phi_{\mathsf{P}}$ which is an instance of a propositional tautology belongs to some $\mathcal{L}^n(\mathsf{S}) = M\mathcal{A}^n(\mathsf{S})$, and hence to Φ^n_{P}. Second, if φ and $\varphi \to \psi$ are in Φ_{P}, and by the induction hypothesis, also in Φ^n_{P} for some $n \in \omega$, then closure of Φ^n_{P} under modus ponens gives $\psi \in \Phi^n_{\mathsf{P}}$. Finally, suppose that $\varphi \in \mathsf{P}(\mathcal{A}(\mathsf{S}), \Phi_{\mathsf{P}})$. By ω-accessibility of P, we can assume that $\varphi \in \mathsf{P}(A, \Psi)$ for some $(A, \Psi) \subseteq (\mathcal{A}(\mathsf{S}), \Phi_{\mathsf{P}})$, with A and Ψ finite. By the induction hypothesis, there exists $n \in \omega$ such that $(A, \Psi) \subseteq (\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}})$. Hence, $\varphi \in \mathsf{P}(A, \Psi) \subseteq \mathsf{P}(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}}) \subseteq \mathsf{Cl}(\mathsf{P}(\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}})) = (\mathcal{A}^{n+1}(\mathsf{S}), \Phi^{n+1}_{\mathsf{P}}) \subseteq \bigcup_{n \in \omega} (\mathcal{A}^n(\mathsf{S}), \Phi^n_{\mathsf{P}})$. This concludes the proof. □

As a result of Proposition 42, we can use induction on n to prove properties of $(\mathcal{A}(\mathsf{S}), \vdash_{\mathsf{P}})$. In the following, we consider soundness and completeness of $(\mathcal{A}(\mathsf{S}), \vdash_{\mathsf{P}})$ w.r.t. the coalgebraic semantics induced by a one-step semantics $[\![\mathsf{S}]\!]^T$ for S, and show that these follow from soundness and completeness conditions involving $[\![\mathsf{S}]\!]^T$ and P.

Definition 43 (*One-Step Soundness and Completeness*). A boolean theory $(A, \vdash)$ is *sound* (*complete*) w.r.t. an interpretation $d : A \to \mathcal{P}X$ if $\vdash \varphi$ implies $d^\sharp(\varphi) = X$ (respectively $d^\sharp(\varphi) = X$ implies $\vdash \varphi$) for any $\varphi \in MA$.

A proof system constructor P for S is *one-step sound* (*one-step complete*) w.r.t. a one-step semantics $[\![\mathsf{S}]\!]^T$ if $(\mathsf{Cl} \circ \mathsf{P})(A, \vdash)$ is sound (respectively complete) w.r.t. $[\![\mathsf{S}]\!]^T(d^\sharp) : \mathsf{S}MA \to \mathcal{P}TX$ whenever $(A, \vdash)$ is sound (complete) w.r.t. $d : A \to \mathcal{P}X$.

Using induction, we can derive soundness and weak completeness in the standard way from their one-step counterparts; due to the lack of compactness, our logics usually fail to be strongly complete.

Theorem 44 (*Soundness and Completeness*). *If the proof system constructor P for S is one-step sound (complete) w.r.t. $[\![\mathsf{S}]\!]^T$, then $(\mathcal{A}(\mathsf{S}), \vdash_{\mathsf{P}})$ is sound (respectively complete) w.r.t. the coalgebraic semantics of $\mathcal{L}(\mathsf{S})$, that is, $\models_T \varphi$ if (only if) $\vdash_{\mathsf{P}} \varphi$ for all $\varphi \in \mathcal{L}(\mathsf{S})$.*

Proof. We assume that $T1 \neq \emptyset$, as otherwise $TX = \emptyset$ for all $X \in \mathsf{Set}$, and the claim is trivial. If the functions $d_n : \mathcal{A}^n(\mathsf{S}) \to \mathcal{P}T^n 1$ with $n \in \omega$ are as in Definition 31, then it follows by induction on n that $\vdash^n_{\mathsf{P}}$ is sound (complete) w.r.t. d_n for $n \in \omega$. Now soundness (completeness) of $(\mathcal{A}(\mathsf{S}), \vdash_{\mathsf{P}})$ w.r.t. the coalgebraic semantics of $\mathcal{L}(\mathsf{S})$ amounts to $\vdash_{\mathsf{P}} \varphi$ implies $[\![\varphi]\!]_C = C$ for any T-coalgebra (C, γ) (respectively $[\![\varphi]\!]_C = C$ for any T-coalgebra (C, γ) implies $\vdash_{\mathsf{P}} \varphi$). Moreover, if $\varphi \in \mathcal{L}^n(\mathsf{S})$, then by Proposition 32, $[\![\varphi]\!]_C = C$ is equivalent to $\hat{\mathcal{P}}\gamma_n(d_n^\sharp(\varphi)) = C$. Thus, assuming that P is one-step sound, it follows that $(\mathcal{A}(\mathsf{S}), \vdash) = \bigcup_{n \in \omega}(\mathcal{A}^n(\mathsf{S}), \vdash^n_{\mathsf{P}})$ is sound w.r.t. the coalgebraic semantics of $\mathcal{L}(\mathsf{S})$. Now assume that P is one-step complete, and let $\varphi \in \mathcal{L}^n(\mathsf{S})$ be such that $[\![\varphi]\!]_C = C$ for any T-coalgebra (C, γ). Consider the coalgebra $(C, \gamma) = (T^n 1, T^n i)$, where $i : 1 \to T1$ is chosen arbitrarily. Then, $(T^n i)_n = id_{T^n 1} : T^n 1 \to T^n 1$. The fact $[\![\varphi]\!]_C = C$ now gives $d_n^\sharp(\varphi) = T^n 1$, and hence, using the completeness of $\vdash^n_{\mathsf{P}}$ w.r.t. d_n, $\vdash^n_{\mathsf{P}} \varphi$. Thus, $\vdash_{\mathsf{P}} \varphi$, which concludes the proof. □

In the case of standard modal logic, the axioms and rules given in Example 38 (with $\vdash'$ replaced by $\vdash$) together with all instances of propositional tautologies and the modus ponens and uniform substitution rules, form a sound and complete proof system. Similarly, for probabilistic transition systems, the axioms and rules given in Example 38 yield a sound and complete proof system: this was proved in [10] using the standard filtration method. However, these results are of limited usefulness here, since in order to be able to derive soundness and completeness results for more complex signature functors, defined in terms of $\mathcal{P}$ and $\mathcal{D}$, we must prove that the proof system constructors $\mathsf{P}_{\mathcal{P}}$ and $\mathsf{P}_{\mathcal{D}}$ are one-step sound and complete.

We now establish one-step soundness and completeness for all the proof system constructors introduced in Example 38. Together with the fact that the combination of proof system constructors preserves one-step soundness and completeness (which we will establish later), this puts us in the position to apply our modular techniques to a large class of probabilistic system types, including probabilistic transition systems and probabilistic automata.

We begin with a simple technical lemma; recall that a *disjunctive clause* over a set A of atoms is a formula of the form $a_1 \vee \cdots \vee a_m \vee \neg a'_1 \vee \cdots \vee \neg a'_n$ with $m, n \geq 0$ and $a_1, \ldots, a_m, \ldots, a'_1, \ldots, a'_n \in A$.

Lemma 45. *Suppose $(A, \Phi_A) \in \mathsf{BTh}$ and let $(A, \vdash) = \mathsf{Cl}(A, \Phi_A)$. If $d : A \to \mathcal{P}X$ is an interpretation, the following are equivalent:*

(1) *$(A, \vdash)$ is sound (complete) w.r.t. d,*
(2) *$d(\varphi) = X$ if (only if) $\vdash \varphi$ for every disjunctive clause φ over A.*

Proof. Follows immediately by converting every formula $\varphi \in MA$ into conjunctive normal form. □

We can now tackle completeness for our basic proof system constructors.

Proposition 46. *The proof system constructors P_D and $\mathsf{P}_{\mathcal{I}d}$ defined in Example 38 are one-step sound and complete w.r.t. $[\![\mathsf{S}_D]\!]$ and $[\![\mathsf{S}_{\mathcal{I}d}]\!]$, respectively.*

Proof. One-step soundness of both proof system constructors follows easily by unfolding their respective definitions. To show one-step completeness, we fix $(A, \vdash)$ that is complete w.r.t. $d : A \to \mathcal{P}X$.

For the one-step completeness of P_D, we have to show that $(\mathsf{Cl} \circ \mathsf{P}_D)(A, \vdash) = (\mathsf{S}_D MA, \vdash')$ is complete w.r.t. $[\![\mathsf{S}_D]\!](d^\sharp) : \mathsf{S}_D MA \to \mathcal{P}D$. Note that $\mathsf{S}_D MA = D$, and hence $([\![\mathsf{S}_D]\!](d^\sharp))^\sharp : MD \to \mathcal{P}D$ is the inductive extension of the mapping $x \mapsto \{x\}$. For $\varphi \in MD$, we abbreviate $([\![\mathsf{S}_D]\!](d^\sharp))^\sharp(\varphi)$ by $[\![\varphi]\!]$. Using this notation, by Lemma 45 it suffices to show that $[\![\bigvee_{i=1}^m \neg\varphi_i \vee \bigvee_{j=1}^n \psi_j]\!] = X$ implies $\vdash' \bigvee_{i=1}^m \neg\varphi_i \vee \bigvee_{j=1}^n \psi_j$, where $\varphi_i, \psi_j \in D$. That is, we have to show that $[\![\bigwedge_{i=1}^m \varphi_i]\!] \subseteq [\![\bigvee_{j=1}^n \psi_j]\!]$ implies $\vdash' \bigwedge_{i=1}^m \varphi_i \to \bigvee_{j=1}^n \psi_j$. Assuming that $[\![\bigwedge_{i=1}^m \varphi_i]\!] \subseteq [\![\bigvee_{j=1}^n \psi_j]\!]$ holds, we have one of the three cases:

Case $m = 0$. Then $[\![\bigwedge_{i=1}^m \varphi_i]\!] = D \subseteq [\![\bigvee_{j=1}^n \psi_j]\!]$, hence D is finite and $\{\psi_j \mid 1 \le j \le n\} = D$. In this case, we have $\vdash' \bigvee_{d\in D} d$, and $\vdash' \bigvee_{j=1}^n \psi_j$ follows by propositional reasoning.

Case $\{\varphi_i \mid 1 \le i \le m\} = \{d\}$ for some $d \in D$. Then there exists $1 \le j \le n$ with $\psi_j = d = \varphi_1$, and since $d \to d$ is a propositional tautology, the claim follows by propositional reasoning.

Case $\{\varphi_i \mid 1 \le i \le m\} \supseteq \{d_0, d_1\}$ for some $d_0 \ne d_1 \in D$. Then $[\![\bigwedge_{i=1}^m \varphi_i]\!] = \emptyset$, and we have, by the definition of P_D, that $\bigwedge_{i=1}^m \varphi_i \to d_0 \wedge d_1 \to \mathsf{ff} \to \bigvee_{j=1}^n \psi_j$, and the proof of the claim is complete.

We now establish the one-step completeness of $\mathsf{P}_{\mathcal{I}d}$, where we have to show that $(\mathsf{Cl} \circ \mathsf{P}_{\mathcal{I}d})(A, \vdash) = (\mathsf{S}_{\mathcal{I}d} MA, \vdash')$ is complete w.r.t. $[\![\mathsf{S}_{\mathcal{I}d}]\!](d^\sharp) : \mathsf{S}_{\mathcal{I}d} MA \to \mathcal{P}X$. Similarly to the proof of the one-step completeness of P_D, and also abbreviating $([\![\mathsf{S}_{\mathcal{I}d}]\!](d^\sharp))^\sharp(\varphi)$ by $[\![\varphi]\!]$, it suffices to show that $[\![\varphi \to \psi]\!] = X$ implies $\vdash' \varphi \to \psi$, where $\varphi = \bigwedge_{i=1}^m \circ\varphi_i$ and $\psi = \bigvee_{j=1}^n \circ\psi_j$ with $\varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_n \in MA$. The assumption that $[\![\varphi \to \psi]\!] = X$ together with the definition of $[\![\mathsf{S}_{\mathcal{I}d}]\!]$ give $\bigcap_{i=1}^m d^\sharp(\varphi_i) \subseteq \bigcup_{j=1}^n d^\sharp(\psi_j)$, or equivalently, $d^\sharp(\bigwedge_{i=1}^m \varphi_i \to \bigvee_{j=1}^n \psi_j) = X$. The completeness of $(A, \vdash)$ w.r.t. d now gives $\vdash \bigwedge_{i=1}^m \varphi_i \to \bigvee_{j=1}^n \psi_j$. Finally, the axioms defining $\mathsf{P}_{\mathcal{I}d}$ can be used to derive first $\vdash' \circ \bigwedge_{i=1}^m \varphi_i \to \circ \bigvee_{j=1}^n \psi_j$, and then, also using propositional reasoning and modus ponens, $\vdash' \bigwedge_{i=1}^m \circ\varphi_i \to \bigvee_{j=1}^n \circ\psi_j$. (Note that the first two axioms in the definition of $\mathsf{P}_{\mathcal{I}d}$ result in the modal operator $\circ$ distributing over all boolean operators.) Thus, $\vdash' \varphi \to \psi$. This concludes the proof. □

We proceed to establish one-step completeness for the powerset functor (as in [24], but with a different proof) and the probability distribution functor.

Proposition 47. *The proof system constructors $\mathsf{P}_{\mathcal{P}}$ and $\mathsf{P}_{\mathcal{P}_\omega}$ defined in Example 38 are one-step sound and complete w.r.t. $[\![\mathsf{S}_{\mathcal{P}}]\!]$ and $[\![\mathsf{S}_{\mathcal{P}_\omega}]\!]$, respectively.*

Proof. The one-step soundness of $\mathsf{P}_{\mathcal{P}}$ w.r.t. $[\![\mathsf{S}_{\mathcal{P}}]\!]$ follows easily from the definitions of $\mathsf{P}_{\mathcal{P}}$ and $[\![\mathsf{S}_{\mathcal{P}}]\!]$. To prove one-step completeness, we fix $(A, \vdash)$ complete w.r.t. $d : A \to \mathcal{P}X$, and show that $(\mathsf{Cl} \circ \mathsf{P}_{\mathcal{P}})(A, \vdash) = (\mathsf{S}_{\mathcal{P}} MA, \vdash')$ is complete w.r.t. $[\![\mathsf{S}_{\mathcal{P}}]\!](d^\sharp) : \mathsf{S}_{\mathcal{P}} MA \to \mathcal{P}\mathcal{P}X$. By Lemma 45, it suffices to show that $([\![\mathsf{S}_{\mathcal{P}}]\!](d^\sharp))^\sharp(\varphi \to \psi) = \mathcal{P}X$ implies $\vdash' \varphi \to \psi$, where $\varphi = \bigwedge_{i=1}^m \Box\varphi_i$ and $\psi = \bigvee_{j=1}^n \Box\psi_j$ with $\varphi_1, \dots, \varphi_m, \psi_1, \dots, \psi_n \in MA$. So assume, for the sake of contradiction, that $d^\sharp(\bigwedge_{i=1}^m \varphi_i) \not\subseteq d^\sharp(\psi_j)$ for any $j \in \{1, \dots, n\}$. By choosing $x_j \in d^\sharp(\bigwedge_{i=1}^m \varphi_i) \setminus d^\sharp(\psi_j)$ for $j \in \{1, \dots, n\}$, we obtain $\{x_1, \dots, x_n\} \in ([\![\mathsf{S}_{\mathcal{P}}]\!](d^\sharp))^\sharp(\Box \bigwedge_{i=1}^m \varphi_i) = ([\![\mathsf{S}_{\mathcal{P}}]\!](d^\sharp))^\sharp(\bigwedge_{i=1}^m \Box\varphi_i) \subseteq ([\![\mathsf{S}_{\mathcal{P}}]\!](d^\sharp))^\sharp(\bigvee_{j=1}^n \Box\psi_j)$. This yields $j_0 \in \{1, \dots, n\}$ such that $\{x_1, \dots, x_n\} \in ([\![\mathsf{S}_{\mathcal{P}}]\!](d^\sharp))^\sharp(\Box\psi_{j_0})$. But this contradicts the fact that $x_{j_0} \notin d^\sharp(\psi_{j_0})$. Hence, there exists some $j \in \{1, \dots, n\}$ such that $d^\sharp(\bigwedge_{i=1}^m \varphi_i) \subseteq d^\sharp(\psi_j)$ (or equivalently, $d^\sharp(\bigwedge_{i=1}^m \varphi_i \to \psi_j) = X$). The completeness of $(A, \vdash)$ w.r.t. d now gives $\vdash \bigwedge_{i=1}^m \varphi_i \to \psi_j$. The last axiom in the definition of $\mathsf{P}_{\mathcal{P}}$ (see Example 38) then gives $\vdash' \Box \bigwedge_{i=1}^m \varphi_i \to \Box\psi_j$, which, together with the second axiom in the definition of $\mathsf{P}_{\mathcal{P}}$ and some suitable use of propositional reasoning, yield $\vdash' \bigwedge_{i=1}^m \Box\varphi_i \to \Box\psi_j$. Some further use of propositional reasoning finally gives $\vdash' \bigwedge_{i=1}^m \Box\varphi_i \to \bigvee_{j=1}^n \Box\psi_j$, that is, $\vdash' \varphi \to \psi$ as required. The one-step soundness and completeness of $\mathsf{P}_{\mathcal{P}_\omega}$ w.r.t. $[\![\mathsf{S}_{\mathcal{P}_\omega}]\!]$ is proved similarly. □

Proposition 48. *The proof system constructor $\mathsf{P}_{\mathcal{D}}$ defined in Example 38 is one-step sound and complete w.r.t. $[\![\mathsf{S}_{\mathcal{D}}]\!]$.*

Proof. The one-step soundness of $\mathsf{P}_{\mathcal{D}}$ w.r.t. $[\![\mathsf{S}_{\mathcal{D}}]\!]$ follows easily from the definitions of $\mathsf{P}_{\mathcal{D}}$ and $[\![\mathsf{S}_{\mathcal{D}}]\!]$ (see also the discussion motivating the last axiom in Example 38).

The one-step completeness of $\mathsf{P}_{\mathcal{D}}$ w.r.t. $[\![\mathsf{S}_{\mathcal{D}}]\!]$ is proved using a version of the *theorem of the alternative*. Before stating this theorem, we fix some notations. If Z is a sub-vector space of $\mathbb{Q}^N$, we write $Z^\perp$ for the orthogonal subspace $\{\bar{z} \in \mathbb{Q}^N \mid \bar{z}z = 0 \text{ for every } z \in Z\}$, where, for $z = (z_1, \dots, z_N)$ and $\bar{z} = (\bar{z}_1, \dots, \bar{z}_N)$, $\bar{z}z = \bar{z}_1 z_1 + \dots + \bar{z}_N z_N$ is the usual dot product. Also, if $q \in \mathbb{Q}$ and I is an interval in $\mathbb{Q}$, we write $q\,I$ for the interval in $\mathbb{Q}$ defined by the set

$\{q * i \mid i \in I\}$. We identify an element $q \in \mathbb{Q}$ with the singleton set $\{q\}$ and write $I < J$, for $I, J \subseteq \mathbb{Q}$ whenever $i < j$ for all $i \in I$ and all $j \in J$. In particular, $I > 0$ is used as a shorthand for $i > 0$ for all $i \in I$. Finally, if $I_1, \ldots, I_N$ are intervals in $\mathbb{Q}$, we write $I_1 + \cdots + I_N$ for the interval $\{i_1 + \cdots + i_n \mid i_1 \in I_1, \ldots, i_N \in I_N\}$. The previously-mentioned result can now be stated as follows.

Theorem 49 (*Rockafellar [26]*). *Let Z be a subspace of $\mathbb{Q}^N$ and $\mathcal{I}_1, \ldots, \mathcal{I}_N$ be intervals in $\mathbb{Q}$. Then, one and only one of the following alternatives holds:*

(*) *There exists a vector $z = (z_1, \ldots, z_N) \in Z$ such that $z_1 \in \mathcal{I}_1, \ldots, z_N \in \mathcal{I}_N$;*
(**) *There exists a vector $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_N) \in Z^\perp$ such that $\bar{z}_1 \mathcal{I}_1 + \cdots + \bar{z}_N \mathcal{I}_N > 0$.*

We now return to the proof of one-step completeness of $\mathsf{P}_\mathcal{D}$. We fix $(A, \vdash)$ complete w.r.t. $d : A \to \mathcal{P}X$, and show that $(\mathsf{Cl} \circ \mathsf{P}_\mathcal{D})(A, \vdash) = (\mathsf{S}_\mathcal{D} MA, \vdash')$ is complete w.r.t. $[\![\mathsf{S}_\mathcal{D}]\!](d^\sharp) : \mathsf{S}_\mathcal{D} MA \to \mathcal{P}\mathcal{D}X$. Again, by Lemma 45, it suffices to show that $([\![\mathsf{S}_\mathcal{D}]\!](d^\sharp))^\sharp(\varphi \to \psi) = \mathcal{D}X$ implies $\vdash' \varphi \to \psi$, where $\varphi = \bigwedge_{i=1}^m L_{\alpha_i}\varphi_i$ and $\psi = \bigvee_{j=1}^n L_{\beta_j}\psi_j$ with $\varphi_1, \ldots, \varphi_m, \psi_1, \ldots, \psi_n \in MA$. This can, in turn, be reduced (through propositional reasoning and the use of the second axiom in the definition of $\mathsf{P}_\mathcal{D}$) to show that $\vdash' L_1 \mathsf{tt} \wedge \varphi \wedge \neg\psi \to \mathsf{ff}$. To show this, let $\alpha_0 = 1$ and $\varphi_0 = \mathsf{tt}$, and consider all (finitely many) formulas of the form $\xi = \varphi_0' \wedge \cdots \wedge \varphi_m' \wedge \psi_1' \wedge \cdots \wedge \psi_n'$, where each φ_i' is either φ_i or $\neg\varphi_i$, each ψ_j' is either ψ_j or $\neg\psi_j$, and such that $\not\vdash \xi \to \mathsf{ff}$. Now let Ξ contain exactly one such ξ from each equivalence class for the equivalence relation $\xi \sim \xi'$ iff $\vdash \xi \leftrightarrow \xi'$. Some immediate properties of Ξ are:

- $\vdash \xi \wedge \xi' \to \mathsf{ff}$ for any $\xi, \xi' \in \Xi$ s.t. $\xi \neq \xi'$,
- $\vdash \bigvee_{\xi \in \Xi} \xi$.

Next, let $L' \subseteq M\mathsf{S}_\mathcal{D} MA$ consist of formulas which contain only φ_is, ψ_js and ξs as atomic sub-formulas, and only multiples of $\frac{1}{q}$ as probability values, where q is the smallest common denominator of $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n$. Showing that $\vdash' L_1 \mathsf{tt} \wedge \varphi \wedge \neg\psi \to \mathsf{ff}$ can be reduced (through propositional reasoning and the use of the fourth axiom in the definition of $\mathsf{P}_\mathcal{D}$) to show that $\vdash' \bigwedge_{i=0}^m L_{\alpha_i}\varphi_i \wedge \bigwedge_{j=1}^n (M_{\beta_j}\psi_j \wedge \neg E_{\beta_j}\psi_j) \to \mathsf{ff}$. So assume, for the sake of contradiction, that this does not hold. We can then construct a maximally consistent set of formulas $\Phi \subseteq L'$ which contains $L_{\alpha_i}\varphi_i$ for $i = 0, \ldots, m$, as well as $M_{\beta_j}\psi_j$ and $\neg E_{\beta_j}\psi_j$ for $j = 1, \ldots, n$. As a result of being maximal, Φ will contain, for each $\zeta \in \{\varphi_0, \ldots, \varphi_m\} \cup \{\psi_1, \ldots, \psi_n\} \cup \Xi$ and each $m \in [0, 1]$ which is a multiple of $\frac{1}{q}$, one of $L_m\zeta$ or $\neg L_m\zeta$, one of $M_m\zeta$ or $\neg M_m\zeta$, and one of $E_m\zeta$ or $\neg E_m\zeta$. We now define

$$m_\zeta = \max\{m \mid L_m\zeta \in \Phi\} \qquad M_\zeta = \min\{m \mid M_m\zeta \in \Phi\}$$

for $\zeta \in \{\varphi_0, \ldots, \varphi_m\} \cup \{\psi_1, \ldots, \psi_n\} \cup \Xi$. The fact that Φ is maximally consistent gives:

- $m_\zeta \le M_\zeta$,
- either $m_\zeta = M_\zeta$ and $E_{m_\zeta}\zeta \in \Phi$, or $m_\zeta \neq M_\zeta$ and $E_\alpha\zeta \notin \Phi$ for any α,
- $m_{\varphi_0} = M_{\varphi_0} = 1$,

where the first and third axioms in the definition of $\mathsf{P}_\mathcal{D}$ are needed to obtain the last statement. We can then apply Rockafellar's theorem to the subspace

$$Z = \left\{ (v_\zeta)_{\zeta \in \{\varphi_0,\ldots,\varphi_m\} \cup \{\psi_1,\ldots,\psi_n\} \cup \Xi} \mid v_\zeta \in \mathbb{Q}, \sum_{\substack{\xi \in \Xi \\ \vdash \xi \to \varphi_i}} v_\xi = v_{\varphi_i}, \sum_{\substack{\xi \in \Xi \\ \vdash \xi \to \psi_j}} v_\xi = v_{\psi_j} \right\},$$

where ξ ranges over Ξ, and to the intervals I_ζ defined by

$$I_\zeta = \begin{cases} (m_\zeta, M_\zeta) & \text{if } m_\zeta \neq M_\zeta \\ \{m_\zeta\} & \text{if } m_\zeta = M_\zeta. \end{cases}$$

The first alternative in Rockafellar's theorem yields a vector

$$(v_\zeta)_{\zeta \in \{\varphi_0,\ldots,\varphi_m\} \cup \{\psi_1,\ldots,\psi_n\} \cup \Xi} \in Z$$

such that $v_\zeta \in I_\zeta$ for each ζ. This, in turn, allows us to construct a probability distribution $\mu : X \to [0, 1]$ which has the property that $\mu([\![\zeta]\!]) \in I_\zeta$ for each ζ (where we use $\mu([\![\zeta]\!])$ as a shorthand for $\sum_{x \in d^\sharp(\zeta)} \mu(x)$), and moreover, satisfies all formulas in Φ: we pick $x_\xi \in d(\xi) \neq \emptyset$ (as $\xi \nvdash \mathsf{ff}$ and $(A, \vdash)$ is complete w.r.t. d) for each $\xi \in \Xi$, and define $\mu(x_\xi) = v_\xi$ for each $\xi \in \Xi$, and $\mu(x) = 0$ elsewhere. The properties of Ξ mentioned earlier together with the soundness of $(A, \vdash)$ w.r.t. d ensure that μ is a probability distribution: $\sum_{\xi \in \Xi} v_\xi = \sum_{\xi \vdash \mathsf{tt}} v_\xi = \sum_{\xi \vdash \varphi_0} v_\xi = v_{\varphi_0} \in \{m_{\varphi_0}\} = \{1\}$. Also, by construction, $\mu([\![\xi]\!]) = v_\xi \in I_\xi$, while for $\varphi \in \{\varphi_1, \ldots, \varphi_m\}$, the fact that $\mu([\![\varphi]\!]) \in I_\varphi$ follows from $\mu([\![\varphi]\!]) = \sum_{\xi \vdash \varphi} \mu([\![\xi]\!]) = \sum_{\xi \vdash \varphi} v_\xi = v_\varphi$, and similarly for $\psi \in \{\psi_1, \ldots, \psi_n\}$. Finally, the fact that μ satisfies all formulas in Φ follows by case analysis on the basic formulas in Φ:

- If $L_\alpha \zeta \in \Phi$, then $\alpha \le m_\zeta$, and hence $\mu([\![\zeta]\!]) \ge m_\zeta \ge \alpha$, i.e. μ satisfies the formula $L_\alpha \zeta$. If $\neg L_\alpha \zeta \in \Phi$, then since Φ is maximally consistent, the fourth axiom in the definition of $\mathsf{P}_\mathcal{D}$ together with propositional reasoning give $M_\alpha \zeta \in \Phi$ and $\neg E_\alpha \zeta \in \Phi$; hence, $\alpha \ge M_\zeta$, and either $M_\zeta = m_\zeta \neq \alpha$ or $M_\zeta \neq m_\zeta$. In the first case, $\mu([\![\zeta]\!]) = M_\zeta < \alpha$, whereas in the second case, $\mu([\![\zeta]\!]) < M_\zeta \le \alpha$. Hence, in both cases, μ satisfies the formula $\neg L_\alpha \zeta$.
- The cases of $M_\alpha \zeta \in \Phi$ and $\neg M_\alpha \zeta \in \Phi$ are treated similarly.
- The cases of $E_\alpha \zeta \in \Phi$ and $\neg E_\alpha \zeta \in \Phi$ are treated using the definition of $E_\alpha \zeta$.

Now since Φ contains $L_{\alpha_i} \varphi_i$ for $i = 0, \ldots, m$, as well as $M_{\beta_j} \psi_j$ and $\neg E_{\beta_j} \psi_j$ for $j = 1, \ldots, n$, we have $\mu \in [\![(\mathsf{S}_\mathcal{D}]\!](d^\sharp))^\sharp(\varphi) \setminus ([\![\mathsf{S}_\mathcal{D}]\!](d^\sharp))^\sharp(\psi)$. Thus, under this alternative, we have arrived at a contradiction.

The second alternative in Rockafellar's theorem yields values a_ζ, with $\zeta \in \{\varphi_0, \ldots, \varphi_m\} \cup \{\psi_1, \ldots, \psi_n\} \cup \Xi$, subject to the following conditions:

(1) $\sum_\zeta a_\zeta v_\zeta = 0$ for all $v \in Z$;
(2) $\sum_\zeta a_\zeta I_\zeta > 0$.

By manipulating these (in)equalities, namely multiplying by the least common denominator of the a_ζs and separating positive coefficients from negative ones, we obtain non-negative integer values b_ζ and b'_ζ, with $\zeta \in \{\varphi_0, \ldots, \varphi_m\} \cup \{\psi_1, \ldots, \psi_n\} \cup \Xi$, such that:

$$\sum_\zeta b_\zeta v_\zeta = \sum_\zeta b'_\zeta v_\zeta \text{ for all } v \in Z \tag{1}$$

$$\sum_\zeta b_\zeta \mathcal{I}_\zeta > \sum_\zeta b'_\zeta \mathcal{I}_\zeta. \tag{2}$$

We then define the formulas $\zeta_1, \ldots, \zeta_k, \zeta'_1, \ldots, \zeta'_l \in \{\varphi_0, \ldots, \varphi_m\} \cup \{\psi_1, \ldots, \psi_n\} \cup \Xi$, by taking b_ζ copies of each ζ in $\zeta_1, \ldots, \zeta_k$, and b'_ζ copies of each ζ in $\zeta'_1, \ldots, \zeta'_l$. We immediately obtain $L_{m_{\zeta_i}} \zeta_i \in \Phi$ for $i = 1, \ldots, k$ and $M_{M_{\zeta'_j}} \zeta'_j \in \Phi$ for $j = 1, \ldots, l$. Moreover, the following hold:

(1) $\vdash \bigwedge_{i=1}^{\max(k,l)} \zeta^{(i)} \leftrightarrow \zeta'^{(i)}$
(2) $\sum_{i=1,\ldots,k} m_{\zeta_i} > \sum_{j=1,\ldots,l} M_{\zeta'_j}$.

To see why the statement in item (1) above is true, note that the equality (1) results in the sum of the characteristic functions of $d^\sharp(\zeta_1), \ldots, d^\sharp(\zeta_k)$ being equal to the sum of the characteristic functions of $d^\sharp(\zeta'_1), \ldots, d^\sharp(\zeta'_l)$: for each $x \in X$, substituting the vector $v \in Z$ defined by $v_\zeta = 1$ if $x \in d^\sharp(\zeta)$ and $v_\zeta = 0$ otherwise into (1) gives the equality of the previously-mentioned functions on the value x. As a result, we have $d^\sharp(\zeta^{(i)}) = d^\sharp(\zeta'^{(i)})$ for $i = 1, \ldots, \max(k, l)$, and hence $d^\sharp(\bigwedge_{i=1}^{\max(k,l)} \zeta^{(i)} \leftrightarrow \zeta'^{(i)}) = X$. Now the completeness of $(A, \vdash)$ w.r.t. $d : A \to \mathcal{P}X$ gives $\vdash \bigwedge_{i=1}^{\max(k,l)} \zeta^{(i)} \leftrightarrow \zeta'^{(i)}$.

The inequality in item (2) above follows directly from the inequality (2), after recalling that I_ζ is either the set $\{m_\zeta\}$ or the open interval (m_ζ, M_ζ), and hence the lower endpoint of the interval $\sum_\zeta b_\zeta \mathcal{I}_\zeta$ is $\sum_{i=1,\ldots,k} m_{\zeta_i}$, while the upper endpoint of the interval $\sum_\zeta b'_\zeta \mathcal{I}_\zeta$ is $\sum_{j=1,\ldots,l} M_{\zeta'_j}$.

Now let $p = M_{\zeta'_1}$ and $q = m_{\zeta_1} + \cdots + m_{\zeta_k} - (M_{\zeta'_2} + \cdots + M_{\zeta'_l})$. Item (1) together with the last axiom defining $\vdash'$ give $\vdash' (\bigwedge_{i=1}^k L_{m_{\zeta_i}} \zeta_i) \wedge (\bigwedge_{j=2}^l M_{M_{\zeta'_j}} \zeta'_j) \to L_q \zeta'_1$. Hence, since Φ is maximally consistent, and since each of $L_{m_{\zeta_i}} \zeta_i$ with $i = 1, \ldots, k$ and $M_{M_{\zeta'_j}} \zeta'_j$ with $j = 2, \ldots, l$ belong to Φ, so does $L_q \zeta'_1$. Moreover, by item (2) we have $p < q$, hence $(1-p)+q > 1$. Now $M_p\zeta_1' \in \Phi$ implies, by the definition of M_p, that $L_{1-p}\neg\zeta_1' \in \Phi$, hence, by the third axiom in the definition of $\mathsf{P}_{\mathcal{D}}$, we have $\neg L_q\zeta_1' \in \Phi$, as $(1-p)+q > 1$. As also $L_q\zeta_1' \in \Phi$, we have arrived at the required contradiction also under the second alternative. This concludes the proof. □

We note that, while the proof of one-step completeness of $\mathsf{P}_{\mathcal{D}}$ resembles the completeness proof in [10] (which also uses Rockafellar's theorem), the one-step completeness of $\mathsf{P}_{\mathcal{D}}$ will, in addition, allow the modular derivation of completeness results for functor combinations.

In what follows, we show how one can combine proof system constructors for simple languages in order to derive proof systems constructors, and hence proof systems, for more complex languages. Moreover, we show that whenever the building blocks of such constructions are one-step sound and complete w.r.t. some given one-step semantics, the resulting proof systems are sound and complete w.r.t. the induced coalgebraic semantics. For ease of exposition, we abbreviate $[\kappa_i]\varphi := \neg\langle\kappa_i\rangle\neg\varphi \in M((\mathsf{S}_1 \oplus \mathsf{S}_2)L)$, where $\varphi \in M\mathsf{S}_i L$ and $i \in \{1, 2\}$.

Definition 50 (*Combinations of Proof System Constructors*). Let $(A_1, \vdash_1)$ and $(A_2, \vdash_2)$ be boolean theories.

(1) We let $(A_1, \vdash_1) \otimes (A_2, \vdash_2) = (MA_1 \otimes MA_2, \vdash_\otimes)$, where $\_ \otimes \_$ is defined on sets as in Definition 10, and the predicate $\vdash_\otimes$ is defined by the following axioms:

$$\vdash_\otimes [\pi_i]\mathsf{ff} \to \mathsf{ff} \qquad \vdash_\otimes [\pi_i](\varphi \to \psi) \leftrightarrow ([\pi_i]\varphi \to [\pi_i]\psi)$$

$$\frac{\vdash_i \varphi \to \psi}{\vdash_\otimes [\pi_i]\varphi \to [\pi_i]\psi}.$$

(2) We let $(A_1, \vdash_1) \oplus (A_2, \vdash_2) = (MA_1 \oplus MA_2, \vdash_\oplus)$, where $\_ \oplus \_$ is defined on sets as in Definition 10, and the predicate $\vdash_\oplus$ is defined by the following axioms:

$$\vdash_\oplus [\kappa_i]\mathsf{tt} \qquad \vdash_\oplus [\kappa_i]\varphi \wedge [\kappa_i]\psi \to [\kappa_i](\varphi \wedge \psi)$$

$$\vdash_\oplus [\kappa_1]\mathsf{ff} \vee [\kappa_2]\mathsf{ff} \qquad \vdash_\oplus [\kappa_i](\varphi \vee \psi) \to [\kappa_i]\varphi \vee [\kappa_i]\psi$$

$$\vdash_\oplus [\kappa_1]\mathsf{ff} \wedge [\kappa_2]\mathsf{ff} \to \mathsf{ff} \qquad \frac{\vdash_i \varphi \to \psi}{\vdash_\oplus [\kappa_i]\varphi \to [\kappa_i]\psi}.$$

(3) For an arbitrary set E, we let $(A_1, \vdash_1) \odot E = ((MA_1) \odot E, \vdash_{\odot E})$, where $\_ \odot E$ is defined on sets as in Definition 10, and the predicate $\vdash_{\odot E}$ is defined by the following axioms:

$$\vdash_{\odot E} [e]\mathsf{ff} \to \mathsf{ff} \qquad \vdash_{\odot E} [e](\varphi \to \psi) \leftrightarrow ([e]\varphi \to [e]\psi)$$

$$\frac{\vdash_1 \varphi \to \psi}{\vdash_{\odot E} [e]\varphi \to [e]\psi}.$$

If P_1 and P_2 are proof system constructors for S_1 and S_2, respectively, define:

$$\begin{aligned}
(\mathsf{P}_1 \otimes \mathsf{P}_2)(A, \vdash) &= (\mathsf{Cl} \circ \mathsf{P}_1)(A, \vdash) \otimes (\mathsf{Cl} \circ \mathsf{P}_2)(A, \vdash) \\
(\mathsf{P}_1 \oplus \mathsf{P}_2)(A, \vdash) &= (\mathsf{Cl} \circ \mathsf{P}_1)(A, \vdash) \oplus (\mathsf{Cl} \circ \mathsf{P}_2)(A, \vdash) \\
(\mathsf{P}_1 \odot E)(A, \vdash) &= (\mathsf{Cl} \circ \mathsf{P}_1)(A, \vdash) \odot E \\
(\mathsf{P}_1 \circledcirc \mathsf{P}_2)(A, \vdash) &= (\mathsf{P}_1 \circ \mathsf{Cl} \circ \mathsf{P}_2)(A, \vdash).
\end{aligned}$$

We note in passing that, in the case of $\vdash_\otimes$ and $\vdash_{\odot E}$, the axiomatisations used here have been chosen so as to minimise the number of axioms required. An alternative, equivalent axiomatisation for $\vdash_\otimes$, which is closer in spirit to the definition of $\vdash_\oplus$, can be given by replacing the first two axioms in the definition of $\vdash_\otimes$ by:

$$\vdash_\otimes [\pi_i]\mathsf{tt} \qquad \vdash_\otimes [\pi_i]\varphi \wedge [\pi_i]\psi \to [\pi_i](\varphi \wedge \psi)$$

$$\vdash_\otimes [\pi_i]\neg\varphi \to \neg[\pi_i]\varphi \qquad \vdash_\otimes [\pi_i](\varphi \vee \psi) \to [\pi_i]\varphi \vee [\pi_i]\psi$$

(and similarly for $\vdash_{\odot E}$).

With these definitions, we obtain that one-step soundness and completeness are preserved by combinations of proof system constructors; for readability we have suppressed the dependency of the one-step semantics on the signature functor.

Proposition 51. *Suppose* P_i *is a proof system constructor for* S_i, *for* $i = 1, 2$, *and* E *is an arbitrary set. Then,* $\mathsf{P}_1 \otimes \mathsf{P}_2$, $\mathsf{P}_1 \oplus \mathsf{P}_2$, $\mathsf{P}_1 \odot E$ *and* $\mathsf{P}_1 \circledcirc \mathsf{P}_2$ *are proof system constructors for* $\mathsf{S}_1 \otimes \mathsf{S}_2$, $\mathsf{S}_1 \oplus \mathsf{S}_2$, $\mathsf{S}_1 \odot E$ *and* $\mathsf{S}_1 \circledcirc \mathsf{S}_2$, *respectively. Moreover, if* P_1 *and* P_2 *are one-step sound (complete) w.r.t.* $[\![\mathsf{S}_1]\!]$ *and* $[\![\mathsf{S}_2]\!]$, *respectively, then* $\mathsf{P}_1 \otimes \mathsf{P}_2$, $\mathsf{P}_1 \oplus \mathsf{P}_2$, $\mathsf{P}_1 \odot E$ *and* $\mathsf{P}_1 \circledcirc \mathsf{P}_2$ *are one-step sound (complete) w.r.t.* $[\![\mathsf{S}_1 \otimes \mathsf{S}_2]\!]$, $[\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!]$, $[\![\mathsf{S}_1 \odot E]\!]$ *and* $[\![\mathsf{S}_1 \circledcirc \mathsf{S}_2]\!]$, *respectively.*

Proof. It is immediate from the definitions that $\mathsf{P}_1 \otimes \mathsf{P}_2$, $\mathsf{P}_1 \oplus \mathsf{P}_2$, $\mathsf{P}_1 \odot E$ and $\mathsf{P}_1 \circledcirc \mathsf{P}_2$ are proof system constructors. Also, the one-step soundness and completeness of $\mathsf{P}_1 \circledcirc \mathsf{P}_2$ follows directly from the definitions.

Checking the one-step soundness of $\mathsf{P}_1 \otimes \mathsf{P}_2$, $\mathsf{P}_1 \oplus \mathsf{P}_2$ and $\mathsf{P}_1 \odot E$ is just a matter of unfolding the respective definitions. For the one-step completeness of $\mathsf{P}_1 \otimes \mathsf{P}_2$, $\mathsf{P}_1 \oplus \mathsf{P}_2$ and $\mathsf{P}_1 \odot E$, we assume that P_i is one-step complete w.r.t. the one-step semantics $[\![\mathsf{S}_i]\!]$, where $[\![\mathsf{S}_i]\!] = [\![\mathsf{S}_i]\!]^{T_i}$ is a one-step semantics for S_i w.r.t. the endofunctor T_i, for $i = 1, 2$. Throughout the proof, we fix $(A, \vdash) \in \mathsf{BTh}$ complete w.r.t. $d : A \to \mathcal{P}X$. We write $\vdash_{\mathsf{P}}$ for the entailment relation on $(\mathsf{Cl} \circ \mathsf{P})(A, \vdash)$, whenever P is a proof system constructor.

We first consider the one-step completeness of $\mathsf{P}_1 \otimes \mathsf{P}_2$ w.r.t. $[\![\mathsf{S}_1 \otimes \mathsf{S}_2]\!]$. We have to show that $(\mathsf{Cl} \circ (\mathsf{P}_1 \otimes \mathsf{P}_2))(A, \vdash)$ is complete w.r.t. $[\![\mathsf{S}_1 \otimes \mathsf{S}_2]\!](d^\sharp) : M\mathsf{S}_1 MA \otimes M\mathsf{S}_2 MA \to \mathcal{P}(T_1 X \times T_2 X)$. By Lemma 45, it suffices to show that $([\![\mathsf{S}_1 \otimes \mathsf{S}_2]\!](d^\sharp))^\sharp(\varphi \to \psi) = T_1 X \times T_2 X$ implies $\vdash_{\mathsf{P}_1 \otimes \mathsf{P}_2} \varphi \to \psi$, where $\varphi = \bigwedge_{i=1}^n [\pi_1]\varphi_i^1 \wedge \bigwedge_{j=1}^m [\pi_2]\varphi_j^2$ and $\psi = \bigvee_{i=1}^l [\pi_1]\psi_i^1 \vee \bigvee_{j=1}^k [\pi_2]\psi_j^2$. Using the fact that $[\pi_i]$, for $i = 1, 2$, distributes over all propositional connectives (as a result of the axioms in the definition of $\vdash_{\mathsf{P}_1 \otimes \mathsf{P}_2}$), this can be reduced further to assuming $\varphi = [\pi_1]\varphi_1 \wedge [\pi_2]\varphi_2$ and $\psi = [\pi_1]\psi_1 \vee [\pi_2]\psi_2$ (where we take φ_1 and φ_2 to be tt if $n = 0$ or $m = 0$, respectively, and ψ_1 and ψ_2 to be ff if $l = 0$ or $k = 0$, respectively). Writing S for $\mathsf{S}_1 \otimes \mathsf{S}_2$, our assumption is now

$$([\![\mathsf{S}]\!](d^\sharp))^\sharp([\pi_1]\varphi_1) \cap ([\![\mathsf{S}]\!](d^\sharp))^\sharp([\pi_2]\varphi_2) \subseteq ([\![\mathsf{S}]\!](d^\sharp))^\sharp([\pi_1]\psi_1) \cup ([\![\mathsf{S}]\!](d^\sharp))^\sharp([\pi_2]\psi_2)$$

and we have to show that $\vdash_{\mathsf{P}_1 \otimes \mathsf{P}_2} [\pi_1]\varphi_1 \wedge [\pi_2]\varphi_2 \to [\pi_1]\psi_1 \vee [\pi_2]\psi_2$.

Unravelling the definition of $[\![\mathsf{S}]\!] = [\![\mathsf{S}_1 \otimes \mathsf{S}_2]\!]$, we obtain that $([\![\mathsf{S}_i]\!](d^\sharp))^\sharp(\varphi_i) \subseteq ([\![\mathsf{S}_i]\!](d^\sharp))^\sharp(\psi_i)$ for $i = 1$ or $i = 2$. By completeness of P_i, we obtain that $\vdash_{\mathsf{P}_i} \varphi_i \to \psi_i$, and from the definition of $\vdash_{\mathsf{P}_1 \otimes \mathsf{P}_2}$, we conclude $\vdash_{\mathsf{P}_1 \otimes \mathsf{P}_2} [\pi_i]\varphi_i \to [\pi_i]\psi_i$, from which we obtain the desired conclusion $\vdash_{\mathsf{P}_1 \otimes \mathsf{P}_2} [\pi_1]\varphi_1 \wedge [\pi_2]\varphi_2 \to [\pi_1]\varphi_1 \vee [\pi_2]\varphi_2$ by propositional reasoning. The one-step completeness of $\mathsf{P}_1 \odot E$ is proved analogously.

We now turn to the completeness of $\mathsf{P}_1 \oplus \mathsf{P}_2$ w.r.t. $[\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!]$. As above, it suffices to show that $([\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!](d^\sharp))^\sharp(\varphi \to \psi) = T_1 X + T_2 X$ implies $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2} \varphi \to \psi$, where $\varphi = \bigwedge_{i=1}^n [\kappa_1]\varphi_i^1 \wedge \bigwedge_{j=1}^m [\kappa_2]\varphi_j^2$ and $\psi = \bigvee_{i=1}^l [\kappa_1]\psi_i^1 \vee \bigvee_{j=1}^k [\kappa_2]\psi_j^2$. We distinguish several cases:

Case 1. $l = k = 0$. Then $\psi = \mathsf{ff}$ and $([\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!](d^\sharp))^\sharp(\varphi) = \emptyset$. Let $\varphi_1 = \bigwedge_{i=1}^n \varphi_i^1$ and $\varphi_2 = \bigwedge_{j=1}^m \varphi_j^2$. Then $([\![\mathsf{S}_1]\!](d^\sharp))^\sharp(\varphi_1) = \emptyset$ and $([\![\mathsf{S}_2]\!](d^\sharp))^\sharp(\varphi_2) = \emptyset$ by the definition of $[\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!]$, hence $\vdash_{\mathsf{P}_1} \varphi_1 \to \mathsf{ff}$ and $\vdash_{\mathsf{P}_2} \varphi_2 \to \mathsf{ff}$ by one-step completeness of P_1 and P_2. Using the last rule in the definition of $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2}$, the axiom $\vdash_\oplus [\kappa_1]\mathsf{ff} \wedge [\kappa_2]\mathsf{ff} \to \mathsf{ff}$ and propositional reasoning, we obtain $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2} [\kappa_1]\varphi_1 \wedge [\kappa_2]\varphi_2 \to \mathsf{ff}$. Given that $[\kappa_i]$ distributes over conjunctions (as a result of the axioms in the definition of $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2}$), this shows $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2} \varphi \to \psi$.

Case 2. $l > 0$ and $k = 0$. Taking $\psi_1 = \bigvee_{i=1}^l \psi_i^1$ and φ_1, φ_2 as above, we have

$$([\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!](d^\sharp))^\sharp([\kappa_1]\varphi_1) \cap ([\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!](d^\sharp))^\sharp([\kappa_2]\varphi_2) \subseteq ([\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!](d^\sharp))^\sharp([\kappa_1]\psi_1)$$

from which we deduce that $([\![\mathsf{S}_1]\!](d^\sharp))^\sharp(\varphi_1) \subseteq ([\![\mathsf{S}_1]\!](d^\sharp))^\sharp(\psi_1)$ by the definition of $[\![\mathsf{S}_1 \oplus \mathsf{S}_2]\!]$. Hence, by one-step completeness of P_1 and P_2, we have $\vdash_{\mathsf{P}_1} \varphi_1 \to \psi_1$, which yields $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2} [\kappa_1]\varphi_1 \to [\kappa_1]\psi_1$, and our claim $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2} \varphi \to \psi$ follows by propositional reasoning, using the distributivity of $[\kappa_1]$ over conjunctions and non-empty disjunctions.

Case 3. $l = 0$ and $k > 0$. Similar.

Case 4. $l > 0$ and $k > 0$. Taking $\varphi_1, \varphi_2, \psi_1$ as above and $\psi_2 = \bigvee_{j=1}^k \varphi_j^2$, we have that $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2} [\kappa_1]\mathsf{ff} \vee [\kappa_2]\mathsf{ff} \to [\kappa_1]\psi_1 \vee [\kappa_2]\psi_2$, hence $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2} [\kappa_1]\psi_1 \vee [\kappa_2]\psi_2$ and therefore $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2} [\kappa_1]\varphi_1 \wedge [\kappa_2]\varphi_2 \to [\kappa_1]\psi_1 \vee [\kappa_2]\psi_2$. The claim that $\vdash_{\mathsf{P}_1 \oplus \mathsf{P}_2} \varphi \to \psi$ now follows by propositional reasoning, using the fact that $[\kappa_i]$ distributes over conjunctions and non-empty disjunctions. □

An important observation is that, if P_1 and P_2 are defined in terms of axioms (as is, for instance, the case for $\mathsf{P}_{\mathcal{P}}$ and $\mathsf{P}_{\mathcal{D}}$), then all their combinations can be described in the same way (namely by incorporating the axioms of Definition 50 together with all propositional tautologies and modus ponens with the original definitions of P_1 and P_2).

Below we only give the full axiomatic definition of $\mathsf{P}_1 \circledcirc \mathsf{P}_2$. The definitions of $\mathsf{P}_1 \otimes \mathsf{P}_2$, $\mathsf{P}_1 \oplus \mathsf{P}_2$ and $\mathsf{P}_1 \odot E$ are similar.

Recall from Example 11 that the composition of syntax constructors can be thought of as introducing an additional sort. In the same way, the composition of proof system constructors can be regarded as introducing a derivability predicate on formulas of this sort: suppose P_1 and P_2 are defined in terms of the rules (possibly with empty premise) $\mathcal{R}_1$ and $\mathcal{R}_2$, respectively. Then, for a boolean theory $(A, \vdash)$, the boolean theory $(\mathsf{P}_1 \circledcirc \mathsf{P}_2)(A, \vdash) = (\mathsf{S}_1 M \mathsf{S}_2 M A, \vdash'')$ is generated using an intermediate derivability predicate $\vdash'$ on $\mathsf{S}_2 M A$, by means of the following axioms:

$$\frac{\vdash \varphi}{\vdash' \rho} \quad \left(\text{if } \frac{\vdash \varphi}{\vdash' \rho} \in \mathcal{R}_2\right) \qquad\qquad \frac{}{\vdash' \rho} \quad (\text{if } \rho \text{ is an instance of a tautology})$$

$$\frac{\vdash' \rho \quad \vdash' \rho \to \rho'}{\vdash' \rho'} \quad (\text{modus ponens for } \vdash') \qquad\qquad \frac{\vdash' \rho}{\vdash'' \varphi'} \quad \left(\text{if } \frac{\vdash' \rho}{\vdash'' \varphi'} \in \mathcal{R}_1\right)$$

where S_1 and S_2 are the syntax constructors associated to P_1 and P_2, respectively, φ ranges over formulas in MA, ρ and ρ' range over formulas in $M\mathsf{S}_2 MA$, and φ' ranges over formulas in $M\mathsf{S}_1 M \mathsf{S}_2 MA$. In particular, the second and third rules account for the presence of Cl in the definition of $\mathsf{P}_1 \circledcirc \mathsf{P}_2$.

As we have already argued in the beginning, a large class of systems, including Kripke structures, (probabilistic) transition systems and probabilistic automata can be modelled as coalgebras of signature functors of the following form:

$$T ::= D \mid \mathcal{I}d \mid \mathcal{P}_\omega \mid \mathcal{P} \mid \mathcal{D} \mid T_1 \times T_2 \mid T_1 + T_2 \mid T_1{}^E \mid T_1 \circ T_2$$

where D and E are arbitrary sets.

We can therefore use Propositions 12, 26, 35 and 51 to derive, for any (probabilistic) system type of the above form, a logic which is sound and complete, and which is also expressive provided that the unbounded powerset functor is not used in the signature functor.

Example 52 (*Probabilistic Automata*). Simple probabilistic automata [29] are modelled coalgebraically using the functor $T = (\mathcal{P} \circ \mathcal{D})^E$. The language $\mathcal{L}_1 = \mathcal{L}(T)$ obtained by applying the modular techniques presented in Section 3 can be described by the following grammar:

$$\begin{aligned} \mathcal{L}_1 \ni \varphi &::= \mathsf{ff} \mid \varphi \to \varphi' \mid [e]\psi && (\psi \in \mathcal{L}_2) \\ \mathcal{L}_2 \ni \psi &::= \mathsf{ff} \mid \psi \to \psi' \mid \Box\xi && (\xi \in \mathcal{L}_3) \\ \mathcal{L}_3 \ni \xi &::= \mathsf{ff} \mid \xi \to \xi' \mid L_p\varphi && (\varphi \in \mathcal{L}_1). \end{aligned}$$

The coalgebraic semantics $\models_1$ of $\mathcal{L}_1$ is obtained automatically from the one-step semantics for $\mathsf{S}_\mathcal{P}$ and $\mathsf{S}_\mathcal{D}$ (as defined in Example 21), using the modular techniques presented in Section 4. The resulting logic is essentially the same as the probabilistic modal logic of [13]. Moreover, if we replace the unbounded powerset functor by its finite variant in the signature functor T and adjust the one-step semantics according to Example 21, Proposition 35 automatically yields a Hennessy–Milner result for this logic (w.r.t. image-finite simple probabilistic automata).

In addition, the techniques described in this section allow us to derive a sound and complete proof system for the above logic, both w.r.t. simple probabilistic automata and w.r.t. their image-finite variants. This proof system can be described by three entailment relations, corresponding to entailment in $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, respectively, as follows:

0. Axioms for all $\vdash_i$ ($\varphi, \psi \in \mathcal{L}_i$):

$$\vdash_i \varphi \quad (\varphi \text{ instance of tautology}) \qquad\qquad \frac{\vdash_i \varphi \quad \vdash_i \varphi \to \psi}{\vdash_i \psi}.$$

1. Axioms for $\vdash_1$ ($\varphi, \psi \in \mathcal{L}_2$, $e \in E$):

$$\vdash_1 [e]\mathsf{ff} \to \mathsf{ff} \qquad \vdash_1 [e](\varphi \to \psi) \leftrightarrow ([e]\varphi \to [e]\psi) \qquad \frac{\vdash_2 \varphi \to \psi}{\vdash_1 [e]\varphi \to [e]\psi}.$$

2. Axioms for $\vdash_2$ ($\varphi, \psi \in \mathcal{L}_3$):

$$\vdash_2 \Box\mathsf{tt} \qquad \vdash_2 \Box\varphi \wedge \Box\psi \to \Box(\varphi \wedge \psi) \qquad \frac{\vdash_3 \varphi \to \psi}{\vdash_2 \Box\varphi \to \Box\psi}.$$

3. Axioms for $\vdash_3$ ($\varphi, \psi, \varphi_i, \psi_j \in \mathcal{L}_1$, $p, q \in \mathbb{Q} \cap [0, 1]$ with $p + q > 1$, $m, n \geq 0$):

$$\vdash_3 L_0\varphi \qquad \vdash_3 L_p\mathsf{tt} \qquad \vdash_3 L_p\varphi \to \neg L_q\neg\varphi \qquad \vdash_3 \neg L_p\varphi \to M_p\varphi$$

$$\frac{\vdash_1 \bigwedge_{k=1}^{\max(m,n)} \varphi^{(k)} \leftrightarrow \psi^{(k)}}{\vdash_3 \left(\bigwedge_{i=1}^{m} L_{p_i}\varphi_i\right) \wedge \left(\bigwedge_{j=2}^{n} M_{q_i}\psi_i\right) \to L_{p_1+\cdots+p_m-(q_2+\cdots+q_n)}\psi_1}.$$

Now Proposition 42 shows that the entailment relation $\vdash_1$ defined above coincides with the entailment relation defined by iteratively applying the corresponding proof system constructor. Since all our constructions preserve completeness, we obtain $\models_T \varphi$ iff $\vdash_1 \varphi$ for all $\varphi \in \mathcal{L}_1$, i.e. completeness of $\vdash_1$ w.r.t. the coalgebraic semantics of $\mathcal{L}_1$.

7. Conclusions

We have studied modular construction principles for coalgebraic logics. When modelling systems coalgebraically, one typically constructs an endofunctor that represents the behaviour of the associated class of systems from a small number of basic ingredients, such as constants, powersets and probability distributions, by means of a small number of operations, viz products, coproducts and functor composition. We have demonstrated that this modular approach carries over to the associated logics. On the logical side, every endofunctor is paired with a proof system constructor, and operations on endofunctors such as products and functor composition give rise to corresponding operations on proof system constructors. By showing that the basic ingredients admit a (sound and) complete proof system, we have shown (soundness and) completeness for combined systems by showing that the operations on proof system constructors preserve (soundness and) completeness. In this way, we have obtained (sound and) complete logics for a wide range of state-based systems, in particular for the probabilistic automata of [29], for which completeness was hitherto an open question.

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