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Advances in Mathematics 207 (2006) 328–419

ADVANCES IN Mathematics

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Infinite-dimensional primitive linearly compact Lie superalgebras

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Received 17 November 2005; accepted 13 February 2006

Available online 18 April 2006

Communicated by Pavel Etingof

Abstract

We classify open maximal subalgebras of all infinite-dimensional linearly compact simple Lie superalgebras. This is applied to the classification of infinite-dimensional Lie superalgebras of vector fields, acting transitively and primitively in a formal neighborhood of a point of a finite-dimensional supermanifold.

Keywords: Linearly compact algebra; Maximal open subalgebra; Primitive Lie superalgebra; Weisfeiler filtration; Growth and size of filtered algebra; Canonical invariant

0. Introduction

A well-known theorem of E. Cartan [7] states that an infinite-dimensional Lie algebra L of vector fields in a neighborhood of a point p of an m-dimensional manifold M acting transitively and primitively in this neighborhood, is formally isomorphic to a member of one of the six series of Lie algebras of formal vector fields:

1. $W_m = \{\sum_{i=1}^m P_i \partial/\partial x_i \mid P_i \in \mathbb{C}[[x_1, \ldots, x_m]]\}$,
2. $S_m = \{X \in W_m \mid div(X) = 0\}$,

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[1] Partially supported by Progetto Giovani Ricercatori CPDG031245.
[2] Partially supported by NSF grants DMS0201017 and DMS0501395.

doi:10.1016/j.aim.2006.02.013

2′. $CS_m = \{X \in W_m \mid div(X) = const\}$,
3. $H_m = \{X \in W_m \mid X\omega_s = 0\}$ $(m = 2k)$, where $\omega_s = \sum_{i=1}^{k} dx_i \wedge dx_{k+i}$ is a symplectic form,
3′. $CH_m = \{X \in W_m \mid X\omega_s = const\,\omega_s\}$ $(m = 2k)$,
4. $K_m = \{X \in W_m \mid X\omega_c = f\omega_c\}$ $(m = 2k+1)$, where $\omega_c = dx_m + \sum_{i=1}^{k} x_i\, dx_{k+i}$ is a contact form and f is a formal power series (depending on X).

Recall that the primitivity of an action means that there are no non-trivial L-invariant fibrations in M. The Lie algebra L has a canonical filtration $L \supset L_0 \supset L_1 \supset \cdots$, where L_j consists of vector fields vanishing up to order j at p, and the formal isomorphism means the isomorphism of the completed Lie algebras with respect to this filtration. The transitivity of the action implies that L_0 contains no non-zero ideals of L, and primitivity implies that L_0 is a maximal subalgebra.

It is easy to see [13] that, in fact, Cartan's theorem gives a classification of infinite-dimensional linearly compact Lie algebras L, which admit a maximal open subalgebra L_0 containing no non-zero ideals of L (recall that the *linear compactness* of L means that L is a topological Lie algebra whose underlying topological space is a topological product of finite-dimensional vector spaces with discrete topology). Such a Lie algebra L is called *primitive*, the subalgebra L_0 is called a fundamental maximal subalgebra, and the pair (L, L_0) is called a *primitive pair*. It is easy to see that all L from the six series contain a unique fundamental maximal subalgebra. Also, the Lie algebras W_m, S_m, H_m and K_m are simple, and the remaining two series CS_m and CH_m are the Lie algebras of derivations of S_m and H_m respectively, obtained by adding the Euler vector field $E = \sum_i x_i \partial/\partial x_i$.

In the present paper we solve the problem of classification of primitive pairs in the Lie superalgebra case. This problem is much more difficult than in the Lie algebra case for several reasons. First, in the Lie algebra case, a primitive L is contained between S and $Der\,S$, where S is simple (cf. Theorem 1.5), which instantly reduces the classification of primitive Lie algebras L to simple ones, but the situation is more complicated in the super case. Second, there are many more simple linearly compact Lie superalgebras than in the Lie algebra case (see [17]). Third, in a sharp contrast to the Lie algebra case, almost all infinite-dimensional simple linearly compact Lie superalgebras contain more than one maximal open subalgebra. Most of the space of the present paper deals with the problem of their classification.

The infinite-dimensional linearly compact simple Lie superalgebras have been classified in [17]. The list consists of ten series $(m \geqslant 1)$: $W(m,n)$, $S(m,n)$ $((m,n) \neq (1,1))$, $H(m,n)$ (m even), $K(m,n)$ (m odd), $HO(m,m)$ $(m \geqslant 2)$, $SHO(m,m)$ $(m \geqslant 3)$, $KO(m,m+1)$, $SKO(m,m+1;\beta)$ $(m \geqslant 2)$, $SHO^{\sim}(m,m)$ (m even), $SKO^{\sim}(m,m+1)$ ($m \geqslant 3$, m odd), and five exceptional Lie superalgebras: $E(1,6)$, $E(3,6)$, $E(3,8)$, $E(4,4)$, $E(5,10)$.

The main idea of [17] is to pick a maximal open subalgebra S_0 of a simple linearly compact Lie superalgebra S, which is invariant with respect to all inner automorphisms of S. The existence of such an *invariant* subalgebra S_0 is a non-trivial fact, the proof of which uses characteristic varieties (cf. [14]). Remarkably, an invariant subalgebra is unique in most, though not all, of the cases. After that the classification procedure is more or less routine. One constructs an irreducible Weisfeiler filtration [22] associated to the pair (S, S_0) and shows, using ideas from [14], that the associated graded Lie superalgebra $Gr\,S = \bigoplus_j \mathfrak{g}_j$ has the property that $[\mathfrak{g}_0, v] = \mathfrak{g}_{-1}$ for any even element $v \in \mathfrak{g}_{-1}$ (which does not hold for a random choice of a maximal open subalgebra S_0). After that one is able to describe all possibilities for the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ and the subalgebra $\bigoplus_{j \leqslant 0} \mathfrak{g}_j$ of $Gr\,S$ [17], and, after some further work, all the possibilities for $Gr\,S$ [10]. Finally, one finds all simple filtered deformations of all these $Gr\,S$ [9].

Recall that one has the following isomorphisms (cf. [17, Remark 6.6]):

$$W(1,1) \cong K(1,2) \cong KO(1,2), \qquad S(2,1) \cong HO(2,2), \qquad SHO^{\sim}(2,2) \cong H(2,1).$$

Besides, $S(2,1) \cong SKO(2,3;0)$. Hence, when discussing $S(m,n)$, $K(m,n)$, $KO(m,m+1)$, $HO(m,m)$ and $SHO^{\sim}(m,m)$, we shall assume that $(m,n) \neq (2,1)$, $(m,n) \neq (1,2)$, $m \geqslant 2$, $m \geqslant 3$ and $m > 3$, respectively. Also we shall assume that $n \geqslant 1$ since the Lie algebra case is trivial.

In the first part of the present paper we give a description of semisimple artinian linearly compact Lie superalgebras in terms of simple ones (Theorem 1.4), which is similar to that in the finite-dimensional case [8,15]. Next, we show that if an infinite-dimensional linearly compact Lie superalgebra L is primitive, then L is artinian semisimple and, moreover, contains an open ideal isomorphic to $S \otimes \Lambda(n)$, where S is a simple linearly compact Lie superalgebra and $\Lambda(n)$ is the Grassmann algebra in n indeterminates, and is contained in $(Der\, S) \otimes \Lambda(n) + 1 \otimes Der\, \Lambda(n)$, so that the projection of L on the second summand acts transitively on $\Lambda(n)$ (Theorem 1.5).

Next, Theorem 1.9 gives a description of fundamental maximal subalgebras in L in terms of those in S. In fact, the situation is slightly more complicated, namely in general $Der\, S = S \rtimes \mathfrak{a}$, where either $\mathfrak{a} \cong gl_2$ or $\dim \mathfrak{a} \leqslant 3$, and we need to classify all maximal among open $\mathfrak{a}_0$-invariant subalgebras of S, for each subalgebra $\mathfrak{a}_0$ of $\mathfrak{a}$. We, thus, arrive at a problem of classification of maximal among $\mathfrak{a}_0$-invariant open subalgebras of each infinite-dimensional simple linearly compact Lie superalgebra S.

If $S = \prod_{j \geqslant -d} \mathfrak{g}_j$ is an *irreducible* grading of S, i.e., $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible and $\mathfrak{g}_{-j} = \mathfrak{g}_{-1}^j$ for all $j \leqslant -2$, then $S_0 = \prod_{j \geqslant 0} \mathfrak{g}_j$ is called a *graded* subalgebra of S. All irreducible gradings (apart for a few omissions) were described in [10,21], and in the present paper we give a detailed proof that these are all. It turns out by inspection (using Proposition 1.11(b)) that for every irreducible grading of a simple infinite-dimensional linearly compact Lie superalgebra S, the corresponding graded subalgebra $S_0 = \prod_{j \geqslant 0} \mathfrak{g}_j$ is maximal.

A surprising discovery of the present paper is a large number of new open maximal subalgebras (which are not graded). The main result of the present paper is a classification of all maximal open $\mathfrak{a}_0$-invariant subalgebras of all infinite-dimensional linearly compact simple Lie superalgebras S, up to conjugation by the group G of inner automorphisms of $Der\, S$. (The group G can be thought of as the unity component of the group of all automorphisms of S.) Unless otherwise specified, by conjugation we always mean the conjugation by G.

An important part of this classification is the classification of all regular maximal open $\mathfrak{a}_0$-invariant subalgebras of S. A subalgebra of S is called *regular* if it is invariant with respect to a maximal torus of $Der\, S$. By Theorem 1.7, all maximal tori in $Der\, S$ are conjugate, hence fixing one "standard" torus T, and classifying all T-invariant maximal open subalgebras we obtain all regular maximal open subalgebras of S, up to conjugation (by G).

The numbers a of graded and b of non-graded regular maximal open subalgebras of S, up to conjugation, are given in Table 1 (the case $\mathfrak{a}_0 = 0$). Thus we see that, with the exception of $K(m,1)$, any simple linearly compact infinite-dimensional Lie superalgebra, which is not a Lie algebra, contains more than one maximal open subalgebra. It turned out that in all cases except for $H(m,n)$ with n positive even, all maximal open subalgebras are regular, but $H(m,n)$ with $n = 2h$ even, contains, up to conjugacy, $h(h+1)/2$ non-regular maximal open subalgebras.

The main tool in the classification of maximal open subalgebras in non-exceptional simple linearly compact Lie superalgebras is a formal analogue of the Frobenius theorem (Theorem 1.1(a)), which implies that a maximal open subalgebra of a transitive subalgebra of $W(m,n)$ consists of vector fields, leaving invariant a conjugate, by a change of variables, of a standard ideal of

Table 1

S	a	b	c	e
$W(1,1)$	2	0	3	1
$W(m,n)$, $(m,n)\neq(1,1)$	$n+1$	0	$n+1$	0
$S(1,2)$	2	0	4	2
$S(m,n)$, $(m,n)\neq(1,2)$	$n+1$	0	$n+1$	0
$K(1,2h)$	$h+1$	0	$h+2$	1
$K(m,2h)$, $m>1$	$h+2$	0	$h+2$	0
$K(m,2h+1)$	$h+1$	0	$h+1$	0
$HO(n,n)$, $n>2$	n	0	$n+1$	1
$SHO(3,3)$	2	0	5	3
$SHO(n,n)$, $n>3$	n	0	$n+1$	1
$H(m,2h)$	$h+2$	$\frac{h}{2}(1+h)$	h^2+2h+2	0
$H(m,2h+1)$	$h+1$	$(h+1)^2$	h^2+3h+2	0
$KO(2,3)$	2	2	4	0
$KO(n,n+1)$, $n>2$	n	n	$2n+2$	2
$SKO(2,3;0)$	2	0	2	0
$SKO(2,3;1)$	2	1	3	0
$SKO(2,3;\beta)$, $\beta\neq 0,1$	3	1	5	1
$SKO(3,4;\beta)$	3	3	$8+8\delta_{3\beta,1}$	$2+8\delta_{3\beta,1}$
$SKO(n,n+1;\beta)$, $n>3$	n	n	$2n+2$	2
$SHO^{\sim}(n,n)$, $n>2$	1	$n-1$	$n+1$	1
$SKO^{\sim}(n,n+1)$	0	$2n-1$	$2n+2$	3
$E(1,6)$	4	0	5	1
$E(3,6)$	3	0	5	2
$E(5,10)$	4	0	6	2
$E(4,4)$	1	3	5	1
$E(3,8)$	3	6	18	9

$\Lambda(m,n)$, that is, an ideal generated by a subspace of the span of all odd indeterminates. This instantly solves the problem in question for $W(m,n)$, but for other non-exceptional simple Lie superalgebras it requires more subtle arguments to show that a conjugate of a standard ideal of $\Lambda(m,n)$ can be replaced by a standard ideal.

In the case of exceptional simple linearly compact Lie superalgebras S we use the notions of growth and size of S (which remain unchanged when passing from S to $Gr\,S$) in order to list possible $Gr\,S$. This allows us to find all maximal open subalgebras of S by analyzing its deviation from a maximal open invariant subalgebra (which is unique in all exceptional superalgebras S).

A posteriori, it follows from the present paper that an open subalgebra of minimal codimension in a linearly compact infinite-dimensional simple Lie superalgebra S is always invariant under all inner automorphisms of S. Moreover, in all cases, but $S=W(1,1)$, $S(1,2)$, $SHO(3,3)$, and $SKO(3,4;1/3)$, such a subalgebra is unique (hence invariant under all automorphisms), and in $S=W(1,1)$, $S(1,2)$, and $SHO(3,3)$ such subalgebras are conjugate by (outer) automorphisms of S. We denote by S_0 the intersection of all open subalgebras of minimal codimension in S, and call it the *canonical subalgebra* of S. The canonical subalgebra is, of course, invariant with respect to the group $Aut\,S$ of all continuous automorphisms of S. Let S_{-1} be a minimal subspace of S, properly containing S_0 and invariant with respect to the group $Aut\,S$, and let $S=S_{-d}\supsetneq S_{-d+1}\supset\cdots\supset S_{-1}\supset S_0\supset\cdots$ be the associated Weisfeiler filtration of S. All members of the Weisfeiler filtration associated to S_0 are invariant with respect to the group $Aut\,S$, hence we have the induced filtration on the superspace $V:=S/S_0=V_{-d}\supset\cdots\supset V_{-1}$, and the induced action of $Aut\,S$ on V preserving this filtration. Note that $Gr\,V$ carries a canonical

$\mathbb{Z}$-graded Lie superalgebra structure, isomorphic to $\bigoplus_{j=-d}^{-1} \mathfrak{g}_j$. A subspace U of V is called *abelian* if $Gr\, U$ is an abelian subalgebra of $Gr\, V$.

Now, it is easy to see that if L_0 is a (proper) open subalgebra of S, its image under the canonical map $S \to V$ is a purely odd abelian subspace of V, denoted by $\pi(L_0)$. Thus, we obtain an $Aut\, S$-equivariant map π from the set of all open subalgebras of S to the set of abelian subspaces of $V_{\bar{1}}$ (the odd part of V).

The G-orbit of $\pi(L_0)$ in $V_{\bar{1}}$ is called the *canonical invariant* of the open subalgebra L_0 of S. A posteriori, it turns out that the canonical invariant uniquely determines an open maximal subalgebra of S, so we have an injective map Π from the set of conjugacy classes (by G) of maximal open subalgebras of S to the set of G-orbits of abelian subspaces of $V_{\bar{1}}$. The number c of elements of the latter set along with the number e of those of them which are not canonical invariants of any open maximal subalgebra are given in Table 1. Looking at this table, we see that in many cases $e = 0$, i.e., the map Π is bijective, and in the remaining cases it is very close to being bijective.

The contents of the paper are as follows. In Section 1 we prove a formal analogue of the Frobenius theorem (Theorem 1.1), establish some general results on the structure of artinian semisimple and primitive infinite-dimensional linearly compact Lie superalgebras (Theorems 1.4 and 1.5), and reduce the classification of primitive pairs (L, L_0) to the case of simple L (Theorem 1.9). We also prove conjugacy of maximal tori in artinian semisimple linearly compact Lie superalgebras (Theorem 1.7), and discuss the notions of growth and size.

In Sections 2 through 10 we give a classification of all maximal open subalgebras (and all $\mathfrak{a}_0$-invariant maximal open subalgebras as well) of all infinite-dimensional simple linearly compact Lie superalgebras. As an immediate application of this long and tedious work, we obtain the list of all irreducible graded infinite-dimensional linearly compact Lie superalgebras which admit a non-trivial simple filtered deformation.

In Section 11 we classify all maximal open subalgebras which are invariant with respect to all inner automorphisms and we discuss the canonical invariant. An a priori proof that the canonical invariant determines a maximal open subalgebra uniquely would considerably shorten the paper, but we were unable to find such a proof.

In Appendix A we prove the solvability of the radical and establish conjugacy of the maximal tori in any linearly compact artinian Lie superalgebra. In Appendix B we give an alternative description of non-graded maximal open subalgebras of all non-exceptional infinite-dimensional simple linearly compact Lie superalgebras.

In a subsequent paper [5] we use the canonical subalgebras to describe automorphisms and forms over an arbitrary field of characteristic zero of all simple infinite-dimensional linearly compact Lie superalgebras.

Throughout the paper all vector spaces and algebras, as well as tensor products, are considered over the field of complex numbers $\mathbb{C}$.

1. General results on semisimple and primitive linearly compact Lie superalgebras

Recall that a *linearly compact space* is a topological vector space which is isomorphic to a topological product of finite-dimensional vector spaces endowed with discrete topology. The basic examples of linearly compact spaces are finite-dimensional vector spaces with the discrete topology, and the space of formal power series $V[[t]]$ over a finite-dimensional vector space V, with the formal topology defined by taking as a fundamental system of neighborhoods of 0 the set $\{t^j V[[t]]\}_{j\in\mathbb{Z}_+}$. We recall Chevalley's principle [13]: if $F_1 \supset F_2 \supset \cdots$ is a sequence of closed subspaces of a linearly compact space such that $\bigcap_j F_j = 0$ and U is an open subspace, then $F_j \subset U$ for $j \gg 0$.

A *linearly compact superalgebra* is a topological superalgebra whose underlying topological space is linearly compact. The basic example of an associative linearly compact superalgebra is $\Lambda(m,n) = \Lambda(n)[[x_1,\dots,x_m]]$, where $\Lambda(n)$ denotes the Grassmann algebra on n anticommuting indeterminates $\xi_1,\dots,\xi_n$, and the superalgebra parity is defined by $p(x_i) = \bar{0}$, $p(\xi_j) = \bar{1}$, with the formal topology defined by the following fundamental system of neighborhoods of 0: $\{(x_1,\dots,x_m,\xi_1,\dots,\xi_n)^j\}_{j\in\mathbb{Z}_+}$. The basic example of a linearly compact Lie superalgebra is $W(m,n) = Der\,\Lambda(m,n)$, the Lie superalgebra of all continuous derivations of the superalgebra $\Lambda(m,n)$. One has:

$$W(m,n) := \left\{ X = \sum_{i=1}^{m} P_i(x,\xi)\frac{\partial}{\partial x_i} + \sum_{j=1}^{n} Q_j(x,\xi)\frac{\partial}{\partial \xi_j} \;\middle|\; P_i, Q_j \in \Lambda(m,n) \right\}.$$

Letting $\deg x_i = \deg \xi_j = 1$, $\deg(\partial/\partial x_i) = \deg(\partial/\partial \xi_j) = -1$, we obtain the *principal* $\mathbb{Z}$-grading $W(m,n) = \prod_{j\geqslant -1} W(m,n)_j$. A subalgebra L of $W(m,n)$ is called *transitive* if the projection of L on $W(m,n)_{-1}$ is onto. Given a vector field $X \in W(m,n)$, we denote by $X(0)$ the projection of X on $W(m,n)_{-1}$.

Given a subspace U of the subspace $\sum_{i=1}^{m}\mathbb{C}x_i + \sum_{j=1}^{n}\mathbb{C}\xi_j$ of $\Lambda(m,n)$, denote by I_U the ideal of $\Lambda(m,n)$ generated by U. Let $W_U = \{X \in W(m,n) \mid XI_U \subset I_U\}$ be the corresponding subalgebra of $W(m,n)$. More generally, for any subalgebra L of $W(m,n)$, let $L_U = \{X \in L \mid XI_U \subset I_U\}$. We shall call I_U a *standard ideal* of $\Lambda(m,n)$ and L_U a *standard subalgebra* of L.

Theorem 1.1.

(a) *Let L be a closed subalgebra of $W(m,n)$, let V be the projection of L on $W(m,n)_{-1} = \sum_i \mathbb{C}\partial/\partial x_i + \sum_j \mathbb{C}\partial/\partial\xi_j$, and let $V^\perp \subset \sum_i \mathbb{C}x_i + \sum_j \mathbb{C}\xi_j$ be the dual of V. Then there exists a continuous automorphism of $\Lambda(m,n)$ such that the induced automorphism of $W(m,n)$ maps L to the subalgebra $W_{V^\perp}$ of $W(m,n)$.*

(b) *The algebra $\Lambda(m,n)$ has no non-trivial closed L-invariant ideals if and only if L is a transitive subalgebra.*

Proof. Making a linear change of variables, we may assume that V is the span of $\partial/\partial x_1,\dots,\partial/\partial x_p, \partial/\partial\xi_1,\dots,\partial/\partial\xi_q$. Also, we may assume that L is invariant with respect to multiplication by elements of $\Lambda(m,n)$. Indeed, $\Lambda(m,n)W_U = W_U$, an ideal of $\Lambda(m,n)$ is L-invariant if and only if it is $\Lambda(m,n)L$-invariant, and L is transitive if and only if $\Lambda(m,n)L$ is transitive.

We turn now to the proof of (a). If $p \geqslant 1$, then L contains a vector field $X_1 = \partial/\partial x_1 + D_1$, where D_1 is an even operator such that $D_1(0) = 0$. Making change of variables (cf. [17, p. 12]), we may assume that $X_1 = \partial/\partial x_1$. Consider $X_2 = \partial/\partial x_2 + D_2 \in L$, $D_2(0) = 0$. Subtracting $f\partial/\partial x_1$ from X_2, we may assume that X_2, hence D_2, do not involve $\partial/\partial x_1$. Next, we show that we may assume that all coefficients of D_2 do not involve x_1. Here we use that L is a subalgebra. Let $D_2 = \sum_{j\geqslant 0} x_1^j \overline{D}_j$, where the $\overline{D}_j$ do not involve x_1. Since

$$[X_1, X_2] = [X_1, D_2] \in L,$$

we see that $\sum_{j\geqslant 0} jx_1^{j-1}\overline{D}_j \in L$, hence, $x_1\sum_{j\geqslant 0} jx_1^{j-1}\overline{D}_j \in L$, then, $D_2 - \sum_{j\geqslant 0} jx_1^j\overline{D}_j \in L$, and we can assume that $\overline{D}_1 = 0$. Repeating this procedure, since L is closed, we get in the limit: $\partial/\partial x_2 + \overline{D}_0 \in L$, where $\overline{D}_0(0) = 0$ and $\overline{D}_0$ does not depend on x_1. Making change of variables, we may assume that $\partial/\partial x_1, \partial/\partial x_2 \in L$. Continuing one gets $\partial/\partial x_1, \dots, \partial/\partial x_p \in L$. If $q \geqslant 1$, let Y_1 be an odd vector field in L whose projection on $W(m,n)_{-1}$ is $\partial/\partial\xi_1$. Up to a change of variables we may assume that $Y_1 = \partial/\partial\xi_1 + \xi_1 D$, where D is an even operator. Since $[Y_1, Y_1] = 2D$, D lies in L, hence $\xi_1 D \in L$ and $\partial/\partial\xi_1 \in L$. Then, arguing as above, we can assume that $\partial/\partial\xi_1, \dots, \partial/\partial\xi_q$ lie in L. Hence L is generated, as a $\Lambda(m,n)$-module, by $\partial/\partial x_1, \dots, \partial/\partial x_p, \partial/\partial\xi_1, \dots, \partial/\partial\xi_q$ and by vector fields X_k which do not involve $\partial/\partial x_1, \dots, \partial/\partial x_p, \partial/\partial\xi_1, \dots, \partial/\partial\xi_q$ and such that $X_k(0) = 0$. As above, we may assume that all coefficients of all X_k do not depend on $x_1, \dots, x_p, \xi_1 \dots, \xi_q$. Therefore the ideal of $\Lambda(m,n)$ generated by $x_{p+1}, \dots, x_m, \xi_{q+1}, \dots, \xi_n$ is L-invariant, which proves (a).

Now we prove (b). The transitivity of L is equivalent to saying that L contains elements $a_i = \partial/\partial x_i + X$ and $b_j = \partial/\partial\xi_j + Y$ for some vector fields X and Y such that $X(0) = 0$ and $Y(0) = 0$, for every i and j. Let I be an L-stable non-zero ideal of $\Lambda(m,n)$. Then I contains a non-zero element $P(x,\xi) \in \Lambda(m,n)$. Since I is stable under the action of the vector fields a_i and b_j, we may assume that $P(0,0) = 1$ and, since I is an ideal, multiplying I by P^{-1}, we find that I contains 1, i.e., $I = \Lambda(m,n)$. Conversely, if L is not transitive, then $V^{\perp} \neq 0$, and we arrive at a contradiction with (a). □

Remark 1.2. Theorem 1.1(a) is an analogue of the Frobenius theorem for the superalgebra $\Lambda(m,n)$. Namely, if the projection V of L on $W(m,n)_{-1}$ has dimension $(p|q)$, then there exists a continuous automorphism φ of $\Lambda(m,n)$ such that the ideal $J_V = (\varphi(x_{p+1}), \dots, \varphi(x_m), \varphi(\xi_{q+1}), \dots, \varphi(\xi_n))$ is L-invariant. In the purely odd case this was proved in [12].

Note that J_V is maximal among L-invariant ideals. Indeed, up to automorphisms, this is equivalent to saying that, if $V = \langle \partial/\partial x_1, \dots, \partial/\partial x_p, \partial/\partial\xi_1, \dots, \partial/\partial\xi_q \rangle$ then $J_V = (x_{p+1}, \dots, x_m, \xi_{q+1}, \dots, \xi_n)$ is maximal among L-invariant ideals. Indeed, if we add a polynomial P to the ideal J_V, we may assume that P depends only on the variables $x_1, \dots, x_p, \xi_1, \dots, \xi_q$. Then, since $\partial/\partial x_i$ and $\partial/\partial\xi_j$ lie in L_0 for every $i = 1, \dots, p$ and $j = 1, \dots, q$, adding P to the ideal adds 1 to it.

Remark 1.3. Let L be an infinite-dimensional linearly compact Lie superalgebra embedded in $W(m,n)$, and let L_0 be a fundamental maximal subalgebra of L such that the projection of L_0 to $W(m,n)_{-1}$ does not contain the even derivations $\partial/\partial x_i$ for any $i = 1, \dots, m$. Then, by Theorem 1.1(a), L_0 stabilizes an ideal J of $\Lambda(m,n)$ which is, up to changes of variables, a standard ideal containing all even indeterminates $x_1, \dots, x_m$. Besides, J is maximal among the $\Lambda(m,n)L_0$-invariant ideals of $\Lambda(m,n)$ by Remark 1.2. Notice that an ideal I of $\Lambda(m,n)$ is L_0-invariant if and only if it is $\Lambda(m,n)L_0$-invariant. Therefore J is also maximal among the L_0-invariant ideals of $\Lambda(m,n)$. It follows that L_0 stabilizes an ideal $J = (x_1 + f_1, \dots, x_m + f_m, \eta_1 + g_1, \dots, \eta_r + g_r)$ for some linear functions η_j in odd indeterminates and even functions f_i and odd functions g_j without constant and linear terms, and that J is maximal among the L_0-invariant ideals of $\Lambda(m,n)$.

Recall that a linearly compact Lie superalgebra L is called *simple* if it is not abelian and contains no closed ideals different from 0 and L; L is called *semisimple* if it contains no non-zero abelian ideals; L is called *artinian* if any descending sequence of closed ideals in L stabilizes.

A subalgebra L_0 of L is called *fundamental* if it is proper, open and contains no non-zero ideals of L. Due to Guillemin's theorem [13] a linearly compact Lie superalgebra is artinian if and only if it contains a fundamental subalgebra (the proof in [13] is given in the Lie algebra case, but it extends verbatim to the super case).

Let L_0 be a fundamental subalgebra of a Lie superalgebra L and let L_{-1} be an $ad\, L_0$-stable subspace of L generating L as a Lie superalgebra. The Weisfeiler filtration [22] associated to the triple $L \supset L_{-1} \supset L_0$ is the filtration of L inductively defined as follows: for $s \geqslant 1$,

$$L_{-(s+1)} = [L_{-1}, L_{-s}] + L_{-s}, \qquad L_s = \{a \in L_{s-1} \mid [a, L_{-1}] \subset L_{s-1}\}.$$

If L_{-1} is a minimal $ad\, L_0$-stable subspace properly containing L_0, then the Weisfeiler filtration is called *irreducible*. If $L_{-1} = L$, the Weisfeiler filtration is called the *canonical filtration*.

A linearly compact Lie superalgebra L is called *primitive* if it contains a fundamental subalgebra L_0 which is a maximal subalgebra. In this case, (L, L_0) is called a *primitive pair*. Note that for a primitive pair (L, L_0) there exists an irreducible Weisfeiler filtration whose 0th term is L_0.

Given a filtered Lie superalgebra $L = L_{-d} \supset \cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots$, we shall denote by $Gr\, L$ the associated $\mathbb{Z}$-graded Lie superalgebra:

$$Gr\, L = \bigoplus_{j \geqslant -d} Gr_j\, L, \quad Gr_j\, L = L_j / L_{j+1}.$$

If $\mathfrak{g} = \bigoplus_{j \geqslant -d} \mathfrak{g}_j$ is a graded Lie superalgebra, we denote by $\bar{\mathfrak{g}} = \prod_{j \geqslant -d} \mathfrak{g}_j$ its completion. Then $\bar{\mathfrak{g}}$ has a natural filtration given by the subspaces

$$\bar{\mathfrak{g}}_i = \prod_{j \geqslant i} \mathfrak{g}_j$$

for $i \geqslant -d$. We shall call such a filtration a *graded* filtration (or, equivalently, a trivial filtered deformation of $\bar{\mathfrak{g}}$, cf. Section 7).

Let L_0 be a fundamental subalgebra of L, let $L = L_{-d} \supset \cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots$ be a Weisfeiler filtration, and let $Gr\, L = \bigoplus_{j \geqslant -d} \mathfrak{g}_j$, where $\mathfrak{g}_j = Gr_j\, L$, be the associated graded superalgebra. Then [22]:

$$\mathfrak{g}_{-j} = \mathfrak{g}_{-1}^j \quad \text{for } j \geqslant 1, \tag{1.1}$$

$$\text{if} \quad x \in \mathfrak{g}_j,\ j \geqslant 0 \text{ and } [x, \mathfrak{g}_{-1}] = 0, \quad \text{then} \quad x = 0. \tag{1.2}$$

If, in addition, the Weisfeiler filtration is irreducible, then

$$\mathfrak{g}_{-1} \text{ is an irreducible } \mathfrak{g}_0\text{-module}. \tag{1.3}$$

A $\mathbb{Z}$-graded Lie superalgebra $\mathfrak{g} = \bigoplus_{j \geqslant -d} \mathfrak{g}_i$ is called *transitive* if properties (1.1) and (1.2) hold, and it is called *irreducible* if, in addition (1.3) holds.

The following theorems describe the artinian semisimple and the infinite-dimensional primitive linearly compact Lie superalgebras. We denote by $Der\, S$ (respectively $Inder\, S$) the Lie superalgebra of all (respectively all inner) continuous derivations of a linearly compact Lie superalgebra S. Recall that the completed tensor product $U \hat{\otimes} V$ of linearly compact spaces U and V is defined as $(U^* \otimes V^*)^*$ [13].

Theorem 1.4. *Let* $S_1, \dots, S_r$ $(r \in \mathbb{N})$ *be simple linearly compact Lie superalgebras, let* $m_1, \dots, m_r, n_1, \dots, n_r$ *be non-negative integers and let* $S = \bigoplus_{i=1}^r (S_i \hat{\otimes} \Lambda(m_i, n_i))$. *Then*

$$Der\, S = \bigoplus_{i=1}^{r} \big((Der\, S_i) \hat{\otimes} \Lambda(m_i, n_i) + 1 \otimes W(m_i, n_i)\big) \tag{1.4}$$

is a linearly compact Lie superalgebra and $S = Inder\, S$ *canonically embeds in* $Der\, S$. *Let* L *be an open subalgebra of* $Der\, S$ *containing* S, *and denote by* F_i *the projection of* L *on* $1 \otimes W(m_i, n_i)$. *Then*:

(a) *L is semisimple if and only if F_i is a transitive subalgebra of $W(m_i, n_i)$ for all $i = 1, \dots, r$.*
(b) *All artinian semisimple linearly compact Lie superalgebras can be obtained as in* (a).
(c) *If L is semisimple, then $Der\, L$ is the normalizer of L in $Der\, S$* (*and is semisimple*).

Proof. It follows traditional lines (cf. [4,8]). Let L be an artinian semisimple linearly compact Lie superalgebra, and let I denote the sum of all its minimal closed ideals. Since L is semisimple, for any (non-zero) minimal closed ideal J one has $[J, J] = J$. Using this, it is standard to show that I is a direct sum of all minimal closed ideals of L. Since L is artinian, it follows that it contains a finite number of (non-zero) minimal closed ideals; denote them by $I_1, \dots, I_r$. We have a homomorphism $\varphi : L \to \bigoplus_j Der\, I_j$ defined by $\varphi(a) = \sum_j (ad\, a)|_{I_j}$. The homomorphism φ is injective since $(\ker \varphi) \cap I = 0$, and therefore, by the artinian property, if $\ker \varphi$ is non-zero, it would contain a (non-zero) minimal closed ideal different from all I_j's. Thus, we have the following inclusions:

$$\bigoplus_{j=1}^{r} I_j \subset L \subset \bigoplus_{j=1}^{r} Der\, I_j. \tag{1.5}$$

Next we use the super analogue of the Cartan–Guillemin theorem [3,13], established in [11], according to which $I_j \cong S_j \hat{\otimes} \Lambda(m_j, n_j)$, where S_j is a simple linearly compact Lie superalgebra and $m_j, n_j \in \mathbb{Z}_+$.

Next, the same argument as in [4] or [8] shows that

$$Der\, I = (Der\, S_j) \hat{\otimes} \Lambda(m_j, n_j) + 1 \otimes W(m_j, n_j),$$

and that L in (1.5) is semisimple if and only if $\Lambda(m_j, n_j)$ contains no non-trivial F_j-invariant ideals. Now (a) and (b) follow from Theorem 1.1. The proof of (c) is the same as in [4] or [8]. □

Theorem 1.5. *If L is an infinite-dimensional primitive Lie superalgebra, then L is artinian semisimple, and, moreover,*

$$S \otimes \Lambda(n) \subset L \subset (Der\, S) \otimes \Lambda(n) + 1 \otimes W(0, n)$$

for some infinite-dimensional simple linearly compact Lie superalgebra S and $n \in \mathbb{Z}_+$, where the projection of L on $W(0, n)$ is a transitive subalgebra.

Proof. By the above mentioned Guillemin's theorem, L is artinian. By another result of Guillemin [13, Proposition 4.1], whose proof works verbatim in the super case, any non-zero closed ideal of L has finite codimension.

In order to show that L is semisimple, choose an irreducible Weisfeiler filtration of L associated with the fundamental maximal subalgebra L_0 of L, and let $\mathfrak{g} = \bigoplus_{j \geqslant -d} \mathfrak{g}_j$ be the associated graded Lie superalgebra. Suppose that L contains a non-zero closed abelian ideal. Then the corresponding ideal $I = \bigoplus_{j \geqslant -d} I_j$ in $\mathfrak{g}$ has finite codimension, and since $\dim \mathfrak{g} = \infty$, we conclude that $I_j \neq 0$ for some $j \geqslant 0$. By the transitivity of $\mathfrak{g}$, $I_0 \neq 0$ and $I_{-1} \neq 0$, and by the irreducibility of the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$, $I_{-1} = \mathfrak{g}_{-1}$. Hence $[I_0, \mathfrak{g}_{-1}] = 0$ (since I is an abelian ideal), which contradicts the transitivity of $\mathfrak{g}$.

Thus, by Theorem 1.4, L contains the ideals $S_i \hat{\otimes} \Lambda(m_i, n_i)$, $i = 1, \ldots, r$. Since $\dim L = \infty$ and all non-zero ideals of L have finite codimension, we conclude that $r = 1$ and

$$S \hat{\otimes} \Lambda(m, n) \subset L \subset (Der\, S) \hat{\otimes} \Lambda(m, n) + 1 \otimes W(m, n),$$

where S is a simple linearly compact Lie superalgebra and the projection F of L on $W(m, n)$ is a transitive subalgebra. It remains to show that $m = 0$.

Since L_0 is a maximal subalgebra of L, and $S \hat{\otimes} \Lambda(m, n)$ is an ideal, we conclude that

$$L = L_0 + \big(S \hat{\otimes} \Lambda(m, n)\big). \tag{1.6}$$

Suppose that $m \geqslant 1$. Since L_0 is an open subalgebra, by Chevalley's principle,

$$S \hat{\otimes} (x_1, \ldots, x_m)^j \Lambda(m, n) \subset L_0 \quad \text{for } j \gg 0. \tag{1.7}$$

By transitivity of F and (1.6), the projection of L_0 on $1 \otimes W(m, n)_{-1}$ is surjective. Hence it follows from (1.7) that $S \hat{\otimes} \Lambda(m, n) \subset L_0$, a contradiction since L_0 contains no non-zero ideals of L. □

An *ad*-diagonalizable subalgebra T of a linearly compact Lie superalgebra L is called a *torus* of L. The following proposition allows one to construct maximal tori.

Proposition 1.6. *Let L be a linearly compact Lie superalgebra with trivial center and let $L = L_{-d} \supset \cdots \supset L_0 \supset L_1 \supset \cdots$ be a filtration of L such that L_0 contains all ad-exponentiable elements of L. Then any torus T of L lies in L_0 and T is a maximal torus in L if and only if its image in L_0/L_1 is a maximal torus. Any maximal torus of L_0/L_1 can be lifted to that of L.*

Proof. Since all elements of T are exponentiable, $T \subset L_0$. Since, obviously, $T \cap L_1 = 0$, T is a maximal torus of L_0 (and hence of L) if and only if its image is a maximal torus of L_0/L_1. □

We do not know examples for which the maximal tori are not conjugate, but we can prove their conjugacy only for the artinian semisimple L (which we shall apply to primitive L). In Appendix A we extend this to the case of an arbitrary artinian L.

Theorem 1.7. *If L is an artinian semisimple linearly compact Lie superalgebra, then all maximal tori of L are conjugate by inner automorphisms of L.*

Proof. We may assume that $\dim L = \infty$. The socle $\bigoplus_{i=1}^{r} S_i \otimes \Lambda(n_i)$ of L (see Theorem 1.4) is invariant with respect to all automorphisms of L. But due to [17], each $Der\, S_i$ contains a fundamental subalgebra S_i^0, which is proper if $\dim S_i = \infty$, and which contains all exponentiable elements of $Der\, S_i$.

Consider the Lie superalgebra

$$\tilde{L} = \bigoplus_{i=1}^{r} \big((Der\, S_i) \hat{\otimes} \Lambda(m_i, n_i)\big) \oplus \big(1 \otimes W(m_i, n_i)\big)$$

containing L. Take the canonical filtration of $Der\, S_i$ defined by S_i^0 and tensor it with the filtration of $\Lambda(m_i, n_i)$ whose jth member is $(x_1, \ldots, x_m, \xi_1, \ldots, \xi_n)^j$; this defines a filtration of $(Der\, S_i) \hat{\otimes} \Lambda(m_i, n_i)$ all of whose exponentiable elements lie in the 0th member of the filtration. These and the principal filtration of $W(m_i, n_i)$ for each i add up to produce a filtration of $\tilde{L}$. Intersecting the members of this filtration with L, we get a filtration of L by open subspaces $L \supset L_0 \supset L_1 \supset \cdots$, such that L_0 contains all exponentiable elements of L. In particular, L_0 contains any two maximal tori T and T' of L. But T and T' are conjugate in L_0 mod L_N for each $N \geqslant 1$ by the conjugacy of maximal tori in any finite-dimensional Lie superalgebra. Taking the limit as $N \to \infty$, we obtain that T and T' are conjugate in L_0. □

We shall use the following (corrected) explicit description of the Lie superalgebras $Der\, S$ for all simple linearly compact Lie superalgebras S, given in [17].

Proposition 1.8. [17, Proposition 6.1] *Let S be a simple infinite-dimensional linearly compact Lie superalgebra. Then $Der\, S = S \rtimes \mathfrak{a}$, where $\mathfrak{a}$ is a finite-dimensional subalgebra, described below*:

(a) *If S is one of the Lie superalgebras $W(m,n)$, $SHO^\sim(m,m)$, $K(m,n)$, $KO(m,m+1)$, $SKO^\sim(m,m+1)$, $E(4,4)$, $E(1,6)$, $E(3,6)$, $E(3,8)$, then $\mathfrak{a} = 0$.*
(b) *If S is one of the Lie superalgebras $S(m,n)$ with $m \geqslant 2$, $(m,n) \neq (2,1)$, $H(m,n)$, $HO(m,m)$ with $m \geqslant 3$, $SKO(m,m+1;\beta)$ with $m \geqslant 2$ and $\beta \neq 1$, $(m-2)/m$, $E(5,10)$, then $\mathfrak{a}$ is a one-dimensional torus of $Der\, S$.*
(c) *If S is one of the Lie superalgebras $S(1,n)$ with $n \geqslant 3$, $SKO(m,m+1;(m-2)/m)$ with $m \geqslant 2$, $SKO(m,m+1;1)$ with $m > 2$, then $\mathfrak{a} = \mathfrak{n} \rtimes \mathfrak{t}_1$, where $\mathfrak{t}_1$ is a one-dimensional torus of $Der\, S$ and $\mathfrak{n}$ is a one-dimensional subalgebra such that $[\mathfrak{t}_1, \mathfrak{n}] = \mathfrak{n}$.*
(d) *If $S = SHO(m,m)$ with $m \geqslant 4$, then $\mathfrak{a} = \mathfrak{n} \rtimes \mathfrak{t}_2$, where $\mathfrak{t}_2$ is a two-dimensional torus of $Der\, S$ and $\mathfrak{n}$ is a one-dimensional subalgebra such that $[\mathfrak{t}_2, \mathfrak{n}] = \mathfrak{n}$.*
(e) *If $S = S(1,2)$ or $S = SKO(2,3;1)$, then $\mathfrak{a} \cong sl_2$.*
(f) *If $S = SHO(3,3)$, then $\mathfrak{a} \cong gl_2$.*

The subalgebra $\mathfrak{a}$ of $Der\, S$ is called the subalgebra of outer derivations *of S.*

The following theorem describes all primitive pairs in terms of simple ones.

Theorem 1.9.

(a) *Let $L = (S \otimes \Lambda(n)) \rtimes F$, where S is a linearly compact Lie superalgebra and F is a transitive subalgebra of $W(0,n)$. Then any fundamental maximal subalgebra L_0 of L is of the form $(S_0 \otimes \Lambda(n)) \rtimes F$, where S_0 is a fundamental maximal subalgebra of S.*

(b) *Let S be a simple infinite-dimensional linearly compact Lie superalgebra. Let $\mathfrak{a}_0$ be a subalgebra of the subalgebra $\mathfrak{a}$ of outer derivations of S and let L_0 be a fundamental maximal subalgebra of $S \rtimes \mathfrak{a}_0$. Then $L_0 = S_0 \rtimes \mathfrak{a}_0$, where S_0 is a maximal among open $\mathfrak{a}_0$-invariant subalgebras of S. Thus, all fundamental maximal subalgebras of $S \rtimes \mathfrak{a}_0$ are $S_0 \rtimes \mathfrak{a}_0$, where S_0 is a maximal among open $\mathfrak{a}_0$-invariant subalgebras of S.*

(c) *Let S be a simple infinite-dimensional linearly compact Lie superalgebra. Let F be a subalgebra of $(\mathfrak{a} \otimes \Lambda(n)) \rtimes W(0,n)$ containing elements f_i, for $i = 1,\ldots,n$, such that $f_i(0) = \partial/\partial\xi_i$. Let $L = (S \otimes \Lambda(n)) \rtimes F$. Then these L exhaust, up to automorphisms, all that occur in a primitive pair. All possible fundamental maximal subalgebras L_0 in L can be obtained as follows. Let $\mathfrak{a}_0 = \{a(0) \mid a(\xi)$ lies in the projection of F on $\mathfrak{a} \otimes \Lambda(n)\} \subset \mathfrak{a}$. Let S_0 be a maximal among $\mathfrak{a}_0$-invariant subalgebras of S. Then $L_0 = (S_0 \otimes \Lambda(n)) \rtimes F$.*

Proof. (a) First, we show that $F \subset L_0$. In the contrary case, consider an irreducible Weisfeiler filtration of L associated to L_0. Then we have: $Gr\,L = Gr(S \otimes \Lambda(n)) \rtimes Gr\,F$. Since $Gr_{-1}\,L$ is irreducible with respect to $Gr_0\,L$ and $Gr_{-1}(S \otimes \Lambda(n))$ is a submodule of $Gr_{-1}\,L$, we conclude that $Gr_{-1}(S \otimes \Lambda(n)) = 0$, i.e., $Gr_{-1}\,L = Gr_{-1}\,F$, hence $Gr_{<0}\,L = Gr_{<0}\,F$. It follows that $S \otimes \Lambda(n) \subset L_0$, which is impossible since $S \otimes \Lambda(n)$ is an ideal of L.

We write elements of $S \otimes \Lambda(n)$ in the form $s(\xi) = \sum_I s_I \xi^I$, where $I = \{i_1,\ldots,i_r\} \subset \{1,\ldots,n\}$, $s_I \in S$, $\xi^I = \xi_{i_1}\ldots\xi_{i_r}$. Let $S_I = \{s_I \mid s(\xi) \in L_0\}$; then $S_0 := S_\emptyset$ is a subalgebra of S. Due to the transitivity of F, we conclude that $S_I \subset S_0$ for all I, hence $S_0 \otimes \Lambda(n) \supset L_0 \cap (S \otimes \Lambda(n))$. Since $F \subset L_0$, we deduce that $(S_0 \otimes \Lambda(n)) + F \supset L_0$. Hence these two subalgebras coincide due to the maximality of L_0. Since L_0 is a fundamental maximal subalgebra of L, S_0 is a fundamental maximal subalgebra of S.

(b) The same argument as in (a) shows that $\mathfrak{a}_0 \subset L_0$. Therefore $L_0 = (L_0 \cap S) \rtimes \mathfrak{a}_0$, and $L_0 \cap S$ is an $\mathfrak{a}_0$-invariant subalgebra of S. By the maximality of L_0 it follows that $L_0 \cap S$ is maximal among the $\mathfrak{a}_0$-invariant subalgebras of S.

(c) Let (L, L_0) be a primitive pair. Then, by Theorem 1.5, $L = (S \otimes \Lambda(n)) \rtimes F$, where S is a simple Lie superalgebra, $\mathfrak{a}$ is the subalgebra of outer derivations of S, and F is a subalgebra of $(\mathfrak{a} \otimes \Lambda(n)) \rtimes W(0,n)$ with transitive projection on $W(0,n)$. Since the projection of F on $W(0,n)$ is transitive, we may assume, up to automorphisms, that F contains some elements f_i, for every $i = 1,\ldots,n$, such that $f_i(0) = \partial/\partial\xi_i$. Indeed, if g_i are elements in F, such that $g_i(0) = \partial/\partial\xi_i + a_i$ for some $a_i \in \mathfrak{a}$, the automorphism $\prod_i(1 + ad(a_i\xi_i))$ brings g_i to f_i such that $f_i(0) = \partial/\partial\xi_i$, $i = 1,\ldots,n$.

The same argument as in (a) shows that $F \subset L_0$, hence $L_0 = L_0 \cap (S \otimes \Lambda(n)) \rtimes F$. Let us write the elements of $S \otimes \Lambda(n)$ in the form $s(\xi)$ as in (a), let S_I be defined as in (a), and let $S_0 = \{s(0) \mid s(\xi) \in L_0\}$. Then S_0 is a subalgebra of S and, since $f_i \in L_0$ for every $i = 1,\ldots,n$, $S_I \subset S_0$ for all I. It follows that $L_0 \subset (S_0 \otimes \Lambda(n)) \rtimes F$, hence, by the maximality of L_0, equality holds.

Likewise, let us write the elements of $\mathfrak{a} \otimes \Lambda(n)$ in the form $a(\xi) = \sum a_I \xi^I$, let $\mathfrak{a}_0$ be as in the statement and let $\mathfrak{a}_I = \{a_I \mid a(\xi) \in$ projection of F on $\mathfrak{a} \otimes \Lambda(n)\}$. Then $\mathfrak{a}_0$ is a subalgebra of $\mathfrak{a}$ and, since L_0 contains the elements f_i, $\mathfrak{a}_I \subset \mathfrak{a}_0$ for all I. It follows that $L_0 \subset S_0 \otimes \Lambda(n) + \mathfrak{a}_0 \otimes \Lambda(n) + F'$, where F' is the projection of F on $W(0,n)$. Since S_0 is $\mathfrak{a}_0$-invariant, the maximality of S_0 among the $\mathfrak{a}_0$-invariant subalgebras of S follows from the maximality of L_0. □

Recall that the growth of an artinian linearly compact Lie superalgebra L is defined as follows. Choose a fundamental subalgebra L_0 of L and construct a Weisfeiler filtration $L = L_{-d} \supset \cdots \supset L_0 \supset L_1 \supset \cdots$, containing L_0 as its 0th member, for some choice of L_{-1} containing L_0 and

Table 2

L	s	L	s	L	s
$W(m,n)$	$(m+n)2^n$	$SHO(n,n)$	2^{n-1}	E(1,6)	32
$S(m,n)$	$(m+n-1)2^n$	$KO(n,n+1)$	2^{n+1}	$E(3,6)$	12
$H(m,n)$	2^n	$SKO(n,n+1;\beta)$	2^n	$E(3,8)$	16
$K(m,n)$	2^n	$SHO^\sim(n,n)$	2^{n-1}	$E(4,4)$	8
$HO(n,n)$	2^n	$SKO^\sim(n,n+1)$	2^n	$E(5,10)$	8

generating L. Consider the function $F(j) = \dim L/L_j$. It depends on the choice of L_0 and on the Weisfeiler filtration, but it is easy to show (see [2,11]), that the leading term of $F(j)$ is independent of these choices. Namely, there exist unique positive real numbers a and g such that $\overline{\lim}_{j\to\infty}\{F(j)/j^g\} = a$. The number g is called the *growth* of L, and is denoted by $g(L)$.

It is easy to see from the classification, that, if L is simple, then $g(L)$ is a positive integer and, moreover, $s(L) := ag(L)!$ is a positive integer. The number $s(L)$ is called the *size* of L. One can think of the growth (respectively size) of L as the minimal number of even variables (respectively minimal number of functions in these variables) involved in vector fields from L. It is also easy to see from the classification that if L is simple and is not a Lie algebra, then $s(L)$ is an even integer, and, moreover, the sizes of the even and the odd parts of L are $\frac{1}{2}s(L)$ (of course, their growths are both equal to $g(L)$). Due to Theorem 1.5, any primitive L contains $S \otimes \Lambda(n)$ as an open ideal, hence $g(L) = g(S)$ and $s(L) = 2^n s(S)$.

If a simple L is of type $X(m,n)$, then $g(L) = m$. The sizes are given in Table 2.

Remark 1.10. If (L, L_0) is a primitive pair, and $Gr\,L$ is its associated graded superalgebra for a Weisfeiler filtration, then $g(L) = g(\overline{Gr\,L})$ and $s(L) = s(\overline{Gr\,L})$. This puts stringent restrictions on the possibilities for $Gr\,L$ for the given primitive pair (L, L_0).

The following proposition allows one to construct graded maximal subalgebras.

Proposition 1.11. *Let* $\mathfrak{g} = \bigoplus_{j\geqslant -d}\mathfrak{g}_j$ *be a* $\mathbb{Z}$*-graded Lie superalgebra and let* $\mathfrak{g}_{\geqslant 0} = \bigoplus_{j\geqslant 0}\mathfrak{g}_j$, $\mathfrak{g}_\pm = \bigoplus_{j>0}\mathfrak{g}_{\pm j}$.

(a) *If* $\mathfrak{g}_{\geqslant 0}$ *is a maximal subalgebra of* $\mathfrak{g}$, *then*:
  (i) $\mathfrak{g}_{-1}$ *is an irreducible* $\mathfrak{g}_0$*-module*;
  (ii) $\mathfrak{g}_-$ *is generated by* $\mathfrak{g}_{-1}$;
  (iii) $\mathfrak{g}_-$ *contains no ideals of* $\mathfrak{g}$ *different from* $\mathfrak{g}_-$ *or zero.*

(b) *If* (i) *and* (ii) *hold and, in addition,*
  (iii)′ $[a, \mathfrak{g}_1] \neq 0$ *for any non-zero* $a \in \mathfrak{g}_j$, $j < -1$,
  then $\mathfrak{g}_{\geqslant 0}$ *is a maximal subalgebra of* $\mathfrak{g}$.

Proof. (a)(i) If V is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_{-1}$ and V_- is the subalgebra of $\mathfrak{g}$ generated by V then $V_- + \mathfrak{g}_{\geqslant 0}$ is a subalgebra of $\mathfrak{g}$.

(ii) If $\mathfrak{g}'_-$ is the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{-1}$, then $\mathfrak{g}'_- + \mathfrak{g}_{\geqslant 0}$ is a subalgebra of $\mathfrak{g}$.

(iii) If I is such an ideal, then $I + \mathfrak{g}_{\geqslant 0}$ is a subalgebra of $\mathfrak{g}$.

(b) Suppose that $\mathfrak{g}_{\geqslant 0}$ is properly contained in a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$. It follows that there exists a non-zero element $a \in \mathfrak{g}_- \cap \mathfrak{g}'$. Now (iii)′ implies that $[a, \mathfrak{g}_1] \neq 0$. It follows that $\mathfrak{g}_{-1} \cap \mathfrak{g}' \neq \{0\}$, therefore, due to (i), $\mathfrak{g}_{-1} \subset \mathfrak{g}'$ and, due to (ii), $\mathfrak{g}' = \mathfrak{g}$. □

Corollary 1.12. *If L is a filtered Lie superalgebra such that $Gr\,L$ has properties* (i), (ii), (iii)′ *of Proposition* 1.11*, then L_0 is a maximal subalgebra of L.*

Remark 1.13. If $\mathfrak{g} = \bigoplus_{i \geqslant -d} \mathfrak{g}_i$ is simple then $\mathfrak{g}_{-d}$ is irreducible. Indeed if V is a $\mathfrak{g}_0$-stable subspace of $\mathfrak{g}_{-d}$ then $V + (\bigoplus_{i > -d} \mathfrak{g}_i)$ is an ideal of $\mathfrak{g}$. In particular any $\mathbb{Z}$-grading of depth 1 of a simple Lie superalgebra is irreducible.

Definition 1.14. Let T be a maximal torus in $Der\,L$. We call an open subalgebra of L regular if it is T-invariant.

Remark 1.15. Let L be a subalgebra of $W(m,n)$ and let I_U be a standard ideal of $\Lambda(m,n)$. If I_U is stabilized by a maximal torus T of $Der\,L$ then the standard subalgebra L_U is regular.

2. Maximal open subalgebras of $W(m,n)$, $S(m,n)$, $K(m,n)$, $HO(n,n)$ and $SHO(n,n)$

2.1. The Lie superalgebras $W(m,n)$ and $S(m,n)$, $m \geqslant 1$

In Section 1 we introduced the Lie superalgebra $W(m,n)$ of continuous derivations of the Lie superalgebra $\Lambda(m,n)$. We shall assume $m \geqslant 1$ (note that $\dim W(0,n) < \infty$). Let us fix the standard maximal torus $T = \langle x_i \partial/\partial x_i, \xi_j \partial/\partial \xi_j \mid i = 1, \ldots, m;\ j = 1, \ldots, n \rangle$ of $W(m,n)$.

The simple Lie superalgebras L considered in this section and in the following three, are subalgebras of $W(m,n)$ such that $Der\,L \subset W(m,n)$ and $T \cap Der\,L$ is a maximal torus of $Der\,L$. Such a maximal torus of $Der\,L$ will be called *standard*.

Remark 2.1. By Theorem 1.7 each regular subalgebra of L is conjugate by G to a subalgebra which is invariant with respect to the standard torus of L. Thus, in order to classify regular subalgebras up to conjugation by G, it suffices to consider the ones that contain T. In what follows, conjugation will always mean conjugation by G, unless otherwise specified. We will often use automorphisms of L defined by changes of variables; each time it will not be difficult to check that they are inner, hence lie in G. Note that when the linear part of a change of variables is the identity then this is always an inner automorphism (cf. [20]).

Remark 2.2. A $\mathbb{Z}$-grading, called the grading of type $(a_1, \ldots, a_m | b_1, \ldots, b_n)$, can be defined on $W(m,n)$ by setting $a_i = \deg x_i = -\deg(\partial/\partial x_i) \in \mathbb{N}$ and $b_i = \deg \xi_i = -\deg(\partial/\partial \xi_i) \in \mathbb{Z}$ (cf. [17, Example 4.1]). The $\mathbb{Z}$-grading of type $(1, \ldots, 1|1, \ldots, 1)$ is the principal grading of $W(m,n)$. In this grading $W(m,n)$ has depth 1 with 0th graded component isomorphic to the Lie superalgebra $gl(m,n)$ and -1st graded component isomorphic to the standard $gl(m,n)$-module $\mathbb{C}^{m|n}$. Since $W(m,n)$ is simple for every $(m,n) \neq (0,1)$, under our hypotheses the principal grading of $W(m,n)$ is irreducible by Remark 1.13. More generally, the gradings of type $(1, \ldots, 1|1, \ldots, 1, 0, \ldots, 0)$ with k zeros, are irreducible for every $k = 0, \ldots, n$ and satisfy the hypotheses of Proposition 1.11(b). It follows, by Proposition 1.11(b), that the corresponding subalgebras $\prod_{j \geqslant 0} W(m,n)_j$ of $W(m,n)$ are maximal. The $\mathbb{Z}$-grading of $W(m,n)$ of type $(1, \ldots, 1|0, \ldots, 0)$ is called *subprincipal*.

Theorem 2.3. *Let $W = W(m,n)$ with $m \geqslant 1$. Then all maximal open subalgebras of W are, up to conjugation, the graded subalgebras of type* $(1, \ldots, 1|1, \ldots, 1, 0, \ldots, 0)$ *with k zeros, for $k = 0, \ldots, n$.*

Proof. Let L_0 be a maximal open subalgebra of W. Since the vector fields $\partial/\partial x_i$ are not exponentiable, L_0 does not contain any vector field of the form $\sum \alpha_i \partial/\partial x_i + X + Y$ for any non-zero linear combination $\sum \alpha_i \partial/\partial x_i$, any $X \in W$ such that $X(0) = 0$ and any $Y \in W(0,n)$. By Theorem 1.1(a), L_0 is conjugate to the subalgebra W_U for some subspace $U = \langle x_1, \ldots, x_m, \xi_1, \ldots, \xi_k \rangle$ of $\Lambda(m,n)$, with $0 \leqslant k \leqslant n$. The subalgebra W_U is in fact the graded subalgebra of type $(1,\ldots,1|1,\ldots,1,0,\ldots,0)$ with $n-k$ zeros. □

Definition 2.4. Let L be a subalgebra of $W(m,n)$. A linear map $Div\colon L \to \Lambda(m,n)$ is called a *divergence* if the action of L on the space $\Lambda(m,n)v$ given by:

$$X(fv) = (Xf)v + (-1)^{p(X)p(f)} f\, Div(X)v, \tag{2.1}$$

is a representation of L. The symbol v is called the *volume form* attached to the divergence Div. Note that $S'L := \{X \in L \mid Div(X) = 0\}$ is a subalgebra of L and Div is a homomorphism of $S'L$-modules.

Definition 2.5. If we have a representation of $L \subset W(m,n)$ on a vector space V, which is also a left module over $\Lambda(m,n)$, compatible with the action of L, and v is a volume form for L, then, for any complex number λ, L acts on the space $V^\lambda := v^\lambda V$, by the *twisted* action defined as follows:

$$X\left(v^\lambda u\right) = \lambda v^\lambda\, Div(X)u + v^\lambda X u.$$

Remark 2.6. The subalgebra $S'L$ consists of vector fields X in L such that $Xv = 0$. Likewise, $CS'L := \{X \in L \mid Div(X) \in \mathbb{C}\}$ is the subalgebra of L consisting of vector fields X in L such that $Xv = cv$ with $c \in \mathbb{C}$.

Remark 2.7. If Div is a divergence and F is an even invertible function in $\Lambda(m,n)$, then the map $Div_F\colon L \to \Lambda(m,n)$ defined by:

$$Div_F(X) = X(F)F^{-1} + Div(X),$$

is also a divergence. If v is the volume form attached to Div, then Fv is the volume form attached to Div_F.

Example 2.8. The function $div\colon W(m,n) \to \Lambda(m,n)$ defined by

$$div\left(\sum_{i=1}^m P_i \frac{\partial}{\partial x_i} + \sum_{j=1}^n Q_j \frac{\partial}{\partial \xi_j}\right) = \sum_{i=1}^m \frac{\partial P_i}{\partial x_i} + \sum_{j=1}^n (-1)^{p(Q_j)} \frac{\partial Q_j}{\partial \xi_j}$$

is a divergence. We will refer to it as the *usual divergence*. It follows, according to Definition 2.4, that the set $S'(m,n) := S'W(m,n) = \{X \in W(m,n) \mid div(X) = 0\}$ is a subalgebra of $W(m,n)$ (cf. [17, Example 4.2]). Moreover, $CS'(m,n) = S'(m,n) + \mathbb{C}\sum_{i=1}^m x_i \partial/\partial x_i$.

Remark 2.9. Let div be the usual divergence (see Example 2.8). Then, for every $X \in W(m,n)$ and any even invertible function $F \in \Lambda(m,n)$, $div(FX) = X(F) + F\, div(X)$. Therefore $div_F(X) = 0$ if and only if $div(FX) = 0$.

Let $S(m,n)=[S'(m,n),S'(m,n)]$. We recall that if $m>1$ then $S(m,n)=S'(m,n)$ is simple. Besides, $S'(1,n)=S(1,n)+\mathbb{C}\xi_1\dots\xi_n\partial/\partial x_1$ and $S(1,n)$ is simple if and only if $n\geqslant 2$ (cf. [17, Example 4.2]). Since $S(2,1)\cong SKO(2,3;0)$, when talking about $S(m,n)$ we shall always assume $(m,n)\neq(2,1)$.

Remark 2.10. Every $\mathbb{Z}$-grading of $W(m,n)$ induces a grading on $S(m,n)$. In particular the $\mathbb{Z}$-gradings of type $(1,\dots,1|1,\dots,1,0,\dots,0)$, with k zeros, induce on $S(m,n)$, by Remark 1.13, irreducible gradings for $m>1$ or $m=1$ and $n\geqslant 2$. As in Remark 2.2, the corresponding subalgebras $\prod_{j\geqslant 0}S(m,n)_j$ of $S(m,n)$ are maximal. The $\mathbb{Z}$-grading of $S(m,n)$ of type $(1,\dots,1|0,\dots,0)$ is called *subprincipal*.

Theorem 2.11. *Let $S=S(m,n)$ or $S=S'(m,n)$ or $S=CS'(m,n)$ with $m>1$ or $m=1$ and $n\geqslant 2$. Then every maximal open subalgebra of S is regular.*

Proof. Let L_0 be a maximal open subalgebra of $S=S'(m,n)$ and let $U=\langle x_1,\dots,x_m,\xi_1,\dots,\xi_k\rangle$ with $0\leqslant k\leqslant n$. Then, by Theorem 1.1, there exists a continuous automorphism φ of $\Lambda(m,n)$ such that $L_0=S\cap\varphi W_U\varphi^{-1}$.

Let ω be the volume form attached to the divergence *div*. Then:

$$\varphi^{-1}S\varphi=\{\varphi^{-1}X\varphi\mid X\omega=0\}=\{Y\mid \varphi Y\varphi^{-1}(\omega)=0\}$$
$$=\{Y\mid Y(f\omega)=0,\ \text{for some invertible } f\in\Lambda(m,n)\}=\{Y\mid f^{-1}Yf\omega=0\}=fSf^{-1}.$$

It follows that:

$$\varphi^{-1}S\varphi\cap W_U=fSf^{-1}\cap W_U=\{fXf^{-1}\mid X\in S,\ fXf^{-1}(I_U)\subset I_U\}$$
$$=\{fXf^{-1}\mid X\in S,\ X(I_U)\subset I_U\}=f(S\cap W_U)f^{-1}.$$

Therefore $L_0=S\cap\varphi W_U\varphi^{-1}=\varphi f(S\cap W_U)f^{-1}\varphi^{-1}$. Since $S\cap W_U$ is a regular subalgebra of $W(m,n)$, its image under an automorphism of $W(m,n)$ is again a regular subalgebra of $W(m,n)$.

The same argument holds if we replace $S'(m,n)$ by $S(m,n)$ or by $CS'(m,n)$. $\square$

Remark 2.12. We recall that $Der\,S(1,2)=S(1,2)+\mathfrak{a}$ with $\mathfrak{a}\cong sl_2$ (cf. Proposition 1.8). Let us denote by e,f,h the standard basis of $\mathfrak{a}\cong sl_2$ defined in [11, Lemma 5.9]. Let $S=\prod_{j\geqslant -2}S_j$ denote the Lie superalgebra $S(1,2)$ with respect to the grading of type $(2|1,1)$. Then $S_0\cong gl_2$ and S_{-1} is isomorphic, as an S_0-module, to the direct sum of two copies of the standard gl_2-module. It follows that, for every irreducible gl_2-submodule U of S_{-1}, $S_U:=U+\prod_{j\geqslant 0}S_j$ is a maximal open subalgebra of S. In particular, if $U=\langle\xi_i\partial/\partial x\mid i=1,2\rangle$ or $U=\langle\partial/\partial\xi_i\mid i=1,2\rangle$, then S_U is the maximal graded subalgebra of type $(1|1,1)$ or $(1|0,0)$, respectively. The subalgebras S_U are not conjugate by inner automorphisms of S, but they are conjugate by inner automorphisms of $Der\,S$, since the subalgebra $\mathfrak{a}$ of outer derivations of S permutes the subspaces U. In particular the graded subalgebras of principal and subprincipal type are conjugate by the (outer) automorphism $\exp(e)\exp(-f)\exp(e)\in G$.

Theorem 2.13.

(a) *Let $S = S(m,n)$ with $m > 1$ or $m = 1$ and $n \geqslant 3$. Then all maximal open subalgebras of S are, up to conjugation, the graded subalgebras of type* $(1,\dots,1|1,\dots,1,0,\dots,0)$ *with k zeros, for $k = 0,\dots,n$.*
(b) *All maximal open subalgebras of $S(1,2)$ are, up to conjugation, the graded subalgebras of type* $(1|1,1)$ *and* $(1|1,0)$.

Proof. Let L_0 be a maximal open subalgebra of S. Then, by Theorem 2.11, L_0 is regular and we can assume, by Remark 2.1, that it is invariant with respect to the standard torus T of $W(m,n)$. In particular L_0 decomposes into the direct product of weight spaces with respect to T. Note that $\mathbb{C}\partial/\partial x_i$, $\mathbb{C}\partial/\partial\xi_i$, $\mathbb{C}\xi_{j_1}\dots\xi_{j_h}\partial/\partial x_i$, $\mathbb{C}\xi_{j_1}\dots\xi_{j_h}\partial/\partial\xi_k$ with $k \neq j_1,\dots,j_h$, are one-dimensional weight spaces. Besides, the vector fields $\partial/\partial x_i$ cannot lie in L_0 since they are not exponentiable (cf. [17, Lemma 1.2]). We may thus assume that one of the following situations occurs:

(1) no element $\partial/\partial\xi_i$ lies in L_0. It follows that the T-invariant complement of L_0 contains the T-invariant complement of the subalgebra of type $(1,\dots,1|1,\dots,1)$, thus L_0 coincides with the subalgebra of type $(1,\dots,1|1,\dots,1)$, since it is maximal;
(2) the elements $\partial/\partial\xi_{k+1},\dots,\partial/\partial\xi_n$ lie in L_0 for some $k = 0,\dots,n-1$, and $\partial/\partial\xi_1,\dots,\partial/\partial\xi_k$ do not. Then the elements $\xi_i\partial/\partial x_j$ and $\xi_i\partial/\partial\xi_h$ cannot lie in L_0 for any $j = 1,\dots,m$, any $i = k+1,\dots,n$ and any $h = 1,\dots,k$, since $[\partial/\partial\xi_i, \xi_i\partial/\partial x_j] = \partial/\partial x_j$ and $[\partial/\partial\xi_i, \xi_i\partial/\partial\xi_h] = \partial/\partial\xi_h$. Similarly, the elements $P\partial/\partial x_j$ and $P\partial/\partial\xi_h$, with $P \in \Lambda(\xi_{k+1},\dots,\xi_n)$, cannot lie in L_0 for any $j = 1,\dots,m$ and any $h = 1,\dots,k$. It follows that L_0 is contained in the graded subalgebra of S of type $(1,\dots,1|1,\dots,1,0,\dots,0)$ with $n-k$ zeros and thus coincides with it since L_0 is maximal.

By Remark 2.12, when $m = 1$ and $n = 2$, the subalgebras of principal and subprincipal type are conjugate by an element of G. □

Corollary 2.14.

(a) *All irreducible $\mathbb{Z}$-gradings of $W(m,n)$ with $m \geqslant 1$, and of $S(m,n)$ with $m > 1$ or $m = 1$ and $n \geqslant 3$, are, up to conjugation, the gradings of type* $(1,\dots,1|1,\dots,1,0,\dots,0)$ *with k zeros, for $k = 0,\dots,n$.*
(b) *All irreducible $\mathbb{Z}$-gradings of $S(1,2)$ are, up to conjugation, the gradings of type* $(1|1,1)$ *and* $(1|1,0)$.

Theorem 2.15. *Let $S = S(m,n)$ with $m > 1$, so that $S(m,n) = S'(m,n)$ and $Der\,S = CS'(m,n) = S(m,n) + \mathbb{C}\sum_{i=1}^m x_i\partial/\partial x_i$. Then all maximal among open $\sum_{i=1}^m x_i\partial/\partial x_i$-invariant subalgebras of S are, up to conjugation, the subalgebras of S listed in Theorem* 2.13(a).

Proof. Let L_0 be a maximal among open $\sum_{i=1}^m x_i\partial/\partial x_i$-invariant subalgebras of S. Then $L_0 + \mathbb{C}\sum_{i=1}^m x_i\partial/\partial x_i$ is a maximal open subalgebra of $CS'(m,n)$, hence it is regular by Theorem 2.11. Then one uses the same arguments as in the proof of Theorem 2.13. □

We recall that if $L = S(1,n)$, with $n \geqslant 3$, then $Der\,L = CS'(1,n) = \mathbb{C}E + S'(1,n)$ where $E = x\partial/\partial x + \sum_{i=1}^n \xi_i\partial/\partial\xi_i$ is the Euler operator and $S'(1,n) = S(1,n) + \mathbb{C}\xi_1\dots\xi_n\partial/\partial x$ (cf.

Proposition 1.8). We are now interested in the subalgebras of $S(1,n)$ which are maximal among the $\mathfrak{a}_0$-invariant subalgebras of $S(1,n)$, for every subalgebra $\mathfrak{a}_0$ of the subalgebra $\mathfrak{a}$ of outer derivations of $S(1,n)$ (cf. Theorem 1.9(b)).

Remark 2.16. By Theorem 2.11 every maximal open subalgebra of $S'(1,n)$ or $CS'(1,n)$, for every $n \geqslant 2$, is regular. Therefore the same argument as in the proof of Theorem 2.13 shows that all fundamental maximal subalgebras of $S'(1,n)$ or $CS'(1,n)$ (i.e., fundamental among maximal subalgebras) are, up to conjugation, the graded subalgebras of type $(1|1,\dots,1,0,\dots,0)$ with k zeros, for $k=0,\dots,n-1$. Indeed, the graded subalgebra of $S'(1,n)$ (respectively $CS'(1,n)$) of type $(1|0,\dots,0)$ is not maximal, since it is contained in $S(1,n)$ (respectively $S(1,n)+\mathbb{C}E$). Notice that the graded subalgebras of principal and subprincipal type of $S'(1,2)$ (respectively $CS'(1,2)$) are not conjugate. By the same arguments, all maximal fundamental subalgebras of $S'(1,n)$ and $CS'(1,n)$ (i.e., maximal among fundamental subalgebras) are, up to conjugation, the graded subalgebras of type $(1|1,\dots,1,0,\dots,0)$ with k zeros, for $k=0,\dots,n$.

In order to distinguish, when needed, a subalgebra of type $(a|b_1,\dots,b_n)$ of $S(1,n)$ from the graded subalgebra of $S'(1,n)$ or $CS'(1,n)$ of the same type, we shall use subscripts. For example $(1|1,\dots,1)_{S'(1,n)}$ will denote the graded subalgebra of $S'(1,n)$ of principal type, so that $(1|1,\dots,1)_{S'(1,n)} = (1|1,\dots,1)_{S(1,n)} + \mathbb{C}\xi_1\dots\xi_n\partial/\partial x$.

Theorem 2.17. *Let $L = S(1,n)$ with $n \geqslant 3$.*

(i) *All maximal among open E-invariant subalgebras of L are, up to conjugation, the graded subalgebras of type $(1|1,\dots,1,0,\dots,0)$ with k zeros, for some $k=0,\dots,n$.*
(ii) *If $\mathfrak{a}_0 = \mathbb{C}\xi_1\dots\xi_n\partial/\partial x$ or $\mathfrak{a}_0 = \mathfrak{a}$, then all maximal among open $\mathfrak{a}_0$-invariant subalgebras of L are, up to conjugation, the graded subalgebras of type $(1|1,\dots,1,\,0,\dots,0)$ with k zeros, for some $k=0,\dots,n-1$.*

Proof. Let L_0 be a maximal among open E-invariant subalgebras of $S(1,n)$. Then $L_0+\mathbb{C}E$ is a fundamental subalgebra of $CS'(1,n)$, hence it is contained in a maximal fundamental subalgebra of $CS'(1,n)$, i.e., by Remark 2.16, in a conjugate of the graded subalgebra of $CS'(1,n)$ of type $(1|1,\dots,1,0,\dots,0)$ with k zeros, for some $k=0,\dots,n$. Suppose $L_0+\mathbb{C}E \subset \varphi((1|1,\dots,1)_{CS'(1,n)}) = \varphi((1|1,\dots,1)_{S(1,n)}) + \mathbb{C}\varphi(E) + \mathbb{C}\varphi(\xi_1\dots\xi_n\partial/\partial x)$ for some inner automorphism φ of $CS'(1,n)$. Since E is contained in $\varphi((1|1,\dots,1)_{CS'(1,n)})$, $\varphi((1|1,\dots,1)_{S(1,n)})$ is an E-invariant subalgebra of $S(1,n)$, hence $L_0 = \varphi((1|1,\dots,1)_{S(1,n)})$ by maximality. If $L_0+\mathbb{C}E$ is contained in a conjugate of the subalgebra of type $(1|1,\dots,1,0,\dots,0)$ with k zeros, for some $k=1,\dots,n$, the argument is similar.

Now let S_0 be a maximal among open $\xi_1\dots\xi_n\partial/\partial x$-invariant subalgebras of $S(1,n)$. Then $S_0+\mathbb{C}\xi_1\dots\xi_n\partial/\partial x$ is a fundamental maximal subalgebra of $S'(1,n)$ containing $\xi_1\dots\xi_n\partial/\partial x$. Likewise, if S_0 is a maximal among open $\mathfrak{a}$-invariant subalgebras of $S(1,n)$, then $S_0 + \mathbb{C}E + \mathbb{C}\xi_1\dots\xi_n\partial/\partial x$ is a fundamental maximal subalgebra of $CS'(1,n)$ containing E and $\xi_1\dots\xi_n\partial/\partial x$. Then statements (ii) and (iii) follow from Remark 2.16. □

Theorem 2.18. *Let $S = S(1,2)$ and let $\mathfrak{b}$ be the 2-dimensional subalgebra of $\mathfrak{a}$ spanned by e and h.*

(i) *If $\mathfrak{a}_0$ is a one-dimensional subalgebra of $\mathfrak{a}$, then all maximal among open $\mathfrak{a}_0$-invariant subalgebras of $S(1,2)$ are, up to conjugation, the subalgebras of type $(1|1,1)$ and $(1|1,0)$.*

(ii) *The graded subalgebra of type* $(1|1,1)$ *is, up to conjugation, the only maximal among open* $\mathfrak{b}$*-invariant subalgebras of* $S(1,2)$*, which is not invariant with respect to* $\mathfrak{a}$.

(iii) *All maximal open among* $\mathfrak{a}$*-invariant subalgebras of* $S(1,2)$ *are, up to conjugation, the subalgebras of type* $(2|1,1)$ *and* $(1|1,0)$.

Proof. By Remark 2.16, the proof of (i) is the same as the proof of (i) and (ii) in Theorem 2.17. Recall that the graded subalgebras of principal and subprincipal type of $S(1,2)$ are conjugate.

Now, using [11, Lemma 5.9] one can check that the graded subalgebras of $S(1,2)$ of type $(1|1,0)$ and $(2|1,1)$ are invariant with respect to $\mathfrak{a}$. On the other hand, the graded subalgebra L_0 of type $(1|1,1)$ is invariant with respect to $\mathfrak{b}$ but it is not $\mathfrak{a}$-invariant. Indeed, one has: $\xi_i\partial/\partial x \in L_0$, $\partial/\partial\xi_j \notin L_0$ and $f(\xi_i\partial/\partial x) = \pm\partial/\partial\xi_j$ with $j \neq i$. Let S_0 be a maximal among $\mathfrak{b}$-invariant subalgebras of $S(1,2)$. Then $S_0 + \mathbb{C}\sum_{i=1}^{2}\xi_i\partial/\partial\xi_i + \mathbb{C}\xi_1\xi_2\partial/\partial x$ is a fundamental maximal subalgebra of $CS'(1,2)$ containing $\sum_{i=1}^{2}\xi_i\partial/\partial\xi_i$ and $\xi_1\xi_2\partial/\partial x$, hence, by Remark 2.16, S_0 is conjugate either to the graded subalgebra of $S(1,2)$ of type $(1|1,1)$ or to the graded subalgebra of type $(1|1,0)$.

Now suppose that $\tilde{S}$ is a maximal among open $\mathfrak{a}$-invariant subalgebras of $S(1,2)$. Then $\tilde{S}$ is invariant with respect to $\mathfrak{b}$, hence $\tilde{S} + \mathfrak{b}$ is contained in a maximal fundamental subalgebra of $CS'(1,2)$ containing $\mathfrak{b}$. It follows that either $\tilde{S}$ is contained in a conjugate of the subalgebra of $S(1,2)$ of type $(1|1,0)$, thus coincides with it by maximality, or it is contained in a conjugate S_U of the subalgebra of principal type. As we noticed in Remark 2.12, S_U is conjugate to the subalgebra of principal type by an automorphism $\varphi = \exp(ad\,a)$ for some $a \in \mathfrak{a}$. Since $\tilde{S}$ is $\mathfrak{a}$-invariant, $\varphi(\tilde{S}) = \tilde{S}$, therefore $\tilde{S}$ is contained in the intersection of S_U with the subalgebra of principal type, i.e., in the graded subalgebra of type $(2|1,1)$. Since the subalgebra of type $(2|1,1)$ is $\mathfrak{a}$-invariant, $\tilde{S}$ coincides with it by maximality. □

2.2. *The Lie superalgebra* $K(2k+1,n)$

Let $k \in \mathbb{Z}_+$ and let $t, p_1, \ldots, p_k, q_1, \ldots, q_k$, be $2k+1$ even indeterminates and $\xi_1, \ldots, \xi_n$ be n odd indeterminates. Consider the differential form $\tau = dt + \sum_{i=1}^{k}(p_i\,dq_i - q_i\,dp_i) + \sum_{j=1}^{n}\xi_j\,d\xi_{n-j+1}$. The contact Lie superalgebra is defined as follows [17, Example 4.4]:

$$K(2k+1,n) := \bigl\{X \in W(2k+1,n) \mid X\tau = f\tau \text{ for some } f \in \Lambda(2k+1,n)\bigr\}.$$

The algebra $K(2k+1,n)$ is simple for every k,n. Recall that we assumed $(k,n) \neq (0,2)$.

Consider the Lie superalgebra $\Lambda(2k+1,n)$ with the following bracket:

$$[f,g] = (2-E)f\frac{\partial g}{\partial t} - \frac{\partial f}{\partial t}(2-E)g + \sum_{i=1}^{k}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) + (-1)^{p(f)}\sum_{i=1}^{n}\frac{\partial f}{\partial\xi_i}\frac{\partial g}{\partial\xi_{n-i+1}}, \tag{2.2}$$

where $E = \sum_{i=1}^{k}(p_i\partial/\partial p_i + q_i\partial/\partial q_i) + \sum_{i=1}^{n}\xi_i\partial/\partial\xi_i$ is the Euler operator. Then the map $\varphi : \Lambda(2k+1,n) \to K(2k+1,n)$ given by:

$$f \mapsto X_f = (2-E)f\frac{\partial}{\partial t} + \frac{\partial f}{\partial t}E + \sum_{i=1}^{k}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) + (-1)^{p(f)}\sum_{i=1}^{n}\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_{n-i+1}},$$

is an isomorphism of Lie superalgebras (cf. [10, §1.2], [6]). We will therefore identify $K(2k+1,n)$ with $\Lambda(2k+1,n)$. The standard maximal torus is $T = \langle t, p_iq_i, \xi_j\xi_{n-j+1} \mid i = 1,\dots,k;\ j = 1,\dots,[n/2]\rangle$.

Remark 2.19. Bracket (2.2) satisfies the following rule:

$$[f, gh] = [f, g]h + (-1)^{p(f)p(g)}g[f, h] + 2\frac{\partial f}{\partial t}gh.$$

Besides we have:

$$X_f(g) = [f, g] + 2\frac{\partial f}{\partial t}g.$$

It follows, in particular, that an ideal $I = (f_1,\dots,f_r)$ of $\Lambda(2k+1,n)$ is stabilized by a function f in $K(2k+1,n)$ if and only if $[f, f_i]$ lies in I for every $i = 1,\dots,r$.

Notice that, if f is an even function independent of t, then $\varphi = \exp ad(f)$ is an automorphism of $\Lambda(2k+1,n)$ with respect to both the Lie bracket and the usual product of polynomials. It follows that a subalgebra L_0 of $K(2k+1,n)$ stabilizes an ideal $I = (f_1,\dots,f_r)$ of $\Lambda(2k+1,n)$ if and only if the subalgebra $\varphi(L_0)$ stabilizes the ideal $J = (\varphi(f_1),\dots,\varphi(f_r))$.

Remark 2.20. A $\mathbb{Z}$-grading of $W(2k+1,n)$ induces a $\mathbb{Z}$-grading on $K(2k+1,n)$ if and only if the differential form τ is homogeneous. It follows that, for every $s = 0,\dots,[n/2]$, the $\mathbb{Z}$-grading of $W(2k+1,n)$ of type $(2,1,\dots,1|2,\dots,2,1,\dots,1,0,\dots,0)$, with $s+1$ 2's and s zeros, induces on $K(2k+1,n)$ a $\mathbb{Z}$-grading of depth 2, where $\mathfrak{g}_0 \cong cspo(2k, n-2s)\otimes\Lambda(s) + W(0,s)$, $\mathfrak{g}_{-1} \cong \mathbb{C}^{2k|n-2s}\otimes\Lambda(s)$ and $\mathfrak{g}_{-2} = [\mathfrak{g}_{-1},\mathfrak{g}_{-1}] \cong \mathbb{C}\otimes\Lambda(s)$. This grading is thus irreducible for every $s = 0,\dots,[n/2]$ when $k = 0$ and n is odd, or $k > 0$, and it is irreducible for every $0 \leqslant s < (n-2)/2$ when $k = 0$ and n is even. Besides, when $k = 0$ and n is even the grading of type $(1|1,\dots,1,0,\dots,0)$ with $n/2$ zeros, is also irreducible by Remark 1.13. One can verify that these irreducible gradings satisfy the hypotheses of Proposition 1.11(b), therefore the corresponding graded subalgebras of $K(2k+1,n)$ are maximal.

The grading of $K(2k+1,n)$ of type $(2,1,\dots,1|1,\dots,1)$ is called *principal*. The grading of $K(2k+1,2h)$ of type $(2,1,\dots,1|2,\dots,2,0,\dots,0)$, with h zeros, is called *subprincipal*.

Remark 2.21. Notice that when $k = 0$, n is even and $s = (n-2)/2$, then the grading of $W(1,n)$ of type $(2|2,\dots,2,1,1,0,\dots,0)$, with $s+1$ 2's and s zeros, induces on $K(1,n)$ a grading which is not irreducible. In particular, the subalgebra $\prod_{j\geqslant 0}K(1,n)_j$ of $K(1,n)$ corresponding to this grading is contained in the maximal subalgebra of type $(1|1,\dots,1,0,\dots,0)$ with $n/2$ zeros.

Remark 2.22. The group of inner automorphisms that preserve the principal grading of $L = K(2k+1,n)$ is isomorphic to $\mathbb{C}^\times(Sp(2k)\times SO(n))$. It follows that when $k = 0$ and n is even the graded subalgebras of L of type $(1|1,\dots,1,0,\dots,0)$ and $(1|1,\dots,1,0,1,0\dots,0)$ with $n/2$ zeros, are not conjugate by an inner automorphism of L. Likewise, when $k>0$ and n is even the subalgebra of L of subprincipal type is not conjugate by an inner automorphism to the subalgebra of type $(2,1,\dots,1|2,\dots,2,0,2,0,\dots,0)$ with $n/2$ zeros.

Remark 2.23. One can define a valuation ν on $\Lambda(m,n)$ (and the induced valuation on $\Lambda(m,n)/\mathbb{C}1$) with values in $\mathbb{Z}$, by assigning the values of ν on the generators $\{x_i,\xi_j \mid i=1,\dots,m;\ j=1,\dots,n\}$ of $\Lambda(m,n)$ as an associative algebra, and by extending ν to $\Lambda(m,n)$ through the usual two rules:

(a) $\nu(f\cdot g)=\nu(f)+\nu(g)$;
(b) $\nu(\sum_i c_i f_i)=\min\nu(f_i)$, if $c_i\in\mathbb{C}^\times$ and f_i are linearly independent monomials.

Example 2.24. Consider the symmetric bilinear form $(\xi_i,\xi_j)=\delta_{i,n-j+1}$ on the vector space $V=\langle\xi_1,\dots,\xi_n\rangle$. Given a subspace U of V we shall denote by U^0 the kernel of the restriction of the bilinear form $(\cdot,\cdot)$ to U. Then $U=U^0\oplus U^1$ where U^1 is a maximal subspace of U with non-degenerate metric. Let $(U^1)^\perp$ be the orthogonal complement of U^1 in V. Then $(U^1)^\perp$ contains U^0 and a subspace $(U^0)'$ non-degenerately paired with U^0. Let us denote by $(U^1)'$ the orthogonal complement of $U^0+(U^0)'$ in $(U^1)^\perp$.

Now suppose that U is a coisotropic subspace of V and consider the ideal $I_{\mathcal{U}}=(t,p_1,\dots,p_k,q_1,\dots,q_k,U)$ of $\Lambda(2k+1,n)$. We define a valuation ν on $\Lambda(2k+1,n)$ as follows:

$$\nu(t)=2,\qquad \nu(p_i)=\nu(q_i)=1,$$
$$\nu(x)=1\quad\text{for } x\in U^1,$$
$$\nu(x)=2\quad\text{for } x\in U^0,\qquad \nu(x)=0\quad\text{for } x\in\big(U^0\big)'.$$

Then the subspaces

$$L_j(U)=\big\{x\in\Lambda(2k,n)\mid \nu(x)\geqslant j+2\big\}$$

define a filtration of $K(2k+1,n)$ where $L_{-1}=I_{\mathcal{U}}$ and $L_0=Stab(I_{\mathcal{U}})$. If n is not even or n is even and $\dim U^0<n/2$, this is in fact the graded filtration of $K(2k+1,n)$ associated, up to conjugation, to the grading of type $(2,1,\dots,1|2,\dots,2,1,\dots,1,0,\dots,0)$ with $s+1$ 2's and s 0's, s being the dimension of U^0. If n is even, $k=0$ (respectively $k>0$), and $U=U^0$ is a maximal isotropic subspace of V, then L_0 is conjugate either to the graded subalgebra of L of type $(1|1,\dots,1,0,\dots,0)$ (respectively $(2,1,\dots,1|2,\dots,2,0,\dots,0)$) or to the graded subalgebra of type $(1|1,\dots,1,0,1,0,\dots,0)$ (respectively $(2,1,\dots,1|2,\dots,2,0,2,0\dots,0)$) with $n/2$ zeros.

Remark 2.25. If $k=0$ and $n=2h$ then the maximal graded subalgebra of $K(1,n)$ of type $(2|2,\dots,2,0,\dots,0)$ is not irreducible since its component of degree -1 does not generate its negative part. Notice that the non-negative part of the irreducible grading of type $(1|1,\dots,1,0,\dots,0)$ with h zeros coincides with the non-negative part of the grading of type $(2|2,\dots,2,0,\dots,0)$. In particular, it stabilizes the ideal $I_{\mathcal{U}}=(t,p_1,\dots,p_k,q_1,\dots,q_k,U^0)$ where $U^0=\langle\xi_1,\dots,\xi_h\rangle$.

Lemma 2.26. *In the associative superalgebra $\Lambda(2k+1,n)$, let us consider an ideal $J = (t, p_1, \dots, p_k, q_1, \dots, q_k, h_1, \dots, h_r)$ where $h_1, \dots, h_r \in \Lambda(0,n)$. Suppose that $h_1 = \eta_1 + F$ and $h_2 = \eta'_1 + G$ where η_1, η'_1 are non-degenerately paired, distinct elements of $V = \langle \xi_1, \dots, \xi_n \rangle$ and F, G contain no constant and linear terms. Then J is conjugate to the ideal $K = (t+T, p_1, \dots, p_k, q_1, \dots, q_k, \eta_1, \eta'_1, f_1, \dots, f_{r-2})$ for some functions T, $f_i \in \Lambda(U)$ where U is the orthogonal complement of $\langle \eta_1, \eta'_1 \rangle$ in V.*

Proof. Multiplying h_1 by some invertible function we can assume that F does not depend on η_1, i.e., $\eta_1 + F = \eta_1 + f_1\eta'_1 + f_2$ where f_1, f_2 lie in $\Lambda(U)$. Also, we can assume that G lies in $\Lambda(U_1)$ where $U_1 = \langle U, \eta_1 \rangle$. Notice that $f_1\eta'_1 + f_1 G$ lies in J, therefore $J = (t, p_1, \dots, p_k, q_1, \dots, q_k, \eta_1 + f_2 - f_1 G, \eta'_1 + G, h_3, \dots, h_r)$ where $f_2 - f_1 G \in \Lambda(U_1)$. Therefore, multiplying $\eta_1 + f_2 - f_1 G$ by an invertible function, we can write $J = (t, p_1, \dots, p_k, q_1, \dots, q_k, \eta_1 + F', \eta'_1 + G, h_3, \dots, h_r)$ where $F' \in \Lambda(U)$.

Now (see Remark 2.19) the automorphism $\exp(ad(\eta'_1 F'))$ maps J to the ideal $J' = (t + T_1, p_1, \dots, p_k, q_1, \dots, q_k, \eta_1, \eta'_1 + H, h'_3, \dots, h'_r)$ where T_1 and the functions h'_i's lie in $\Lambda(0,n)$, and $H \in \Lambda(U)$. Then, similarly as above, the automorphism $\exp(ad(\eta_1 H))$, maps J' to the ideal $K = (t + T_2, p_1, \dots, p_k, q_1, \dots, q_k, \eta_1, \eta'_1, f_1, \dots, f_{r-2})$, for some $T_2, f_1, \dots, f_{r-2} \in \Lambda(0,n)$. Since η_1, η'_1 lie in K, it is immediate to see that we can assume $T_2, f_1, \dots, f_{r-2} \in \Lambda(U)$. □

Lemma 2.27. *In the associative superalgebra $\Lambda(2k+1,n)$, let us consider an ideal $J = (t, p_1, \dots, p_k, q_1, \dots, q_k, h_1, \dots, h_r)$ where $h_1, \dots, h_r \in \Lambda(0,n)$. Suppose that $h_1 = \eta_1 + F$ where η_1 is an element of V non-degenerately paired with itself, and F contains no constant and linear terms. Then J is conjugate to the ideal $K = (t+T, p_1, \dots, p_k, q_1, \dots, q_k, \eta_1, f_1, \dots, f_{r-1})$ for some functions T, $f_i \in \Lambda(U)$ where U is the orthogonal complement of $\langle \eta_1 \rangle$ in V.*

Proof. One uses the same argument as in the first part of the proof of Lemma 2.26. □

Lemma 2.28. *Let L_0 be a maximal open subalgebra of $L = K(m,n) \cong \Lambda(m,n)$ and let I be an ideal of $\Lambda(m,n)$ stabilized by L_0. Suppose that I is maximal among the L_0-invariant ideals. Then $L_0 \subset I$.*

Proof. Let (L_0) be the ideal generated by L_0. Every invertible element of $\Lambda(m,n)$ is not exponentiable therefore (L_0) contains no invertible element of $\Lambda(m,n)$. It follows that $(L_0) + I$ is a proper ideal of $\Lambda(m,n)$ containing I, and it is L_0-invariant. By the maximality of I it follows $L_0 \subset I$. □

Lemma 2.29. *Let $J = (t + f_0, p_1 + f_1, q_1 + h_1, \dots, p_k + f_k, q_k + h_k)$ be an ideal of $\Lambda(2k+1,n)$, for some even functions f_i, h_j containing no linear and constant terms. Then $J = (t + f'_0, p_1 + f'_1, q_1 + h'_1, \dots, p_k + f'_k, q_k + h'_k)$ with f'_i, h'_j in $\Lambda(0,n)$.*

Proof. Suppose $f_0 = t + t\phi_1 + \phi_2$ with ϕ_2 independent of t and $n_2 = \deg \phi_2 > 1$. Then $f_0 - f_0\phi_1 = t - t\phi_1^2 - \phi_2\phi_1 + \phi_2$. Then the coefficients of t in the second and in the third term have degree $2n_1$ and $n_1 + n_2 - 1 > n_1$, respectively, where $n_1 = \deg \phi_1$. Hence in the limit we get $t + \psi$ for some function ψ independent of t. Similarly we can make f_0, and f_j, h_j independent of all even variables for every $j = 1, \dots, k$. □

Theorem 2.30. *Let L_0 be a maximal open subalgebra of $L = K(2k+1, n)$. Then L_0 is conjugate to the standard subalgebra $L_{\mathcal{U}}$ of L stabilizing the ideal $I_{\mathcal{U}} = (t, p_1, q_1, \ldots, p_k, q_k, U)$ of $\Lambda(2k+1, n)$, for some coisotropic subspace U of $V = \langle \xi_1, \ldots, \xi_n \rangle$.*

Proof. By Remark 1.3 L_0 stabilizes an ideal of the form

$$J = (t + f_0, p_1 + f_1, q_1 + h_1, \ldots, p_k + f_k, q_k + h_k, \nu_1 + g_1, \nu_2 + g_2, \ldots, \nu_s + g_s)$$

for some linear functions ν_j in odd indeterminates, and even functions f_i, h_i and odd functions g_j without constant and linear terms, and J is maximal among the L_0-invariant ideals of $\Lambda(2k+1, n)$.

By Lemma 2.29 we can assume f_0 and, similarly, f_i, h_i in $\Lambda(0, n)$ for every i. Therefore the automorphism $\exp(ad(f_1 q_1))$ maps J to

$$J_1 = \left(t + f_0', p_1, q_1 + h_1', p_2 + f_2', q_2 + h_2', \ldots, p_k + f_k', q_k + h_k', \nu_1 + g_1', \nu_2 + g_2', \ldots, \nu_s + g_s'\right).$$

As above we can make h_1' independent of even variables. It follows that the automorphism $\exp(ad(-h_1' p_1))$ maps J_1 to $J_2 = (t + f_0'', p_1, q_1, p_2 + f_2'', q_2 + h_2'', \ldots, p_k + f_k'', q_k + h_k'', \nu_1 + g_1'', \nu_2 + g_2'', \ldots, \nu_s + g_s'')$. The same procedure applied to all generators $p_i + f_i''$ and $q_j + h_j''$ shows that J is in fact conjugate to the ideal

$$I = (t + T_0, p_1, \ldots, p_k, q_1, \ldots, q_k, \nu_1 + \ell_1, \nu_2 + \ell_2, \ldots, \nu_s + \ell_s)$$

where $\nu_1, \ldots, \nu_s$ are linearly independent vectors in V and $T_0, \ell_1, \ldots, \ell_s$ are functions in $\Lambda(0, n)$ without constant and linear terms.

Let $U = \langle \nu_1, \ldots, \nu_s \rangle$. Then, using the notation introduced in Example 2.24, by Lemmas 2.26 and 2.27,

$$I = \left(t + T_1, p_1, \ldots, p_k, q_1, \ldots, q_k, U^1, \eta_1 + \ell_1, \ldots, \eta_r + \ell_r\right)$$

where $U^0 = \langle \eta_1, \ldots, \eta_r \rangle$ and $T_1, \ell_1, \ldots, \ell_r \in \Lambda((U^1)^\perp)$. Let $(U^0)' = \langle \eta_1', \ldots, \eta_r' \rangle$ with $(\eta_i, \eta_j') = \delta_{i,j}$.

Denote by I' the ideal $I' = (t + T_1, \eta_1 + \ell_1, \ldots, \eta_r + \ell_r) \subset I$. Then, each function f in L_0 (thus stabilizing I) stabilizes the ideal $K = (I, [I', I'])$. Indeed, for every $g, h \in I'$ we have:

$$\left[f, [g, h]\right] = \left[[f, g], h\right] \pm \left[g, [f, h]\right] \in [I, I']$$

and $[I, I'] \subset K$ since T_1 and all odd generators of I' are orthogonal to U^1. Notice that K is generated by the generators of I and by the commutators between every pair of generators of I'. Therefore K is a proper ideal of $\Lambda(2k+1, n)$ since among its generators there is no invertible element. By the maximality of I among the ideals stabilized by L_0 we have $I = K$.

Let us rewrite the ideal I as follows:

$$I = \left(t + T_1, p_1, q_1, \ldots, p_k, q_k, U^1, \eta_1, \ldots, \eta_{h-1}, \eta_h + \ell_h, \ldots, \eta_r + \ell_r\right)$$

where $h = \min\{i = 1, \ldots, r \mid \ell_i \neq 0\}$.

We first show that the functions ℓ_h can be made independent of $\eta_1', \ldots, \eta_{h-1}'$. Indeed, let $\eta_h + \ell_h = \eta_h + \eta_1' \phi_1 + \phi_2$ where ϕ_1, ϕ_2 do not depend on η_1'. Then $\phi_1 = [\eta_1, \eta_h + \ell_h] \in$

$[I', I'] \subset K = I$, thus $I = (t + T_1, p_1, q_1, \dots, p_k, q_k, U^1, \eta_1, \dots, \eta_{h-1}, \eta_h + \phi_2, \eta_{h+1} + \ell'_{h+1}, \dots, \eta_r + \ell'_r)$, where $\phi_2 \in \Lambda((U^1)^\perp)$ does not depend on η'_1. Arguing in the same way for the variables $\eta'_2, \dots, \eta'_{h-1}$, we get

$$I = \big(t + T_1, p_1, q_1, \dots, p_k, q_k, U^1, \eta_1, \dots, \eta_{h-1}, \eta_h + \phi, \eta_{h+1} + \ell_{h+1}, \dots, \eta_r + \ell_r\big)$$

where ϕ does not depend on $\eta'_1, \dots, \eta'_{h-1}$. Besides, multiplying $\eta_h + \phi$ by an invertible function, we can assume that ϕ does not depend on η_h. Now we can write $\phi = \eta'_h \psi_1 + \psi_2$ with ψ_1, ψ_2 independent of $\eta'_1, \dots, \eta'_h$. Therefore, applying the automorphism $\exp(ad(\psi_2 \eta'_h))$ we can write

$$I = \big(t + T_2, p_1, q_1, \dots, p_k, q_k, U^1, \eta_1, \dots, \eta_{h-1}, \eta_h + \eta'_h \psi_1, \eta_{h+1} + \ell'_{h+1}, \dots, \eta_r + \ell'_r\big)$$

for some $T_2, \ell'_j \in \Lambda(0, n)$. Then $\psi_1 = \frac{1}{2}[\eta_h + \eta'_h \psi_1, \eta_h + \eta'_h \psi_1] \in [I', I'] \subset K = I$. Therefore, up to conjugation,

$$I = \big(t + T_2, p_1, q_1, \dots, p_k, q_k, U^1, \eta_1, \dots, \eta_{h-1}, \eta_h, \eta_{h+1} + \ell'_{h+1}, \dots, \eta_r + \ell'_r\big).$$

Arguing as above for $\ell'_{h+1}, \dots, \ell'_s$, we can assume, up to conjugation, that I has the following form:

$$I = \big(t + f, p_1, q_1, \dots, p_k, q_k, U^1, \eta_1, \dots, \eta_r\big) = (t + T, p_1, q_1, \dots, p_k, q_k, U)$$

for some function f in $\Lambda(0, n)$. Notice that, in fact, we can assume $f \in \Lambda((U^0)' \oplus (U^1)')$, since $U = U^0 \oplus U^1 \subset I$. Now suppose $f = \eta'_1 \varphi_1 + \varphi_2$ with φ_1, φ_2 independent of η'_1. Then $[t + f, \eta_1] = -\eta_1 + \varphi_1 \in [I', I'] \subset K = I$, thus we can replace $t + f$ with $t + \varphi_2$ (here $I' = (t + f, U^0)$). Similarly, we can make f independent of $\eta'_2, \dots, \eta'_r$, i.e., $f \in \Lambda((U^1)')$ with no linear and constant terms. In particular, if U is coisotropic then $f = 0$ and I is a standard ideal.

Suppose that U is not coisotropic and consider the ideal $Y = (t, p_1, \dots, p_k, q_1, \dots, q_k, U + (U^1)')$. Note that if U is not coisotropic then the subspace $U + (U^1)'$ is coisotropic. Let $L'_{-2} \supset L'_{-1} \supset \cdots$ be the filtration of $K(2k + 1, n)$ associated to the ideal Y as in Example 2.24, where $L'_0 = Stab(Y)$. Then the completion of the graded superalgebra $Gr\, L$ associated to this filtration is isomorphic to $K(2k + 1, n)$ with respect to the grading of type $(2|2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$ with $s + 1$ 2's and s 0's, where $s = \dim U^0$. In particular we have:

$$Gr_{-2}\, L = \Lambda\big((U^0)'\big), \qquad Gr_{-1}\, L = \big(\langle p_i, q_i\rangle \oplus U^1 \oplus \big(U^1\big)'\big) \otimes \Lambda\big((U^0)'\big).$$

We want to show that L_0 is contained in L'_0. Suppose that $X \in K(2k+1, n)$ stabilizes I. Then we can write

$$X = X_{-2} + X_{-1} + X_0$$

with $X_{-2} \in Gr_{-2}\, L$, $X_{-1} \in Gr_{-1}\, L$ and $X_0 \in \prod_{j \geqslant 0} Gr_j\, L$. In fact, since L_0 is open, $X_{-2} \in \Lambda((U^0)')/\mathbb{C}$.

Note that $t \in Gr_0\, L$, $f \in \prod_{j \geqslant 0} Gr_j\, L$, $U^0 \subset Gr_0\, L$, $U^1 \subset Gr_{-1}\, L$ and $p_i, q_i \in Gr_{-1}\, L$. It follows that $I \subset Gr_{\geqslant -1} = L'_{-1}$.

Now, since $X \in Stab(I)$, we have:

$$\big[X, U^0\big] \subset I \subset L'_{-1} \quad \Rightarrow \quad \big[X_{-2}, U^0\big] = 0$$

and, since $X_{-2} \in \Lambda((U^0)')/\mathbb{C}$, it follows $X_{-2} = 0$.

Similarly,

$$[X, U^1] \subset I \quad \Rightarrow \quad [X_{-1}, U^1] = 0$$

and

$$[X, p_i] \subset I \;\Rightarrow\; [X_{-1}, p_i] = 0, \quad [X, q_i] \subset I \;\Rightarrow\; [X_{-1}, q_i] = 0, \quad \forall i = 1, \dots, k,$$

hence $X_{-1} \in (U^1)' \otimes \Lambda((U^0)')$. Therefore $X = X_{-1} + X_0 \in L_0$ with $X_{-1} \in (U^1)' \otimes \Lambda((U^0)') \subset Gr_{-1} L$ and $X_0 \in \prod_{j \geqslant 0} Gr_j L$. By Lemma 2.28, $L_0 \subset I$, hence $X \in I$. It follows $X_{-1} = 0$, i.e., $L_0 \subset \prod_{j \geqslant 0} Gr_j L$. Indeed if $X_{-1} \neq 0$ then $\nu(X) = 1$ but

$$I \cap \{x \mid \nu(x) = 1\} = (\langle p_i, q_i \rangle + U^1) \otimes \Lambda((U^0)') + \prod_{j \geqslant 0} Gr_j L.$$

By the maximality of L_0 the statement follows. □

Theorem 2.31.

(i) *All maximal open subalgebras of $K(1, 2h)$ $(h > 1)$ are, up to conjugation, the graded subalgebras of type $(1|1, \dots, 1, 0, \dots, 0)$ and $(1|1, \dots, 1, 0, 1, 0, \dots, 0)$ with h zeros, and the graded subalgebras of type $(2|2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$ with $s + 1$ 2's and s zeros, for $s = 0, \dots, h - 2$.*

(ii) *If $k > 0$ and n is even, all maximal open subalgebras of $K(2k+1, n)$ are, up to conjugation, the graded subalgebras of type $(2, 1, \dots, 1|2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$ with $s + 1$ 2's and s zeros, for $s = 0, \dots, n/2$ and the graded subalgebra of type $(2, 1, \dots, 1|2, \dots, 2, 0, 2, 0, \dots, 0)$ with $n/2$ zeros.*

(iii) *If n is odd, all maximal open subalgebras of $K(2k+1, n)$ are, up to conjugation, the graded subalgebras of type $(2, 1, \dots, 1|2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$ with $s + 1$ 2's and s zeros, for $s = 0, \dots, [n/2]$.*

Proof. By Theorem 2.30 every maximal open subalgebra of $K(2k + 1, n)$ is conjugate to the standard subalgebra associated to the ideal $I_{\mathcal{U}} = (t, p_1, \dots, p_k, q_1, \dots, q_k, U)$ of $\Lambda(2k + 1, n)$, for some coisotropic subspace U of $V = \langle \xi_1, \dots, \xi_n \rangle$. Now the statement follows from Example 2.24 and Remarks 2.20, 2.21, 2.22 and 2.25. □

Corollary 2.32.

(i) *All irreducible $\mathbb{Z}$-gradings of $K(1, 2h)$ are, up to conjugation, the grading of type $(2|2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$, with $s + 1$ 2's and s zeros, for $s = 0, \dots, h - 2$ and the gradings of type $(1|1, \dots, 1, 0, \dots, 0)$ and $(1|1, \dots, 1, 0, 1, 0, \dots, 0)$ with h zeros.*

(ii) *All irreducible $\mathbb{Z}$-gradings of $K(2k+1, n)$ where $k > 0$ and n is even are, up to conjugation, the gradings of type $(2, 1, \dots, 1|2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$ with $s + 1$ 2's and s zeros, for $s = 0, \dots, n/2$ and the grading of type $(2, 1, \dots, 1|2, \dots, 2, 0, 2, 0, \dots, 0)$ with $n/2$ zeros.*

(iii) *All irreducible $\mathbb{Z}$-gradings of $K(2k + 1, n)$ where n is odd are, up to conjugation, the gradings of type $(2, 1, \dots, 1|2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$ with $s + 1$ 2's and s zeros, for $s = 0, \dots, [n/2]$.*

We take the opportunity here to describe the embedding of the Lie superalgebra $S(1,2)$ in $K(1,4)$ and to correct Proposition 4.1.2 in [10].

Remark 2.33. Consider the Lie superalgebra $K(1,4)$ with its principal grading. Then $\mathfrak{g}_0 = cso_4$ and we want to study $\mathfrak{g}_1$ as a $\mathfrak{g}_0$-module. $\mathfrak{g}_1$ is spanned by the elements $t\xi_i$, for $i = 1, \dots, 4$, and $\xi_i\xi_j\xi_k$ for $i, j, k = 1, \dots, 4$, $i \neq j \neq k$, thus it is the direct sum of two isomorphic irreducible representations of so_4: $V = \langle t\xi_i \rangle$ and $W = \langle \xi_i\xi_j\xi_k \rangle$, each of which is isomorphic to the standard so_4-module. Note that $[W, W] = \langle \xi_1\xi_2\xi_3\xi_4 \rangle$ and $[\mathfrak{g}_{-2}, W] = 0$. It follows that $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + W + [W, W]$ is isomorphic to $\hat{H}(0,4) + \mathbb{C}t$ where t is the grading operator (see [10, §1.2]). As it was noticed in [10, Proposition 4.1.2], for every $\lambda \in \mathbb{C}$, the subspace $V_\lambda = \langle \xi_1 t + \lambda\xi_1\xi_2\xi_3, \xi_2 t + \lambda\xi_2\xi_1\xi_4, \xi_3 t + \lambda\xi_4\xi_1\xi_3, \xi_4 t - \lambda\xi_4\xi_2\xi_3 \rangle$ is an irreducible $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$, but, differently from what is claimed in [10, Proposition 4.1.2], $\dim([V_\lambda, V_\lambda]) = 1$ for every $\lambda \in \mathbb{C}$. Besides, for every $\lambda \neq \pm 1$, $[\mathfrak{g}_{-1}, V_\lambda] = cso_4 = \mathfrak{g}_0$, while, if $\lambda = 1$ or $\lambda = -1$, then $[\mathfrak{g}_{-1}, V_\lambda] = gl_2$. Therefore, for every $\lambda \neq \pm 1$, $\mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0 + V_\lambda + [V_\lambda, V_\lambda]$ is a simple, 17-dimensional Lie superalgebra, isomorphic to the Lie superalgebra $D(2,1;\alpha)$ for some α (cf. [15, Remark 2.5.7]). If $\lambda = 1$ or $\lambda = -1$, the Lie superalgebra $L := \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + gl_2 + V_\lambda + V_\lambda^2$ has dimension 14 and it is isomorphic to $sl(2,2)/\mathbb{C}1$, and the copy of sl_2, lying in $\mathfrak{g}_0$ and outside of L, acts on L by outer derivations.

Now consider the Lie superalgebra $S(1,2) = \sum_{j \geqslant -2} \mathfrak{h}_j$ with respect to the grading of type $(2|1,1)$. Then the positive part of this grading is not generated by $\mathfrak{h}_1$. On the contrary, $(\mathfrak{h}_1)^2$ has dimension 1 and $\mathfrak{h}_{-2} + \mathfrak{h}_{-1} + \mathfrak{h}_0 + \mathfrak{h}_1 + (\mathfrak{h}_1)^2 \cong sl(2,2)/\mathbb{C}1$. We have the following embedding of $S(1,2)$ in $K(1,4)$:

$$S(1,2) \cong \mathbb{C}[t + \xi_1\xi_4 + \xi_2\xi_3]\Lambda(\xi_1, \xi_2) + \mathbb{C}[t - \xi_1\xi_4 - \xi_2\xi_3]\Lambda(\xi_3, \xi_4).$$

Another description of this important embedding is given in [5, Remark 5.12].

2.3. *The Lie superalgebras $HO(n,n)$ and $SHO(n,n)$*

Let $x_1, \dots, x_n$ be n even indeterminates and $\xi_1, \dots, \xi_n$ be n odd indeterminates, and let us consider the differential form $\sigma = \sum_{i=1}^n dx_i\, d\xi_i$. The odd Hamiltonian superalgebra is defined as follows (cf. [1]):

$$HO(n,n) := \{X \in W(n,n) \mid X\sigma = 0\}.$$

It is a simple Lie superalgebra if and only if $n \geqslant 2$. The Lie superalgebra $HO(n,n)$ contains the subalgebra

$$SHO'(n,n) := S'HO(n,n) = \{X \in HO(n,n) \mid div(X) = 0\}$$

(see Definition 2.4 and Example 2.8).

Its derived algebra $SHO(n,n) = [SHO'(n,n), SHO'(n,n)]$ is simple if and only if $n \geqslant 3$.

The Lie superalgebra $HO(n,n)$ can be realized as follows (cf. [10, §1.3]): in $\Lambda(n,n)$ one can consider the Lie superalgebra structure defined by the Buttin bracket:

$$[f,g] := \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} \right).$$

Then the map $\Lambda(n,n) \to HO(n,n)$ given by:

$$f \mapsto \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} \right)$$

defines a surjective homomorphism of Lie superalgebras whose kernel consists of constant functions. Hence we will identify $HO(n,n)$ with $\Lambda(n,n)/\mathbb{C}1$ with the Buttin bracket, with reversed parity. Under this identification

$$SHO'(n,n) = \{ f \in \Lambda(n,n) \mid \Delta(f) = 0 \} / \mathbb{C}1 =: \Lambda^{\Delta}(n,n)/\mathbb{C}1,$$

where $\Delta = \sum_{i=1}^{n} \partial^2/(\partial x_i \partial \xi_i)$ is the odd Laplace operator, and $SHO(n,n)$ is the span of all monomials in $SHO'(n,n)$ except for $\xi_1 \dots \xi_n$.

Since $HO(2,2) \cong S(2,1)$ and since $SHO(n,n)$ is simple if and only if $n \geqslant 3$, when talking about $HO(n,n)$ and $SHO(n,n)$ we shall assume $n \geqslant 3$. Consider the maximal torus $T = \langle x_i \xi_i \mid i = 1, \dots, n \rangle$ of $HO(n,n)$. Recall that $Der\, HO(n,n) = HO(n,n) + \mathbb{C}E$ where $E = \sum_{i=1}^{n} (x_i \partial/\partial x_i + \xi_i \partial/\partial \xi_i)$ is the Euler operator. Besides, if $n \geqslant 4$ then $Der\, SHO(n,n) = SHO'(n,n) + \mathbb{C}E + \mathbb{C}\Phi$ where $\Phi = \sum_{i=1}^{n} x_i \xi_i$ (with $\sum_{i=1}^{n} (-x_i \partial/\partial x_i + \xi_i \partial/\partial \xi_i)$ the corresponding vector field) (cf. Proposition 1.8). Finally, $Der\, SHO(3,3) = SHO(3,3) + \mathfrak{a}$ where $\mathfrak{a} \cong gl_2$ and a maximal torus of $\mathfrak{a}$ is spanned by E and Φ (cf. [10, Remark 4.4.1]).

Remark 2.34. The $\mathbb{Z}$-grading of type $(1, \dots, 1|0, \dots, 0)$ of $W(n,n)$ induces on $HO(n,n)$ (respectively $SHO(n,n)$) a grading of depth 1 (called the *subprincipal* grading) which is irreducible by Remark 1.13.

Consider the gradings induced on $HO(n,n)$ (respectively $SHO(n,n)$) by the $\mathbb{Z}$-gradings of type $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0)$ of $W(n,n)$, with k 2's and k zeros. For any fixed k, $0 \leqslant k \leqslant n-2$, the 0th graded component of $HO(n,n)$ (respectively $SHO(n,n)$) with respect to this grading is isomorphic to the Lie superalgebra $\tilde{P}(n-k) \otimes \Lambda(k) + W(0,k)$ (respectively $P(n-k) \otimes \Lambda(k) + W(0,k)$) and the -1st graded component is isomorphic to $\mathbb{C}^{n-k|n-k} \otimes \Lambda(k)$ where $\mathbb{C}^{n-k|n-k}$ is the standard $P(n-k)$-module (cf. [15]). Therefore for every $k = 0, \dots, n-2$ these are irreducible gradings of $HO(n,n)$ (respectively $SHO(n,n)$). If $k > 0$ then the grading of type $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0)$ with k 2's and k zeros, has depth 2, its -1st graded component generates its negative part and property (iii)$'$ of Proposition 1.11(b) is satisfied. It follows that the subalgebras of $HO(n,n)$ (respectively $SHO(n,n)$) of type $(1, \dots, 1|0, \dots, 0)$ and $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0)$ with k 2's and k zeros, for $k = 0, \dots, n-2$, are maximal. (All claims hold also for the Lie superalgebra $HO(2,2)$.)

The $\mathbb{Z}$-grading induced on $HO(n,n)$ (respectively $SHO(n,n)$) by the principal grading of $W(n,n)$ is also called *principal*.

Remark 2.35. The $\mathbb{Z}$-grading of type $(1, 2, \dots, 2|1, 0, \dots, 0)$ of $HO(n,n)$ is not irreducible. Indeed the 0th graded component of $HO(n,n)$ with respect to this grading is the subspace $\langle x_1^2, x_1\xi_1, x_i \mid i = 2, \dots, n \rangle \otimes \Lambda(\xi_2, \dots, \xi_n)$ and its -1st graded component is $\langle x_1, \xi_1 \rangle \otimes \Lambda(\xi_2, \dots, \xi_n)$. It follows that the subspace $\langle x_1 \rangle \otimes \Lambda(\xi_2, \dots, \xi_n)$ of $HO(n,n)_{-1}$ is $HO(n,n)_0$ stable.

Likewise, the $\mathbb{Z}$-grading of type $(1, 2, \dots, 2|1, 0, \dots, 0)$ of $SHO(n,n)$ fails to be irreducible. Indeed, with respect to this grading, $HO(n,n)_{-1} = SHO(n,n)_{-1}$.

The subalgebra of type $(1,2,\dots,2|1,0,\dots,0)$ of $HO(n,n)$ (respectively $SHO(n,n)$) is contained in the maximal subalgebra of type $(1,\dots,1|0,\dots,0)$.

Remark 2.36. Let $L = HO(n,n)$ or $L = SHO(n,n)$. Then the graded subalgebra L_k of L of type $(1,\dots,1,2,\dots,2|1,\dots,1,0,\dots,0)$ with $n-k$ 2's and $n-k$ zeros, is, for every $k=1,\dots,n$, the standard subalgebra L_U of L stabilizing the ideal $I_U = (x_1,\dots,x_n,\xi_1,\dots,\xi_k)$. Indeed, for every k, $L_k \subset L_U$ since L_k is contained in the graded subalgebra of $W(n,n)$ of type $(1,\dots,1|1,\dots,1,0,\dots,0)$ with $n-k$ zeros, which stabilizes I_U (cf. the proof of Theorem 2.3). If $k \neq 1$, then, by Remark 2.34, L_k is a maximal subalgebra of L, thus $L_k = L_U$ for every $k \neq 1$.

Now suppose $k=1$. By Remark 1.15, L_U is regular and, up to conjugation, we can assume that it is invariant with respect to the standard torus $T + \mathbb{C}E$ of $Der\,HO(n,n)$. Therefore L_U decomposes into the direct product of weight spaces with respect to $T+\mathbb{C}E$. Consider the $\mathbb{Z}$-grading of L of type $(1,2,\dots,2|1,0,\dots,0)$. Then the negative part of this grading is $\mathfrak{g}_- = (\langle 1, x_1, \xi_1\rangle \otimes \Lambda(\xi_2,\dots,\xi_n))/\mathbb{C}1$. Notice that $\mathbb{C}\xi_{i_1}\dots\xi_{i_h}$, with $i_1 \neq \dots \neq i_h$, and $\mathbb{C}x_1\xi_{j_1}\dots\xi_{j_h}$, with $1 \neq j_1 \neq \dots \neq j_h$, are one-dimensional weight spaces with respect to $T+\mathbb{C}E$. Therefore, in order to prove that L_U is contained in L_1 (hence $L_U = L_1$) it is sufficient to show that, for every $f \in \mathfrak{g}_-$, f does not lie in L_U. Notice that L_U contains the elements $x_2,\dots,x_n$ but it does not contain neither the elements ξ_i for any $i=1,\dots,n$, nor the element x_1, since these elements do not stabilize the ideal I_U. It follows that the elements $\xi_i\xi_j$ cannot lie in L_U for any $i \neq j$. Indeed, $[x_j, \xi_i\xi_j] = -\xi_i$. Likewise, by induction on $k=1,\dots,n$, the elements $\xi_{i_1}\dots\xi_{i_k}$ cannot lie in L_U for any $k=1,\dots,n$. Now, suppose that $x_1\xi_j$ lies in L_U for some $j \neq 1$. Then L_U contains the element $[x_j, x_1\xi_j] = x_1$ and this contradicts our assumptions. It follows that L_U cannot contain the elements $x_1\xi_j$ and, similarly, the elements $x_1\xi_{j_1}\dots\xi_{j_k}$ for any $j_1 \neq \dots \neq j_k \neq 1$. L_U is therefore contained in L_1, hence $L_U = L_1$.

Finally, the graded subalgebra of L of type $(1,\dots,1|0,\dots,0)$ is the standard subalgebra of L stabilizing the ideal $(x_1,\dots,x_n)$.

Remark 2.37. We recall that $Der\,SHO(3,3) = SHO(3,3) + \mathfrak{a}$ with $\mathfrak{a} \cong gl_2$ (cf. Proposition 1.8, [10, Remark 4.4.1]). The subalgebra $\mathfrak{a}$ of outer derivations is generated by the Euler operator E and by a copy of sl_2 with Chevalley basis $\{e,h,f\}$ where

$$e = ad\left(\xi_1\xi_3\frac{\partial}{\partial x_2} - \xi_2\xi_3\frac{\partial}{\partial x_1} - \xi_1\xi_2\frac{\partial}{\partial x_3}\right) \quad \text{and} \quad h = ad\left(\frac{2}{3}\sum_{i=1}^{3}\left(\xi_i\frac{\partial}{\partial \xi_i} - x_i\frac{\partial}{\partial x_i}\right)\right).$$

In order to describe the action of the derivation f one can proceed as in [11, Lemma 5.9]. Here it is convenient, as before, to identify $SHO'(3,3)$ with the set of elements g in $\Lambda(3,3)/\mathbb{C}1$ such that $\Delta(g)=0$, and $SHO(3,3)$ with the subspace consisting of elements not containing the monomial $\xi_1\xi_2\xi_3$. Under this identification $e = ad(\xi_1\xi_2\xi_3)$ and $h = ad(\frac{2}{3}\sum_{i=1}^{3} x_i\xi_i)$. Let us consider $SHO(3,3)$ with its principal grading. With respect to this grading, $SHO(3,3)_j = (SHO(3,3)_1)^j$, for $j>1$, therefore it is sufficient to define the derivation f on the local part $SHO(3,3)_{-1} \oplus SHO(3,3)_0 \oplus SHO(3,3)_1$ of $SHO(3,3)$. One has:

$$f(\xi_1\xi_2) = -\frac{4}{3}x_3, \quad f(\xi_1\xi_3) = \frac{4}{3}x_2, \quad f(\xi_2\xi_3) = -\frac{4}{3}x_1, \quad f(x_1\xi_2\xi_3) = -\frac{1}{3}x_1^2,$$

$$f(x_2\xi_1\xi_3) = \frac{1}{3}x_2^2, \quad f(x_3\xi_1\xi_2) = -\frac{1}{3}x_3^2, \quad f(x_1\xi_1\xi_2 - x_3\xi_3\xi_2) = -\frac{2}{3}x_1x_3,$$

$$f(x_2\xi_2\xi_1 - x_3\xi_3\xi_1) = \frac{2}{3}x_2x_3, \quad f(x_1\xi_1\xi_3 - x_2\xi_2\xi_3) = \frac{2}{3}x_1x_2,$$

and $f = 0$ elsewhere on $SHO(3,3)_{-1} \oplus SHO(3,3)_0 \oplus SHO(3,3)_1$.

Remark 2.38. Let $S = \prod_{j\geqslant -2} S_j$ denote the Lie superalgebra $SHO(3,3)$ with respect to the grading of type $(2,2,2|1,1,1)$. Then $S_0 \cong sl_3$ and S_{-1} is isomorphic, as an S_0-module, to the direct sum of two copies of the standard sl_3-module. It follows that, for every irreducible sl_3-submodule U of S_{-1}, $S_U := U + \prod_{j\geqslant 0} S_j$ is a maximal open subalgebra of S. In particular, if $U = \langle \xi_i\xi_j \mid i,j = 1,2,3\rangle$ or $U = \langle x_i \mid i = 1,2,3\rangle$, then S_U is the maximal graded subalgebra of type $(1,1,1|1,1,1)$ or $(1,1,1|0,0,0)$, respectively. The subalgebras S_U are not conjugate by inner automorphisms of S, but they are conjugate by inner automorphisms of $Der\, S$, since the copy of sl_2 of outer derivations of S described in Remark 2.37, permutes the subspaces U. In particular the graded subalgebras of principal and subprincipal type are conjugate by the automorphism $\exp(e)\exp(\frac{-3}{4}f)\exp(e) \in G$.

Remark 2.39. Let $1 \leqslant i < j \leqslant n$. Then the change of indeterminates that exchanges x_i with x_j and ξ_i with ξ_j preserves the form σ.

Remark 2.40. Let $\eta = \alpha_{i_1}\xi_{i_1} + \cdots + \alpha_{i_k}\xi_{i_k}$ for some $k \leqslant n$, with $\alpha_{i_j} \in \mathbb{C}$, $\alpha_{i_j} \neq 0$. According to Remark 2.39 we can assume $\eta = \alpha_1\xi_1 + \cdots + \alpha_k\xi_k$ with $\alpha_i \neq 0$ for $i = 1,\dots,k$. Then the following change of indeterminates preserves the form σ:

$$\begin{array}{ll}
x'_1 = \frac{1}{\alpha_1}x_1, & \xi'_1 = \eta, \\
x'_2 = x_2 - \frac{\alpha_2}{\alpha_1}x_1, & \xi'_2 = \xi_2, \\
\vdots & \vdots \\
x'_k = x_k - \frac{\alpha_k}{\alpha_1}x_1, & \xi'_k = \xi_k, \\
x'_i = x_i, & \xi'_i = \xi_i \quad \forall i > k.
\end{array}$$

Theorem 2.41. *Let $L = HO(n,n)$ and let L_0 be a maximal open subalgebra of L. Then L_0 is conjugate to a standard subalgebra of L.*

Proof. Let $L = HO(n,n)$. By Remark 1.3 L_0 stabilizes an ideal of the form

$$J = (x_1 + f_1, \dots, x_n + f_n, \eta_1 + g_1, \dots, \eta_s + g_s)$$

for some linear functions η_j in odd indeterminates, and even functions f_i and odd functions g_j without constant and linear terms, and J is maximal among the L_0-invariant ideals of $\Lambda(n,n)$. By Remark 2.40, up to changes of indeterminates, we can write

$$J = (x_1 + F_1, \dots, x_n + F_n, \xi_1 + G_1, \dots, \xi_s + G_s)$$

for some even functions F_i and odd functions G_j without constant and linear terms, where we can assume G_j independent of $\xi_1,\dots,\xi_s$ for every $j = 1,\dots,s$.

Suppose that $x_1 + F_1 = x_1 + \xi_1F'_1 + F''_1$ with F'_1 and F''_1 independent of ξ_1. Then we can replace $x_1 + F_1$ by $x_1 + F_1 - (\xi_1 + G_1)F'_1 = x_1 + H_1$ with H_1 independent of ξ_1. Similarly we can make every function F_i independent of ξ_j for every $j = 1, \dots, s$. Besides, as in Lemma 2.29, since the ideal J is closed, we can make the functions F_i and G_i independent of all even variables, i.e., $F_i, G_i \in \Lambda(0, n)$. It follows that the automorphism $\exp(ad(\xi_1 F_1))$ maps J to the ideal

$$I = \big(x_1, x_2 + F_2', \dots, x_n + F_n', \xi_1 + G_1, \xi_2 + G_2, \dots, \xi_s + G_s\big).$$

Arguing in the same way for every function F_j' with $1 \leqslant j \leqslant s$, we have, up to automorphisms,

$$I = (x_1, \dots, x_s, x_{s+1} + h_{s+1}, \dots, x_n + h_n, \xi_1 + G_1, \dots, \xi_s + G_s)$$

for some functions $h_i \in \Lambda(\xi_{s+1}, \dots, \xi_n)$ with no constant and linear terms. Now the automorphism $\exp(ad(-x_1 G_1))$ sends I to the ideal

$$I_1 = (x_1, \dots, x_s, x_{s+1} + h_{s+1}, \dots, x_n + h_n, \xi_1, \xi_2 + G_2, \dots, \xi_s + G_s).$$

Analogous automorphisms for G_i, $i = 1, \dots, s$, yield to the ideal

$$Y = (x_1, \dots, x_s, x_{s+1} + h_{s+1}, \dots, x_n + h_n, \xi_1, \xi_2, \dots, \xi_s).$$

Consider the ideal $Y' = (x_{s+1} + h_{s+1}, \dots, x_n + h_n) \subset Y$. Then, each function f in L_0 (thus stabilizing Y) stabilizes the ideal $K = (Y, [Y', Y'])$, i.e., the ideal generated by the generators of Y and by the commutators between every pair of generators of Y'. Indeed, for every $g, h \in Y'$ we have:

$$\big[f, [g, h]\big] = \big[[f, g], h\big] \pm \big[g, [f, h]\big] \in [Y, Y']$$

and $[Y, Y'] \subset K$ since all generators of Y outside Y' commute with the generators of Y'. Notice that K is a proper ideal of $\Lambda(2k+1, n)$ since among its generators there is no invertible element. By the maximality of J among the ideals stabilized by L_0 we have $Y = K$.

Suppose that $h_{s+1} = \xi_{s+1}\psi_1 + \psi_2$ with ψ_1 and ψ_2 independent of ξ_{s+1}. Then, applying the automorphism $\exp(ad(\xi_{s+1}\psi_2))$, we can assume

$$Y = \big(x_1, \dots, x_s, x_{s+1} + \xi_{s+1}\psi_1, \dots, x_n + h_n', \xi_1, \xi_2, \dots, \xi_s\big).$$

Now $\psi_1 = \frac{1}{2}[x_{s+1} + \xi_{s+1}\psi_1, x_{s+1} + \xi_{s+1}\psi_1] \in [Y', Y'] \subset K = Y$, therefore

$$Y = \big(x_1, \dots, x_s, x_{s+1}, x_{s+2} + h_{s+2}', \dots, x_n + h_n', \xi_1, \xi_2, \dots, \xi_s\big).$$

Arguing in the same way for every function h_j' we end up with a standard ideal. □

Theorem 2.42.

(a) *Let $L = HO(n, n)$, or $SHO(n, n)$ with $n > 3$. Then all maximal open subalgebras of L are, up to conjugation, the graded subalgebras of type* $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0)$ *with k 2's and k zeros, for $k = 0, \dots, n-2$ and the graded subalgebra of type* $(1, \dots, 1|0, \dots, 0)$.
(b) *All maximal open subalgebras of $SHO(3, 3)$ are, up to conjugation, the graded subalgebras of type* $(1, 1, 1|1, 1, 1)$ *and* $(1, 1, 2|1, 1, 0)$.

Proof. Let $L = HO(n,n)$ and let L_0 be a maximal open subalgebra of L. By Theorem 2.41, L_0 is, up to conjugation, the standard subalgebra of L stabilizing the ideal $I_{\mathcal{U}} = (x_1, \dots, x_n, \xi_1, \dots, \xi_s)$ for some $s = 0, \dots, n$. The statement then follows using Remarks 2.34–2.36.

Let now $L = SHO(n,n)$ and let L_0 be a maximal open subalgebra of L. The same argument as in Theorem 2.11 shows that L_0 is regular and we can assume, by Remark 2.1, that it is invariant with respect to the standard torus $T + \mathbb{C}E$ of $Der\, HO(n,n)$. It follows that L_0 decomposes into the direct product of weight spaces with respect to $T + \mathbb{C}E$. As we noticed in Remark 2.36, $\mathbb{C}\xi_{i_1} \dots \xi_{i_h}$, with $i_1 \neq \dots \neq i_h$, and $\mathbb{C}x_1\xi_{j_1} \dots \xi_{j_h}$, with $1 \neq j_1 \neq \dots \neq j_h$, are one-dimensional weight spaces with respect to $T + \mathbb{C}E$. Besides, note that the elements ξ_i cannot lie in L_0 since they are not exponentiable.

We may assume that one of the following two cases holds:

(1) $x_1, \dots, x_n$ lie in L_0. Since $[x_i, \xi_i\xi_h] = \xi_h$, it follows that the $(T + \mathbb{C}E)$-invariant complement of L_0 contains the elements $\xi_i\xi_h$ for every $i, h = 1, \dots, n$. Arguing inductively, since $[x_{i_1}, \xi_{i_1} \dots \xi_{i_h}] = \xi_{i_2} \dots \xi_{i_h}$, one shows that L_0 cannot contain any element lying in the negative part of the grading of type $(1, \dots, 1|0, \dots, 0)$, therefore L_0 is contained in the maximal graded subalgebra of L of type $(1, \dots, 1|0, \dots, 0)$, thus L_0 coincides with this graded subalgebra, by maximality;

(2) $x_1, \dots, x_k$ do not lie in L_0 for some $k = 2, \dots, n$, and $x_{k+1}, \dots, x_n$ lie in L_0. Then the $(T + \mathbb{C}E)$-invariant complement of L_0 contains the elements $\xi_h P$ for $h = 1, \dots, n$ and $P \in \Lambda(\xi_{k+1}, \dots, \xi_n)$. Likewise, since $[x_{i_1}, x_j\xi_{i_1} \dots \xi_{i_h}] = x_j\xi_{i_2} \dots \xi_{i_h}$, the $(T + \mathbb{C}E)$-invariant complement of L_0 contains the elements $x_j P$, for $j = 1, \dots, k$ and $P \in \Lambda(\xi_{k+1}, \dots, \xi_n)$. Therefore the $(T + \mathbb{C}E)$-invariant complement of L_0 contains the $(T + \mathbb{C}E)$-invariant complement of the graded subalgebra of type $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0)$ with $n - k$ 2's and $n - k$ zeros. Hence L_0 coincides with this graded subalgebra of L.

Note that any open regular subalgebra of L containing $x_2, \dots, x_n$ and not containing x_1, is not a maximal subalgebra of L. Indeed any such a subalgebra is contained in the graded subalgebra of type $(1, 2, \dots, 2|1, 0, \dots, 0)$ which is not maximal by Remark 2.35.

By Remark 2.38, the subalgebras of principal and subprincipal type of $SHO(3,3)$ are conjugate by an element of G. □

Corollary 2.43.

(a) *All irreducible $\mathbb{Z}$-gradings of $HO(n,n)$ and of $SHO(n,n)$ with $n > 3$, are, up to conjugation, the gradings of type* $(1, \dots, 1, 2, \dots, 2|\ 1, \dots, 1, 0, \dots, 0)$ *with k 2's and k zeros, for* $k = 0, \dots, n-2$ *and the grading of type* $(1, \dots, 1|0, \dots, 0)$.

(b) *All irreducible $\mathbb{Z}$-gradings of $SHO(3,3)$ are, up to conjugation, the gradings of type* $(1,1,1|1,1,1)$ *and* $(1,1,2|1,1,0)$.

Remark 2.44. By Remark 1.3, the proof of Theorem 2.41 works verbatim if we replace $L = HO(n,n)$ with $Der\, L$ and L_0 with a fundamental maximal subalgebra of $Der\, L$. Therefore every fundamental maximal subalgebra of $Der\, L$ is conjugate to the standard subalgebra of $Der\, L$ stabilizing the ideal $I_{\mathcal{U}} = (x_1, \dots, x_n, \xi_1, \dots, \xi_s)$ of $\Lambda(n,n)$, for some $s = 0, \dots, n$.

Theorem 2.45. *Let $L = HO(n,n)$. Then all maximal among E-invariant subalgebras of L are, up to conjugation, the subalgebras of L listed in Theorem* 2.42(a).

Proof. By Remark 2.44 every fundamental maximal subalgebra of $Der\,L$ is conjugate to the standard subalgebra of $Der\,L$ stabilizing the ideal $I_{\mathcal{U}} = (x_1, \ldots, x_n, \xi_1, \ldots, \xi_s)$ of $\Lambda(n,n)$, for some $s = 0, \ldots, n$. Therefore, by Remarks 2.34–2.36, all fundamental maximal subalgebras of $Der\,L$ are, up to conjugation, the subalgebras $L_0 + \mathbb{C}E$ where L_0 is one of the maximal open subalgebras of L listed in Theorem 2.42(a). If S_0 is a maximal among open E-invariant subalgebras of L, then $S_0 + \mathbb{C}E$ is a fundamental maximal subalgebra of $Der\,L$ and the thesis follows. □

As in the case of the Lie superalgebra $S(1,n)$, we are now interested in the subalgebras of $SHO(n,n)$ which are maximal among its $\mathfrak{a}_0$-invariant subalgebras, for any subalgebra $\mathfrak{a}_0$ of the subalgebra $\mathfrak{a}$ of outer derivations of $SHO(n,n)$.

Remark 2.46. The same arguments as in Theorem 2.11 show that every maximal open subalgebra of $SHO'(n,n)$ and $CSHO'(n,n)$ is regular. Therefore the same arguments as for $SHO(n,n)$ in Theorem 2.42, show that all fundamental among maximal subalgebras of $SHO'(n,n)$ or $CSHO'(n,n)$ with $n \geqslant 3$, are, up to conjugation, the graded subalgebras of type $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0)$ with k 0's and k 2's, for some $k = 0, \ldots, n-2$. Indeed the graded subalgebra of $SHO'(n,n)$ (respectively $CSHO'(n,n)$) of type $(1, \ldots, 1|0, \ldots, 0)$ is not maximal, since it is contained in $SHO(n,n)$ (respectively $SHO(n,n) + \mathbb{C}\Phi + \mathbb{C}E$). Notice that the graded subalgebras of principal and subprincipal type of $SHO'(3,3)$ (respectively $CSHO'(3,3)$) are not conjugate. By the same arguments, all maximal among fundamental subalgebras of $SHO'(n,n)$ and $CSHO'(n,n)$ are, up to conjugation, the graded subalgebra of type $(1, \ldots, 1|0, \ldots, 0)$ and the graded subalgebras of type $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0)$ with k 0's and k 2's, for some $k = 0, \ldots, n-2$.

Theorem 2.47. *Let $L = SHO(n,n)$ with $n \geqslant 4$.*

- (i) *If $\mathfrak{a}_0$ is a torus of $\mathfrak{a}$, then all maximal among open $\mathfrak{a}_0$-invariant subalgebras of L are, up to conjugation, the graded subalgebra of type $(1, \ldots, 1|0, \ldots, 0)$ and the graded subalgebras of type $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0)$ with k 0's and k 2's, for some $k = 0, \ldots, n-2$.*
- (ii) *If $\mathfrak{a}_0 = \mathbb{C}\xi_1 \ldots \xi_n \rtimes \mathfrak{t}$, where $\mathfrak{t}$ is a torus of $\mathfrak{a}$, then all maximal among open $\mathfrak{a}_0$-invariant subalgebras of L are, up to conjugation, the graded subalgebras of type $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0)$, with k 2's and k zeros, for $k = 0, \ldots, n-2$.*

Proof. One uses Remark 2.46 and the same arguments as in the proof of Theorem 2.17. □

Theorem 2.48. *Let $L = SHO(3,3)$ and let $\mathfrak{b} = \mathbb{C}e + \mathbb{C}h \subset \mathfrak{a} \cong gl_2$.*

- (i) *If $\mathfrak{a}_0$ is a one-dimensional subalgebra of $\mathfrak{a}$ or a two-dimensional torus of $\mathfrak{a}$, then all maximal among open $\mathfrak{a}_0$-invariant subalgebras of $SHO(3,3)$ are, up to conjugation, the subalgebras of type $(1,1,1|1,1,1)$ and $(1,1,2|1,1,0)$.*
- (ii) *If $\mathfrak{a}_0 = \mathbb{C}e \rtimes \mathfrak{t}$, where $\mathfrak{t}$ is a torus of $\mathfrak{a}$, then the graded subalgebra of type $(1,1,1|1,1,1)$ is, up to conjugation, the only maximal among open $\mathfrak{a}_0$-invariant subalgebras of $SHO(3,3)$, which is not invariant with respect to $\mathfrak{a}$.*
- (iii) *If $\mathfrak{a}_0 = sl_2$ or $\mathfrak{a}_0 = \mathfrak{a}$, then all maximal among open $\mathfrak{a}_0$-invariant subalgebras of $SHO(3,3)$ are, up to conjugation, the subalgebras of type $(1,1,2|1,1,0)$ and $(2,2,2|1,1,1)$.*

Proof. By Remark 2.46, the proof of (i) is the same as the proof of (i) and (ii) in Theorem 2.17. Recall that the graded subalgebras of principal and subprincipal type of $SHO(3,3)$ are conjugate.

Now, using Remark 2.37, one verifies that the graded subalgebras of $SHO(3,3)$ of type $(1,1,2|1,1,0)$ and $(2,2,2|1,1,1)$ are invariant with respect to $\mathfrak{a}$ (see also [17, Example 5.5], [10, Remark 4.4.1]). On the other hand the maximal graded subalgebra L_0 of $SHO(3,3)$ of type $(1,1,1|1,1,1)$ is invariant with respect to the action of h, e and E, but it is not $\mathfrak{a}$-invariant, indeed: $\xi_i\xi_j \in L_0$, $x_k \notin L_0$ and $f(\xi_i\xi_j) = \pm\frac{4}{3}x_k$ with $k \neq i, j$.

Let S_0 be a maximal among open $\mathfrak{b}$-invariant subalgebras of $SHO(3,3)$, then $S_0 + \mathbb{C}\xi_1\xi_2\xi_3 + \mathbb{C}\sum_{i=1}^3 x_i\xi_i$ is a fundamental subalgebra of $CSHO'(3,3)$, hence it is contained in a maximal among fundamental subalgebras of $CSHO'(3,3)$ containing $\xi_1\xi_2\xi_3$ and $\sum_{i=1}^3 x_i\xi_i$. It follows, by Remark 2.46, that S_0 is conjugate either to the graded subalgebra of type $(1,1,1|1,1,1)$ or to the subalgebra of type $(1,1,2|1,1,0)$. A similar argument holds if S_0 is maximal among open $\mathfrak{a}_0$-invariant subalgebras, with $\mathfrak{a}_0 = \mathbb{C}e + \mathfrak{t}$ where $\mathfrak{t}$ is a one-dimensional torus of $\mathfrak{a}$. Likewise, if S_0 is a maximal among open $\mathfrak{b} + \mathbb{C}E$-invariant subalgebras of $SHO(3,3)$, then $S_0 + \mathbb{C}\xi_1\xi_2\xi_3 + \mathbb{C}\sum_{i=1}^3 x_i\xi_i + \mathbb{C}E$ is a fundamental maximal subalgebra of $CSHO'(3,3)$, hence, by Remark 2.46, it is conjugate either to the graded subalgebra of type $(1,1,1|1,1,1)$ or to the subalgebra of type $(1,1,2|1,1,0)$.

Finally, let $\mathfrak{a}_0 = sl_2$ or $\mathfrak{a}_0 = \mathfrak{a}$, and let S' be a maximal among open $\mathfrak{a}_0$-invariant subalgebras of $SHO(3,3)$. Then S' is $\mathfrak{b}$-invariant, hence $S' + \mathfrak{b}$ is contained in a maximal among fundamental subalgebras of $CSHO'(3,3)$ containing $\mathfrak{b}$. It follows that S' is contained either in a conjugate of the subalgebra of type $(1,1,2|1,1,0)$, thus coincides with it by maximality, or in a conjugate S_U of the subalgebra of type $(1,1,1|1,1,1)$. As we noticed in Remark 2.38, S_U is conjugate to the subalgebra of principal type by an automorphism $\varphi = \exp(ad\, a)$ for some $a \in \mathfrak{a}$. Since S' is $\mathfrak{a}$-invariant, $\varphi(S') = S'$, therefore S' is contained in the intersection of S_U with the graded subalgebra of principal type, i.e., it is contained in the graded subalgebra of $SHO(3,3)$ of type $(2,2,2|1,1,1)$, thus coincides with it by maximality. □

3. Maximal open subalgebras of $H(2k,n)$

Let $p_1, \ldots, p_k, q_1, \ldots, q_k$ be $2k > 0$ even indeterminates and $\xi_1, \ldots, \xi_n$ be n odd indeterminates. Consider the differential form $\omega = 2\sum_{i=1}^k dp_i \wedge dq_i + \sum_{i=1}^n d\xi_i\, d\xi_{n-i+1}$. The Hamiltonian superalgebra $H(2k,n)$ is the Lie superalgebra defined as follows [15]:

$$H(2k,n) = \{X \in W(2k,n) \mid X\omega = 0\}.$$

Let us consider the Lie superalgebra $\Lambda(2k,n)$ with the following bracket:

$$[f,g] = \sum_{i=1}^k \left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right) - (-1)^{p(f)}\sum_{i=1}^n \frac{\partial f}{\partial \xi_i}\frac{\partial g}{\partial \xi_{n-i+1}}. \tag{3.1}$$

Then the map

$$\Lambda(2k,n) \to H(2k,n)$$
$$f \mapsto \sum_{i=1}^k \left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)}\sum_{i=1}^n \frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \xi_{n-i+1}}$$

defines a surjective homomorphism whose kernel consists of constant functions (cf. [10, §1.2]). We will therefore identify $H(2k,n)$ with $\Lambda(2k,n)/\mathbb{C}1$ with bracket (3.1). Consider the maximal torus $T = \langle p_iq_i, \xi_j\xi_{n-j+1} \mid i = 1, \ldots, k;\ j = 1, \ldots, [n/2]\rangle$ of $H(2k,n)$.

Remark 3.1. The $\mathbb{Z}$-grading of type $(a_1,\dots,a_{2k}|b_1,\dots,b_n)$ of $W(2k,n)$ induces a grading of $H(2k,n)$ if and only if the differential form ω is homogeneous in this grading (cf. [16]).

The $\mathbb{Z}$-grading of type $(1,\dots,1|2,\dots,2,1,\dots,1,0,\dots,0)$ of $W(2k,n)$, with t 2's and t zeros, induces an irreducible grading on $H(2k,n)$ for every t such that $0\leqslant t\leqslant [n/2]$, where $\mathfrak{g}_0\cong spo(2k,n-2t)\otimes\Lambda(t)+W(0,t)$ and $\mathfrak{g}_{-1}\cong\mathbb{C}^{2k|n-2t}\otimes\Lambda(t)$. One can check that when $t>0$, $[\mathfrak{g}_{-1},\mathfrak{g}_{-1}]\neq 0$ thus it coincides with $\mathfrak{g}_{-2}$ by Remark 1.13. Besides, property (iii)′ of Proposition 1.11(b) is satisfied. Therefore the subalgebras $\prod_{j\geqslant 0}H(2k,n)_j$ of $H(2k,n)$ corresponding to the gradings of type $(1,\dots,1|2,\dots,2,1,\dots,1,0,\dots,0)$, with t 2's and t zeros, for $0\leqslant t\leqslant[n/2]$, are maximal open subalgebras of $H(2k,n)$.

The $\mathbb{Z}$-grading of $H(2k,n)$ induced by the principal grading of $W(2k,n)$ is also called *principal*. The $\mathbb{Z}$-grading induced on $H(2k,2h)$ by the $\mathbb{Z}$-grading of type $(1,\dots,1|2,\dots,2,0,\dots,0)$, with h zeros, is called *subprincipal*.

Remark 3.2. The $\mathbb{Z}$-gradings of $H(2k,2h)$ of type $(1,\dots,1|2,\dots,2,\ 0,\dots,0)$ and $(1,\dots,1|2,\dots,2,0,2,0,\dots,0)$, with h zeros, are not conjugate by an element of G, but are conjugate by an outer automorphism.

Example 3.3. Let us identify $L=H(2k,n)$ with $\Lambda(2k,n)/\mathbb{C}1$. Let V be the n-dimensional odd vector space spanned by $\xi_1,\dots,\xi_n$, with the bilinear form $(\xi_i,\xi_j)=\delta_{i,n-j+1}$. Let us fix a subspace U of V and let us repeat the same construction as in Example 2.24.

We define a valuation on $\Lambda(2k,n)/\mathbb{C}1$ with values in $\mathbb{Z}_+$ by letting

$$\nu(p_i)=\nu(q_i)=1,$$
$$\nu(x)=2\quad\text{for } x\in U^0,\qquad \nu(x)=0\quad\text{for } x\in\left(U^0\right)',$$
$$\nu(x)=1\quad\text{for } x\in U^1,\qquad \nu(x)=0\quad\text{for } x\in\left(U^1\right)'.$$

Consider the following subspaces of L:

$$L_j(U)=\left\{x\in\Lambda(2k,n)/\mathbb{C}1\mid\nu(x)\geqslant j+2\right\}+\Lambda\left(\left(U^1\right)'\right)/\mathbb{C}1\quad\text{for } j\leqslant 0,$$
$$L_j(U)=\left\{x\in\Lambda(2k,n)/\mathbb{C}1\mid\nu(x)\geqslant j+2\right\}\quad\text{for } j>0.$$

These subsets define, in fact, a filtration of $H(2k,n)$ for every subspace U of V, as one can verify using the definition of bracket (3.1). Notice that this filtration has depth 1 if and only if U is non-degenerate, including $U=0$.

Let us denote by s the dimension of U and by s_i the dimension of U^i for $i=0,1$. Then $\overline{Gr\,L}\cong H(2k,n-r_1)\otimes\Lambda(r_1)+H(0,r_1)$ with respect to the grading of type $(1,\dots,1|2,\dots,2,1,\dots,1,0,\dots,0)$ of $H(2k,n-r_1)$, with s_0 2's and s_0 zeros, where $r_1=n-2s_0-s_1=\dim(U^1)'$, and $\deg(\tau)=0$ for every $\tau\in\Lambda(r_1)$. This is an irreducible grading of $H(2k,n-r_1)$ for every choice of U (cf. Remark 3.1), and, by Corollary 1.12, $L_0(U)$ is a maximal open subalgebra of L.

Let us consider the standard ideal $I_{\mathcal{U}}=(p_1,\dots,p_k,q_1,\dots,q_k,U)$ of $\Lambda(2k,n)$. Notice that $I_{\mathcal{U}}=\{x\in\Lambda(2k,n)/\mathbb{C}1\mid\nu(x)\geqslant 1\}$. It follows that $L_0(U)$ stabilizes $I_{\mathcal{U}}$ hence, due to its maximality, $L_0(U)$ is the standard subalgebra of $H(2k,n)$ corresponding to the ideal $I_{\mathcal{U}}$.

Remark 3.4. $L_0(U)$ is a maximal graded subalgebra of L if and only if U is a coisotropic subspace of V.

Remark 3.5. If U is conjugate to a subspace of V spanned by $\xi_{i_1},\dots,\xi_{i_t}$ for some $i_1,\dots,i_t$, then U is stable under the action of the maximal torus T. It follows that in this case $L_0(U)$ is regular. If n is odd, then any subspace of V is conjugate to $\langle\xi_{i_1},\dots,\xi_{i_t}\rangle$ for some $i_1,\dots,i_t$, and the same holds when n is even for any subspace of V whose non-degenerate part has even dimension.

Remark 3.6. If n is even and s_1 is odd then any maximal torus of L has dimension $k+n/2$ and any maximal torus of $\overline{Gr\,L}$ has dimension $k+n/2-1$. It follows that, under these hypotheses, $L_0(U)$ is not a regular subalgebra of L. For example any one-dimensional non-degenerate subspace U of V gives rise to a maximal open subalgebra $L_0(U)$ of $H(2k,2t)$ which is not regular.

Lemma 3.7. *Let us consider an ideal $J=(h_1,\dots,h_r)$ of $\Lambda(0,n)$. Suppose that $h_1=\eta_1+F$ and $h_2=\eta_1'+G$ where η_1, η_1' are non-degenerately paired elements of V and F, G contain no constant and linear terms. Then J is conjugate to an ideal $K=(\eta_1,\eta_1',f_1,\dots,f_{r-2})$ for some functions $f_i\in\Lambda(U)$ where U is the orthogonal complement of $\langle\eta_1,\eta_1'\rangle$ in V.*

Proof. Up to multiplying h_1 by some invertible function we can assume that F does not depend on η_1, i.e., $\eta_1+F=\eta_1+f_1\eta_1'+f_2$ where f_1, f_2 lie in $\Lambda(U)$. Also, we can assume that G lies in $\Lambda(U_1)$ where $U_1=\langle U,\eta_1\rangle$. Notice that $f_1\eta_1'+f_1G$ lies in J, therefore $J=(\eta_1+f_2-f_1G,\eta_1'+G,h_3,\dots,h_r)$ where $f_2-f_1G\in\Lambda(U_1)$. Therefore, up to multiplying $\eta_1+f_2-f_1G$ by an invertible function, we can write $J=(\eta_1+F',\eta_1'+G,h_3,\dots,h_r)$ where $F'\in\Lambda(U)$.

Now the automorphism $\exp(ad(\eta_1'F'))$ maps J to the ideal $J'=(\eta_1,\eta_1'+H,h_3',\dots,h_r')$ where the h_i''s lie in $\Lambda(U_1)$ and H lies in $\Lambda(U)$. Then, similarly as above, the automorphism $\exp(ad(\eta_1H))$, maps J' to the ideal $K=(\eta_1,\eta_1',f_1,\dots,f_{r-2})$, since $H\in\Lambda(U)$. Since η_1, η_1' lie in K, we can assume $f_1,\dots,f_{r-2}\in\Lambda(U)$. □

Remark 3.8. Notice that if $\eta_1\in V$ is non-degenerately paired with itself, one can prove, arguing as in the proof of Lemma 3.7, that if the ideal J contains an element of the form η_1+F, then, up to automorphisms, $J=(\eta_1,f_1,\dots,f_{r-1})$ where the f_i's lie in $\Lambda(U)$, U being the orthogonal complement of $\langle\eta_1\rangle$ in V.

Theorem 3.9. *Let L_0 be a maximal open subalgebra of $L=H(2k,n)$. Then L_0 is conjugate to a standard subalgebra of L.*

Proof. By Remark 1.3 L_0 stabilizes an ideal of the form

$$J=(p_1+f_1,q_1+h_1,\dots,p_k+f_k,q_k+h_k,\eta_1+g_1,\eta_2+g_2,\dots,\eta_r+g_r)$$

for some linear functions η_j in odd indeterminates and even functions f_i, h_i and odd functions g_j without constant and linear terms, and this ideal is maximal among the L_0-invariant ideals of $\Lambda(2k,n)$. As in Lemma 2.29 we can assume f_i, h_i and g_j in $\Lambda(0,n)$.

Note that the automorphism $\exp(ad(q_1 f_1))$ maps J to $J_1 = (p_1, q_1 + h'_1, p_2 + f'_2, q_2 + h'_2, \ldots, p_k + f'_k, q_k + h'_k, \eta_1 + g'_1, \eta_2 + g'_2, \ldots, \eta_r + g'_r)$. As above we can assume h'_1 independent of all even variables. It follows that the automorphism $\exp(ad(-p_1 h'_1))$ maps J_1 to $J_2 = (p_1, q_1, p_2 + f''_2, q_2 + h''_2, \ldots, p_k + f''_k, q_k + h''_k, \eta_1 + g''_1, \eta_2 + g''_2, \ldots, \eta_r + g''_r)$. The same arguments applied to all generators $p_i + f''_i$ and $q_j + h''_j$ show that J is in fact conjugate to the ideal

$$I = (p_1, p_2, \ldots, p_k, q_1, \ldots, q_k, \eta_1 + \ell_1, \eta_2 + \ell_2, \ldots, \eta_r + \ell_r)$$

where $\eta_1, \ldots, \eta_r$ are linearly independent vectors in V and $\ell_1, \ldots, \ell_r$ are functions in $\Lambda(0, n)$ without constant and linear terms. Since from now on we shall work only with odd indeterminates, with an abuse of notation we shall simply write

$$I = (\eta_1 + \ell_1, \eta_2 + \ell_2, \ldots, \eta_r + \ell_r).$$

Let $U = \langle \eta_1, \ldots, \eta_r \rangle \subset V$. Let $U^0 = \langle \nu_1, \ldots, \nu_s \rangle$ be the kernel of the restriction of the bilinear form $(\cdot,\cdot)$ to U. Then, as in Example 3.3, $U = U^0 \oplus U^1$ where U^1 is a maximal subspace of U with non-degenerate metric. Then, by Lemma 3.7 and Remark 3.8, $I = (U^1, \nu_1 + \ell_1, \ldots, \nu_s + \ell_s)$ where $\ell_1, \ldots, \ell_s \in \Lambda((U^1)^\perp)$. In particular, $(U^1)^\perp$ contains U^0 and a subspace $(U^0)'$ non-degenerately paired with U^0. Let $(U^0)' = \langle \nu'_1, \ldots, \nu'_s \rangle$ with $(\nu_i, \nu'_j) = \delta_{i,j}$.

Now, if $\ell_i = 0$ for every $i = 1, \ldots, s$, then I is standard. Suppose that at least one of the ℓ_j's is not zero, i.e.,

$$I = \big(U^1, \nu_1, \ldots, \nu_{k-1}, \nu_k + \ell_k, \ldots, \nu_s + \ell_s\big)$$

with $k = \min\{i = 1, \ldots, s \mid \ell_i \neq 0\}$.

Denote by I' the ideal $I' = (\nu_1, \ldots, \nu_{k-1}, \nu_k + \ell_k, \ldots, \nu_s + \ell_s) \subset I$. Then, each function f in L_0 (thus stabilizing I) stabilizes the ideal $K = (I, [I', I'])$. Indeed, for every $g, h \in I'$ we have:

$$\big[f, [g, h]\big] = \big[[f, g], h\big] \pm \big[g, [f, h]\big] \in [I, I']$$

and $[I, I'] \subset K$ since every generator of I' is orthogonal to U^1. Notice that K is generated by the generators of I and by the brackets between every pair of generators of I'. Therefore K is a proper ideal of $\Lambda(0, n)$ since among its generators there is no invertible element. By the maximality of I among the ideals stabilized by L_0 we have $I = K$.

We first show that the function ℓ_k can be made independent of $\nu'_1, \ldots, \nu'_{k-1}$. Indeed, let $\nu_k + \ell_k = \nu_k + \nu'_1\phi_1 + \phi_2$ where ϕ_1, ϕ_2 do not depend on ν'_1. Then $\phi_1 = [\nu_1, \nu_k + \ell_k] \in [I', I'] \subset K = I$, thus $I = (U^1, \nu_1, \ldots, \nu_{k-1}, \nu_k + \phi_2, \nu_{k+1} + \ell_{k+1}, \ldots, \nu_s + \ell_s)$, where $\phi_2 \in \Lambda((U^1)^\perp)$ does not depend on ν'_1. Arguing in the same way with the variables $\nu'_2, \ldots, \nu'_{k-1}$ we get

$$I = \big(U^1, \nu_1, \ldots, \nu_{k-1}, \nu_k + \phi, \nu_{k+1} + \ell_{k+1}, \ldots, \nu_s + \ell_s\big)$$

where ϕ does not depend on $\nu'_1, \ldots, \nu'_{k-1}$.

Besides, multiplying $\nu_k + \phi$ by an invertible function, we can assume that ϕ does not depend on ν_k. Now we can write $\phi = \nu'_k\psi_1 + \psi_2$ with ψ_1, ψ_2 not depending on $\nu'_1, \ldots, \nu'_k$. Therefore, applying the automorphism $\exp(ad(\nu_k'\psi_2))$ to I, we can assume $\psi_2 = 0$. Then $\psi_1 = \frac{1}{2}[\nu_k + \phi, \nu_k + \phi] \in [I', I'] \subset K = I$. Therefore

$$I = (U^1, \nu_1, \ldots, \nu_{k-1}, \nu_k, \nu_{k+1} + \ell_{k+1}, \ldots, \nu_r + \ell_r).$$

Arguing as above for $\ell_{k+1}, \ldots, \ell_r$, we end up with a standard ideal. □

Theorem 3.10. *All maximal open subalgebras of $L = H(2k, n)$ are, up to conjugation, the following*:

(a) *if $n = 2h + 1$*:
   (i) *the $\mathbb{Z}$-graded subalgebras of type* $(1, \ldots, 1|2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$ *with t 2's and t zeros for $0 \leqslant t \leqslant h$*;
   (ii) *the regular (non-graded) subalgebras $L_0(U)$ constructed in Example* 3.3 *where U is not coisotropic*;

(b) *if $n = 2h$*:
   (i) *the $\mathbb{Z}$-graded subalgebras of type* $(1, \ldots, 1|2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$ *with t 2's and t zeros for $0 \leqslant t \leqslant h$, and the $\mathbb{Z}$-graded subalgebra of type* $(1, \ldots, 1|2, \ldots, 2, 0, 2, 0, \ldots, 0)$ *with h zeros*;
   (ii) *the regular (non-graded) subalgebras $L_0(U)$ constructed in Example* 3.3 *where U is not coisotropic and* $\dim U^1$ *is even*;
   (iii) *the non-regular subalgebras $L_0(U)$ constructed in Example* 3.3, *where* $\dim U^1$ *is odd.*

Proof. By Theorem 3.9, every maximal open subalgebra of L is conjugate to the standard subalgebra of L stabilizing the ideal $I_{\mathcal{U}}$ of $\Lambda(2k, n)$, for some subspace $\mathcal{U} = \langle p_1, \ldots, p_k, q_1, \ldots, q_k, U\rangle$ of $\sum_{i=1}^k (\mathbb{C}p_i + \mathbb{C}q_i) + \sum_{j=1}^n \mathbb{C}\xi_j$, where U is a subspace of V. Then the statement follows from Example 3.3 and Remarks 3.1, 3.4, 3.5 and 3.6. □

Corollary 3.11. *All irreducible $\mathbb{Z}$-gradings of $H(2k, n)$ are, up to conjugation, the $\mathbb{Z}$-gradings of type* $(1, \ldots, 1|2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$ *with t 2's and t zeros, for $t = 0, \ldots, [n/2]$, and the $\mathbb{Z}$-grading of type* $(1, \ldots, 1|2, \ldots, 2, 0, 2, 0, \ldots, 0)$ *with $n/2$ zeros if n is even.*

We recall that $Der\, H(2k, n) = H(2k, n) + \mathbb{C}E$ where $E = \sum_{i=1}^k (p_i \partial/\partial p_i + q_i \partial/\partial q_i) + \sum_{j=1}^n \xi_j \partial/\partial \xi_j$ is the Euler operator (cf. Proposition 1.8). We now aim to classify all fundamental maximal subalgebras of $Der\, H(2k, n)$.

Remark 3.12. All members of the filtration

$$H(2k, n) = L_{-d}(U) \supset \cdots \supset L_0(U) \supset \cdots$$

of $H(2k, n)$, constructed in Example 3.3, are invariant with respect to the Euler operator, for every choice of the subspace U. It follows that we can construct a filtration

$$Der\, L = L'_{-d}(U) \supset \cdots \supset L'_0(U) \supset \cdots$$

of $Der\, L$ by setting $L'_k(U) = L_k(U)$ for every $k \neq 0$, and $L'_0(U) = L_0(U) + \mathbb{C}E$. Then the completion of the graded Lie superalgebra associated to this filtration is isomorphic to $H(2k, n - r_1) \otimes \Lambda(r_1) + H(0, r_1) + \mathbb{C}(E_1 + E_2)$, with respect to the grading of type $(1, \dots, 1|2, \dots, 2, 1, \dots, 1, 0, \dots, 0)$ of $H(2k, n - r_1)$, with s_0 2's and s_0 zeros, where s_0 and r_1 are defined as in Example 3.3, and where E_1 and E_2 are the Euler operators of $H(2k, n - r_1)$ and $H(0, r_1)$, respectively. It follows that $L_0'(U)$ is a fundamental maximal subalgebra of $Der\, L$. By Remark 3.3, this is, in fact, the standard subalgebra of $Der\, L$ stabilizing the ideal $I_{\mathcal{U}} = (p_1, \dots, p_k, q_1, \dots, q_k, U)$.

Remark 3.13. The proof of Theorem 3.9 works verbatim if we replace $L = H(2k, n)$ with $Der\, L = H(2k, n) + \mathbb{C}E$. Therefore, every fundamental maximal subalgebra of $Der\, L$ is conjugate to a standard subalgebra.

Theorem 3.14. *Let $L = H(2k, n)$. Then all maximal among E-invariant subalgebras of L are, up to conjugation, the maximal open subalgebras of L listed in Theorem* 3.10.

Proof. By Remark 3.13 every fundamental maximal subalgebra of $Der\, L$ is conjugate to a standard subalgebra. Therefore, by Remark 3.12, all maximal fundamental subalgebras of $Der\, L$ are, up to conjugation, the subalgebras $L_0 + \mathbb{C}E$, where L_0 is one of the maximal open subalgebras of L listed in Theorem 3.10. Let S_0 be a maximal among open E-invariant subalgebras of L. Then $S_0 + \mathbb{C}E$ is a fundamental maximal subalgebra of $Der\, L$. Hence the thesis. □

4. Maximal open subalgebras of $KO(n, n+1)$ and $SKO(n, n+1; \beta)$

Let $x_1, \dots, x_n$ be n even indeterminates and $\xi_1, \dots, \xi_n, \xi_{n+1} = \tau$ be $n+1$ odd indeterminates. Consider the differential form $\Omega = d\tau + \sum_{i=1}^n (\xi_i\, dx_i + x_i\, d\xi_i)$. The odd contact superalgebra is defined as follows [1]:

$$KO(n, n+1) = \{X \in W(n, n+1) \mid X\Omega = f\Omega,\ f \in \Lambda(n, n+1)\}.$$

It is a simple Lie superalgebra for every $n \geqslant 1$.

Define the following bracket on $\Lambda(n, n+1)$ (cf. [10, §1.4]):

$$[f, g] = (2 - E) f \frac{\partial g}{\partial \tau} + (-1)^{p(f)} \frac{\partial f}{\partial \tau} (2 - E) g - \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial x_i} \right) \tag{4.1}$$

where $E = \sum_{i=1}^n (x_i \partial/\partial x_i + \xi_i \partial/\partial \xi_i)$ is the Euler operator. Then the map $\rho : \Lambda(n, n+1) \to KO(n, n+1)$,

$$\rho(f) = X_f := (2 - E) f \frac{\partial}{\partial \tau} - (-1)^{p(f)} \frac{\partial f}{\partial \tau} E - \sum_{i=1}^n \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial x_i} \right)$$

is an isomorphism between $KO(n, n+1)$ and $\Lambda(n, n+1)$ with reversed parity. We will therefore identify $KO(n, n+1)$ with $\Lambda(n, n+1)$ with reversed parity. Then the standard maximal torus is $T = \langle \tau, x_i \xi_i \mid i = 1, \dots, n \rangle$.

Remark 4.1. Bracket (4.1) satisfies the following rule:

$$[f, gh] = [f, g]h + (-1)^{p(X_f)p(g)} g[f, h] - 2(-1)^{p(f)} \frac{\partial f}{\partial \tau} gh.$$

It follows, in particular, that an ideal $I = (f_1, \dots, f_r)$ of $\Lambda(n, n+1)$ is stabilized by a function f in $KO(n, n+1)$ if and only if $[f, f_i]$ lies in I for every $i = 1, \dots, r$.

Besides, if f is an odd function independent of τ, then $\varphi = \exp(ad(f))$ is an automorphism of $\Lambda(n, n+1)$ with respect to both the Lie bracket and the usual product of polynomials. It follows that a subalgebra L_0 of $KO(n, n+1)$ stabilizes an ideal $I = (f_1, \dots, f_r)$ of $\Lambda(n, n+1)$ if and only if the subalgebra $\varphi(L_0)$ stabilizes the ideal $J = (\varphi(f_1), \dots, \varphi(f_r))$.

For $\beta \in \mathbb{C}$ let $div_\beta := \Delta + (E - n\beta)\partial/\partial\tau$, where $\Delta = \sum_{i=1}^n \partial^2/(\partial x_i \partial \xi_i)$ is the odd Laplace operator, and let

$$SKO'(n, n+1; \beta) = \{f \in \Lambda(n, n+1) \mid div_\beta(f) = 0\} =: \Lambda^\beta(n, n+1)$$

(cf. [19], [17, Example 4.9], [10, §1.4]).

Remark 4.2. If $f, g \in \Lambda(n, n+1)$, then:

$$div_\beta([f, g]) = X_f(div_\beta(g)) - (-1)^{p(X_f)p(X_g)} X_g(div_\beta(f)).$$

It follows that the function $div_\beta : KO(n, n+1) \to \Lambda(n, n+1)$ is a divergence (see Definition 2.4). Therefore $SKO'(n, n+1; \beta)$ is a subalgebra of the Lie superalgebra $\Lambda(n, n+1)$ with bracket (4.1). According to Remark 2.6,

$$SKO'(n, n+1; \beta) = S'KO(n, n+1) = \{X \in KO(n, n+1) \mid X\omega_\beta = 0\}$$

where ω_β is the volume form attached to the divergence div_β.

Let $SKO(n, n+1; \beta)$ denote the derived algebra of $SKO'(n, n+1; \beta)$. Then $SKO(n, n+1; \beta)$ is simple for $n \geqslant 2$ and coincides with $SKO'(n, n+1; \beta)$ unless $\beta = 1$ or $\beta = (n-2)/n$. The Lie superalgebra $SKO(n, n+1; 1)$ (respectively $SKO(n, n+1; (n-2)/n)$) consists of the elements of $SKO'(n, n+1; 1)$ (respectively $SKO'(n, n+1; (n-2)/n)$) not containing the monomial $\tau\xi_1 \dots \xi_n$ (respectively $\xi_1 \dots \xi_n$).

Since the Lie superalgebra $KO(1, 2)$ is isomorphic to the Lie superalgebra $W(1, 1)$ (cf. [17, Remark 6.6]), and since $SKO(n, n+1; \beta)$ is simple for $n \geqslant 2$, when talking about $KO(n, n+1)$ and $SKO(n, n+1; \beta)$ we shall assume $n \geqslant 2$.

Remark 4.3. The $\mathbb{Z}$-grading of type $(1, \dots, 1|0, \dots, 0, 1)$ of $W(n, n+1)$ induces on $KO(n, n+1)$ (respectively $SKO(n, n+1; \beta)$) a grading of depth 1 which is irreducible by Remark 1.13. This grading is called the *subprincipal* grading of $KO(n, n+1)$ (respectively $SKO(n, n+1; \beta)$).

The $\mathbb{Z}$-grading of type $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0, 2)$ of $W(n, n+1)$, with $t+1$ 2's and t zeros, induces, for every $t = 0, \dots, n-2$, an irreducible grading on $\mathfrak{g} = KO(n, n+1)$ (respectively $SKO(n, n+1; \beta)$ for $(t, \beta) \neq (n-2, (n-2)/n)$) where $\mathfrak{g}_0$ is isomorphic to the Lie superalgebra $c\tilde{P}(n-t) \otimes \Lambda(t) + W(0, t)$ (respectively $\tilde{P}(n-t) \otimes \Lambda(t) + W(0, t)$), $\mathfrak{g}_{-1}$ is isomorphic to $\mathbb{C}^{n-t|n-t} \otimes \Lambda(t)$ and $\mathfrak{g}_{-2}$ is isomorphic to $\mathbb{C} \otimes \Lambda(t)$. When $\mathfrak{g} = SKO(n, n+1; (n-2)/n)$ and $t = n-2$, $\mathfrak{g}_0$ does not contain the element $\xi_1 \dots \xi_n$, and the grading is irreducible if and only if $n > 2$. These gradings satisfy the hypotheses of Proposition 1.11(b), therefore the corresponding graded subalgebras of $KO(n, n+1)$ and $SKO(n, n+1; \beta)$ are maximal.

The grading of type $(1,\dots,1|1,\dots,1,2)$ is called the *principal* grading of $KO(n,n+1)$ (respectively $SKO(n,n+1;\beta)$).

Remark 4.4. The $\mathbb{Z}$-grading of type $(1,2,\dots,2|1,0,\dots,0,2)$ of $W(n,n+1)$ induces on $KO(n,n+1)$ (respectively $SKO(n,n+1;\beta)$) a grading which is not irreducible. In fact the corresponding subalgebra $\prod_{j\geqslant 0}\mathfrak{g}_j$ of $KO(n,n+1)$ (respectively $SKO(n,n+1;\beta)$) is contained in the subalgebra of type $(1,\dots,1|0,\dots,0,1)$.

Remark 4.5. The subspaces $\mathbb{C}1$, $\mathbb{C}x_i$, $\mathbb{C}\xi_{i_1}\dots\xi_{i_h}$, $\mathbb{C}x_k\xi_{j_1}\dots\xi_{j_h}$, with $k\neq j_1,\dots,j_h$, $\mathbb{C}\xi_{i_1}\dots\xi_{i_h}\otimes T$, and $\mathbb{C}x_k\xi_{j_1}\dots\xi_{j_h}\otimes T$ with $k\neq j_1,\dots,j_h$, are T-weight spaces of $KO(n,n+1)$.

Remark 4.6. Let $L=KO(n,n+1)$. Then the graded subalgebra L_k of L of type $(1,\dots,1,2,\dots,2|1,\dots,1,0,\dots,0,2)$ with $n-k+1$ 2's and $n-k$ zeros, is, for every $k=1,\dots,n$, the standard subalgebra L_U of L stabilizing the ideal $I_U=(x_1,\dots,x_n,\xi_1,\dots,\xi_k,\tau)$. Indeed, for every k, $L_k\subset L_U$ since L_k is contained in the graded subalgebra of $W(n,n+1)$ of type $(1,\dots,1|1,\dots,1,0,\dots,0,1)$ with $n-k$ zeros, which stabilizes I_U (cf. the proof of Theorem 2.3). Since, for every $k\neq 1$, L_k is a maximal subalgebra of L (cf. Remark 4.3), $L_k=L_U$.

Now suppose $k=1$. Notice that L_U contains the standard torus T of $KO(n,n+1)$, hence it is regular and decomposes into the direct product of T-weight spaces. The subspace $S=\langle 1,x_1,\xi_1\rangle\otimes\Lambda(\xi_2,\dots,\xi_n)$ is a T-invariant complementary subspace of the subalgebra L_1 and, according to Remark 4.5, the subspaces $\mathbb{C}1$, $\mathbb{C}\xi_{j_1}\dots\xi_{j_h}$, $\mathbb{C}x_1\xi_{i_1}\dots\xi_{i_h}$, with $1\neq i_1\neq\dots\neq i_h$, are one-dimensional T-weight spaces. Therefore, in order to prove that $L_U\subset L_1$, it is sufficient to show that no element of S lies in L_U. Notice that L_U contains the elements $x_2,\dots,x_n$ but it does not contain the elements 1, x_1, ξ_i for any $i=1,\dots,n$. Since $[x_{i_1},\xi_{i_1}\dots\xi_{i_h}]=-\xi_{i_2}\dots\xi_{i_h}$ and $[x_{i_1},x_j\xi_{i_1}\dots\xi_{i_h}]=-x_j\xi_{i_2}\dots\xi_{i_h}$, it follows that S is contained in the T-invariant complementary subspace of L_U, therefore $L_U\subset L_1$, hence $L_U=L_1$.

Likewise, the graded subalgebra of L of type $(1,\dots,1|0,\dots,0,1)$ is the standard subalgebra of L stabilizing the ideal $(x_1,\dots,x_n,\tau)$.

Example 4.7. Throughout this example we shall identify $KO(n,n+1)$ with $\Lambda(n,n+1)$ as described at the beginning of this paragraph. On $\Lambda(n,n+1)$ we define a valuation with values in $\mathbb{Z}_+$ by setting:

$$\nu(x_i)=1,\quad \nu(\xi_i)=0,\quad i=1,\dots,n,\qquad \nu(\tau)=0$$

(see Remark 2.23). Consider the following subspaces of $KO(n,n+1)$:

$$L_i=\{f\in\Lambda(n,n+1)\mid \nu(f)\geqslant i+1\}+\Lambda(\tau)\quad\text{for } i=-1,0,$$
$$L_i=\{f\in\Lambda(n,n+1)\mid \nu(f)\geqslant i+1\}\quad\text{for } i>0.$$

Using commutation rules (4.1) one can check that the subspaces L_i define in fact a filtration of $KO(n,n+1)$ of depth 1 whose associated graded superalgebra $Gr\,L$ has the following structure:

$$Gr_{-1}L=\Lambda(\xi_1,\dots,\xi_n,\tau)/\Lambda(\tau),$$
$$Gr_0L=\langle x_i\rangle\otimes\Lambda(\xi_1,\dots,\xi_n,\tau)+\Lambda(\tau),$$
$$Gr_jL=\langle f\in\mathbb{C}[[x_1,\dots,x_n]]\mid \deg(f)=j+1\rangle\otimes\Lambda(\xi_1,\dots,\xi_n,\tau)\quad\text{for } j\geqslant 1.$$

It follows that $\overline{Gr\,L} \cong HO(n,n) \otimes \Lambda(\eta) + \mathbb{C}\partial/\partial\eta + \mathbb{C}(E - 2 + 2\eta\partial/\partial\eta)$ with respect to the grading of type $(1,\dots,1|0,\dots,0)$ of $HO(n,n)$, where $E = \sum_{i=1}^n (x_i\partial/\partial x_i + \xi_i\partial/\partial\xi_i)$, and $\deg(\eta) = 0$. Since this grading is irreducible (cf. Remark 2.34) and satisfies property (iii)′ of Proposition 1.11(b), L_0 is a maximal subalgebra of $KO(n, n+1)$ by Corollary 1.12.

Note that the subalgebra L_0 stabilizes the ideal $I_U = (x_1,\dots,x_n)$ of $\Lambda(n,n)$, hence, due to its maximality, L_0 is the standard subalgebra of $KO(n, n+1)$ corresponding to the ideal I_U.

Example 4.8. Throughout this example we shall identify $KO(n, n+1)$ with $\Lambda(n, n+1)$ as above. Let us fix an integer t such that $1 \leqslant t \leqslant n$ and let us define a valuation on $\Lambda(n, n+1)$ by setting:

$$\nu(x_i) = 1, \quad \nu(\xi_i) = 1, \quad \text{for } i = 1,\dots,t,$$
$$\nu(\tau) = 0, \qquad \nu(x_i) = 2, \quad \nu(\xi_i) = 0, \quad \text{for } i = t+1,\dots,n.$$

Consider the following subspaces of $KO(n, n+1)$:

$$L_i(t) = \{ f \in \Lambda(n, n+1) \mid \nu(f) \geqslant i+2 \} + \Lambda(\tau) \quad \text{for } i \leqslant 0,$$
$$L_i(t) = \{ f \in \Lambda(n, n+1) \mid \nu(f) \geqslant i+2 \} \quad \text{for } i > 0.$$

Using commutation rules (4.1) one verifies that the subspaces $L_i(t)$ define in fact a filtration of $KO(n, n+1)$. This filtration has depth 1 if $t = n$, otherwise it has depth 2. We have: $\overline{Gr\,L} \cong HO(n,n) \otimes \Lambda(\eta) + \mathbb{C}\partial/\partial\eta + \mathbb{C}(E - 2 + 2\eta\partial/\partial\eta)$ with respect to the grading of type $(1,\dots,1,2,\dots,2|1,\dots,1,0,\dots,0)$ of $HO(n,n)$, with $n-t$ 2's and $n-t$ zeros, and $\deg(\eta) = 0$. Since this grading is irreducible for every $t = 2,\dots,n$ (cf. Remark 2.34) and satisfies property (iii)′ of Proposition 1.11(b), by Corollary 1.12 $L_0(t)$ is a maximal (regular) subalgebra of $KO(n, n+1)$ for every $t = 2,\dots,n$. On the contrary, the subalgebra $L_0(1)$ is contained in the subalgebra L_0 of $KO(n, n+1)$ constructed in Example 4.7, hence it is not maximal.

Note that $L_0(t)$ is contained in the graded subalgebra of $W(n, n+1)$ of type $(1,\dots,1|1,\dots,1,0,\dots,0)$ with $n+t$ 1's, therefore it stabilizes the ideal $I_{\mathcal{U}} = (x_1,\dots,x_n,\xi_1,\dots,\xi_t)$ of $\Lambda(n, n+1)$. It follows that, for every $t = 2,\dots,n$, $L_0(t)$ is the standard subalgebra of $KO(n, n+1)$ corresponding to the ideal $I_{\mathcal{U}}$, due to its maximality.

Likewise, $L_0(1)$ is the standard subalgebra $L_{\mathcal{U}_1}$ of $KO(n, n+1)$ stabilizing the ideal $I_{\mathcal{U}_1} = (x_1,\dots,x_n,\xi_1)$. Indeed, $L_{\mathcal{U}_1}$ contains the standard torus T of $KO(n, n+1)$, hence it is regular and decomposes into the direct product of T-weight spaces. By definition $L_{\mathcal{U}_1}$ contains the elements $1, x_2,\dots,x_n$ and does not contain the elements x_1 and ξ_j for any $j = 1,\dots,n$. Notice that $Gr_{<0} L := L_{-2}(1)/L_0(1) = (\langle 1, x_1, \xi_1\rangle \otimes \Lambda(\xi_2,\dots,\xi_n,\tau))/\Lambda(\tau)$. Then the same arguments as in Remark 4.6 show that no element in $(\langle 1, x_1, \xi_1\rangle \otimes \Lambda(\xi_2,\dots,\xi_n))/\mathbb{C}1$ lies in $L_{\mathcal{U}_1}$. Now suppose that an element of the form $\xi_{i_1}\dots\xi_{i_k}\tau + \varphi$ lies in $L_{\mathcal{U}_1}$ for some $\varphi \in \Lambda(n,n)$, where by $\Lambda(n,n)$ we mean the subalgebra of $\Lambda(n, n+1)$ generated by all even indeterminates and by the odd indeterminates except τ. Then $L_{\mathcal{U}_1}$ contains the element $[1, \xi_{i_1}\dots\xi_{i_k}\tau + \varphi] = \pm 2\xi_{i_1}\dots\xi_{i_k}$ and this is a contradiction. Therefore $L_{\mathcal{U}_1}$ cannot contain any element of the form $\xi_{i_1}\dots\xi_{i_k}\tau + \varphi$ for any function $\varphi \in \Lambda(n,n)$ and, similarly, it cannot contain any element of the form $x_1\xi_{i_1}\dots\xi_{i_k}\tau + \varphi$ for any $i_1 \neq \dots \neq i_k \neq 1$ and any function $\varphi \in \Lambda(n,n)$. By Remark 4.5 it follows that $L_{\mathcal{U}_1}$ is contained in $L_0(1)$, hence $L_{\mathcal{U}_1} = L_0(1)$.

Remark 4.9. Let $1 \leqslant i < j \leqslant n$. Then the change of indeterminates that leaves τ invariant and exchanges x_i with x_j and ξ_i with ξ_j, preserves the form Ω.

Remark 4.10. Let $\eta = \alpha_{i_1}\xi_{i_1} + \cdots + \alpha_{i_k}\xi_{i_k}$ for some $k \leqslant n$, with $\alpha_{i_j} \in \mathbb{C}$, $\alpha_{i_j} \neq 0$. According to Remark 4.9, up to changes of variables, we can assume $\eta = \alpha_1\xi_1 + \cdots + \alpha_k\xi_k$ with $\alpha_i \neq 0$ for $i = 1, \ldots, k$. Then the following change of indeterminates preserves the form Ω:

$$\begin{aligned}
&\tau' = \tau, \\
&x_1' = \tfrac{1}{\alpha_1}x_1, && \xi_1' = \eta, \\
&x_2' = x_2 - \tfrac{\alpha_2}{\alpha_1}x_1, && \xi_2' = \xi_2, \\
&\quad\vdots && \quad\vdots \\
&x_k' = x_k - \tfrac{\alpha_k}{\alpha_1}x_1, && \xi_k' = \xi_k, \\
&x_i' = x_i, && \xi_i' = \xi_i \quad \forall i > k.
\end{aligned}$$

Theorem 4.11. *Let L_0 be a maximal open subalgebra of $L = KO(n, n+1)$. Then L_0 is conjugate to a standard subalgebra of L.*

Proof. By Remark 1.3 L_0 stabilizes an ideal of the form

$$J = (x_1 + f_1, \ldots, x_n + f_n, \eta_1 + g_1, \ldots, \eta_s + g_s)$$

for some linear functions η_j in odd indeterminates, and even functions f_i and odd functions g_j without constant and linear terms, and J is maximal among the L_0-invariant ideals of $\Lambda(n, n+1)$.

We distinguish the following two cases:

Case 1. η_i lies in $\Lambda(\xi_1, \ldots, \xi_n)$ for every $i = 1, \ldots, s$. By Remark 4.10, up to changes of indeterminates, we have:

$$J = (x_1 + F_1, \ldots, x_n + F_n, \xi_1 + G_1, \ldots, \xi_s + G_s)$$

for some even functions F_i and odd functions G_j without constant and linear terms, where the functions G_j's are independent of $\xi_1, \ldots, \xi_s$, for every $j = 1, \ldots, s$ and where, since the ideal J is closed, we can assume the functions F_i and G_i independent of all even indeterminates, i.e., $F_i, G_i \in \Lambda(0, n+1)$ (cf. Lemma 2.29).

Suppose that $x_1 + F_1 = x_1 + \xi_1 F_1' + F_1''$ with F_1' and F_1'' independent of ξ_1. Then we can replace $x_1 + F_1$ by $x_1 + F_1 - (\xi_1 + G_1)F_1' = x_1 + H_1$ with H_1 independent of ξ_1. Similarly we can make every function F_i independent of ξ_j for every $j = 1, \ldots, s$.

Now suppose $x_1 + F_1 = x_1 + \tau\varphi_0 + \varphi_1$ with φ_0 and φ_1 independent of τ. Notice that, although the map $ad(\tau\xi_1\varphi_0)$ is not a derivation of $\Lambda(n, n+1)$ with respect to the usual product, the map $\psi := ad(\tau\xi_1\varphi_0) + 2\xi_1\varphi_0 id$ is a derivation, as one can verify using Remark 4.1. Thus $\exp(\psi)$ is an automorphism of $\Lambda(n, n+1)$ with respect both to bracket (4.1) and to the usual product. Notice that $\exp(\psi)(x_1 + F_1) = x_1 + \Phi_1$ for some function Φ_1 independent of τ. Thus, up to automorphisms, we can assume F_1 and, similarly, every function F_i, for every $i = 1, \ldots, s$, independent of τ. As a consequence, the map $\exp(ad(-\xi_1 F_1))$ is an automorphism of $\Lambda(n, n+1)$, mapping

J to the ideal

$$I = \big(x_1, x_2 + F'_2, \ldots, x_n + F'_n, \xi_1 + G_1, \ldots, \xi_s + G_s\big).$$

Arguing in the same way for every function F'_j with $1 \leqslant j \leqslant s$, we have, up to automorphisms,

$$I = (x_1, \ldots, x_s, x_{s+1} + h_{s+1}, \ldots, x_n + h_n, \xi_1 + G_1, \ldots, \xi_s + G_s)$$

for some functions $h_{s+1}, \ldots, h_n \in \Lambda(\xi_{s+1}, \ldots, \xi_n, \tau)$.

Suppose $G_1 = \tau\rho_0 + \rho_1$ with ρ_0, ρ_1 independent of τ. Then $\exp(ad(x_1\tau\rho_0) + 2x_1\rho_0 id)$ is an automorphism of $\Lambda(n, n+1)$ mapping the ideal I to

$$I' = \big(x_1, \ldots, x_s, x_{s+1} + h_{s+1}, \ldots, x_n + h_n, \xi_1 + \rho_1, \xi_2 + G'_2, \ldots, \xi_s + G'_s\big)$$

where ρ_1 is independent of τ. Arguing in the same way for every function G_j we can assume, up to automorphisms, that

$$I = (x_1, \ldots, x_s, x_{s+1} + h_{s+1}, \ldots, x_n + h_n, \xi_1 + \rho_1, \ldots, \xi_s + \rho_s)$$

where ρ_j lies in $\Lambda(\xi_{s+1}, \ldots, \xi_n)$ for every j. It follows that the map $\exp(ad(-x_1\rho_1))$ is an automorphism of L sending the ideal I to the ideal

$$Y = \big(x_1, \ldots, x_s, x_{s+1} + h'_{s+1}, \ldots, x_n + h'_n, \xi_1, \xi_2 + \rho'_2, \ldots, \xi_s + \rho'_s\big),$$

for some functions $h'_i \in \Lambda(\xi_{s+1}, \ldots, \xi_n, \tau)$, $\rho'_j \in \Lambda(\xi_{s+1}, \ldots, \xi_n)$. Analogous automorphisms yield to the ideal

$$Y' = \big(x_1, \ldots, x_s, x_{s+1} + h''_{s+1}, \ldots, x_n + h''_n, \xi_1, \ldots, \xi_s\big),$$

for some functions $h''_i \in \Lambda(\xi_{s+1}, \ldots, \xi_n, \tau)$.

Let $h''_{s+1} = \xi_{s+1}\psi_1 + \psi_2$ for some ψ_1, ψ_2 independent of ξ_{s+1}. By the same argument as above we can assume ψ_2 independent of τ and, applying the automorphism $\exp(ad(\xi_{s+1}\psi_2))$, we can assume $\psi_2 = 0$. Now the proof can be concluded as in the case of the Lie superalgebra $HO(n,n)$ (cf. Theorem 2.41). Namely, let $Y'' = (x_{s+1} + h_{s+1}, \ldots, x_n + h_n) \subset Y'$. Then, each function f in L_0 (thus stabilizing Y) stabilizes the ideal $K = (Y', [Y'', Y''])$, i.e., the ideal generated by the generators of Y' and by the commutators between every pair of generators of Y''. Indeed, for every $g, h \in Y''$ we have:

$$\big[f, [g,h]\big] = \big[[f,g], h\big] \pm \big[g, [f,h]\big] \in [Y', Y'']$$

and $[Y', Y''] \subset K$. Notice that K is a proper ideal of $\Lambda(2k+1, n)$ since among its generators there is no invertible element. By the maximality of J among the ideals stabilized by L_0 we have $Y' = K$.

Now $\frac{1}{2}[x_{s+1} + \xi_{s+1}\psi_1, x_{s+1} + \xi_{s+1}\psi_1] = -\psi_1 + \xi_{s+1}\tilde{\varphi} \in [Y'', Y''] \subset K = Y'$, therefore $(\psi_1 - \xi_{s+1}\tilde{\varphi})\xi_{s+1} = \psi_1\xi_{s+1}$ lies in Y'. It follows that

$$Y' = \big(x_1, \ldots, x_s, x_{s+1}, x_{s+2} + h'_{s+2}, \ldots, x_n + h'_n, \xi_1, \xi_2, \ldots, \xi_s\big).$$

Arguing in the same way for every function h'_j, we end up with a standard ideal.

Case 2. There exists one i such that $\eta_i = \tau + \eta$ with $\eta \in \Lambda(\xi_1, \dots, \xi_n)$, i.e., up to changes of indeterminates,

$$J = (x_1 + f_1, \dots, x_n + f_n, \xi_1 + g_1, \dots, \xi_{s-1} + g_{s-1}, \tau + \eta_s + g_s)$$

for some linear function η_s in $\Lambda(\xi_1, \dots, \xi_n)$, and even functions f_i and odd functions g_j without constant and linear terms. We can assume f_i, g_j and η_s in $\Lambda(\xi_s, \dots, \xi_n)$.

Besides, arguing similarly as above and as in the proof of Theorem 2.41, one shows that, up to automorphisms,

$$J = (x_1, \dots, x_{s-1}, x_s + h_s, \dots, x_n + h_n, \xi_1, \dots, \xi_{s-1}, \tau + \eta_s + H)$$

for some functions $h_i, \eta_s, H \in \Lambda(\xi_s, \dots, \xi_n)$.

(i) Suppose $\eta_s = 0$. Denote by J' the ideal $J' = (x_s + h_s, \dots, x_n + h_n, \tau + H) \subset J$. Then, each function f in L_0 (thus stabilizing J) stabilizes the ideal $K = (J, [J', J'])$, i.e., the ideal generated by the generators of J and by the commutators between every pair of generators of J'. Indeed, for every $g, h \in J'$ we have:

$$\bigl[f, [g, h]\bigr] = \bigl[[f, g], h\bigr] \pm \bigl[g, [f, h]\bigr] \in [J, J']$$

and $[J, J'] \subset K$. Notice that K is a proper ideal of $\Lambda(n, n+1)$ since among its generators there is no invertible element. By the maximality of J among the ideals stabilized by L_0 we have $J = K$.

Suppose that $h_s = \xi_s \psi_1 + \psi_2$ with ψ_1 and ψ_2 independent of ξ_s. Then applying the automorphism $\exp(ad(-\xi_s \psi_2))$ we can assume

$$J = \bigl(x_1, \dots, x_{s-1}, x_s + \xi_s \psi_1, x_{s+1} + h'_{s+1}, \dots, x_n + h'_n, \xi_1, \xi_2, \dots, \xi_{s-1}, \tau + H'\bigr).$$

Now $\psi_1 = -\frac{1}{2}[x_{s+1} + \xi_{s+1}\psi_1, x_{s+1} + \xi_{s+1}\psi_1] \in [J', J'] \subset K = J$, therefore

$$J = \bigl(x_1, \dots, x_{s-1}, x_s, x_{s+1} + h'_{s+1}, \dots, x_n + h'_n, \xi_1, \xi_2, \dots, \xi_{s-1}, \tau + H'\bigr).$$

Repeating a similar argument for every function h'_j and, finally, for the function H', we end up with the standard ideal $J = (x_1, \dots, x_n, \xi_1, \xi_2, \dots, \xi_{s-1}, \tau)$.

(ii) If $\eta_s \neq 0$, by Remark 4.10, we can assume:

$$J = (x_1, \dots, x_{s-1}, x_s + h_s, \dots, x_n + h_n, \xi_1, \dots, \xi_{s-1}, \tau + \xi_s + H).$$

Thus the automorphism $\exp(ad(x_s \tau) + 2x_s id)$ maps J to the ideal

$$J' = \bigl(x_1, \dots, x_{s-1}, x_s + h'_s, \dots, x_n + h'_n, \xi_1, \dots, \xi_{s-1}, \xi_s + \tau\rho + H'\bigr)$$

where H' is independent of τ and $\deg(\rho) \geqslant 1$. In the limit, since J' is closed, we get the ideal

$$J'' = (x_1, \dots, x_{s-1}, x_s + h_s, \dots, x_n + h_n, \xi_1, \dots, \xi_{s-1}, \xi_s + M)$$

where M is independent of τ. We thus proceed as in Case 1. □

Theorem 4.12. *All maximal open subalgebras of $L = KO(n, n+1)$ are, up to conjugation, the following*:

(i) *the graded subalgebra of type* $(1, \dots, 1|0, \dots, 0, 1)$;
(ii) *the graded subalgebras of type* $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0, 2)$ *with* $n - t + 1$ *2's and* $n - t$ *zeros, for* $t = 2, \dots, n$;
(iii) *the non-graded subalgebra* L_0 *described in Example* 4.7 *and the non-graded subalgebras* $L_0(t)$ *described in Example* 4.8 *for* $t = 2, \dots, n$.

Proof. Let L_0 be a maximal open subalgebra of L. By Theorem 4.11, L_0 is, up to conjugation, the standard subalgebra of L stabilizing either the ideal $I_{\mathcal{U}} = (x_1, \dots, x_n, \xi_1, \dots, \xi_s)$ for some $s = 0, \dots, n$, or the ideal $I_{\mathcal{U}'} = (x_1, \dots, x_n, \xi_1, \dots, \xi_t, \tau)$ for some $t = 0, \dots, n$. The statement then follows using Remarks 4.6, 4.3, 4.4, and Examples 4.7, 4.8. □

Corollary 4.13. *All irreducible $\mathbb{Z}$-gradings of $KO(n, n+1)$ are, up to conjugation, the grading of type* $(1, \dots, 1|0, \dots, 0, 1)$ *and the gradings of type* $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0, 2)$ *with* $t + 1$ *2's and* t *zeros, for* $t = 0, \dots, n - 2$.

We shall now focus on the Lie superalgebra $SKO(n, n+1; \beta)$ introduced at the beginning of this section.

Remark 4.14. The $\mathbb{Z}$-grading of type $(1, \dots, 1| -1, \dots, -1, 0)$ of $W(n, n+1)$ induces on $\mathfrak{g} = SKO(n, n+1; \beta)$ a $\mathbb{Z}$-grading $\mathfrak{g} = \prod_j \mathfrak{g}_j$, where

$$\mathfrak{g}_{-1} = \left\{ f \in \bigoplus_{h=0}^{n-2} \langle x_{i_1} \dots x_{i_h} \xi_{j_1} \dots \xi_{j_{h+1}} \rangle \otimes \langle 1, \tau, \Phi \rangle \;\middle|\; div_\beta(f) = 0 \right\}$$

and

$$\mathfrak{g}_0 = \left\{ f \in \bigoplus_{h=0}^{n-1} \langle x_{i_1} \dots x_{i_h} \xi_{j_1} \dots \xi_{j_h} \rangle \otimes \langle 1, \tau, \Phi \rangle \;\middle|\; div_\beta(f) = 0 \right\} + \langle 1, \tau + \beta\Phi \rangle$$

where $\Phi = \sum_{i=1}^n x_i \xi_i$. One can check that $S = \{ f \in \bigoplus_{h=1}^{n-2} \langle x_{i_1} \dots x_{i_h} \xi_{j_1} \dots \xi_{j_{h+1}} \rangle \otimes \langle 1, \tau, \Phi \rangle \mid div_\beta(f) = 0 \}$, is a $\mathfrak{g}_0$ stable subspace of $\mathfrak{g}_{-1}$. Notice that $S = 0$ if and only if $n = 2$. It follows that for $n > 2$ the grading of type $(1, \dots, 1|-1, \dots, -1, 0)$ induces on $SKO(n, n+1, \beta)$ a grading which is not irreducible.

Now suppose $n = 2$ and $\beta \neq 0$. Then the $\mathbb{Z}$-grading of type $(1, 1|-1, -1, 0)$ has depth 2. One has: $\mathfrak{g}_0 \cong sl_2 \otimes \Lambda(1) + W(0, 1)$, $\mathfrak{g}_{-1} \cong \mathbb{C}^2 \otimes \Lambda(1)$, where $\mathbb{C}^2$ is the standard sl_2-module, and $\mathfrak{g}_{-2} = \mathbb{C}\xi_1\xi_2 = [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]$. It follows that the grading of type $(1, 1|-1, -1, 0)$ of $\mathfrak{g} = SKO(2, 3; \beta)$ is irreducible. The $\mathfrak{g}_0$-module $\mathfrak{g}_1$ consists of the elements $f \in \langle x_i, x_i x_j \xi_k \rangle \otimes \langle 1, \tau, \Phi \rangle$ such that $div_\beta(f) = 0$. Notice that $\mathfrak{g}_1$ is not irreducible: it has an irreducible $\mathfrak{g}_0$-submodule $S \cong S^3(\mathbb{C}^2) \otimes \Lambda(1)$ and $\mathfrak{g}_1 / S \cong \mathbb{C}^2 \otimes \Lambda(1)$. Besides, for every $j > 1$, $\mathfrak{g}_j = \mathfrak{g}_1^j$. One can check that property (iii)$'$ of Proposition 1.11(b) is satisfied, hence $\prod_{j \geqslant 0} \mathfrak{g}_j$ is a maximal subalgebra of $\mathfrak{g}$.

Finally, if $n = 2$ and $\beta = 0$ the grading of type $(1, 1|-1, -1, 0)$ has depth 1, hence it is irreducible by Remark 1.13.

Remark 4.15. When $\beta \neq 0, -1$ the even part of the Lie superalgebra $SKO(2,3;\beta)$ is isomorphic to $W(2,0)$ and its odd part is isomorphic to $\Omega^0(2)^{-1/(\beta+1)} \oplus \Omega^0(2)^{-\beta/(\beta+1)}$ (cf. Definition 2.5). It follows that, when $\beta = 1$, $SKO(2,3;\beta)_{\bar{1}}$ is the direct sum of two irreducible $SKO(2,3;\beta)_{\bar{0}}$-submodules each of which is isomorphic to $\Omega^0(2)^{-1/2}$ (cf. [10, Proposition 5.3.4]). Therefore if $S = SKO(2,3;1)$, then $Der\, S = S + \mathfrak{a}$ with $\mathfrak{a} \cong sl_2$ (cf. [17, Proposition 6.1], Proposition 1.8). Let e, h, f be the standard basis of $\mathfrak{a}$ where $e = ad(\xi_1\xi_2\tau)$ and $h = ad(-\tau + \sum_{i=1}^{2} x_i\xi_i)$. We will denote by $\mathfrak{b}$ the subalgebra of $\mathfrak{a}$ spanned by e and h.

Remark 4.16. Let $S = SKO(2,3;1)$. Let us denote by S_0 the intersection between the graded subalgebras of S of type $(1,1|1,1,2)$ and $(1,1|-1,-1,0)$, and let $S = S_{-2} \supset S_{-1} \supset S_0 \supset \cdots$ be the Weisfeiler filtration associated to S_0, where $S_{-1} = \langle 1, x_i, \xi_1\xi_2, \xi_i(\tau + \Phi) \mid i = 1,2\rangle + S_0$. Then $Gr\,S$ is a graded Lie superalgebra of depth 2 where $Gr_0\,S \cong S(0,2) + \mathbb{C}E$ and $Gr_{-1}\,S$ is isomorphic, as a $Gr_0\,S$-module, to the direct sum of two copies of $\Lambda(2)/\mathbb{C}1$. Let V be the subspace of $Gr_{-1}\,S$ spanned by the elements $\xi_1\xi_2$ and $\xi_i(\tau + \Phi)$ for $i = 1,2$. Then V is a $Gr_0\,S$-submodule of $Gr_{-1}\,S$ and $\prod_{j\geqslant 0} Gr_j\,S + V$ is the graded subalgebra of S of type $(1,1|1,1,2)$. Likewise, for every $\gamma \in \mathbb{C}$, the subspace $V_\gamma = \langle 1 + \gamma\xi_1\xi_2, -2x_1 + \gamma\xi_2(\tau + \Phi), 2x_2 + \gamma\xi_1(\tau + \Phi)\rangle$ is a $Gr_0\,S$-submodule of $Gr_{-1}\,S$ and $\prod_{j\geqslant 0} Gr_j\,S + V_0$ is the graded subalgebra of S of type $(1,1|-1,-1,0)$. Notice that, for every $\gamma \neq 0$, the automorphism $\exp(\frac{\gamma}{2}e)$ maps V_γ to V_0. It follows that every subalgebra $S_\gamma := \prod_{j\geqslant 0} Gr_j\,S + V_\gamma$, with $\gamma \in \mathbb{C}$, is conjugate to the maximal subalgebra of type $(1,1|-1,-1,0)$. On the other hand, the grading of type $(1,1|-1,-1,0)$ is conjugate to the grading of type $(1,1|1,1,2)$ by the automorphism $\exp(e)\exp(-f)\exp(e)$. Therefore the maximal subalgebras of S of type $(1,1|-1,-1,0)$ and $(1,1|1,1,2)$ lie in the same G-orbit. This orbit consists of the subalgebras S_γ, with $\gamma \in \mathbb{C}$, and of the subalgebra of principal type, and the intersection of any pair of subalgebras in this orbit is the subalgebra $\prod_{j\geqslant 0} Gr_j\,S$. Notice that $\prod_{j\geqslant 0} Gr_j\,S$ is contained also in the (maximal) subalgebra of type $(1,1|0,0,1)$.

Remark 4.17. If $\beta \neq -1$, then the subalgebra of $SKO'(n, n+1;\beta)$ consisting of the elements $f \in \langle P\xi_k, Q\tau \mid P, Q \in \mathbb{C}[[x_1,\ldots,x_n]]\rangle$ such that $div_\beta(f) = 0$ is isomorphic to W_n.

Remark 4.18. The even part of the Lie superalgebra $SKO(2,3;0)$ is isomorphic to $W(2,0)$ and its odd part is isomorphic to $\Omega^0(2)^{-1} \oplus \Omega^0(2)/\mathbb{C}1$. The outer derivation $D = ad(\xi_1\xi_2)$ of $SKO(2,3;0)$ can then be described as follows. Let $p: \Omega^0(2) \to \Omega^0(2)/\mathbb{C}1$ be the natural projection. Then:

$$D(X) = p\big(div(X)\big) \quad \text{if } X \in W(2,0),$$
$$D(f) = df \quad \text{if } f \in \Omega^0(2)^{-1},$$
$$D(f) = 0 \quad \text{if } f \in \Omega^0(2)/\mathbb{C}1.$$

The image of D is thus given by $(\Omega^1(2)_{\text{closed}})^{-1} + \Omega^0(2)/\mathbb{C}1$ where $(\Omega^1(2)_{\text{closed}})^{-1}$ can be identified with $S(2,0)$ via contraction with the volume form $dx_1 \wedge dx_2$.

Remark 4.19. Let us describe the structure of the Lie superalgebra $SKO(2,3;-1)$. Its even part is not simple: it has a commutative ideal consisting of elements in $\Omega^0(2)(\tau - \Phi)$. We have:

$$SKO(2,3;-1)_{\bar{0}} \cong \Omega^0(2) \rtimes S(2,0), \qquad SKO(2,3;-1)_{\bar{1}} \cong \Omega^0(2) + \Omega^0(2).$$

Here $S(2,0)$ acts on each odd copy of $\Omega^0(2)$ in the natural way, and the even functions in $\Omega^0(2)$ act by multiplication on one copy and by $-$multiplication on the other.

Example 4.20. Throughout this example we shall consider the Lie superalgebra $S' = SKO'(n, n+1;\beta)$ for $n > 2$, and we shall identify it with $\Lambda^\beta(n, n+1)$ as explained at the beginning of this section.

Notice that $\Lambda^\beta(n,n+1) \subset \Lambda^\Delta(n,n) \otimes \langle 1, \tau, \Phi\rangle$, where $\Lambda^\Delta(n,n) = \{f \in \Lambda(n,n) \mid \Delta(f) = 0\}$ and $\Phi = \sum_{i=1}^n x_i\xi_i$. We define a valuation ν on $\Lambda^\Delta(n,n) \otimes \langle 1, \tau, \Phi\rangle$ (hence on $\Lambda^\beta(n,n+1)$) by setting:

$$\nu(1) = \nu(\tau) = \nu(\Phi) = 0,$$

$$\nu(x_i) = 1 \quad \forall i = 1, \dots, n, \qquad \nu(\xi_{i_1} \dots \xi_{i_k}) = 0 \quad \forall k < n, \qquad \nu(\xi_1 \dots \xi_n) = -1$$

and we extend it on $\Lambda(0,n)$ by property (b) in Remark 2.23, on $\mathbb{C}[[x_1, \dots, x_n]]$ by properties (a) and (b) in Remark 2.23, and finally on $\Lambda^\Delta(n,n) \otimes \langle 1, \tau, \Phi\rangle$ by setting $\nu(\sum_i P_i(x)Q_i(\xi)\eta_i) = \min_i(\nu(P_i(x)) + \nu(Q_i(\xi)))$ where $P_i(x) \in \mathbb{C}[[x_1, \dots, x_n]]$, $Q_i(\xi) \in \Lambda(0,n)$ and $\eta_i \in \langle 1, \tau, \Phi\rangle$.

Then the following subspaces define a filtration of $SKO'(n, n+1;\beta)$:

$$S'_j = \{f \in \Lambda^\beta(n,n+1) \mid \nu(f) \geqslant j+1\} + \langle 1, \tau + \beta\Phi\rangle \quad \text{if } j \leqslant 0,$$
$$S'_j = \{f \in \Lambda^\beta(n,n+1) \mid \nu(f) \geqslant j+1\} \quad \text{if } j > 0.$$

This filtration has depth 2, with $Gr_{-2}\, S' = \langle \xi_1 \dots \xi_n\rangle$ if $\beta \neq 1$ and $Gr_{-2}\, S' = \langle \xi_1 \dots \xi_n, \xi_1 \dots \xi_n\tau\rangle$ if $\beta = 1$. In fact $Gr_{-2}\, S'$ is an ideal of $Gr\, S'$, since for any $g \in Gr_j\, S'$, $j \geqslant 1$, and any $f \in Gr_{-2}\, S'$, $\nu([f,g]) = \nu(g) - 1$, hence $[f,g]$ lies in S'_{j-1}, i.e., $[f,g] = 0$ in $Gr\, S'$. We have:

$$\overline{Gr\, S'}/Gr_{-2}\, S' \cong SHO(n,n) \otimes \Lambda(\eta) + \mathbb{C}\frac{\partial}{\partial\eta} + \mathbb{C}\left(E - 2 - \beta\, ad(\Phi) + 2\eta\frac{\partial}{\partial\eta}\right)$$

with respect to the grading of type $(1, \dots, 1|0, \dots, 0)$ on $SHO(n,n)$ and $\deg(\eta) = 0$. $\prod_{j \geqslant 0} Gr_j\, S'$ is thus not a maximal subalgebra of $\overline{Gr\, S'}$ since it is contained in $\prod_{j \geqslant 0} Gr_j\, S' + Gr_{-2}\, S'$. Nevertheless, note that, for every β, S'_0 is contained in $S = SKO(n, n+1;\beta)$, and $S'_0 + \mathbb{C}\xi_1 \dots \xi_n$ generates the whole S. It follows that, for every $\beta \neq 1$, $(n-2)/n$, since $S = S'$, S'_0 is a maximal open subalgebra of S. If $\beta = 1$ or $\beta = (n-2)/n$, S'_0 is not a maximal subalgebra of S' but it is a maximal subalgebra of S.

Finally, for every β, $S'_0 = SKO(n, n+1;\beta) \cap L_0$ where L_0 is the standard subalgebra of $KO(n, n+1)$ constructed in Example 4.7. It follows that S'_0 is the standard subalgebra of $SKO(n, n+1;\beta)$ stabilizing the ideal $I_U = (x_1, \dots, x_n)$.

Example 4.21. Let t be an integer such that $1 \leqslant t \leqslant n$ and let us consider the valuation ν on $\Lambda(n, n+1)$ defined in Example 4.8. Consider the following subspaces of $S' = SKO'(n, n+1;\beta)$:

$$S'_i(t) = \{f \in \Lambda^\beta(n,n+1) \mid \nu(f) \geqslant i+2\} + \langle 1, \tau + \beta\Phi\rangle \quad \text{if } i \leqslant 0,$$
$$S'_i(t) = \{f \in \Lambda^\beta(n,n+1) \mid \nu(f) \geqslant i+2\} \quad \text{if } i > 0.$$

By commutation rules (4.1), these subspaces define in fact a filtration of $SKO'(n, n+1; \beta)$, having depth 2 if $t \neq n$ and depth 1 if $t = n$. Then, if $\beta \neq 1$,

$$\overline{Gr\, S'} \cong SHO(n,n) \otimes \Lambda(\eta) + \mathbb{C}\xi_1 \dots \xi_n + \mathbb{C}\frac{\partial}{\partial \eta} + \mathbb{C}\left(E - 2 - \beta\, ad(\Phi) + 2\eta \frac{\partial}{\partial \eta}\right)$$

and, if $\beta = 1$,

$$\overline{Gr\, S'} \cong SHO'(n,n) \otimes \Lambda(\eta) + \mathbb{C}\frac{\partial}{\partial \eta} + \mathbb{C}\left(E - 2 - \beta\, ad(\Phi) + 2\eta \frac{\partial}{\partial \eta}\right),$$

with respect to the grading of type $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0)$ of $SHO'(n,n)$, with $n - t$ 2's and $n - t$ zeros, and $\deg(\eta) = 0$. When $n > 2$ these gradings are irreducible for every $t = 2, \dots, n$ (cf. Remark 2.46) and satisfy property (iii)$'$ of Proposition 1.11(b). Therefore, by Corollary 1.12, when $n > 2$, $S'_0(t)$ is a maximal subalgebra of $SKO'(n, n+1; \beta)$ for every $t = 2, \dots, n$.

Let $S = SKO(n, n+1; \beta)$ and let $S_j(t) := S'_j(t) \cap S$. If $\beta \neq 1, (n-2)/n$, then $S = S'$, hence $S_0(t)$ is, for every $t = 2, \dots, n$, a maximal open subalgebra of S. If $\beta = (n-2)/n$ or $\beta = 1$, then the subspaces $S_j(t)$ define a filtration of S such that:

$$\overline{Gr\, S} \cong SHO(n,n) \otimes \Lambda(\eta) + \mathbb{C}\frac{\partial}{\partial \eta} + \mathbb{C}\left(E - 2 - \beta\, ad(\Phi) + 2\eta \frac{\partial}{\partial \eta}\right)$$

or

$$\overline{Gr\, S} \cong SHO(n,n) \otimes \Lambda(\eta) + \mathbb{C}\xi_1 \dots \xi_n + \mathbb{C}\frac{\partial}{\partial \eta} + \mathbb{C}\left(E - 2 - \beta\, ad(\Phi) + 2\eta \frac{\partial}{\partial \eta}\right),$$

respectively, with respect to the grading of type $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0)$ of $SHO(n,n)$, with $n - t$ 2's and $n - t$ zeros, and $\deg(\eta) = 0$. It follows that, if $n > 2$, then $S_0(t)$ is a maximal open subalgebra of S, for every $t = 2, \dots, n$ (cf. Remark 2.34).

Notice that the grading of principal type of $W(n,n)$ induces an irreducible grading on $SHO'(n,n)$ also for $n = 2$, but it induces on $SHO(2,2)$ a grading which is not irreducible. It follows that $S'_0(2)$ is a fundamental maximal subalgebra of $SKO'(2, 3; \beta)$ for every β, but $S_0(2)$ is a maximal subalgebra of $SKO(2, 3; \beta)$ if and only if $\beta \neq 0$. When $\beta = 0$ the subalgebra $S_0(2)$ of $SKO(2, 3; 0)$ is indeed contained in the graded subalgebra of type $(1, 1| -1, -1, 0)$.

Finally, note that $S_0(t) = L_0(t) \cap SKO(n, n+1; \beta)$ where $L_0(t)$ is the subalgebra of $KO(n, n+1)$ constructed in Example 4.8. It follows that $S_0(t)$ stabilizes the ideal $I_{\mathcal{U}} = (x_1, \dots, x_n, \xi_1, \dots, \xi_t)$ of $\Lambda(n, n+1)$.

Remark 4.22. Let $S = SKO(n, n+1; \beta)$ and consider its grading of principal type: $S = \prod_{j \geqslant -2} S_j$. Then τ acts on S_j by multiplication by j. By Remark 4.5, $(\mathbb{C}\xi_{i_1} \dots \xi_{i_h} \otimes T) \cap S_h$, and $(\mathbb{C}x_k \xi_{j_1} \dots \xi_{j_h} \otimes T) \cap S_{h+1}$ with $k \neq j_1, \dots, j_h$, are T-weight spaces of $SKO(n, n+1; \beta)$.

Remark 4.23. The same arguments as in the proof of Theorem 2.11 show, by Remark 4.2, that every maximal open subalgebra of $SKO(n, n+1; \beta)$, $SKO'(n, n+1; \beta)$ and $CSKO'(n, n+1; \beta) = SKO'(n, n+1; \beta) + \mathbb{C}\Phi$ is regular.

Theorem 4.24. *Let $S = SKO(n, n+1; \beta)$. Then all maximal open subalgebras of S are, up to conjugation, the following*:

(a) *if $n = 2$ and $\beta \neq 0, 1$*:
  (i) *the graded subalgebras of type* $(1, 1|0, 0, 1)$, $(1, 1|1, 1, 2)$ *and* $(1, 1|-1, -1, 0)$;
  (ii) *the non-graded subalgebra $S_0(2)$ constructed in Example* 4.21;
(b) *if $n = 2$ and $\beta = 1$*:
  (i) *the graded subalgebras of type* $(1, 1|0, 0, 1)$, $(1, 1|1, 1, 2)$;
  (ii) *the non-graded subalgebra $S_0(2)$ constructed in Example* 4.21;
(c) *if $n = 2$ and $\beta = 0$*:
  (i) *the graded subalgebras of type* $(1, 1|0, 0, 1)$ *and* $(1, 1|-1, -1, 0)$;
(d) *if $n > 2$*:
  (i) *the graded subalgebra of type* $(1, \ldots, 1|0, \ldots, 0, 1)$ *and the graded subalgebras of type* $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0, 2)$ *with $n - t + 1$ 2's and $n - t$ zeros, for $t = 2, \ldots, n$*;
  (ii) *the non-graded subalgebra S_0' described in Example* 4.20 *and the non-graded subalgebras $S_0(t)$ described in Example* 4.21 *for $t = 2, \ldots, n$.*

Proof. Let L_0 be a maximal open subalgebra of S. By Remark 4.23, L_0 is regular. Therefore, by Remark 2.1 and Proposition 1.8, we can assume that L_0 is invariant with respect to the standard torus T of $KO(n, n+1)$. It follows that L_0 decomposes into the direct product of weight spaces with respect to T. Notice that $\mathbb{C}1$, $\mathbb{C}x_i$, $\mathbb{C}\xi_{i_1} \ldots \xi_{i_h}$, $\mathbb{C}x_j\xi_{i_1} \ldots \xi_{i_h}$, with $j \neq i_1 \neq \cdots \neq i_h$, are one-dimensional T-weight spaces (see Remark 4.5). Besides, note that the elements ξ_i cannot lie in L_0 since the corresponding vector fields $\rho(\xi_i) = \xi_i \partial/\partial\tau + \partial/\partial x_i$ are not exponentiable.

Let us first assume $n = 2$. We distinguish two cases:

Case I. 1 does not lie in L_0. We may assume that one of the following possibilities occurs:

(1) No x_i lies in L_0. Then the T-invariant complement of L_0 contains the T-invariant complement of the maximal graded subalgebra of S of type $(1, 1|1, 1, 2)$, hence L_0 is contained in the graded subalgebra of principal type. If $\beta = 0$ then the subalgebra of principal type is not maximal therefore this contradicts the maximality of L_0. If $\beta \neq 0$ then L_0 coincides with the graded subalgebra of type $(1, 1|1, 1, 2)$ by maximality.
(2) The elements x_1, x_2 lie in L_0. Then the T-invariant complement of L_0 contains the T-invariant complement of the maximal graded subalgebra of L of type $(1, 1|0, 0, 1)$. Since L_0 is maximal it coincides with this graded subalgebra.

Notice that if L_0 contains x_2 (respectively x_1) then, due to its maximality, it contains also x_1 (respectively x_2). Indeed, any open regular subalgebra of S containing x_2 and not containing 1 and x_1 (respectively containing x_1 and not containing 1 and x_2) is contained in the subalgebra of type $(1, 2|1, 0, 2)$ (respectively $(2, 1|0, 1, 2)$) which is not maximal by Remark 4.4.

Case II. 1 lies in L_0. Since the elements ξ_i's do not lie in L_0, the elements $\xi_i\tau + \varphi$ cannot lie in L_0 for any $\varphi \in \Lambda(2, 2)$, where by $\Lambda(n, n)$ we mean the subalgebra of $\Lambda(n, n+1)$ generated by all even indeterminates and all odd indeterminates except τ. Indeed, by commutation rules (4.1), we have: $[1, \xi_i\tau + \varphi] = -2\xi_i$. Note that if $\beta = 0$ then the grading of type $(1, 1|-1, -1, 0)$ has depth 1 with -1st graded component spanned by the elements ξ_i and $\xi_i(\tau - \Phi)$ for $i = 1, 2$.

It follows that if $\beta = 0$, then L_0 is contained in the graded subalgebra of $SKO(2,3;0)$ of type $(1,1|-1,-1,0)$, thus coincides with it, due to its maximality.

Now suppose $\beta \neq 0$. Since, for every i, $\mathbb{C}x_i$ is a one-dimensional weight space of $SKO(n,n+1;\beta)$, we may assume that one of the following situations holds:

(1) No x_i lies in L_0. Then the same arguments as in Example 4.8 show that L_0 coincides with the subalgebra $S_0(2)$ constructed in Example 4.21;
(2) x_1, x_2 lie in L_0. Then L_0 is contained in the graded subalgebra of S of type $(1,1|-1,-1,0)$. Since L_0 is maximal the two subalgebras coincide.

Notice that if $1, x_2$ lie in L_0, by the maximality of L_0, also $x_1 \in L_0$. Indeed, any open regular subalgebra of S containing the elements $1, x_2$ and not containing x_1 is contained in the maximal subalgebra of type $(1,1|-1,-1,0)$.

Finally, as we pointed out in Remark 4.15, when $\beta = 1$, the subalgebras of type $(1,1|-1,-1,0)$ and $(1,1|1,1,2)$ are conjugate by an element of G.

Let us now suppose $n > 2$. We distinguish two cases:

Case I. 1 does not lie in L_0. We may assume that one of the following possibilities occurs:

(1) No x_i lies in L_0. Then the T-invariant complement of L_0 contains the T-invariant complement of the maximal graded subalgebra of S of type $(1,\dots,1|1,\dots,1,2)$. By the maximality of L_0 it follows that L_0 coincides with the graded subalgebra of type $(1,\dots,1|1,\dots,1,2)$;
(2) the elements $x_{t+1},\dots,x_n$ lie in L_0 for some $t = 2,\dots,n-1$, and the elements $x_1,\dots,x_t$ do not. It follows, using commutation rules (4.1), that the T-invariant complement of L_0 contains the T-invariant complement of the maximal graded subalgebra of L of type $(1,\dots,1,2,\dots,2|1,\dots,1,0,\dots,0,2)$ with $n-t+1$ 2's and $n-t$ zeros. Since L_0 is maximal it coincides with this graded subalgebra;
(3) the elements x_i lie in L_0 for every i. Then the T-invariant complement of L_0 contains the T-invariant complement of the maximal graded subalgebra of S of type $(1,\dots,1|0,\dots,0,1)$. It follows that L_0 coincides with this subalgebra.

Notice that if L_0 contains the elements $x_2,\dots,x_n$ then, due to its maximality, it contains also x_1. Indeed, any regular subalgebra of S containing $x_2,\dots,x_n$ and not containing 1 and x_1 is contained in the subalgebra of type $(1,2,\dots,2|1,0,\dots,0,2)$ which is not maximal by Remark 4.4.

Case II. 1 lies in L_0. We may assume that one of the following situations holds:

(1) For some $t = 2,\dots,n$ the elements $x_1,\dots,x_t$ do not lie in L_0 and $x_{t+1},\dots,x_n$ do. Then the same arguments as in Example 4.8 show that L_0 coincides with the subalgebra $S_0(t)$ constructed in Example 4.21;
(2) $x_1,\dots,x_n$ lie in L_0. Then the same arguments as in Example 4.7 show that L_0 is contained in the subalgebra S_0' of S constructed in Example 4.20. Since L_0 is maximal the two subalgebras coincide.

Notice that if $1, x_2,\dots,x_n$ lie in L_0, by the maximality of L_0, also $x_1 \in L_0$. Indeed any open regular subalgebra of S containing the elements $1, x_2,\dots,x_n$ and not containing x_1 is contained in the maximal subalgebra constructed in Example 4.20. □

Corollary 4.25. *All irreducible $\mathbb{Z}$-gradings of $SKO(n,n+1;\beta)$ are, up to conjugation, the following*:

(i) *the gradings of type* $(1,1|0,0,1)$, $(1,1|1,1,2)$ *and* $(1,1|-1,-1,0)$, *if* $n=2$, $\beta\neq 0,1$;
(ii) *the gradings of type* $(1,1|0,0,1)$ *and* $(1,1|1,1,2)$ *if* $n=2$, $\beta=1$;
(iii) *the gradings of type* $(1,1|0,0,1)$, $(1,1|-1,-1,0)$ *if* $n=2$, $\beta=0$;
(iv) *the gradings of type* $(1,\dots,1,2,\dots,2|1,\dots,1,0,\dots,0,2)$ *with* $t+1$ *2's and* t *zeros, for* $t=0,\dots,n-2$ *and* $(1,\dots,1|0,\dots,0,1)$, *if* $n>2$.

We recall that if $S=SKO(n,n+1;\beta)$ with $n\geqslant 2$ and $\beta\neq 1,(n-2)/n$, then $Der\,S=S+\mathbb{C}\Phi$ with $\Phi=\sum_{i=1}^n x_i\xi_i$; if $S=SKO(n,n+1;(n-2)/n)$ with $n\geqslant 2$, then $Der\,S=S+\mathbb{C}\Phi+\mathbb{C}\xi_1\dots\xi_n$; if $S=SKO(n,n+1;1)$ with $n>2$, then $Der\,S=S+\mathbb{C}\Phi+\mathbb{C}\xi_1\dots\xi_n\tau$; finally, if $S=SKO(2,3;1)$ then $Der\,S=S+sl_2$ (cf. Proposition 1.8, Remark 4.15).

Theorem 4.26. *Let* $S=SKO(n,n+1;\beta)$ *with* $n\geqslant 2$ *and* $\beta\neq 1$, $(n-2)/n$, *so that* $SKO(n,n+1;\beta)=SKO'(n,n+1;\beta)$ *and* $Der\,S=CSKO'(n,n+1;\beta)$. *Then all maximal among open* Φ*-invariant subalgebras of* S *are, up to conjugation, the subalgebras of* S *listed in Theorem* 4.24(a) *and* (d).

Proof. Let L_0 be a maximal among open Φ-invariant subalgebras of S. Then $L_0+\mathbb{C}\Phi$ is a maximal open subalgebra of $CSKO'(n,n+1;\beta)$, hence it is regular by Remark 4.23. Then one uses the same arguments as in the proof of Theorem 4.24. □

We shall now classify the open subalgebras of $S=SKO(n,n+1;(n-2)/n)$ and $S=SKO(n,n+1;1)$, which are maximal among the $\mathfrak{a}_0$-invariant subalgebras of S, for every subalgebra $\mathfrak{a}_0$ of $\mathfrak{a}$.

Remark 4.27. By Remark 4.23 every maximal open subalgebra of $SKO'(n,n+1;\beta)$ or $CSKO'(n,n+1;\beta)$ is regular. Therefore the same arguments as in the proof of Theorem 4.24 show that all fundamental among maximal subalgebras of $SKO'(n,n+1;(n-2)/n)$ (respectively $CSKO'(n,n+1;(n-2)/n)$), with $n>2$, are, up to conjugation, the graded subalgebras of type $(1,\dots,1,2,\dots,2|1,\dots,1,\ 0,\dots,0,2)$, with $n-t+1$ 2's and $n-t$ zeros, and the non-graded subalgebras $S_0'(t)$ (respectively $S_0'(t)+\mathbb{C}\Phi$) constructed in Example 4.21, for $t=2,\dots,n$. Indeed, the graded subalgebra of $SKO'(n,n+1;(n-2)/n)$ (respectively $CSKO'(n,n+1;(n-2)/n)$) of type $(1,\dots,1|0,\dots,0,1)$ and the subalgebra S_0' constructed in Example 4.20, are not maximal, since they are contained in $SKO(n,n+1;(n-2)/n)$ (respectively $SKO(n,n+1;(n-2)/n)+\mathbb{C}\Phi$). By the same arguments, all maximal among fundamental subalgebras of $SKO'(n,n+1;(n-2)/n)$ and $CSKO'(n,n+1;(n-2)/n)$, for $n>2$, are, up to conjugation, the graded subalgebra of subprincipal type, the graded subalgebras of type $(1,\dots,1,2,\dots,2|1,\dots,1,\ 0,\dots,0,2)$, with $n-t+1$ 2's and $n-t$ zeros, the non-graded subalgebras $S_0'(t)$ constructed in Example 4.21, for $t=2,\dots,n$, and, the subalgebra S_0' constructed in Example 4.20.

Likewise, all fundamental among maximal subalgebras of $SKO'(2,3;0)$ (respectively $CSKO'(2,3;0)$) are, up to conjugation, the graded subalgebra of type $(1,1|1,1,2)$ and the subalgebra $S_0'(2)$ (respectively $S_0'(2)+\mathbb{C}\Phi$). All maximal among fundamental subalgebras of $SKO'(2,3;0)$ (respectively $CSKO'(2,3;0)$) are the graded subalgebras of type $(1,1|1,1,2)$, $(1,1|0,0,1)$, $(1,1|-1,-1,0)$ and the non-graded subalgebra $S_0'(2)$ (respectively $S_0'(2)+\mathbb{C}\Phi$).

Theorem 4.28. *Let $S = SKO(n, n+1; (n-2)/n)$ with $n \geqslant 2$.*

(i) *All maximal among open Φ-invariant subalgebras of S are, up to conjugation, the maximal open subalgebras listed in Theorem* 4.24(c) *and* (d).
(ii) *If $\mathfrak{a}_0 = \mathbb{C}\xi_1 \dots \xi_n$ or $\mathfrak{a}_0 = \mathfrak{a}$, then all maximal among $\mathfrak{a}_0$-invariant open subalgebras of S are, up to conjugation, the graded subalgebras of type $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0, 2)$, with $n - t + 1$ 2's and $n - t$ zeros, and the non-graded subalgebras $S_0(t)$ constructed in Example* 4.21*, for $t = 2, \dots, n$.*

Proof. One uses Remark 4.27 and the same arguments as in the proof of Theorem 2.17. □

Remark 4.29. By Remark 4.23 every maximal open subalgebra of $SKO'(n, n+1; \beta)$ or $CSKO'(n, n+1; \beta)$, for every $n \geqslant 2$, is regular. Therefore the same arguments as in the proof of Theorem 4.24 show that all fundamental among maximal subalgebras of $SKO'(n, n+1; 1)$ (respectively $CSKO'(n, n+1; 1)$) are, up to conjugation, the graded subalgebras of type $(1, \dots, 1|0, \dots, 0, 1)$ and $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0, 2)$ with $n - t + 1$ 2's and $n - t$ zeros, and the non-graded subalgebras $S_0'(t)$ (respectively $S_0'(t) + \mathbb{C}\Phi$) constructed in Example 4.21, for $t = 2, \dots, n$. By the same arguments, all maximal among fundamental subalgebras of $SKO'(n, n+1; 1)$ (respectively $CSKO'(n, n+1; 1)$) are, up to conjugation, all the subalgebras listed above and the subalgebra S_0' constructed in Example 4.20, if $n > 2$, or the graded subalgebra of type $(1, 1|-1, -1, 0)$ if $n = 2$. Note that the subalgebras of $SKO'(2, 3; 1)$ or $CSKO'(2, 3; 1)$ of type $(1, 1|1, 1, 2)$ and $(1, 1|-1, -1, 0)$ are not conjugate.

Theorem 4.30. *Let $S = SKO(n, n+1; 1)$ with $n > 2$.*

(i) *All maximal among open Φ-invariant subalgebras of S are, up to conjugation, the maximal open subalgebras listed in Theorem* 4.24(d).
(ii) *If $\mathfrak{a}_0 = \mathbb{C}\xi_1 \dots \xi_n \tau$ or $\mathfrak{a}_0 = \mathfrak{a}$, then all maximal among $\mathfrak{a}_0$-invariant open subalgebras of S are, up to conjugation, the graded subalgebras of type $(1, \dots, 1|0, \dots, 0)$ and $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0, 2)$, with $n - t + 1$ 2's and $n - t$ zeros, and the non-graded subalgebras $S_0(t)$ constructed in Example* 4.21*, for $t = 2, \dots, n$.*

Proof. One uses Remark 4.29 and the same arguments as in the proof of Theorem 2.17. □

Theorem 4.31. *Let $S = SKO(2, 3; 1)$ and let $\mathfrak{b} = \mathbb{C}e + \mathbb{C}h \subset \mathfrak{a} \cong sl_2$.*

(i) *If $\mathfrak{a}_0$ is a one-dimensional subalgebra of $\mathfrak{a}$, then all maximal among open $\mathfrak{a}_0$-invariant subalgebras of S are, up to conjugation, the maximal subalgebras listed in Theorem* 4.24(b).
(ii) *The graded subalgebra of type $(1, 1|1, 1, 2)$ is, up to conjugation, the only maximal among open $\mathfrak{b}$-invariant subalgebras of S, which is not invariant with respect to $\mathfrak{a}$.*
(iii) *All maximal among open $\mathfrak{a}$-invariant subalgebras of S are, up to conjugation, the graded subalgebra of type $(1, 1|0, 0, 1)$ and the non-graded subalgebra $S_0(2)$ constructed in Example* 4.21.

Proof. By Remark 4.29, the proof of (i) is the same as the proof of (i) and (ii) in Theorem 2.17. Recall that the graded subalgebras of type $(1, 1|1, 1, 2)$ and $(1, 1|-1, -1, 0)$ are conjugate.

Now, using [10, Proposition 5.3.4] one can check that the maximal graded subalgebra of $SKO(2,3;1)$ of type $(1,1|0,0,1)$ and the subalgebra $S_0(2)$ constructed in Example 4.21 are invariant with respect to $\mathfrak{a}$. On the other hand, the maximal subalgebra L_0 of S of type $(1,1|1,1,2)$ is invariant with respect to $\mathfrak{b}$ but it is not $\mathfrak{a}$-invariant. Indeed L_0 contains $\xi_1\xi_2$, it does not contain 1, but $f(\xi_1\xi_2)=1$. Let M_0 be a maximal among open $\mathfrak{b}$-invariant subalgebras of $SKO(2,3;1)$, then $M_0+\mathbb{C}\xi_1\xi_2\tau+\mathbb{C}\Phi$ is a fundamental maximal subalgebra of $CSKO'(2,3;1)$ containing $\xi_1\xi_2\tau$ and Φ, hence, by Remark 4.29, M_0 is conjugate to the graded subalgebra of type $(1,1|1,1,2)$, or to the subalgebra of type $(1,1|0,0,1)$, or to the subalgebra $S_0(2)$.

Now suppose that $\tilde{S}$ is a maximal among open $\mathfrak{a}$-invariant subalgebras of $SKO(2,3;1)$. Then $\tilde{S}$ is $\mathfrak{b}$-invariant, hence it is conjugate either to the graded subalgebra of type $(1,1|0,0,1)$, or to the subalgebra $S_0(2)$ constructed in Example 4.21. Indeed, otherwise, $\tilde{S}$ is contained either in the subalgebra of type $(1,1|1,1,2)$ or in a conjugate S_γ of it (see Remark 4.16). Since $\tilde{S}$ is $\mathfrak{a}$-invariant, it is invariant with respect to all outer automorphisms of S, hence it is contained in the intersection of all the subalgebras in the orbit of the subalgebra of principal type. It follows, by Remark 4.16, that $\tilde{S}$ is contained in the subalgebra of type $(1,1|0,0,1)$. This contradicts the maximality of $\tilde{S}$ among $\mathfrak{a}$-invariant subalgebras. □

5. Maximal open subalgebras of $SHO^\sim(n,n)$ and $SKO^\sim(n,n+1)$

5.1. The Lie superalgebra $SHO^\sim(n,n)$

Let n be even. The Lie superalgebra $SHO^\sim(n,n)$ is the subalgebra of $HO(n,n)$ defined as follows:

$$SHO^\sim(n,n)=\{X\in HO(n,n)\mid X(F\omega)=0\}$$

where ω is the volume form associated to the usual divergence and $F=1-2\xi_1\dots\xi_n$. By Remark 2.7, $SHO^\sim(n,n)$ consists of vector fields X in $HO(n,n)$ such that $div_F(X)=0$ or, equivalently, by Remark 2.9, such that $div(FX)=0$.

Using the isomorphism between $HO(n,n)$ and $\Lambda(n,n)/\mathbb{C}1$ with the Buttin bracket, it is possible to realize $SHO^\sim(n,n)$ as follows (cf. [18, §2]):

$$SHO^\sim(n,n)=\big((1+\xi_1\dots\xi_n)\Lambda^\Delta(n,n)\big)/\mathbb{C}1$$

where $\Lambda^\Delta(n,n)=\{f\in\Lambda(n,n)\mid\Delta(f)=0\}$ and Δ is the odd Laplacian. Equivalently, $SHO^\sim(n,n)$ can be identified with the space $\Lambda(n,n)^\Delta/\mathbb{C}1$ with the following deformed bracket [9, §5]:

$$\begin{aligned}
[f,g]&=[\xi_1\dots\xi_n,fg]_{ho}\quad\text{if } f,g\in\mathbb{C}[[x_1,\dots,x_n]],\\
[x_i,\xi_j]&=\delta_{ij}\xi_1\dots\xi_n,\\
[f,g]&=[f,g]_{ho}\quad\text{otherwise},
\end{aligned}\tag{5.1}$$

where $[\cdot,\cdot]_{ho}$ denotes the bracket in $HO(n,n)$.

The superalgebra $SHO^\sim(n,n)$ is simple for $n\geqslant 2$ (n even) [17, Example 6.2]. Since, as we recalled in the introduction, $SHO^\sim(2,2)\cong H(2,1)$, when dealing with $SHO^\sim(n,n)$ we will assume $n>2$.

Remark 5.1. A $\mathbb{Z}$-grading of $W(n,n)$ induces a $\mathbb{Z}$-grading on $SHO^{\sim}(n,n)$ if and only if $\deg x_i + \deg \xi_i = \text{const}$ and $\sum_{i=1}^{n} \deg \xi_i = 0$. In particular the $\mathbb{Z}$-grading of type $(1,\ldots,1|0,\ldots,0)$ induces on $SHO^{\sim}(n,n)$ a grading of depth 1 which is irreducible by Remark 1.13.

In what follows we will identify $SHO^{\sim}(n,n)$ with $\Lambda(n,n)^{\Delta}/\mathbb{C}1$ with bracket (5.1). Then its standard maximal torus is $T = \langle x_i\xi_i - x_{i+1}\xi_{i+1} \mid i = 1,\ldots,n-1\rangle$.

Example 5.2. On $\Lambda(n,n)$, for any fixed integer t such that $1 \leqslant t \leqslant n$, let us define the following valuation ν:

$$\nu(x_i) = 1, \quad \nu(\xi_i) = 1 \quad \text{for } i = 1,\ldots,t,$$

$$\nu(x_i) = 2, \quad \nu(\xi_i) = 0 \quad \text{for } i = t+1,\ldots,n.$$

Let us define the following filtration of $L = SHO^{\sim}(n,n)$:

$$L_j(t) = \{x \in \Lambda^{\Delta}(n,n)/\mathbb{C}1 \mid \nu(x) \geqslant j+2\}.$$

Then $\overline{Gr\,L} \cong SHO'(n,n)$ with respect to the $\mathbb{Z}$-grading of type $(1,\ldots,1,2,\ldots,2|1,\ldots,1,0,\ldots,0)$ with $n-t$ 2's and $n-t$ zeros. Since this grading is irreducible for every $t = 2,\ldots,n$ (cf. Remarks 2.34, 2.46), it follows, using Corollary 1.12, that $L_0(t)$ is a maximal regular subalgebra of L for every $t = 2,\ldots,n$.

Remark 5.3. Let $\Sigma_0 := \langle x_{i_1}\ldots x_{i_k}\xi_{i_1}\ldots\xi_{i_k} \mid k = 1,\ldots,n\rangle$. All elements of $SHO^{\sim}(n,n)$ lying in Σ_0 have T-weights equal to zero.

Let $i_1 \neq \cdots \neq i_h$ and $\{i_1,\ldots,i_h,j_1,\ldots,j_{n-h}\} = \{1,\ldots,n\}$. Then $\{f \in \langle \xi_{i_1}\ldots\xi_{i_h}, x_{j_1}\ldots x_{j_{n-h}}\rangle \otimes \Sigma_0 \mid \Delta(f) = 0\}$ is a weight space with respect to T. Likewise, if $i_1 \neq \cdots \neq i_h \neq j$, then $\{f \in \langle x_j\xi_{i_1}\ldots\xi_{i_h}, x_jx_{j_1}\ldots x_{j_{n-h}}\rangle \otimes \Sigma_0 \mid \Delta(f) = 0\}$ is a weight space with respect to T.

Theorem 5.4. *Let $L = SHO^{\sim}(n,n)$ with $n > 2$ even. All maximal open subalgebras of L are, up to conjugation, the following*:

(i) *the graded subalgebra of type* $(1,\ldots,1|0,\ldots,0)$;
(ii) *the non-graded subalgebras $L_0(t)$ constructed in Example* 5.2, *for $t = 2,\ldots,n$.*

Proof. Let L_0 be a maximal open subalgebra of L. The same argument as in the proof of Theorem 2.11 shows that L_0 is regular hence we can assume, by Remark 2.1, that it is invariant with respect to the torus T. It follows that L_0 decomposes into the direct product of T-weight spaces. Note that the elements $\sum_j \alpha_j\xi_j + f$ cannot lie in L_0 for any non-zero linear combination $\sum_j \alpha_j\xi_j$ and any odd function $f \in \Lambda^{\Delta}(n,n)/\mathbb{C}1$ with no linear terms, since the elements ξ_j are not exponentiable. We may therefore assume that one of the following situations occurs:

(1) the elements $x_i + \varphi_i$ lie in L_0 for some elements φ_i with no linear terms, for every $i = 1,\ldots,n$. Then the elements $\xi_i\xi_j + \psi$ do not lie in L_0 for any ψ in the T-weight space of $\xi_i\xi_j$, $\psi \notin \mathbb{C}\xi_i\xi_j$, since, for such a ψ, by Remark 5.3, $[x_i + \varphi_i, \xi_i\xi_j + \psi] = \xi_j + \eta$ for some function $\eta \in \Lambda^{\Delta}(n,n)/\mathbb{C}1$ without linear terms. It follows that L_0 does not contain any element $\xi_i\xi_j + \psi$ for any $\psi \notin \mathbb{C}\xi_i\xi_j$. The same argument shows, by induction on $k = 1,\ldots,n$,

that L_0 does not contain the elements $\xi_{i_1}\ldots\xi_{i_k}+\psi_k$ for any function $\psi_k\notin\mathbb{C}\xi_{i_1}\ldots\xi_{i_k}$, for any $k=1,\ldots,n$. L_0 is therefore contained in the maximal graded subalgebra of L of type $(1,\ldots,1|0,\ldots,0)$, hence coincides with it since it is maximal;

(2) there exists some $t=2,\ldots,n$ such that the elements $x_1+\varphi_1,\ldots,x_t+\varphi_t$ do not lie in L_0 for any functions $\varphi_1,\ldots,\varphi_t$ without linear terms, and $x_{t+1}+\varphi_{t+1},\ldots,x_n+\varphi_n$ lie in L_0 for some functions $\varphi_{t+1},\ldots,\varphi_n$ with no linear terms. Then arguing as in (1) and using Remark 5.3, one shows that L_0 is contained in the subalgebra $L_0(t)$ constructed in Example 5.2. Thus $L_0=L_0(t)$ due to the maximality of L_0.

Notice that if $x_2+\varphi_2,\ldots,x_n+\varphi_n$ lie in L_0 for some functions $\varphi_2,\ldots,\varphi_n$ with no linear terms, then also $x_1+\varphi_1$ lies in L_0 for some $\varphi_1\in\Lambda^{\Delta}(n,n)/\mathbb{C}1$ with no linear terms. Indeed, any open T-invariant subalgebra of L containing $x_2+\varphi_2,\ldots,x_n+\varphi_n$ and not containing $x_1+\varphi$ for any function $\varphi\in\Lambda^{\Delta}(n,n)/\mathbb{C}1$ with no linear terms, is properly contained in the maximal graded subalgebra of type $(1,\ldots,1|0,\ldots,0)$, hence it is not maximal. □

Corollary 5.5. *The Lie superalgebra $SHO^{\sim}(n,n)$ has, up to conjugation, only one irreducible $\mathbb{Z}$-grading: the grading of type* $(1,\ldots,1|0,\ldots,0)$.

5.2. The Lie superalgebra $SKO^{\sim}(n,n+1)$

Let n be odd. The Lie superalgebra $SKO^{\sim}(n,n+1)$ is the subalgebra of $KO(n,n+1)$ defined as follows:

$$SKO^{\sim}(n,n+1)=\{X\in KO(n,n+1)\mid X(F\omega_\beta)=0\}$$

where ω_β is the volume form attached to the divergence div_β for $\beta=(n+2)/n$ and $F=1+\xi_1\ldots\xi_n\tau$. By Remark 2.7, $SKO^{\sim}(n,n+1)$ consists of vector fields X in $KO(n,n+1)$ such that $X(F)F^{-1}+div_\beta(X)=0$, where $\beta=(n+2)/n$.

Using the isomorphism between $KO(n,n+1)$ and $\Lambda(n,n+1)$ with bracket (4.1), it is possible to realize $SKO^{\sim}(n,n+1)$ as follows (cf. [18, §2]):

$$SKO^{\sim}(n,n+1)=(1+\xi_1\ldots\xi_n\tau)\Lambda^{\Delta'}(n,n+1)$$

where $\Lambda^{\Delta'}(n,n+1)=\{f\in\Lambda(n,n+1)\mid\Delta'(f)=0\}$ and $\Delta':=div_{(n+2)/n}=\Delta+(E-(n+2))\partial/\partial\tau$. Equivalently, $SKO^{\sim}(n,n+1)$ can be identified with the space $\Lambda(n,n+1)^{\Delta'}$ with the following deformed bracket:

$$[f,g]=[f,g]_{ko}+\alpha(fg)\tag{5.2}$$

where $[\cdot,\cdot]_{ko}$ denotes the bracket in the Lie superalgebra $KO(n,n+1)$ and $\alpha(b)=[\xi_1\ldots\xi_n\tau,b]_{ko}-2b\xi_1\ldots\xi_n$ if b is a monomial in the x_i, and $\alpha(b)=0$ for all other monomials ([9], [17, Example 6.3]). The superalgebra $SKO^{\sim}(n,n+1)$ is simple for $n\geqslant3$ (n odd).

Remark 5.6. If $F=1+\xi_1\ldots\xi_n\tau$ and $\beta\neq(n+2)/n$, then $\{X\in KO(n,n+1)\mid X(F\omega_\beta)=0\}=\{X\in SKO(n,n+1)\mid X(F)=0\}$. In particular this is a proper subalgebra of $KO(n,n+1)$ which is not transitive.

In what follows we will identify $SKO^{\sim}(n, n+1)$ with $\Lambda(n, n+1)^{\Delta'}$ with bracket (5.2). Then the standard maximal torus is $T = \langle \tau + \frac{n+2}{n}\Phi,\ x_i\xi_i - x_{i+1}\xi_{i+1} \mid i = 1, \dots, n-1 \rangle$, where $\Phi = \sum_{i=1}^{n} x_i\xi_i$.

Example 5.7. Let us define the following valuation ν on $\Lambda(n, n+1)$:

$$\nu(x_i) = 1, \qquad \nu(\xi_i) = 0, \qquad \nu(\tau) = 1$$

and let us consider the following filtration of $L = SKO^{\sim}(n, n+1)$:

$$L_j = \{x \in \Lambda^{\Delta'}(n, n+1) \mid \nu(x) \geqslant j+1\}.$$

Then $\overline{Gr\,L} \cong SKO'(n, n+1; (n+2)/n)$ with respect to the $\mathbb{Z}$-grading of type $(1, \dots, 1|0, \dots, 0, 1)$. It follows, using Corollaries 1.12 and 4.25, that L_0 is a maximal open subalgebra of L.

Example 5.8. On $\Lambda(n, n+1)$, for any fixed integer t, $1 \leqslant t \leqslant n$, let us define the following valuation ν:

$$\nu(x_i) = 1, \quad \nu(\xi_i) = 1 \quad \text{for } i = 1, \dots, t,$$
$$\nu(x_i) = 2, \quad \nu(\xi_i) = 0 \quad \text{for } i = t+1, \dots, n, \qquad \nu(\tau) = 2,$$

where by τ we denoted the $(n+1)$th odd indeterminate of $\Lambda(n, n+1)$. Let us define the following filtration of $L = SKO^{\sim}(n, n+1)$:

$$L_j(t) = \{x \in \Lambda^{\Delta'}(n, n+1) \mid \nu(x) \geqslant j+2\}.$$

Then $\overline{Gr\,L} \cong SKO'(n, n+1; (n+2)/n)$ with respect to the $\mathbb{Z}$-grading of type $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0, 2)$ with $n-t+1$ 2's and $n-t$ zeros. It follows, using Corollaries 1.12 and 4.25, that $L_0(t)$ is a maximal regular subalgebra of L for every $t = 2, \dots, n$.

Example 5.9. Let us fix an integer t such that $2 \leqslant t \leqslant n$. Let us consider on $\Lambda^{\Delta'}(n, n+1)$ the same valuation as the one defined in Example 4.21 and let us consider the subspaces $S_i(t)$ of $L = SKO^{\sim}(n, n+1)$ defined as follows:

$$S_i(t) = \{f \in \Lambda^{\Delta'}(n, n+1) \mid \nu(f) \geqslant i+2\} + \left\langle 1, \tau + \frac{n+2}{n}\Phi \right\rangle \quad \text{if } i \leqslant 0,$$
$$S_i(t) = \{f \in \Lambda^{\Delta'}(n, n+1) \mid \nu(f) \geqslant i+2\} \quad \text{if } i > 0.$$

The subspaces $S_i(t)$ define a filtration of L having depth 1 if $t = n$ and having depth 2 if $t < n$. One has:

$$\overline{Gr\,L} \cong SHO(n, n) \otimes \Lambda(\eta) + \mathfrak{a}$$

with respect to the grading of type $(1,\dots,1,2,\dots,2|1,\dots,1,0,\dots,0)$ of $SHO(n,n)$, with $n-t$ 2's and $n-t$ zeros, and $\deg(a)=0$ for every $a\in\mathfrak{a}$, where

$$\mathfrak{a}=\mathbb{C}\left(\frac{\partial}{\partial\eta}-\xi_1\dots\xi_n\otimes\eta\right)+\mathbb{C}\xi_1\dots\xi_n+\mathbb{C}\left(E-2+\frac{n+2}{n}\Phi+2\eta\frac{\partial}{\partial\eta}\right).$$

Since the grading of type $(1,\dots,1,2,\dots,2|1,\dots,1,0,\dots,0)$, with $n-t$ 2's and $n-t$ zeros, is an irreducible grading of $SHO(n,n)$ for $t=2,\dots,n$, $S_0(t)$ is a maximal subalgebra of L for every $t=2,\dots,n$, by Corollary 1.12.

Remark 5.10. The subspaces $\mathbb{C}1$, $\mathbb{C}x_i$, $\mathbb{C}\xi_{i_1}\dots\xi_{i_h}$ and $\mathbb{C}x_j\xi_{i_1}\dots\xi_{i_h}$ with $j\neq i_1\neq\cdots\neq i_h$, are one-dimensional T-weight spaces of $SKO^{\sim}(n,n+1)$. Besides, the subspaces $\{f\in\langle\xi_{i_1}\dots\xi_{i_h}\tau, x_j\xi_j\xi_{i_1}\dots\xi_{i_h}\rangle \mid \Delta'(f)=0\}$ and $\{f\in\langle x_k\xi_{i_1}\dots\xi_{i_h}\tau, x_kx_j\xi_j\xi_{i_1}\dots\xi_{i_h}, k\neq i_1,\dots,i_h\rangle \mid \Delta'(f)=0\}$ are T-weight spaces.

Theorem 5.11. *Let $L=SKO^{\sim}(n,n+1)$ with n odd, $n\geqslant 3$. All maximal open subalgebras of L are, up to conjugation, the (non-graded) subalgebras L_0, $L_0(t)$, and $S_0(t)$, with $t=2,\dots,n$, constructed in Examples* 5.7, 5.8, *and* 5.9, *respectively.*

Proof. Let L_0 be a maximal open subalgebra of L. The same argument as in the proof of Theorem 2.11 shows that L_0 is regular. Therefore, by Remark 2.1, we can assume that L_0 is invariant with respect to the torus T of $SKO^{\sim}(n,n+1)$. It follows that L_0 decomposes into the direct product of T-weight spaces.

Note that the elements ξ_i cannot lie in L_0 since they are not exponentiable.

We distinguish two cases:

Case I. 1 does not lie in L_0. We may assume that one of the following cases occurs:

(1) the elements $x_1,\dots,x_n$ lie in L_0. It follows that the T-invariant complement of L_0 contains the subalgebra $\Lambda(\xi_1,\dots,\xi_n)$, i.e., the T-invariant complement of the maximal subalgebra constructed in Example 5.7. Since L_0 is maximal, it coincides with the subalgebra constructed in Example 5.7;
(2) there exists some $t=2,\dots,n$ such that the elements $x_1,\dots,x_t$ do not lie in L_0 and the elements $x_{t+1},\dots,x_n$ do. It follows that the T-invariant complement of L_0 contains the subspace $\langle 1,\xi_j,x_j \mid j=1,\dots,t\rangle\otimes\Lambda(\xi_{t+1},\dots,\xi_n)$, i.e., the T-invariant complement of the subalgebra $L_0(t)$ of L constructed in Example 5.8. By the maximality of L_0 we conclude that L_0 coincides with $L_0(t)$.

Notice that if the elements $x_2,\dots,x_n$ lie in L_0, then also x_1 does. Indeed any open regular subalgebra of L containing $x_2,\dots,x_n$ and not containing x_1 and 1 is contained in the maximal subalgebra constructed in Example 5.7.

Case II. 1 lies in L_0. Using the definition of the deformed bracket defined in $SKO^{\sim}(n,n+1)$, one has:

$$\bigl[1,[1,x_i]\bigr]=\pm 2\xi_1\dots\hat{\xi}_i\dots\xi_n$$

where by $\xi_1 \dots \hat{\xi_i} \dots \xi_n$ we mean the product of all ξ_j's except ξ_i. It follows that, if L_0 contains 1, then it cannot contain the elements $x_{i_1}, \dots, x_{i_{n-1}}$ for $i_1 \neq \dots \neq i_{n-1}$, because the subalgebra generated by $1, x_{i_1}, \dots, x_{i_n}$ contains the elements ξ_j's which are not exponentiable. We may therefore assume that L_0 contains the elements $x_{t+1}, \dots, x_n$ for some $t = 2, \dots, n$ and does not contain $x_1, \dots, x_t$. Using Remark 5.10 and the same arguments as in the proof of Theorem 5.4, one then shows that L_0 is contained in the subalgebra $S_0(t)$ constructed in Example 5.9. By the maximality of L_0, $L_0 = S_0(t)$. □

Corollary 5.12. *The Lie superalgebra* $SKO^{\sim}(n, n+1)$ *has no irreducible* $\mathbb{Z}$*-gradings.*

6. Maximal regular subalgebras of $E(1,6)$ and $E(3,6)$

6.1. The Lie superalgebra $E(1,6)$

Let us consider the contact Lie superalgebra $K(1,6)$ and let us identify it with the polynomial superalgebra $\Lambda(1,6)$ with the contact bracket via the isomorphism $\varphi : \Lambda(1,6) \to K(1,6)$, as described in Section 2. In this case, since the number of odd indeterminates is 6, let us denote them by ξ_i and η_i for $i = 1,2,3$, and choose the contact form $\tau' = dt + \sum_{i=1}^{3} (\xi_i \, d\eta_i + \eta_i \, d\xi_i)$.

The $\mathbb{Z}$-grading of type $(2|1,1,1,1,1,1)$ of $W(1,6)$ induces on $K(1,6)$ the irreducible grading $K(1,6) = \prod_{j \geqslant -2} \mathfrak{g}_j$ where $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \oplus \mathbb{C}c$, $[\mathfrak{g}_0, \mathfrak{g}_0] \cong sl_4$ and $\mathfrak{g}_{-1} \cong \Lambda^2 \mathbb{C}^4$, where $\mathbb{C}^4$ denotes the standard sl_4-module, $\mathfrak{g}_1 \cong \mathfrak{g}_{-1}^* \oplus \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-$, as $[\mathfrak{g}_0, \mathfrak{g}_0]$-modules, with $\mathfrak{g}_1^+ \cong S^2\mathbb{C}^4$ and $\mathfrak{g}_1^- \cong S^2(\mathbb{C}^4)^*$.

The Lie superalgebra $E(1,6)$ is the graded subalgebra of $K(1,6)$ generated by $\mathfrak{g}_{-1} + \mathfrak{g}_0 + (\mathfrak{g}_{-1}^* + \mathfrak{g}_1^+)$ (cf. [17, Example 5.2], [10, §4.2], [21, §3]). It follows that the $\mathbb{Z}$-grading of type $(2|1,1,1,1,1,1)$ induces on $E(1,6)$ an irreducible grading, called the *principal* grading, where ξ_3 is the highest weight vector of $\mathfrak{g}_{-1} = \langle \xi_i, \eta_i \rangle$ and $t\eta_3$, $\xi_1\eta_2\eta_3$ are the lowest weight vectors of $\mathfrak{g}_{-1}^* = \langle t\xi_i, t\eta_i \rangle$ and $\mathfrak{g}_1^+$, respectively. Notice that

$$\mathfrak{g}_1^+ = \langle \xi_1\xi_2\xi_3, \xi_1\eta_2\eta_3, \xi_2\eta_1\eta_3, \xi_3\eta_1\eta_2, \xi_1(\xi_2\eta_2 + \xi_3\eta_3), \xi_2(\xi_1\eta_1 + \xi_3\eta_3), \eta_3(\xi_1\eta_1 - \xi_2\eta_2),$$
$$\xi_3(\xi_1\eta_1 + \xi_2\eta_2), \eta_2(\xi_1\eta_1 - \xi_3\eta_3), \eta_1(\xi_2\eta_2 - \xi_3\eta_3) \rangle$$

and $\mathfrak{g}_1^-$ is obtained from $\mathfrak{g}_1^+$ exchanging ξ_i with η_i for every $i = 1,2,3$. The standard maximal torus is $T = \langle t, \xi_i\eta_i \mid i = 1,2,3 \rangle$.

Remark 6.1. The $\mathbb{Z}$-gradings of $E(1,6)$ are parametrized, up to conjugation, by elements $(a|b_1, b_2, b_3, b_4, b_5, b_6)$ such that $a = \deg t = -\deg(\partial/\partial t) \in \mathbb{N}$, $b_i = \deg \xi_i = -\deg(\partial/\partial \xi_i) \in \mathbb{Z}$ for $i = 1,2,3$, $b_{i+3} = \deg \eta_i = -\deg(\partial/\partial \eta_i) \in \mathbb{Z}$ and $b_i + b_{3+i} = a$ (cf. [10, §5.4]). The $\mathbb{Z}$-gradings of type $(1|1,1,1,0,0,0)$ and $(1|1,1,0,0,0,1)$ of $K(1,6)$ induce on $E(1,6)$ irreducible gradings by Remark 1.13, since $E(1,6)$ is a simple Lie superalgebra. These two gradings are not conjugate since the negative part of $(1|1,1,1,0,0,0)$ is generated by the elements $1, \eta_i, \eta_i\eta_j$ for $i, j = 1,2,3$, and has therefore dimension $(4|3)$, while the negative part of $(1|1,1,0,0,0,1)$ is generated by the elements $1, \eta_1, \eta_2, \xi_3, \xi_3\eta_2, \xi_3\eta_1, \eta_1\eta_2, \xi_3\eta_1\eta_2$, and has therefore dimension $(4|4)$.

Remark 6.2. Let us consider the $\mathbb{Z}$-grading induced on $E(1,6)$ by the grading of type $(2|2,1,1,0,1,1)$ of $K(1,6)$. With respect to this grading $E(1,6)_0 \cong gl_2 \otimes \Lambda(1) \oplus W(0,1) \oplus sl_2$

and $E(1,6)_{-1}$ is isomorphic, as an $E(1,6)_0$-module, to $\mathbb{C}^4 \otimes \Lambda(1)$ where $\mathbb{C}^4$ is the standard so_4-module. In particular, $E(1,6)_{-1}$ is an irreducible $E(1,6)_0$-module. Besides, $E(1,6)_{-2} = [E(1,6)_{-1}, E(1,6)_{-1}] = \Lambda(1)$.

Theorem 6.3. *All maximal open regular subalgebras of* $L = E(1,6)$ *are, up to conjugation, the graded subalgebras of type* $(2|1,1,1,1,1,1)$, $(2|2,1,1,0,1,1)$, $(1|1,1,1,0,0,0)$, $(1|1,1,0,0,0,1)$.

Proof. Let L_0 be a maximal open regular subalgebra of L. By Remark 2.1, we can assume that L_0 is invariant with respect to the standard torus T of $E(1,6)$. Therefore L_0 decomposes into the direct product of T-weight spaces. Notice that $\mathbb{C}1$, $\mathbb{C}\xi_i$, $\mathbb{C}\eta_i$, for $i = 1,2,3$, $\mathbb{C}\xi_i\eta_j$, $\mathbb{C}\xi_i\xi_j$, $\mathbb{C}\eta_i\eta_j$, for $i \neq j$, $\mathbb{C}\xi_i\eta_j\eta_k$, for $i \neq j \neq k$, and $\mathbb{C}\xi_1\xi_2\xi_3$, are one-dimensional T-weight spaces. Note also that the vector field $\partial/\partial t$ cannot lie in L_0 since it is not exponentiable. It follows that, the elements ξ_i and η_i cannot lie both in L_0 for any fixed i, since $[\xi_i, \eta_i] = -1$ and $\varphi(1) = 2\partial/\partial t$. We may therefore assume, up to conjugation, that one of the following cases occurs:

(1) L_0 contains no ξ_i and no η_i. Then the T-invariant complement of L_0 contains the T-invariant complement of the maximal subalgebra $\bar{\mathfrak{g}}_{\geqslant 0}$ of L of type $(2|1,1,1,1,1,1)$, hence $L_0 = \bar{\mathfrak{g}}_{\geqslant 0}$;
(2) ξ_1 lies in L_0, $\xi_i \notin L_0$ for any $i \neq 1$, $\eta_j \notin L_0$ for any j. It follows that the T-invariant complement of L_0 contains the T-invariant complement of the maximal subalgebra $\bar{\mathfrak{g}}'_{\geqslant 0}$ of L of type $(2|2,1,1,0,1,1)$, hence $L_0 = \bar{\mathfrak{g}}'_{\geqslant 0}$;
(3) the elements ξ_i lie in L_0 for every $i = 1,2,3$. It follows that the T-invariant complement of L_0 contains the T-invariant complement of the maximal subalgebra $\bar{\mathfrak{g}}''_{\geqslant 0}$ of L of type $(1|1,1,1,0,0,0)$, hence $L_0 = \bar{\mathfrak{g}}''_{\geqslant 0}$;
(4) $\xi_1, \xi_2, \eta_3 \in L_0$ and the elements $\xi_3, \eta_1, \eta_2 \notin L_0$. Then L_0 is the maximal graded subalgebra of L of type $(1|1,1,0,0,0,1)$.

Notice that if ξ_1, ξ_2 lie in L_0 and $\eta_1, \eta_2, \eta_3, \xi_3$ do not, then the T-invariant complement of L_0 contains the T-invariant complement of both the graded subalgebras of type $(1|1,1,1,0,0,0)$ and $(1|1,1,0,0,0,1)$, and this is impossible since it contradicts the maximality of L_0. □

Corollary 6.4. *All irreducible* $\mathbb{Z}$*-gradings of* $E(1,6)$ *are, up to conjugation, the gradings of type* $(2|1,1,1,1,1,1)$, $(2|2,1,1,0,1,1)$, $(1|1,1,1,0,0,0)$ *and* $(1|1,1,0,0,0,1)$.

6.2. The Lie superalgebra $E(3,6)$

The Lie superalgebra $E(3,6)$ has the following structure: $E(3,6)_{\bar{0}} = W_3 \oplus \Omega^0(3) \otimes sl_2$ and $E(3,6)_{\bar{1}} \cong \Omega^1(3)^{-1/2} \otimes \mathbb{C}^2$ as an $E(3,6)_{\bar{0}}$-module (cf. Definition 2.5 and [10, §4.4]). The bracket between two odd elements is defined as follows: we identify $\Omega^2(3)^{-1}$ with W_3 (via contraction of vector fields with the volume form) and $\Omega^3(3)^{-1}$ with $\Omega^0(3)$. Then, for $\omega_1, \omega_2 \in \Omega^1(3)^{-1/2}$, $u_1, u_2 \in \mathbb{C}^2$, we have:

$$[\omega_1 \otimes u_1, \omega_2 \otimes u_2] = (\omega_1 \wedge \omega_2) \otimes (u_1 \wedge u_2) + \frac{1}{2}(d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2) \otimes u_1 \cdot u_2$$

where $u_1 \cdot u_2$ denotes an element in the symmetric square of $\mathbb{C}^2$, i.e., an element in sl_2, and $u_1 \wedge u_2$ an element in the skew-symmetric square of $\mathbb{C}^2$, i.e., a complex number. In order to simplify notation we will write the elements of $E(3,6)$ omitting the $\otimes$ sign. Let us denote by H, E, F

the standard basis of sl_2 and by $\{v_1, v_2\}$ the standard basis of $\mathbb{C}^2$. Then $E = v_1^2/2$, $F = -v_2^2/2$, $H = -v_1 \cdot v_2$ and $v_1 \wedge v_2 = 1$. Let us fix the maximal torus $T = \langle H, x_i\partial/\partial x_i, i = 1, 2, 3\rangle$.

Remark 6.5. The $\mathbb{Z}$-gradings of $E(3, 6)$ are parametrized by quadruples $(a_1, a_2, a_3, \varepsilon)$ where $a_i = \deg x_i = -\deg(\partial/\partial x_i) \in \mathbb{N}$, $\varepsilon = \deg v_1 = -\deg v_2 \in \frac{1}{2}\mathbb{Z}$ and the following relations hold [10, §5.4]:

$$\varepsilon + \frac{1}{2}\sum_{i=1}^{3} a_i \in \mathbb{Z}, \qquad \deg d = -\frac{1}{2}\sum_{i=1}^{3} a_i, \qquad \deg E = -\deg F = 2\varepsilon, \qquad \deg H = 0.$$

The grading of type $(2, 2, 2, 0)$ is called the *principal* grading of $E(3, 6)$: it has depth 2 and its 0th graded component is isomorphic to $sl_3 \oplus sl_2 \oplus \mathbb{C}$ (cf. [17, Example 5.4]). $E(3, 6)_{-1}$ and $E(3, 6)_1$ are isomorphic, as $[E(3, 6)_0, E(3, 6)_0]$-modules, to $\mathbb{C}^3 \boxtimes \mathbb{C}^2$ and $S^2\mathbb{C}^3 \boxtimes \mathbb{C}^2 \oplus (\mathbb{C}^3)^* \boxtimes \mathbb{C}^2$, respectively, where $\mathbb{C}^3$ and $\mathbb{C}^2$ denote the standard sl_3 and sl_2-modules, respectively. In particular $E(3, 6)_{-1} = \langle dx_i\, v_j \mid i = 1, 2, 3;\ j = 1, 2\rangle$ has highest weight vector $dx_1\, v_1$; $E(3, 6)_1 = \langle x_i\, dx_j\, v_k \mid i, j = 1, 2, 3, k = 1, 2\rangle$ has lowest weight vectors $x_3\, dx_3\, v_2$ and $(x_2\, dx_3 - x_3\, dx_2)v_2$. Notice that the elements $dx_i\, v_1$ and $dx_i\, v_2$ lie in $E(3, 6)_{-1}$ for every $i = 1, 2, 3$. It follows that $[E(3, 6)_{-1}, E(3, 6)_{-1}] \neq 0$ since, $[dx_i\, v_1, dx_j\, v_2] = \partial/\partial x_k$ for $i \neq j \neq k$. By Remark 1.13, $[E(3, 6)_{-1}, E(3, 6)_{-1}] = E(3, 6)_{-2}$.

Let us now consider the $\mathbb{Z}$-grading of type $(2, 1, 1, 0)$: this is an irreducible grading of depth 2 whose 0th graded component is spanned by the elements: E, F, H, $x_1\partial/\partial x_1$, $x_i\partial/\partial x_j$, $x_i x_j\partial/\partial x_1$, $dx_1\, v_h$, $x_i\, dx_j\, v_h$ for $i, j = 2, 3$ and $h = 1, 2$. One can check that $E(3, 6)_0 = [E(3, 6)_0, E(3, 6)_0] + \mathbb{C}c$, where $c = 2x_1\partial/\partial x_1 + x_2\partial/\partial x_2 + x_3\partial/\partial x_3$, and $[E(3, 6)_0, E(3, 6)_0]$ is isomorphic to $sl_2 \otimes \Lambda(2) + W(0, 2)$. Besides, $E(3, 6)_{-1} = \langle x_i\partial/\partial x_1, \partial/\partial x_i, dx_i\, v_1, dx_i\, v_2, i = 2, 3\rangle$ is isomorphic, as a $E(3, 6)_0$-module, to $\mathbb{C}^2 \otimes \Lambda(2)$ where $\mathbb{C}^2$ is the standard sl_2-module. Note that $[E(3, 6)_{-1}, E(3, 6)_{-1}] \neq 0$ thus $[E(3, 6)_{-1}, E(3, 6)_{-1}] = E(3, 6)_{-2}$ by Remark 1.13.

Finally, the grading of type $(1, 1, 1, 1/2)$ is irreducible by Remark 1.13, since it has depth 1.

The $\mathbb{Z}$-gradings of type $(2, 2, 2, 0)$, $(2, 1, 1, 0)$ and $(1, 1, 1, 1/2)$ satisfy the hypotheses of Proposition 1.11(b), therefore the corresponding graded subalgebras $\prod_{j \geqslant 0} E(3, 6)_j$ of $E(3, 6)$ are maximal.

Theorem 6.6. *All maximal open regular subalgebras of* $L = E(3, 6)$ *are, up to conjugation, the graded subalgebras of type* $(2, 2, 2, 0)$*,* $(2, 1, 1, 0)$*,* $(1, 1, 1, 1/2)$*.*

Proof. Let L_0 be a maximal open regular subalgebra of L. By Remark 2.1, we can assume that L_0 is invariant with respect to the maximal torus T of $E(3, 6)$. Therefore L_0 decomposes into the direct product of T-weight spaces. Note that $\mathbb{C}\partial/\partial x_j$, $\mathbb{C}x_i\partial/\partial x_j$ for $i \neq j$, $\mathbb{C}\, dx_i\, v_k$ and $\mathbb{C}F$ are one-dimensional weight spaces. The vector fields $\partial/\partial x_i$ cannot lie in L_0 since they are not exponentiable. It follows that if $dx_i\, v_1$ lies in L_0 for some i then $dx_j\, v_2$ cannot lie in L_0 for any $j \neq i$, since, for $i \neq j$, $[dx_i\, v_1, dx_j\, v_2] = \epsilon(ijk)\partial/\partial x_k$, where $k \neq i, j$ and $\epsilon(ijk)$ is the sign of the permutation ijk. One can check that if $dx_1\, v_1$ lies in L_0 then, due to the maximality of L_0, either $dx_i\, v_1$ lies in L_0 for every $i = 1, 2, 3$, or $dx_1\, v_2$ does. We may therefore assume that one of the following cases occurs:

(1) L_0 contains the elements $dx_1\, v_1$ and $dx_1\, v_2$. It follows that the T-invariant complement of L_0 contains the T-invariant complement of the maximal graded subalgebra $\bar{\mathfrak{g}}_{\geqslant 0}$ of L of type $(2, 1, 1, 0)$. Thus $L_0 = \bar{\mathfrak{g}}_{\geqslant 0}$.

(2) L_0 contains the elements $dx_i\, v_1$ for every $i = 1, 2, 3$. As a consequence the elements $dx_i\, v_2$, $i = 1, 2, 3$, and F lie in the T-invariant complement of L_0. It follows that L_0 is the maximal graded subalgebra of type $(1, 1, 1, 1/2)$.
(3) L_0 does not contain the elements $dx_i v_k$ for any i, k. It follows that L_0 is the maximal graded subalgebra of L of type $(2, 2, 2, 0)$. □

Corollary 6.7. *All irreducible gradings of $E(3, 6)$ are, up to conjugation, the gradings of type $(2, 2, 2, 0)$, $(2, 1, 1, 0)$ and $(1, 1, 1, 1/2)$.*

7. On primitive pairs and filtered deformations

Proposition 7.1. *Let L be an artinian semisimple linearly compact Lie superalgebra. If L has a completed irreducible grading then*:

$$L = S \otimes \Lambda(n) + F \tag{7.1}$$

where S is a simple linearly compact Lie superalgebra, F is a subalgebra of $\mathfrak{a} \otimes \Lambda(n) + W(0, n)$ whose projection on $W(0, n)$ is transitive, and $\mathfrak{a}$ is the subalgebra of outer derivations of S. Let $\mathfrak{a}_0 = \{a(0) \mid a(\xi) \in$ projection of F on $\mathfrak{a} \otimes \Lambda(n)\} \subset \mathfrak{a}$. Then the irreducible grading of L is obtained by extending to L an irreducible grading of $S + \mathfrak{a}_0$ through the condition $\deg(\tau) = 0$ for every $\tau \in \Lambda(n)$.

Proof. By Theorem 1.4 we have:

$$\bigoplus_{i=1}^{r} \big(S_i \mathbin{\hat{\otimes}} \Lambda(m_i, n_i)\big) \subset L \subset \bigoplus_{i=1}^{r} \big((Der\, S_i) \mathbin{\hat{\otimes}} \Lambda(m_i, n_i) + 1 \otimes W(m_i, n_i)\big).$$

Suppose that L has a completed irreducible grading $L = \prod_j \mathfrak{g}_j$. Since $S_i \mathbin{\hat{\otimes}} \Lambda(m_i, n_i)$ is an ideal of L, $(S_i \mathbin{\hat{\otimes}} \Lambda(m_i, n_i)) \cap \mathfrak{g}_{-1}$ is either 0 or the whole $\mathfrak{g}_{-1}$ for each i. Hence $r = 1$ and $L = S \otimes \Lambda(m, n) + F$ where F is a subalgebra of $\mathfrak{a} \otimes \Lambda(m, n) + W(m, n)$ whose projection on $W(m, n)$ is transitive by Theorem 1.4.

We recall that a $\mathbb{Z}$-grading of the Lie superalgebra L is defined by an ad-diagonalisable element D of $Der\, L$, i.e, by a one-dimensional torus (cf. [10, §5.4]). The subalgebra $\tilde{L} = S \mathbin{\hat{\otimes}} \Lambda(m, n)$ of L is D-invariant. But all maximal tori of $Der\, L$ are conjugate by Theorem 1.7, hence we may assume that D lies in the standard torus of $Der\, L$, which is the sum of a maximal torus of $Der\, S$ and the standard maximal torus of $W(m, n)$. This means that the grading of L is obtained by taking a grading of S (thus of $S + \mathfrak{a}_0$) and extending it to L by letting $\deg x_i = s_i$, $\deg \xi_j = t_j$. Let $L_0 = \prod_{j \geqslant 0} \mathfrak{g}_j$. Then the same argument as in the proof of Theorem 1.9(a) shows that F is contained in L_0, since L_0 is fundamental. In particular all even elements of L_0 are exponentiable, hence the transitivity of the projection of F on $W(m, n)$ implies $m = 0$. Finally, by the irreducibility of the grading, $t_j = 0$ for every j and the grading of $S + \mathfrak{a}_0$ is irreducible. □

Corollary 7.2. *Let (L, L_0) be a primitive pair and consider its irreducible Weisfeiler filtration. Then the completion of the associated graded superalgebra, divided by the maximal ideal in its negative part, is a semisimple Lie superalgebra of the form* (7.1).

A linearly compact Lie superalgebra L whose associated graded is $\mathfrak{g}$ is called a *filtered deformation* of the completion $\bar{\mathfrak{g}}$ of $\mathfrak{g}$. Of course, $\bar{\mathfrak{g}}$ is a filtered deformation of $\bar{\mathfrak{g}}$, called the *trivial* filtered deformation; note that $\bar{\mathfrak{g}}$ is simple if and only if $\mathfrak{g}$ is. If L is simple, it is called a *simple* filtered deformation of $\bar{\mathfrak{g}}$. If $\bar{\mathfrak{g}}$ is the only simple filtered deformation of $\bar{\mathfrak{g}}$, we shall say that $\bar{\mathfrak{g}}$ has no simple filtered deformations.

Remark 7.3. We recall that if $\mathfrak{g} = \bigoplus_{j=-d}^{\infty} \mathfrak{g}_j$ is a graded Lie superalgebra and $\mathfrak{g}_0$ contains an element z such that $ad(z)|_{\mathfrak{g}_j} = j\,Id$, then $\bar{\mathfrak{g}}$ has no non-trivial filtered deformations (cf. [9, Corollary 2.2]). It follows that the Lie superalgebras $\bar{\mathfrak{g}}$ of the form (7.1) listed below have no non-trivial filtered deformations, since they contain the grading operator:

(a) $\bar{\mathfrak{g}} = S \otimes \Lambda(t) + F$ with $S = W(m,n)$, $K(2k+1,n)$, $KO(n,n+1)$, $E(1,6)$, $E(4,4)$, $E(3,6)$ or $E(3,8)$ and $t \geqslant 0$;
(b) $\bar{\mathfrak{g}} = S(1,2) \otimes \Lambda(t) + F$ with respect to the $\mathbb{Z}$-grading of $S(1,2)$ of type $(1|1,0)$, where $t \geqslant 0$. Here the grading operator is $z = x\partial/\partial x + \xi_1\partial/\partial\xi_1$;
(c) $\bar{\mathfrak{g}} = Der\,S(1,2) \otimes \Lambda(t) + F'$ with respect to the $\mathbb{Z}$-grading of $Der\,S(1,2)$ of type $(2|1,1)$, where $t \geqslant 0$ and $F' \subset W(0,t)$. Here the grading operator is $z = 2x\partial/\partial x + \xi_1\partial/\partial\xi_1 + \xi_2\partial/\partial\xi_2$.

Proposition 7.4. *Let $L = \prod_j L_j$ be a completed irreducible grading of the Lie superalgebra L of the form* (7.1) *with $n > 0$ and $S = S(m,h)$, for some $m > 2$. Then L has no simple filtered deformations.*

Proof. Suppose that $L = \overline{Gr\,M}$ for some Lie superalgebra M. We want to show that $M \neq [M,M]$ is not simple. Let $S = \prod_{j \geqslant -1} S_j$ be the corresponding completed irreducible grading of S and let $\tilde{S}$ be a maximal reductive subalgebra of S_0. Notice that, since $m > 2$, $\mathfrak{a}$ is a one-dimensional torus, therefore the subspaces $S_j\tau$ are $\tilde{S}$-submodules of L for every j and every element $\tau \in \Lambda(n)$, and F is a trivial $\tilde{S}$-module. We claim that $\mathfrak{a}_{\bar{1}}$ is not contained in $[M,M]$. Indeed, $\mathfrak{a}_{\bar{1}}$ can be obtained only from $[S_{-1}, \Lambda(n)S_{-1}]$, $[S_{-1}, \Lambda(n)S_0]$, $[S_0, \Lambda(n)S_{-1}]$, but under our hypotheses $S_{-1} \otimes S_{-1}$ and $S_{-1} \otimes S_0$ do not contain any one-dimensional $\tilde{S}$-submodule. Thus the thesis follows. $\square$

Theorem 7.5. *All maximal open subalgebras of $L = E(1,6)$ are, up to conjugation, the graded subalgebras listed in Theorem* 6.3.

Proof. Suppose that L_0 is a maximal open subalgebra of L which is not graded. Consider the Weisfeiler filtration associated to L_0 and its associated graded Lie superalgebra $Gr\,L$. Then, by Proposition 7.1, $\overline{Gr\,L}$ is of the form (7.1) and its growth and size are the same as those of L.

From Table 2 we see that the growth of $L = E(1,6)$ is 1 and its size is 32. Hence for $\overline{Gr\,L}$ of the form (7.1) the growth of S is 1 and size$(S)2^n = 32$. So it follows from Table 2 that $S = W(1,h)$, $K(1,h)$, $S(1,h)$ or $E(1,6)$ and $n = 0$ in the last case. If $S = W(1,h)$ or $K(1,h)$, then, by Remark 7.3(a), $E(1,6) = L = \overline{Gr\,L} = S \otimes \Lambda(n) + F$ for some $n \geqslant 0$ and some finite-dimensional subalgebra F of $W(0,n)$, which is impossible. If $S = S(1,h)$, then size$(\overline{Gr\,L}) = h2^h 2^n = 32$ if and only if $h = 2$ and $n = 2$. Then, by Remark 7.3(b) and (c), $S = S(1,2)$ with respect to the $\mathbb{Z}$-grading of principal type. Since a maximal torus of $Der\,S(1,2)$ has dimension 3, $\overline{Gr_{\geqslant 0}\,L}$ contains a torus $\hat{T}$ of dimension less than or equal to 3 containing the standard torus of $S(1,2)$. It follows that L_0 contains a torus $\tilde{T}$, which is the lift of $\hat{T}$, of dimension 2 or 3. The weights of $\tilde{T}$ on L/L_0 coincide with the weights of $\hat{T}$ on $Gr\,L/Gr_{\geqslant 0}\,L$. Since the dimension of a maximal torus of L is 4, $Gr_{<0}\,L$ contains a $\tilde{T}$-weight space of weight 0 of dimension greater than or equal to 1. But $S(1,2)_{-1}$ does not contain any weight vector of weight zero with respect to the standard torus of $S(1,2)$. Hence we get a contradiction. It follows that $S = E(1,6)$ and $\overline{Gr\,L} = E(1,6)$. Hence L_0 is a regular subalgebra of $E(1,6)$ and the theorem follows from Theorem 6.3. □

Theorem 7.6. *All maximal open subalgebras of $L = E(3,6)$ are, up to conjugation, the graded subalgebras listed in Theorem* 6.6.

Proof. Suppose that L_0 is a maximal open subalgebra of L which is not graded. Consider the Weisfeiler filtration associated to L_0 and its associated graded Lie superalgebra $Gr\,L$. Then, by Proposition 7.1, $\overline{Gr\,L}$ is of the form (7.1) and its growth and size are the same as those of L.

From Table 2 we see that the growth of L is 3 and its size is 12. Hence for $\overline{Gr\,L}$ of the form (7.1) the growth of S is 1 and $\text{size}(S)2^n = 12$. So it follows from Table 2 that $S = W(3,h)$, $K(3,h)$, $S(3,h)$ or $E(3,6)$ and $n = 0$ in the last case. If $S = W(1,h)$ or $K(1,h)$, then, by Remark 7.3(a), $E(3,6) = L = \overline{Gr\,L} = S \otimes \Lambda(n) + F$ for some $n \geqslant 0$ and some finite-dimensional subalgebra F of $W(0,n)$, which is impossible. If $S = S(3,h)$, then $n = 0$ by Proposition 7.4, and $\text{size}(S) = (2+h)2^h \neq 12$. Thus $S = E(3,6)$ and $\overline{Gr\,L} = E(3,6)$. Hence L_0 is a regular subalgebra of $E(3,6)$ and the theorem follows from Theorem 6.6. □

8. Maximal open subalgebras of $E(5,10)$

The Lie superalgebra $E(5,10)$ has the following structure (cf. [10, §4.3, 5.3]): $E(5,10)_{\bar{0}} \cong S_5 = S(5,0)$ and $E(5,10)_{\bar{1}} = d\Omega^1(5)$. $E(5,10)_{\bar{0}}$ acts on $E(5,10)_{\bar{1}}$ in the natural way and if $\omega_1, \omega_2 \in d\Omega^1(5)$ then $[\omega_1, \omega_2] = \omega_1 \wedge \omega_2$ where the identification between $\Omega^4(5)$ and W_5 is used. Let us fix the maximal torus $T = \langle x_i\partial/\partial x_i - x_{i+1}\partial/\partial x_{i+1} \mid i = 1,2,3,4\rangle$.

As in Section 1, for every vector field $X = \sum_{i=1}^5 P_i\partial/\partial x_i$ in S_5, we shall set $X(0) = \sum_{i=1}^5 P_i(0)\partial/\partial x_i$. Likewise, for every 2-form $\omega = \sum P_{ij}\,dx_i \wedge dx_j$ in $d\Omega^1(5)$, we shall set $\omega(0) = \sum P_{ij}(0)\,dx_i \wedge dx_j$.

Remark 8.1. The $\mathbb{Z}$-gradings of $E(5,10)$ are parametrized by quintuples of positive integers (a_1,a_2,a_3,a_4,a_5) such that $\sum_{i=1}^5 a_i \in 2\mathbb{N}$ where $a_i = \deg x_i = -\deg(\partial/\partial x_i)$ and $\deg d = -\frac{1}{4}\sum_{i=1}^5 a_i$ [10, §5.4].

If we define $\deg x_i = -\deg(\partial/\partial x_i) = 2$ and $\deg(dx_i) = -1/2$ we get a consistent irreducible grading of $E(5,10)$, called the *principal* grading of $E(5,10)$, with respect to which $E(5,10)_0 = sl_5$. One can check that $E(5,10)_{-1} \cong \Lambda^2\mathbb{C}^5$, where $\mathbb{C}^5$ is the standard sl_5-module, it is spanned by the 2-forms $dx_i \wedge dx_j$ and it has highest weight vector $dx_1 \wedge dx_2$; $E(5,10)_1$ is isomorphic to the highest component of $\mathbb{C}^5 \otimes \Lambda^2\mathbb{C}^5$, i.e., to the irreducible sl_5-module of highest weight $\pi_1 + \pi_2$, and has lowest weight vector $x_5\,dx_4 \wedge dx_5$. Notice that the 2-forms $dx_i \wedge dx_j$ lie in $E(5,10)_{-1}$ for every i, j, thus $[E(5,10)_{-1}, E(5,10)_{-1}] \neq 0$ since $[dx_i \wedge dx_j, dx_h \wedge dx_k] = \partial/\partial x_t$ for $i \neq j \neq h \neq k \neq t$. It follows from Remark 1.13 that $[E(5,10)_{-1}, E(5,10)_{-1}] = E(5,10)_{-2}$.

Let us consider the $\mathbb{Z}$-grading of type $(2,1,1,1,1)$: this is an irreducible grading of $E(5,10)$ of depth 2 whose 0th graded component is spanned by the elements $x_i\partial/\partial x_i - x_{i+1}\partial/\partial x_{i+1}$ for $i = 1,2,3,4$, $x_i\partial/\partial x_j$ for $i \neq j \neq 1$, $x_ix_j\partial/\partial x_1$ for $i, j \neq 1$, $dx_1 \wedge dx_i$ for $i \neq 1$, and by closed 1-forms in $\langle x_i\,dx_j \wedge dx_k \mid i,j,k \neq 1\rangle$. $E(5,10)_0$ is isomorphic to $S(0,4) + \mathbb{C}Z$, where

Z is the grading operator on $S(0,4)$ with respect to its principal grading, and $E(5,10)_{-1} = \langle x_i\partial/\partial x_1, \partial/\partial x_i, dx_i \wedge dx_j \mid i, j \neq 1\rangle$ is an irreducible $E(5,10)_0$-module with highest weight vector $x_2\partial/\partial x_1$. Finally, $E(5,10)_{-2} = [E(5,10)_{-1}, E(5,10)_{-1}] = \langle \partial/\partial x_1\rangle$.

Let us now consider the $\mathbb{Z}$-grading of type $(3,3,2,2,2)$: this is an irreducible grading of depth 3 whose 0th graded component is spanned by the following elements: $x_i\partial/\partial x_i - x_{i+1}\partial/\partial x_{i+1}$ for $i = 1,\dots,4$, $x_i\partial/\partial x_j$ for $i, j = 1, 2$ and $i, j = 3, 4, 5$, $i \neq j$, $dx_1 \wedge dx_2$ and the closed 2-forms in $\langle x_i\, dx_k \wedge dx_t \mid i, k, t = 3, 4, 5\rangle$. $E(5,10)_0$ is isomorphic to $(sl_3 \otimes \Lambda(1) + W(0,1)) \oplus sl_2$ and $E(5,10)_{-1} = \langle x_i\partial/\partial x_1, x_i\partial/\partial x_2, dx_1 \wedge dx_i, dx_2 \wedge dx_i \mid i = 3, 4, 5\rangle$ is isomorphic to $\mathbb{C}^3 \otimes \Lambda(1) \boxtimes \mathbb{C}^2$ where $\mathbb{C}^3$ and $\mathbb{C}^2$ denote the standard sl_3 and sl_2-modules, respectively. Finally, we note that $E(5,10)_{-2} = \langle \partial/\partial x_i, dx_i \wedge dx_j \mid i, j = 3, 4, 5\rangle$ and $E(5,10)_{-3} = \langle \partial/\partial x_i \mid i = 1, 2\rangle$. Therefore $[E(5,10)_{-1}, E(5,10)_{-1}] = E(5,10)_{-2}$ since $[dx_1 \wedge dx_i, dx_2 \wedge dx_j] = \partial/\partial x_k$ for $i \neq j \neq k$ and $[x_i\partial/\partial x_1, dx_1 \wedge dx_j] = dx_i \wedge dx_j$ for $i \neq j$. Besides, $[E(5,10)_{-2}, E(5,10)_{-1}] = E(5,10)_{-3}$ since $[E(5,10)_{-2}, E(5,10)_{-1}] \neq 0$.

Let us finally consider the $\mathbb{Z}$-grading of type $(2,2,2,1,1)$: this is an irreducible grading of depth 2 whose 0th graded component is isomorphic to $sl_2 \otimes \Lambda(\xi_1, \xi_2, \xi_3) + \langle \xi_i\partial/\partial\xi_j, \partial/\partial\xi_j, \xi_j(\sum_{k=1}^{3}\xi_k\partial/\partial\xi_k) \mid i, j = 1, 2, 3\rangle$. Besides, the -1st graded component of $E(5,10)$ with respect to this grading is isomorphic, as an $E(5,10)_0$-module, to $\mathbb{C}^2 \otimes \Lambda(3)$ where $\mathbb{C}^2$ is the standard sl_2-module. Since $[E(5,10)_{-1}, E(5,10)_{-1}] \neq 0$, $[E(5,10)_{-1}, E(5,10)_{-1}] = E(5,10)_{-2}$ by Remark 1.13.

The gradings of type $(2,1,1,1,1)$, $(3,3,2,2,2)$, $(2,2,2,1,1)$, $(2,2,2,2,2)$ satisfy the hypotheses of Proposition 1.11. It follows that the corresponding subalgebras $\prod_{j\geqslant 0} E(5,10)_j$ are maximal subalgebras of $E(5,10)$.

Remark 8.2. Let us consider the even elements $x_i\partial/\partial x_j$ for $i \neq j$, and the odd elements $dx_i \wedge dx_j$. Then the weight of $x_i\partial/\partial x_j$ with respect to T is different from the weight of $x_h\partial/\partial x_k$ for every $(h,k) \neq (i,j)$. Likewise, the weight of $dx_i \wedge dx_j$ with respect to T is different from the weight of $dx_h \wedge dx_k$ for every $(h,k) \neq (i,j)$.

Theorem 8.3. *Let L_0 be a maximal open T-invariant subalgebra of $L = E(5,10)$. Then L_0 is conjugate to one of the graded subalgebras of type* $(2,1,1,1,1)$, $(3,3,2,2,2)$, $(2,2,2,1,1)$, $(2,2,2,2,2)$.

Proof. Since L_0 is T-invariant, it decomposes into the direct product of weight spaces with respect to T. We analyze what T-weight vectors outside the maximal graded subalgebra of $E(5,10)$ of principal type may lie in L_0.

The elements $\partial/\partial x_i + Y$ cannot lie in L_0 for any vector field Y such that $Y(0) = 0$, since they are not exponentiable. It follows that if $i \neq j \neq k \neq h$, then the elements $x_\omega = dx_i \wedge dx_j + \omega$ and $x_\sigma = dx_k \wedge dx_h + \sigma$ cannot lie both in L_0 for any ω and σ in $E(5,10)_{\bar 1}$ such that $\omega(0) = \sigma(0) = 0$. Indeed, if x_ω and x_σ lie in L_0 then $[x_\omega, x_\sigma] = \partial/\partial x_s + Y$ for some vector field Y such that $Y(0) = 0$.

Now suppose that L_0 contains the odd element $x = dx_1 \wedge dx_2 + \varphi$ for some $\varphi \in E(5,10)_{\bar 1}$ such that $\varphi(0) = 0$. It follows that $dx_3 \wedge dx_4 + \omega$ and, similarly, $dx_3 \wedge dx_5 + \omega$ and $dx_4 \wedge dx_5 + \omega$ cannot lie in L_0 for any $\omega \in E(5,10)_{\bar 1}$ such that $\omega(0) = 0$.

Now, either (i) $dx_1 \wedge dx_j + \rho$ lies in L_0 for some $j \neq 2$ and some $\rho \in E(5,10)_{\bar 1}$ such that $\rho(0) = 0$, and we may assume that ρ has the same weight as $dx_1 \wedge dx_j$, or (ii) L_0 contains no element of the form $dx_1 \wedge dx_j + \mu$ for any $j \neq 2$ and any $\mu \in E(5,10)_{\bar 1}$ such that $\mu(0) = 0$. Let us analyze these two possibilities:

(i) Up to conjugation we can assume $j = 3$. Since $dx_1 \wedge dx_3 + \rho$ lies in L_0, L_0 contains no element of the form $dx_2 \wedge dx_4 + \omega$ and $dx_2 \wedge dx_5 + \omega$ for any $\omega \in E(5,10)_{\bar 1}$ such that $\omega(0) = 0$. The following two possibilities may then occur:

(i1) $dx_2 \wedge dx_3 + \omega$ does not lie in L_0 for any $\omega \in E(5,10)_{\bar 1}$ such that $\omega(0) = 0$. It follows that L_0 contains no vector field of the form $x_i \partial/\partial x_1 + Y$ for any $i \neq 1$ and any Y such that $Y(0) = 0$ of order greater than or equal to 2. Indeed, if such a vector field lies in L_0 then, if $i \neq 1, 2$, $[x_i \partial/\partial x_1 + Y, dx_1 \wedge dx_2 + \varphi] = dx_i \wedge dx_2 + \tau$ lies in L_0, for some form τ such that $\tau(0) = 0$, in contradiction to our assumptions. Similarly, if $x_2 \partial/\partial x_1 + Y$ lies in L_0 for some Y such that $Y(0) = 0$ of order greater than or equal to 2, then $[x_2 \partial/\partial x_1 + Y, dx_1 \wedge dx_3 + \varphi] = dx_2 \wedge dx_3 + \tau$ lies in L_0, for some τ such that $\tau(0) = 0$, in contradiction to our assumptions.

Using Remark 8.2, we can conclude that L_0 is contained in the maximal graded subalgebra of L of type $(2, 1, 1, 1, 1)$ and, due to its maximality, it coincides with it.

(i2) $dx_2 \wedge dx_3 + \tau$ lies in L_0 for some $\tau \in E(5,10)_{\bar 1}$ such that $\tau(0) = 0$. Then L_0 contains no element of the form $dx_1 \wedge dx_i + \omega$ for any $i = 4, 5$ and any $\omega \in E(5,10)_{\bar 1}$ such that $\omega(0) = 0$. As a consequence, the vector fields $x_i \partial/\partial x_j + Y$ cannot lie in L_0 for any $i = 4, 5$, $j = 1, 2, 3$, and any Y such that $Y(0) = 0$ of order greater than or equal to 2.

Notice that L_0 does not contain the elements $x_4\, dx_4 \wedge dx_5 + \sigma$ and $x_5\, dx_4 \wedge dx_5 + \sigma$ for any $\sigma \in E(5,10)_{\bar 1}$ such that $\sigma(0) = 0$ of order greater than or equal to 2. Indeed if $x_4\, dx_4 \wedge dx_5 + \sigma$ lies in L_0 for some σ such that $\sigma(0) = 0$ of order greater than or equal to 2, then $[dx_1 \wedge dx_2 + \varphi, x_4\, dx_4 \wedge dx_5 + \sigma] = x_4 \partial/\partial x_3 + Z$ lies in L_0, for some Z such that $Z(0) = 0$ of order greater than or equal to 2, in contradiction to our assumptions. Similarly for the elements $x_5\, dx_4 \wedge dx_5 + \sigma$.

Note that if a 2-form σ has the same weight as $x_4\, dx_4 \wedge dx_5$ (respectively $x_5\, dx_4 \wedge dx_5$), then $\sigma(0) = 0$ of order greater than or equal to 2. It follows, using Remark 8.2, that L_0 is contained in the graded subalgebra of L of type $(2, 2, 2, 1, 1)$ and thus coincides with it.

(ii) $dx_1 \wedge dx_j + \mu$ does not lie in L_0 for any $j \neq 2$ and any μ such that $\mu(0) = 0$. Then two possibilities may occur:

(ii1) $dx_2 \wedge dx_t + \nu$ lies in L_0 for some $t \neq 1, 2$ and some $\nu \in E(5,10)_{\bar 1}$ such that $\nu(0) = 0$. Then, exchanging x_1 with x_2 and x_3 with x_t, we are again in case (i1).

(ii2) $dx_2 \wedge dx_t + \nu$ does not lie in L_0 for any $t \neq 1, 2$ and any ν such that $\nu(0) = 0$. It follows that the vector fields $x_i \partial/\partial x_1 + Z$ and $x_i \partial/\partial x_2 + Z$ cannot lie in L_0 for any $i = 3, 4, 5$ and any Z such that $Z(0) = 0$ of order greater than or equal to 2. By Remark 8.2, L_0 is the graded subalgebra of L of type $(3, 3, 2, 2, 2)$.

We are now ready to prove the statement. Up to conjugation we can assume that one of the following cases occurs:

(1) the elements $dx_i \wedge dx_j + \omega$ do not lie in L_0 for any i, j, and any $\omega \in E(5,10)_{\bar 1}$ such that $\omega(0) = 0$. Then, by Remark 8.2, L_0 is the maximal graded subalgebra of L of type $(2, 2, 2, 2, 2)$.
(2) $dx_1 \wedge dx_2 + \varphi$ lies in L_0 for some $\varphi \in E(5,10)_{\bar 1}$ such that $\varphi(0) = 0$ and the elements $dx_i \wedge dx_j + \sigma$ do not for any $(i, j) \neq (1, 2)$ and any σ such that $\sigma(0) = 0$. Then L_0 is the maximal graded subalgebra of type $(3, 3, 2, 2, 2)$;
(3) the elements $dx_1 \wedge dx_2 + \varphi$, $dx_1 \wedge dx_3 + \rho$ lie in L_0 for some $\varphi, \rho \in E(5,10)_{\bar 1}$ such that $\varphi(0) = 0 = \rho(0)$ but $dx_2 \wedge dx_3 + \omega$ does not lie in L_0 for any $\omega \in E(5,10)_{\bar 1}$ such that $\omega(0) = 0$. Then L_0 is the graded subalgebra of L of type $(2, 1, 1, 1, 1)$;

(4) the elements $dx_1 \wedge dx_2 + \varphi$, $dx_1 \wedge dx_3 + \rho$ and $dx_2 \wedge dx_3 + \tau$ lie in L_0 for some $\varphi, \rho, \tau \in E(5,10)_{\bar{1}}$ such that $\varphi(0) = \rho(0) = \tau(0) = 0$. Then L_0 is the graded subalgebra of L of type $(2,2,2,1,1)$. □

Corollary 8.4. *All irreducible gradings of* $E(5,10)$ *are, up to conjugation, the gradings of type* $(2,1,1,1,1)$, $(3,3,2,2,2)$, $(2,2,2,1,1)$ *and* $(2,2,2,2,2)$.

Theorem 8.5. *All maximal open subalgebras of* $L = E(5,10)$ *are, up to conjugation, the graded subalgebras of type* $(2,1,1,1,1)$, $(3,3,2,2,2)$, $(2,2,2,1,1)$ *and* $(2,2,2,2,2)$.

Proof. Let L_0 be a maximal open subalgebra of L and let $Gr\, L$ be the graded Lie superalgebra associated to the Weisfeiler filtration corresponding to L_0. Then $\overline{Gr\, L}$ has growth equal to 5 and size equal to 8 (see Table 2), and, by Proposition 7.1, it is of the form (7.1). It follows from Table 2 that $S = S(5,h)$, $K(5,h)$ or $E(5,10)$, and $n = 0$ in the last case. Hence, by Proposition 7.4 and Remark 7.3, $n = 0$ in the first two cases as well, so $S \subset \overline{Gr\, L} \subset Der\, S$, where S is as above. If $S = K(5,h)$, then, by Remark 7.3, $E(5,10) = L = \overline{Gr\, L} = S$, which is impossible. If $S = S(5,h)$, then $\mathrm{size}(S) = (4+h)2^h \neq 8$. Thus $S = E(5,10)$. In particular $\overline{Gr\, L}$ contains a torus of dimension 4, thus L_0 contains a torus of dimension 4, and, up to conjugation, we may assume that this is the maximal torus T. Now the result follows from Theorem 8.3. □

We recall that if $L = E(5,10)$ then $Der\, L = E(5,10) + \mathbb{C}Z$ where Z is the grading operator of L with respect to its principal grading.

Remark 8.6. The same arguments as in the proof of Theorem 8.3 show that all maximal open regular subalgebras of $Der\, L$ are, up to conjugation, its graded subalgebras of type $(2,1,1,1,1)$, $(3,3,2,2,2)$, $(2,2,2,1,1)$ and $(2,2,2,2,2)$.

Theorem 8.7. *All maximal among* Z*-invariant subalgebras of* $L = E(5,10)$ *are, up to conjugation, the graded subalgebras listed in Theorem* 8.5.

Proof. The same considerations on growth and size as in Theorem 8.5 show that every fundamental maximal subalgebra of $Der\, L$ is regular. If L_0 is a maximal among Z-invariant subalgebras of L, then $L_0 + \mathbb{C}Z$ is a fundamental maximal subalgebra of $Der\, L$, hence it is regular. The thesis then follows from Remark 8.6. □

9. Maximal open subalgebras of $E(4,4)$

The Lie superalgebra $E(4,4)$ has the following structure [10, §5.3]: $E(4,4)_{\bar{0}} = W_4$ and $E(4,4)_{\bar{1}} \cong \Omega^1(4)^{-1/2}$ as an $E(4,4)_{\bar{0}}$-module (cf. Definition 2.5). Besides, for $\omega_1, \omega_2 \in E(4,4)_{\bar{1}}$:

$$[\omega_1, \omega_2] = d\omega_1 \wedge \omega_2 + \omega_1 \wedge d\omega_2.$$

Let us fix the maximal torus $T = \langle x_i \partial/\partial x_i \mid i = 1,2,3,4 \rangle$ of L and let $T' = \langle x_i \partial/\partial x_i - x_{i+1}\partial/\partial x_{i+1} \mid i = 1,2,3 \rangle$.

If we set $\deg x_i = 1 = -\deg(\partial/\partial x_i)$ and $\deg d = -2$ we obtain an irreducible $\mathbb{Z}$-grading of $E(4,4) = \prod_{j \geqslant -1} E(4,4)_j$, called the *principal* grading of $E(4,4)$, such that the $E(4,4)_0$-module $E(4,4)_{-1}$ is isomorphic to the $\hat{p}(4)$-module $\mathbb{C}^{4|4}$. Then, by Proposition 1.11, $L_0 = \prod_{j \geqslant 0} E(4,4)_j$ is a maximal open subalgebra of $E(4,4)$, which is graded.

Remark 9.1. The Lie superalgebra $L = E(4,4)$ is a free finite type module over $\mathbb{C}[[x_1,\dots,x_4]]$. Let $\{b_i\}$ be a set of free generators of L as a module over $\mathbb{C}[[x_1,\dots,x_4]]$ so that every element $a \in L$ can be written as $a = \sum_i P_i b_i$ with $P_i \in \mathbb{C}[[x_1,\dots,x_4]]$. Then we can define a valuation ν on L by assigning the value of ν on any formal power series, as in Remark 2.23, and on any b_i, and defining $\nu(a) = \min_i\{\nu(P_i) + \nu(b_i)\}$.

We shall give below three examples of maximal regular subalgebras of $L = E(4,4)$ which are not graded, making use of Remark 9.1. In all these examples $\partial/\partial x_i$ and dx_i, with $i = 1,2,3,4$, will be the generators of L as a $\mathbb{C}[[x_1,x_2,x_3,x_4]]$-module.

Example 9.2. Throughout this example, the valuation ν will be defined as follows:

$$\nu(\partial/\partial x_i) = -1, \quad \nu(dx_i) = -1 \quad \text{for } i = 1,2,3,$$
$$\nu(\partial/\partial x_4) = -2, \qquad \nu(dx_4) = 0;$$

besides, for every $P \in \mathbb{C}[[x_1,x_2,x_3,x_4]]$, $\nu(P)$ will denote the order of vanishing at $t = 0$ of the formal power series $P(t,t,t,t^2) \in \mathbb{C}[[t]]$.

Let us consider the following filtration $L = L_{-2} \supset L_{-1} \supset L_0 \supset \cdots$ of L:

$$(L_j)_{\bar 0} = \{X \in W_4 \mid \nu(X) \geqslant j,\ div(X) \in \mathbb{C}\} + \{Y \in W_4 \mid \nu(Y) \geqslant j+1\},$$
$$(L_j)_{\bar 1} = \{\omega \in \Omega^1(4) \mid \nu(\omega) \geqslant j,\ d\omega = 0\} + \{\sigma \in \Omega^1(4) \mid \nu(\sigma) \geqslant j+1\}.$$

Then $Gr\, L$ has the following structure:

$$(Gr_j\, L)_{\bar 0} = \{X \in W_4 \mid \nu(X) = j,\ div(X) \in \mathbb{C}\} + \{Y \in W_4 \mid \nu(Y) = j+1\} / \{Y \mid div(Y) \in \mathbb{C}\},$$
$$(Gr_j\, L)_{\bar 1} = \{\omega \in d\Omega^0(4) \mid \nu(\omega) = j\} + \{\sigma \in \Omega^1(4) \mid \nu(\sigma) = j+1\} / d\Omega^0.$$

$\overline{Gr\, L}$ is isomorphic to the Lie superalgebra $SHO(4,4) + \mathbb{C}E$ with the irreducible $\mathbb{Z}$-grading of type $(1,1,1,2|1,1,1,0)$, where $E = \sum_{i=1}^4 x_i \partial/\partial x_i + \sum_{i=1}^4 x_i \partial/\partial \xi_i$ is the Euler operator. The hypotheses of Corollary 1.12 are then satisfied. It follows that L_0 is a maximal subalgebra of L.

Example 9.3. Throughout this example, the valuation ν will be defined as follows:

$$\nu(\partial/\partial x_i) = -1, \quad \nu(dx_i) = -1 \quad \text{for } i = 1,2,$$
$$\nu(\partial/\partial x_i) = -2, \quad \nu(dx_i) = 0, \quad \text{for } i = 3,4;$$

besides, for every $P \in \mathbb{C}[[x_1,x_2,x_3,x_4]]$, $\nu(P)$ will denote the order of vanishing at $t = 0$ of the formal power series $P(t,t,t^2,t^2) \in \mathbb{C}[[t]]$.

Let us consider the following filtration $L = L_{-2} \supset L_{-1} \supset L_0 \supset \cdots$ of L:

$$(L_j)_{\bar 0} = \{X \in W_4 \mid \nu(X) \geqslant j,\ div(X) \in \mathbb{C}\} + \{Y \in W_4 \mid \nu(Y) \geqslant j+2\},$$
$$(L_j)_{\bar 1} = \{\omega \in \Omega^1(4) \mid \nu(\omega) \geqslant j,\ d\omega = 0\} + \{\sigma \in \Omega^1(4) \mid \nu(\sigma) \geqslant j+2\}.$$

It follows that $Gr\, L = \bigoplus_{j \geqslant -2} Gr_j\, L$ has the following structure:

$$(Gr_j\,L)_{\bar{0}} = \{Y \in W_4 \mid \nu(Y) = j+2\}/\{Y \mid div(Y) \in \mathbb{C}\} + \{X \in W_4 \mid \nu(X) = j,\ div(X) \in \mathbb{C}\},$$
$$(Gr_j\,L)_{\bar{1}} = \{\omega \in d\Omega^0(4) \mid \nu(\omega) = j\} + \{\omega \in \Omega^1(4) \mid \nu(\omega) = j+2\}/d\Omega^0.$$

$\overline{Gr\,L}$ is isomorphic to $SHO(4,4) + \mathbb{C}E$ with respect to its irreducible grading of type $(1,1,2,2|1,1,0,0)$. By Corollary 1.12, L_0 is therefore a maximal subalgebra of L.

Example 9.4. Throughout this example, the valuation ν will be defined as follows:

$$\nu(\partial/\partial x_i) = -1, \quad \nu(dx_i) = 0 \quad \text{for } i = 1,2,3,4;$$

besides, for every $P \in \mathbb{C}[[x_1, x_2, x_3, x_4]]$, $\nu(P)$ will denote the order of vanishing of P at 0.

If we define L_j as in Example 9.3 we obtain a filtration of L of depth 1. In this case $\overline{Gr\,L}$ is isomorphic to $SHO(4,4) + \mathbb{C}E$ with the irreducible grading of type $(1,1,1,1|0,0,0,0)$. It follows that L_0 is a maximal subalgebra of L.

Remark 9.5. (i) The vector fields $x_i\partial/\partial x_j$ and $x_h\partial/\partial x_k$, with $i \neq j$ and $h \neq k$, have the same weights with respect to T' if and only if $(i,j) = (h,k)$.

(ii) The vector fields $x_i\partial/\partial x_j$ and $x_h x_k\partial/\partial x_k$ have never the same weights with respect to T', for any i, j, h, k.

Remark 9.6. (i) The 1-forms dx_i and dx_j have the same weights with respect to T' if and only if $i = j$.

(ii) The 1-forms dx_i and $x_j\,dx_k$ have never the same weights with respect to T', for any i, j, k.

(iii) The 1-forms $x_i\,dx_j$ and $x_h\,dx_k$ have the same weights with respect to T' if and only if $\{i,j\} = \{k,h\}$.

Theorem 9.7. *Let L_0 be a maximal open T'-invariant subalgebra of $L = E(4,4)$. Then L_0 is a regular subalgebra of L which is conjugate either to the graded subalgebra of type* $(1,1,1,1)$*, or to one of the non-graded subalgebras constructed in Examples* 9.2*,* 9.3*,* 9.4.

Proof. We first notice that the vector fields $\partial/\partial x_i + Y$ such that $Y(0) = 0$ cannot lie in L_0 since they are not exponentiable. Likewise, no non-zero linear combination of vector fields $\partial/\partial x_i$ can lie in L_0.

We distinguish two cases:

1. The elements $dx_i + \omega$ do not lie in L_0 for any i and any form ω such that $\omega(0) = 0$. By Remark 9.6(i), no non-zero linear combination of the forms dx_i lies in L_0. It follows that L_0 is contained in the maximal graded subalgebra of type $(1,1,1,1)$, hence they coincide, due to the maximality of L_0.
2. $dx_i + \omega$ lies in L_0 for some i and some ω such that $\omega(0) = 0$. Up to conjugation we can assume $i = 4$, i.e., $dx_4 + \omega \in L_0$ for some ω such that $\omega(0) = 0$. Then, up to conjugation, the following possibilities may occur:

(a) $dx_i + \varphi \notin L_0$ for any $i \neq 4$ and any 1-form φ such that $\varphi(0) = 0$.

Suppose that the vector field $x_i\partial/\partial x_4 + Y$, such that $i \neq 4$ and Y has a zero in 0 of order greater than or equal to 2, lies in L_0. Then $[x_i\partial/\partial x_4 + Y, dx_4 + \omega] = dx_i + \omega' \in L_0$ for some ω'

such that $\omega'(0) = 0$, thus contradicting our hypotheses. It follows that $x_i\partial/\partial x_4 + Y$ does not lie in L_0 for any $i \neq 4$ and any Y such that $Y(0) = 0$ of order greater than or equal to 2. Besides, by Remark 9.5(i), no non-zero linear combination of the vector fields $x_i\partial/\partial x_4$ lies in L_0.

Now suppose that the form $x_i\, dx_j + \alpha x_j\, dx_i + \sigma$ lies in L_0, for some $i \neq j \neq 4$, some $\alpha \neq 1$ and some σ such that $\sigma(0) = 0$ of order greater than or equal to 2. Then $[x_i\, dx_j + \alpha x_j\, dx_i + \sigma, dx_4 + \omega] = (1 - \alpha)\partial/\partial x_k + Y \in L_0$ for some $k \neq i, j, 4$ and some Y such that $Y(0) = 0$, contradicting our hypotheses. It follows that no 1-form $\tau + \sigma$ such that $\tau \in \langle x_i\, dx_j \mid i \neq j \neq 4\rangle$ and $d\tau \neq 0$, and $\sigma(0) = 0$ of order greater than or equal to 2, lies in L_0. By Remark 9.6, L_0 is contained in the maximal regular subalgebra of $E(4, 4)$ constructed in Example 9.2, thus coincides with it.

(b) $dx_3 + \varphi \in L_0$ for some φ such that $\varphi(0) = 0$ and $dx_i + \psi \notin L_0$ for every $i \neq 3, 4$, and every ψ such that $\psi(0) = 0$.

Arguing as in (a), one shows that the vector fields $x_i\partial/\partial x_4 + Y$ and $x_i\partial/\partial x_3 + Y$ do not lie in L_0 for every $i = 1, 2$ and any Y such that $Y(0) = 0$ of order greater than or equal to 2. Likewise, the 1-forms $\tau + \sigma$ such that $\tau \in \langle x_i\, dx_j \mid i, j = 1, 2, i \neq j\rangle$ and $d\tau \neq 0$ do not lie in L_0 for any σ such that $\sigma(0) = 0$ of order greater than or equal to 2.

Now suppose that $x_i\, dx_4 + \alpha x_4\, dx_i + \tilde{\omega} \in L_0$ for some $i = 1, 2$, some $\alpha \neq 1$ and some $\tilde{\omega}$ such that $\tilde{\omega}(0) = 0$ of order greater than or equal to 2. Then $[x_i\, dx_4 + \alpha x_4\, dx_i + \tilde{\omega}, dx_3 + \varphi] = (1 - \alpha)\partial/\partial x_j + Y \in L_0$ for some vector field Y such that $Y(0) = 0$, contradicting our hypotheses. Therefore the 1-forms $\tau + \tilde{\omega}$ such that $\tau \in \langle x_i\, dx_4, x_4\, dx_i \mid i = 1, 2\rangle$ and $d\tau \neq 0$, do not lie in L_0 for any $\tilde{\omega}$ such that $\tilde{\omega}(0) = 0$ of order greater than or equal to 2.

Likewise, the 1-forms $\tau + \sigma$ such that $\tau \in \langle x_i\, dx_3, x_3\, dx_i \mid i = 1, 2\rangle$ and $d\tau \neq 0$, do not lie in L_0 for any σ such that $\sigma(0) = 0$ of order greater than or equal to 2.

Finally, suppose that a vector field $X + Z$ such that $X(0) = 0$ of order greater than or equal to 2 and $div(X) = \alpha x_1 + \beta x_2 \neq 0$, and $Z(0) = 0$ of order greater than or equal to 3, lies in L_0. Then $[X + Z, dx_4 + \omega] = [X, dx_4] + \sigma \in L_0$, where $[X, dx_4]$ is a non-closed 1-form in $\langle x_i\, dx_4, x_4\, dx_i \mid i = 1, 2\rangle$ and $\sigma(0) = 0$ of order greater than or equal to 2. This contradicts our hypotheses. Therefore no such a vector field $X + Z$ lies in L_0. It follows that L_0 is the maximal regular subalgebra of L constructed in Example 9.3.

(c) $dx_3 + \varphi \in L_0$ and $dx_2 + \psi \in L_0$, for some φ and ψ such that $\varphi(0) = 0$ and $\psi(0) = 0$, and $dx_1 + \tilde{\varphi} \notin L_0$ for every $\tilde{\varphi}$ such that $\tilde{\varphi}(0) = 0$.

We will show that, since L_0 is maximal, this case cannot in fact occur. Indeed, arguing as in (a) and (b) one shows that the 1-forms $\tau + \sigma$, where $\tau \in \langle x_i\, dx_j\rangle$, $d\tau \neq 0$, do not lie in L_0 for any σ such that $\sigma(0) = 0$ of order greater than or equal to 2. It follows that the vector fields $X + Z$ where $div(X) \in \langle x_1, x_2, x_3, x_4\rangle$ and $div(X) \neq 0$ do not lie in L_0 for any Z such that $Z(0) = 0$ of order greater than or equal to 3. Therefore L_0 is contained in the maximal subalgebra of L constructed in Example 9.4. In fact, since we assumed that $dx_1 + \tilde{\varphi} \notin L_0$ for every $\tilde{\varphi}$ such that $\tilde{\varphi}(0) = 0$, L_0 is properly contained in the maximal subalgebra of L constructed in Example 9.4. This contradicts the maximality of L_0.

(d) $dx_i + \omega_i$ lies in L_0 for every i and some ω_i such that $\omega_i(0) = 0$.

Arguing as above, one shows that L_0 is the subalgebra of L constructed in Example 9.4. □

Corollary 9.8. *The Lie superalgebra* $E(4,4)$ *has, up to conjugation, only one irreducible grading, that of type* $(1,1,1,1)$.

Theorem 9.9. *All maximal open subalgebras of* $L = E(4,4)$ *are, up to conjugation, the following*:

(i) *the graded subalgebra of type* $(1,1,1,1)$;
(ii) *the non-graded subalgebras constructed in Examples* 9.2, 9.3, 9.4.

Proof. Let L_0 be a maximal open subalgebra of L and let $Gr\,L$ be the graded Lie superalgebra associated to the Weisfeiler filtration corresponding to L_0. Then $\overline{Gr\,L}$ has growth equal to 4 and size equal to 8 and, by Proposition 7.1, it is of the form (7.1). Using Table 2 we see that either $n = 0$ and $S = S(4,1)$, $H(4,3)$, $SHO(4,4)$, $SHO^{\sim}(4,4)$, $E(4,4)$, or $n > 0$ and $S = W(4,0)$ or $S = H(4,h)$ for $h < 3$. Remark 7.3 shows that the case $n > 0$, $S = W(4,0)$ cannot hold.

If $S = SHO(4,4)$ then S contains a maximal torus $\hat{T}$ of dimension 3, thus L_0 contains a torus $\tilde{T}$ of dimension 3 which is the lift of $\hat{T}$. In particular, the weights of $\tilde{T}$ on L/L_0 coincide with the weights of $\hat{T}$ on $Gr\,L/Gr_{\geqslant 0}\,L$. Since L is transitive, these weights determine the torus $\tilde{T}$ completely. Therefore we may assume, up to conjugation, that L_0 contains the standard torus T' of S_4. By Theorem 9.7, L_0 is thus conjugate to one of the non-graded subalgebras constructed in Examples 9.2, 9.3, 9.4. Likewise, if $S = E(4,4)$, then S contains a maximal torus of dimension 4, hence L_0 contains a torus of dimension 4, i.e., it is regular. By Theorem 9.7, L_0 is thus conjugate to the graded subalgebra of type $(1,1,1,1)$.

If $S = SHO^{\sim}(4,4)$, then S contains a maximal torus $\hat{T}$ of dimension 3, hence we may assume, as above, that L_0 contains the standard torus T' of S_4. Then, by Theorem 9.7, $\overline{Gr\,L}$ is of the form (7.1) with either $S = SHO(4,4)$ or $S = E(4,4)$ and this is impossible. By the same argument, if $S = S(4,1)$, one gets a contradiction.

Finally, we will show that the case $S = H(4,h)$ cannot hold for any $h \leqslant 3$. Indeed, suppose $S = H(4,h)$. If $\overline{Gr_{\geqslant 0}\,L}$ contains a torus of dimension 4 then L_0 is regular and, by Theorem 9.7, $\overline{Gr\,L}$ is of the form (7.1) with $S = E(4,4)$ or $S = SHO(4,4)$, contradicting our assumptions. Therefore $\overline{Gr_{\geqslant 0}\,L}$ contains a maximal torus $\hat{T}$ of dimension $k < 4$, containing the standard torus T_h of $H(4,h)$. Then L_0 contains a maximal torus $\tilde{T}$ of dimension k (which is the lift of $\hat{T}$) and the even part of $Gr_{<0}\,L$ contains a $\hat{T}$-weight subspace of weight 0 of dimension $4-k$. Consider the Lie superalgebra $H(4,h) \otimes \Lambda(3-h)$ with respect to an irreducible grading of $H(4,h)$. Then the negative part of this grading contains a non-trivial even T_h-weight subspace of weight 0 if and only if $h = 1$. Therefore we conclude that $h = 1$. Notice that $H(4,1)$ has, up to conjugation, only one irreducible grading (that of principal type) and this is of depth 1. In this case $Gr_{-1}\,L$ contains a two-dimensional even T_h-weight subspace V of weight 0. Since L is transitive the weights of $\hat{T}$ on $Gr_{-1}\,L$ determine $\hat{T}$ completely and we can assume, up to conjugation, that the lift $\tilde{T}$ of $\hat{T}$ is contained in the standard torus of L. It follows that the standard torus of L contains some non-zero element $\sum_i a_i x_i \partial/\partial x_i$ whose projection on $Gr_{-1}\,L$ lies in V. Since $Gr_{-1}\,L$ is commutative and $\partial/\partial x_j$ is not exponentiable for any j, hence it cannot lie in L_0, it follows that there exist some vector fields P and Q in W_4, such that $P(0) = 0$ of order greater than or equal to 2, and $Q(0) = 0$ of order greater than or equal to 1, such that the commutators $[\sum_i a_i x_i \partial/\partial x_i + P, \partial/\partial x_j + Q]$ lie in L_0 for every $j = 1, \ldots, 4$. But this is impossible since $[\sum_i a_i x_i \partial/\partial x_i + P, \partial/\partial x_j + Q] = -a_j \partial/\partial x_j + R$ for some $R \in W_4$ such that $R(0) = 0$. We conclude that S cannot be the Lie superalgebra $H(4,h)$ for any h. □

10. Maximal open subalgebras of $E(3,8)$

The Lie superalgebra $L = E(3,8)$ has the following structure [6,10]: it has even part $E(3,8)_{\bar{0}} = W_3 + \Omega^0(3) \otimes sl_2 + d\Omega^1(3)$ and odd part $E(3,8)_{\bar{1}} = \Omega^0(3)^{-1/2} \otimes \mathbb{C}^2 + \Omega^2(3)^{-1/2} \otimes \mathbb{C}^2$. W_3 acts on $\Omega^0(3) \otimes sl_2 + d\Omega^1(3)$ in the natural way while, for $X, Y \in W_3$, $f, g \in \Omega^0(3)$, $A, B \in sl_2$, $\omega_1, \omega_2 \in d\Omega^1(3)$, we have:

$$[X,Y] = XY - YX - \frac{1}{2}d\big(div(X)\big) \wedge d\big(div(Y)\big),$$
$$[f \otimes A, \omega_1] = 0,$$
$$[f \otimes A, g \otimes B] = fg \otimes [A,B] + df \wedge dg\,\mathrm{tr}(AB), \qquad [\omega_1, \omega_2] = 0.$$

Besides, for $X \in W_3$, $f \in \Omega^0(3)^{-1/2}$, $g \in \Omega^0(3)$, $v \in \mathbb{C}^2$, $A \in sl_2$, $\omega \in d\Omega^1(3)$, $\sigma \in \Omega^2(3)^{-1/2}$,

$$[X, f \otimes v] = \left(X.f + \frac{1}{2}d(div\,X) \wedge df\right) \otimes v,$$
$$[g \otimes A, f \otimes v] = (gf + dg \wedge df) \otimes Av, \qquad [g \otimes A, \sigma \otimes v] = g\sigma \otimes Av,$$
$$[\omega, f \otimes v] = f\omega \otimes v, \qquad [\omega, \sigma \otimes v] = 0.$$

Here W_3 acts on $\Omega^2(3)$ by Lie derivative.

Finally, we identify W_3 with $\Omega^2(3)^{-1}$ and $\Omega^0(3)$ with $\Omega^3(3)^{-1}$. Besides, we identify $\Omega^2(3)^{-1/2}$ with $W_3^{1/2}$ and we denote by X_ω the vector field corresponding to the 2-form ω under this identification. Then, for $\omega_1, \omega_2 \in \Omega^2(3)^{-1/2}$, $f_1, f_2 \in \Omega^0(3)^{-1/2}$, $u_1, u_2 \in \mathbb{C}^2$, we have:

$$[\omega_1 \otimes u_1, \omega_2 \otimes u_2] = \big(X_{\omega_1}(\omega_2) - (div\,X_{\omega_2})\omega_1\big)u_1 \wedge u_2,$$
$$[f_1 \otimes u_1, f_2 \otimes u_2] = df_1 \wedge df_2 \otimes u_1 \wedge u_2,$$
$$[f_1 \otimes u_1, \omega_1 \otimes u_2] = \big(f_1\omega_1 + df_1 \wedge d(div\,X_{\omega_1})\big) \otimes u_1 \wedge u_2 - \frac{1}{2}(f_1\,d\omega_1 - \omega_1\,df_1) \otimes u_1 \cdot u_2,$$

where, as in the description of $E(3,6)$, $u_1 \cdot u_2$ denotes an element in the symmetric square of $\mathbb{C}^2$, i.e., an element in sl_2, and $u_1 \wedge u_2$ an element in the skew-symmetric square of $\mathbb{C}^2$, i.e., a complex number. (Note that the last formula is corrected as compared to [6].) Let $\{v_1, v_2\}$ be the standard basis of $\mathbb{C}^2$ and E, F, H the standard basis of sl_2. We shall simplify notation by writing elements of L omitting the $\otimes$ sign. Let us fix the maximal torus $T = \langle H, x_i\partial/\partial x_i, i = 1,2,3\rangle$.

Remark 10.1. The $\mathbb{Z}$-gradings of $E(3,8)$ are parametrized by quadruples $(a_1, a_2, a_3, \epsilon)$ where $a_i = \deg x_i \in \mathbb{N}$ and $\epsilon = \deg v_1 \in \mathbb{Z}$ [10, §5.4]. The following relations hold:

$$\deg v_2 = -\epsilon - \sum_{i=1}^{3} a_i, \qquad \deg E = -\deg F = 2\epsilon + \sum_{i=1}^{3} a_i, \qquad \deg d = \deg H = 0.$$

The grading of type $(2,2,2,-3)$ is called the *principal* grading of $E(3,8)$ (cf. [17, Example 5.4]). It is an irreducible consistent $\mathbb{Z}$-grading of depth 3. Its 0th graded component is isomorphic to $sl_3 \oplus sl_2 \oplus \mathbb{C}$ and is spanned by the elements $x_i\partial/\partial x_j$, E, H and F. $E(3,8)_{-1}$ is spanned by the elements $x_i v_1$ and $x_i v_2$ and is isomorphic, as an $E(3,8)_0$-module, to

$\mathbb{C}^3 \boxtimes \mathbb{C}^2 \boxtimes \mathbb{C}(-1)$ where $\mathbb{C}^k$ denotes the standard sl_k-module. Besides, $E(3,8)_{-2} = \langle \partial/\partial x_i \rangle$ and $E(3,8)_{-3} = \langle v_1, v_2 \rangle$. It is then immediate to verify that g_{-1} generates g_-, since, for $i \neq j$, $[x_i v_1, x_j v_2] = \partial/\partial x_k$ and $[\partial/\partial x_k, x_k v_h] = v_h$.

Let us now consider the grading of type $(2,1,1,-2)$. This is an irreducible grading of depth 2 whose 0th graded component is spanned by the following elements: E, F, H, $x_1\partial/\partial x_1$, $x_i x_j \partial/\partial x_1$, $x_i\partial/\partial x_j$, $x_1 v_k$, $x_i x_j v_k$, and $dx_2 \wedge dx_3\, v_k$, for $i, j = 2, 3$, $k = 1, 2$; it follows that $E(3,8)_0 = [E(3,8)_0, E(3,8)_0] + \mathbb{C}c$ where $c = 2x_1\partial/\partial x_1 + x_2\partial/\partial x_2 + x_3\partial/\partial x_3$ is central in $E(3,8)_0$ and $[E(3,8)_0,\ E(3,8)_0] \cong sl_2 \otimes \Lambda(2) + W(0,2)$. Besides, $E(3,8)_{-1} = \langle x_i v_1, x_i v_2, x_i\partial/\partial x_1, \partial/\partial x_i, i = 2,3 \rangle$ is isomorphic, as an $E(3,8)_0$-module, to $\mathbb{C}^2 \otimes \Lambda(2)$ where $\mathbb{C}^2$ is the standard sl_2-module; finally, by Remark 1.13, $E(3,8)_{-2} = [E(3,8)_{-1},\ E(3,8)_{-1}]$ since $[E(3,8)_{-1},\ E(3,8)_{-1}] \neq 0$.

Now let us consider the grading of type $(1,1,1,-1)$. This is an irreducible grading of depth 2 whose 0th graded component is spanned by the elements $x_i\partial/\partial x_j$, H, $x_i F$, $x_i x_j v_2$, $x_i v_1$, $dx_i \wedge dx_j\, v_2$. One can verify that $E(3,8)_0 \cong W(0,3) + \mathbb{C}Z$ where Z is the grading operator of $W(0,3)$ with respect to its principal grading. Besides, $E(3,8)_{-1} = \langle \partial/\partial x_i, F, v_1, x_i v_2 \rangle$ is an irreducible $E(3,8)_0$-module with highest weight vector F. Finally, by Remark 1.13, $E(3,8)_{-2} = [E(3,8)_{-1},\ E(3,8)_{-1}]$ since $[E(3,8)_{-1},\ E(3,8)_{-1}] \neq 0$.

The gradings of type $(2,2,2,-3)$, $(2,1,1,-2)$ and $(1,1,1,-1)$ satisfy the hypotheses of Proposition 1.11, therefore the corresponding subalgebras $\prod_{j \geqslant 0} E(3,8)_j$ are maximal subalgebras of $E(3,8)$, which are graded, hence regular.

We shall give below six examples of maximal regular subalgebras of L which are not graded.

Remark 10.2. We can view the Lie superalgebra $L = E(3,8)$ as a submodule of a (non-free) module M over $\mathbb{C}[[x_1, x_2, x_3]]$. In order to define a valuation on L we can fix a set of generators $\{b_i\}$ of M so that every element $a \in L$ can be written as $a = \sum_i P_i b_i$ with $P_i \in \mathbb{C}[[x_1, x_2, x_3]]$, and assign the value of ν on any formal power series and any b_i. Then we define $\nu(a) = \min_{a=\sum_i P_i b_i}(\min_i\{\nu(P_i) + \nu(b_i)\})$.

Example 10.3. Throughout this example, for every $P \in \mathbb{C}[[x_1, x_2, x_3]]$, $\nu(P)$ will be the order of vanishing of P at 0. Let us fix the following set of elements $\{b_i\}$ (see Remark 10.2):

$$\partial/\partial x_i,\ E,\ H,\ F,\ dx_i \wedge dx_j,\ v_1,\ v_2,\ x_i v_1,\ dx_i \wedge dx_j\, v_1,\ dx_i \wedge dx_j\, v_2 \quad (i, j = 1, 2, 3),$$

and let us set:

$$\nu(\partial/\partial x_i) = -1, \quad \nu(E) = 1, \quad \nu(H) = 0, \quad \nu(F) = -2, \quad \nu(dx_i \wedge dx_j) = 1,$$
$$\nu(v_1) = 0, \quad \nu(v_2) = -2, \quad \nu(x_i v_1) = 0, \quad \nu(dx_i \wedge dx_j\, v_1) = 1, \quad \nu(dx_i \wedge dx_j\, v_2) = -1.$$

Let us consider the following filtration $L_{-2} \supset L_{-1} \supset L_0 \supset \cdots$ of L:

$$(L_j)_{\bar{0}} = \{X \in W_3 \mid \nu(X) \geqslant j,\ div(X) \in \mathbb{C}\} + \left\{X + \frac{1}{2} div(X) H \,\middle|\, X \in W_3,\ \nu(X) \geqslant j\right\}$$
$$+ \{X \in W_3 \mid \nu(X) \geqslant j+1\} + \{\omega \in d\Omega^1(3) \mid \nu(\omega) \geqslant j\}$$
$$+ \langle fE, fF \in \Omega^0(3) \otimes sl_2 \mid \nu(fE) \geqslant j,\ \nu(fF) \geqslant j \rangle,$$

$$(L_j)_{\bar 1}=\{f\in\Omega^0(3)\otimes\mathbb{C}^2 \mid \nu(f)\geqslant j\}+\{\omega v_1\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_1)\geqslant j,\ d\omega=0\}$$
$$+\{\omega v_1\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_1)\geqslant j+1\}+\{\omega v_2\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_2)\geqslant j\}.$$

Then $Gr\,L$ has the following structure:

$$(Gr_j\,L)_{\bar 0}=\{X\in W_3\mid \nu(X)=j,\ div(X)\in\mathbb{C}\}+\left\{X+\frac12 div(X)H\ \middle|\ \nu(X)=j\right\}$$
$$+\langle X\in W_3,\ fH\in\Omega^0(3)\otimes sl_2\mid \nu(X)=j+1=\nu(fH)\rangle$$
$$\Big/\left\langle Y, X+\frac12 div(X)H\ \middle|\ div(Y)\in\mathbb{C}\right\rangle$$
$$+\{\omega\in d\Omega^1(3)\mid \nu(\omega)=j\}+\langle fE, fF\in\Omega^0(3)\otimes sl_2\mid \nu(fE)=j=\nu(fF)\rangle,$$
$$(Gr_j\,L)_{\bar 1}=\{f\in\Omega^0(3)\otimes\mathbb{C}^2\mid \nu(f)=j\}+\{\omega v_1\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_1)=j,\ d\omega=0\}$$
$$+\{\omega v_1\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_1)=j+1\}/\{\omega v_1\mid d\omega=0\}$$
$$+\{\omega v_2\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_2)=j\}.$$

It follows that $\overline{Gr\,L}\cong SKO(3,4;-1/3)\otimes\Lambda(\xi)+\mathfrak{a}$ with respect to the irreducible grading of type $(1,1,1|1,1,1,2)$ of $SKO(3,4;-1/3)$ and $\deg\xi=0$, with $\mathfrak{a}=\mathbb{C}(\partial/\partial\xi)+\mathbb{C}(Z+\xi\partial/\partial\xi)$, where Z is the grading operator of $SKO(3,4;-1/3)$ with respect to its principal grading. By Corollary 1.12, L_0 is a maximal subalgebra of L.

Example 10.4. Let us fix the same set $\{b_i\}$ as in Example 10.3. Throughout this example, for every $P\in\mathbb{C}[[x_1,x_2,x_3]]$, $\nu(P)$ will be the order of vanishing at $t=0$ of the formal power series $P(t^2,t,t)\in\mathbb{C}[[t]]$. Besides we set:

$$\nu(\partial/\partial x_1)=-2,\quad \nu(\partial/\partial x_2)=\nu(\partial/\partial x_3)=-1,\quad \nu(E)=0,\quad \nu(H)=0,\quad \nu(F)=-2,$$
$$\nu(v_1)=0,\quad \nu(v_2)=-2,\quad \nu(x_1v_1)=0,\quad \nu(x_2v_1)=\nu(x_3v_1)=-1,$$
$$\nu(dx_2\wedge dx_3)=0,\quad \nu(dx_2\wedge dx_3\,v_1)=0,\quad \nu(dx_2\wedge dx_3\,v_2)=-2,$$
$$\nu(dx_1\wedge dx_i)=1,\quad \nu(dx_1\wedge dx_i\,v_1)=1,\quad \nu(dx_1\wedge dx_i\,v_2)=-1,\quad \text{for } i=2,3.$$

Let us consider the filtration $L=L_{-2}\supset L_{-1}\supset L_0\supset\cdots$ of L where:

$$(L_j)_{\bar 0}=\{X\in W_3\mid \nu(X)\geqslant j,\ div(X)\in\mathbb{C}\}+\left\{X+\frac12 div(X)H\ \middle|\ X\in W_3,\ \nu(X)\geqslant j\right\}$$
$$+\{X\in W_3\mid \nu(X)\geqslant j+2\}+\{\omega\in d\Omega^1(3)\mid \nu(\omega)\geqslant j\}$$
$$+\langle fE, fF\in\Omega^0(3)\otimes sl_2,\ \nu(fE)\geqslant j,\ \nu(fF)\geqslant j\rangle,$$
$$(L_j)_{\bar 1}=\{f\in\Omega^0(3)\otimes\mathbb{C}^2\mid \nu(f)\geqslant j\}+\{\omega v_1\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_1)\geqslant j,\ d\omega=0\}$$
$$+\{\omega v_1\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_1)\geqslant j+2\}+\{\omega v_2\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_2)\geqslant j\}.$$

Then $Gr\,L$ has the following structure:

$$(Gr_j\, L)_{\bar 0} = \{X \in W_3 \mid \nu(X) = j,\ div(X) \in \mathbb{C}\} + \left\{X + \frac{1}{2} div(X)H \;\middle|\; \nu(X) = j\right\}$$
$$+ \langle X \in W_3,\ fH \in \Omega^0(3) \otimes sl_2 \mid \nu(X) = j + 2 = \nu(fH)\rangle$$
$$\Big/ \left\langle X + \frac{1}{2} div(X)H,\ Y \;\middle|\; div(Y) \in \mathbb{C}\right\rangle$$
$$+ \{\omega \in d\Omega^1(3) \mid \nu(\omega) = j\} + \langle fE, fF \mid \nu(fE) = j = \nu(fF)\rangle,$$
$$(Gr_j\, L)_{\bar 1} = \{f \in \Omega^0(3) \otimes \mathbb{C}^2 \mid \nu(f) = j\} + \{\omega v_1 \in \Omega^2 \otimes \mathbb{C}^2 \mid \nu(\omega v_1) = j,\ d\omega = 0\}$$
$$+ \{\omega v_1 \in \Omega^2 \otimes \mathbb{C}^2 \mid \nu(\omega v_1) = j + 2\} / \{\omega v_1 \mid d\omega = 0\}$$
$$+ \{\omega v_2 \in \Omega^2 \otimes \mathbb{C}^2 \mid \nu(\omega v_2) = j\}.$$

It follows that $\overline{Gr\, L} \cong SKO(3,4;-1/3) \otimes \Lambda(\xi) + \mathfrak{a}$ with respect to the irreducible grading of type $(2,1,1|0,1,1,2)$ of $SKO(3,4;-1/3)$ and $\deg \xi = 0$, with $\mathfrak{a} = \mathbb{C}(\partial/\partial\xi) + \mathbb{C}(Z + 2\xi\partial/\partial\xi)$, where Z is the grading operator of $SKO(3,4;-1/3)$ with respect to the grading of type $(2,1,1|0,1,1,2)$. By Corollary 1.12, L_0 is a maximal subalgebra of L.

Example 10.5. Let us fix the same set $\{b_i\}$ as in Examples 10.3, 10.4. Throughout this example, for every $P \in \mathbb{C}[[x_1, x_2, x_3]]$, $\nu(P)$ will denote the order of vanishing of P at 0. Besides, we set:

$$\nu(\partial/\partial x_i) = -1, \quad \nu(E) = -1, \quad \nu(H) = 0, \quad \nu(F) = -1, \quad \nu(dx_i \wedge dx_j) = 0,$$
$$\nu(v_1) = 0, \quad \nu(v_2) = -1, \quad \nu(x_i v_1) = -1, \quad \nu(dx_i \wedge dx_j\, v_1) = 0, \quad \nu(dx_i \wedge dx_j\, v_2) = -1.$$

Now, if we define L_j as in Example 10.4, we obtain a filtration of L of depth 1. In this case $\overline{Gr\, L} \cong SKO(3,4;-1/3) \otimes \Lambda(\xi) + \mathfrak{a}$ with respect to the irreducible grading of type $(1,1,1|0,0,0,1)$ of $SKO(3,4;-1/3)$ and $\deg \xi = 0$, with $\mathfrak{a} = \mathbb{C}(\partial/\partial\xi) + \mathbb{C}(Z + 2\xi\partial/\partial\xi)$, where Z is the grading operator of $SKO(3,4;-1/3)$ with respect to the grading of type $(1,1,1|0,0,0,1)$. By Corollary 1.12, L_0 is a maximal subalgebra of L.

Example 10.6. Throughout this example, for every $P \in \mathbb{C}[[x_1, x_2, x_3]]$, $\nu(P)$ will be the order of vanishing at $t = 0$ of the formal power series $P(t^2, t, t) \in \mathbb{C}[[t]]$. Let us fix the following elements:

$$\partial/\partial x_i,\ E,\ H,\ F,\ x_i E,\ x_i H,\ x_i F,\ dx_i \wedge dx_j,$$
$$v_1,\ v_2,\ x_i v_1,\ x_i v_2,\ dx_i \wedge dx_j\, v_1,\ dx_i \wedge dx_j\, v_2 \quad (i, j = 1, 2, 3),$$

and let us set, for $t = 2, 3$, $h = 1, 2$:

$$\nu(\partial/\partial x_1) = -2, \quad \nu(\partial/\partial x_t) = -1, \quad \nu(E) = \nu(H) = \nu(F) = 0,$$
$$\nu(x_1 E) = \nu(x_1 H) = \nu(x_1 F) = 0, \quad \nu(x_t E) = \nu(x_t H) = \nu(x_t F) = -1,$$
$$\nu(v_h) = 0, \quad \nu(x_1 v_h) = 0, \quad \nu(x_t v_h) = -1,$$
$$\nu(dx_i \wedge dx_j) = \nu(\partial/\partial x_k), \quad \nu(dx_i \wedge dx_j\, v_h) = \nu(\partial/\partial x_k), \quad \text{for } i \neq j \neq k.$$

Let us consider the following filtration $L = L_{-2} \supset L_{-1} \supset L_0 \supset \cdots$ of L where

$$(L_j)_{\bar 0}=\{X\in W_3 \mid \nu(X)\geqslant j,\ div(X)\in\mathbb{C}\}+\{X\in W_3\mid \nu(X)\geqslant j+2\}$$
$$+\{g\in\Omega^0(3)\otimes sl_2\mid \nu(g)\geqslant j\}+\{\omega\in d\Omega^1(3)\mid \nu(\omega)\geqslant j\},$$
$$(L_j)_{\bar 1}=\{f\in\Omega^0(3)\otimes\mathbb{C}^2\mid \nu(f)\geqslant j\}+\langle\omega v_h\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\omega v_h)\geqslant j,\ div(X_\omega)\in\mathbb{C}\rangle$$
$$+\{\sigma\in\Omega^2\otimes\mathbb{C}^2\mid \nu(\sigma)\geqslant j+2\}.$$

Then $Gr\,L$ has the following structure:

$$(Gr_j\,L)_{\bar 0}=\{X\in W_3\mid \nu(X)=j,\ div(X)\in\mathbb{C}\}+\{X\in W_3\mid \nu(X)=j+2\}/\{X\mid div(X)\in\mathbb{C}\}$$
$$+\{g\in\Omega^0(3)\otimes sl_2\mid \nu(g)=j\}+\{\omega\in d\Omega^1(3)\mid \nu(\omega)=j\},$$
$$(Gr_j\,L)_{\bar 1}=\{f\in\Omega^0(3)\otimes\mathbb{C}^2\mid \nu(f)=j\}+\langle\omega v_h\in\Omega^2(3)\otimes\mathbb{C}^2\mid \nu(\omega)=j,\ div(X_\omega)\in\mathbb{C}\rangle$$
$$+\langle\omega v_h\in\Omega^2(3)\otimes\mathbb{C}^2\mid \nu(X_\omega)=j+2\rangle/\langle\omega v_h\mid div(X_\omega)\in\mathbb{C}\rangle.$$

It follows that $\overline{Gr\,L}\cong SHO(3,3)\otimes\Lambda(\eta_1,\eta_2)+\mathfrak{b}$ with respect to the grading of type $(2,1,1|0,1,1)$ of $SHO(3,3)$ and $\deg\eta_i=0$, with

$$\mathfrak{b}\cong\mathbb{C}(\partial/\partial\eta_1)+\mathbb{C}(\partial/\partial\eta_2)+sl_2+\mathbb{C}(Z+2\eta_1\partial/\partial\eta_1+2\eta_2\partial/\partial\eta_2)$$
$$+\mathbb{C}(-4e\eta_1+4\eta_1\eta_2\partial/\partial\eta_1+(2h-Z)\eta_2)+\mathbb{C}(4f\eta_2+4\eta_1\eta_2\partial/\partial\eta_2+(2h+Z)\eta_1),$$

where Z is the grading operator of $SHO(3,3)$ with respect to its grading of type $(2,1,1|0,1,1)$. Here sl_2 has generators $e-\eta_2\partial/\partial\eta_1$, $f-\eta_1\partial/\partial\eta_2$ and $h+\eta_2\partial/\partial\eta_2-\eta_1\partial/\partial\eta_1$, where e, f, h is the Chevalley basis of the copy of sl_2 of outer derivations of $SHO(3,3)$ described in Remark 2.37. By Corollary 1.12, L_0 is a maximal subalgebra of L.

Example 10.7. Throughout this example, for every element $P\in\mathbb{C}[[x_1,x_2,x_3]]$, $\nu(P)$ will denote the order of vanishing of P at 0. Let us fix the following set of elements of L:

$$\partial/\partial x_i,\ E,\ H,\ F,\ x_iE,\ dx_i\wedge dx_j,$$
$$v_1,\ v_2,\ x_iv_1,\ x_iv_2,\ dx_i\wedge dx_j\,v_h,\quad \text{for } i,j=1,2,3,$$

and let us set:

$$\nu(\partial/\partial x_i)=-1,\quad \nu(E)=0,\quad \nu(H)=0,\quad \nu(F)=-2,\quad \nu(x_iE)=0,$$
$$\nu(v_1)=0=\nu(v_2),\quad \nu(x_iv_1)=0,\quad \nu(x_iv_2)=-1,$$
$$\nu(dx_i\wedge dx_j)=\nu(\partial/\partial x_k),\quad \nu(dx_i\wedge dx_j\,v_h)=\nu(\partial/\partial x_k),\quad \text{for } i\neq j\neq k,\ h=1,2.$$

Let us consider the following filtration $L=L_{-2}\supset L_{-1}\supset L_0\supset\cdots$ of L where

$$(L_j)_{\bar 0}=\{X\in W_3\mid \nu(X)\geqslant j,\ div(X)\in\mathbb{C}\}$$
$$+\left\{X-\frac{1}{2}div(X)H\mid X\in W_3,\ \nu(X)\geqslant j \text{ and } div(X)\in\mathbb{C},\ \text{or } \nu(X)\geqslant j+1\right\}$$

$$+\{X\in W_3 \mid \nu(X)\geqslant j+2\}+\langle fE, fF\in \Omega^0(3)\otimes sl_2 \mid \nu(fE)\geqslant j,\ \nu(fF)\geqslant j\rangle$$
$$+\{\omega\in d\Omega^1(3)\mid \nu(\omega)\geqslant j\},$$
$$(L_j)_{\bar 1}=\{f\in \Omega^0(3)\otimes\mathbb{C}^2 \mid \nu(f)\geqslant j\}+\{\omega v_1\in \Omega^2(3)\otimes\mathbb{C}^2 \mid \nu(\omega)\geqslant j,\ div(X_\omega)\in\mathbb{C}\}$$
$$+\{\omega v_1\in \Omega^2(3)\otimes\mathbb{C}^2 \mid \nu(\omega)\geqslant j+1\}+\{\omega v_2\in \Omega^2(3)\otimes\mathbb{C}^2 \mid \nu(X_\omega)\geqslant j,\ d\omega=0\}$$
$$+\{\omega v_2\in \Omega^2\otimes\mathbb{C}^2 \mid \nu(X_\omega)\geqslant j+2\}.$$

Then $Gr\,L$ has the following structure:

$$(Gr_j\,L)_{\bar 0}=\{X\in W_3\mid \nu(X)=j,\ div(X)\in\mathbb{C}\}+\left\{X-\frac{1}{2}div(X)H \;\middle|\; \nu(X)=j,\ div(X)\in\mathbb{C}\right\}$$
$$+\left\{X-\frac{1}{2}div(X)H \;\middle|\; \nu(X)=j+1\right\}\Big/\left\langle X, X-\frac{1}{2}div(X)H \;\middle|\; div(X)\in\mathbb{C}\right\rangle$$
$$+\{X\in W_3,\ fH\in\Omega^0(3)\otimes sl_2 \mid \nu(X)=j+2=\nu(fH)\}$$
$$\Big/\left\langle Y, X-\frac{1}{2}div(X)H \;\middle|\; div(Y)\in\mathbb{C}\right\rangle$$
$$+\langle fE, fF\in\Omega^0(3)\otimes sl_2 \mid \nu(fE)=j=\nu(fF)\rangle+\{\omega\in d\Omega^1 \mid \nu(\omega)=j\},$$
$$(Gr_j\,L)_{\bar 1}=\{f\in\Omega^0(3)\otimes\mathbb{C}^2 \mid \nu(f)=j\}+\{\omega v_1\in\Omega^2(3)\otimes\mathbb{C}^2 \mid \nu(X_\omega)=j,\ div(X_\omega)\in\mathbb{C}\}$$
$$+\{\omega v_1\in\Omega^2(3)\otimes\mathbb{C}^2 \mid \nu(X_\omega)=j+1\}/\{\omega v_1 \mid div(X_\omega)\in\mathbb{C}\}$$
$$+\{\omega v_2\in\Omega^2(3)\otimes\mathbb{C}^2 \mid \nu(X_\omega)=j,\ d\omega=0\}$$
$$+\{\omega v_2\in\Omega^2(3)\otimes\mathbb{C}^2 \mid \nu(X_\omega)=j+2\}/\{\omega v_2 \mid d\omega=0\}.$$

Note that $Gr_{-2}\,L=\langle F,\omega v_2\rangle$, where $\omega\in\langle x_i\,dx_j\wedge dx_k\rangle/d\Omega^1(3)$, is an ideal of $Gr\,L$ and $\overline{Gr\,L}/Gr_{-2}\,L\cong SHO(3,3)\otimes\Lambda(\eta_1,\eta_2)+\mathfrak{b}$, with respect to the irreducible grading of type $(1,1,1|0,0,0)$ of $SHO(3,3)$ and $\deg\eta_i=0$, with $\mathfrak{b}=\mathbb{C}(\partial/\partial\eta_1)+\mathbb{C}(\partial/\partial\eta_2)+\mathbb{C}(Z+\eta_1\partial/\partial\eta_1+2\eta_2\partial/\partial\eta_2)+\mathbb{C}(\eta_2\partial/\partial\eta_1)+\mathbb{C}(h+\eta_1\partial/\partial\eta_1-\eta_2\partial/\partial\eta_2)+\mathbb{C}(3\eta_1\eta_2\partial/\partial\eta_1-(2h+Z)\eta_2)$, where Z is the grading operator of $SHO(3,3)$ with respect to its grading of subprincipal type. It follows that $Gr_{\geqslant 0}\,L$ is not a maximal subalgebra of $Gr\,L$, since it is contained in $Gr_{\geqslant 0}\,L+Gr_{-2}\,L$. Nevertheless, L_0 is a maximal subalgebra of L, since, for every non-trivial subspace V of $Gr_{-2}\,L$, L_0+V generates the whole algebra L.

Example 10.8. Let us fix the same set of elements as in Example 10.6. Throughout this example, for every $P\in\mathbb{C}[[x_1,x_2,x_3]]$, $\nu(P)$ will denote twice the order of vanishing of P at 0. Besides, we set:

$$\nu(\partial/\partial x_i)=-2,\quad \nu(E)=\nu(H)=\nu(F)=0,\quad \nu(x_iE)=\nu(x_iH)=\nu(x_iF)=-1,$$
$$\nu(v_1)=0=\nu(v_2),\quad \nu(x_iv_1)=-1=\nu(x_iv_2),$$
$$\nu(dx_i\wedge dx_j)=\nu(\partial/\partial x_k),\quad \nu(dx_i\wedge dx_j\,v_h)=\nu(\partial/\partial x_k)\quad \text{for } i\neq j\neq k,\ h=1,2.$$

Let us consider the filtration $L=L_{-2}\supset L_{-1}\supset L_0\supset\cdots$ of L where:

$$\begin{aligned}L_{2j} = &\{X \in W_3 \mid \nu(X) \geqslant 2j,\ div(X) \in \mathbb{C}\} + \{X \in W_3 \mid \nu(X) \geqslant 2j+4\} \\ &+ \{g \in \Omega^0(3) \otimes sl_2 \mid \nu(g) \geqslant 2j\} + \{\omega \in d\Omega^1(3) \mid \nu(\omega) \geqslant 2j\} \\ &+ \{f \in \Omega^0(3) \otimes \mathbb{C}^2 \mid \nu(f) \geqslant 2j\} + \langle \omega v_h \in \Omega^2(3) \otimes \mathbb{C}^2 \mid \nu(\omega) \geqslant 2j,\ div(X_\omega) \in \mathbb{C}\rangle \\ &+ \{\sigma \in \Omega^2(3) \otimes \mathbb{C}^2 \mid \nu(\sigma) \geqslant 2j+4\}, \\ L_{2j+1} = &\{X \in W_3 \mid \nu(X) \geqslant 2j+2,\ div(X) \in \mathbb{C}\} + \{X \in W_3,\ \nu(X) \geqslant 2j+4\} \\ &+ \{g \in \Omega^0(3) \otimes sl_2 \mid \nu(g) \geqslant 2j+1\} + \{\omega \in d\Omega^1(3) \mid \nu(\omega) \geqslant 2j+1\} \\ &+ \{f \in \Omega^0(3) \otimes \mathbb{C}^2 \mid \nu(f) \geqslant 2j+1\} \\ &+ \langle \omega v_h \in \Omega^2(3) \otimes \mathbb{C}^2 \mid \nu(\omega) \geqslant 2j+2,\ div(X_\omega) \in \mathbb{C}\rangle \\ &+ \{\sigma \in \Omega^2(3) \otimes \mathbb{C}^2 \mid \nu(\sigma) \geqslant 2j+4\}.\end{aligned}$$

Then $Gr\,L$ has the following structure:

$$\begin{aligned}Gr_{2j}\,L = &\{X \in W_3 \mid \nu(X) = 2j,\ div(X) \in \mathbb{C}\} + \{g \in \Omega^0(3) \otimes sl_2 \mid \nu(g) = 2j\} \\ &+ \{\omega \in d\Omega^1(3) \mid \nu(\omega) = 2j\} + \{f \in \Omega^0(3) \otimes \mathbb{C}^2 \mid \nu(f) = 2j\} \\ &+ \langle \omega v_h \in \Omega^2(3) \otimes \mathbb{C}^2 \mid \nu(X_\omega) = 2j,\ div(X_\omega) \in \mathbb{C}\rangle, \\ Gr_{2j+1}\,L = &\{X \in W_3 \mid \nu(X) = 2j+4\}/\{X \mid div(X) \in \mathbb{C}\} + \{g \in \Omega^0(3) \otimes sl_2 \mid \nu(g) = 2j+1\} \\ &+ \{\omega \in d\Omega^1(3) \mid \nu(\omega) = 2j+1\} + \{f \in \Omega^0(3) \otimes \mathbb{C}^2 \mid \nu(f) = 2j+1\} \\ &+ \langle \omega v_h \in \Omega^2(3) \otimes \mathbb{C}^2 \mid \nu(X_\omega) = 2j+4\rangle/\langle \omega v_h \mid div(X_\omega) \in \mathbb{C}\rangle.\end{aligned}$$

It follows that $\overline{Gr\,L} \cong SHO(3,3) \otimes \Lambda(\eta_1,\eta_2) + \mathfrak{b}$ with respect to the grading of type $(2,2,2|1,1,1)$ on $SHO(3,3)$ and $\deg \eta_i = 0$, with

$$\begin{aligned}\mathfrak{b} = &\mathbb{C}(\partial/\partial\eta_1) + \mathbb{C}(\partial/\partial\eta_2) + sl_2 + \mathbb{C}(Z + 3\eta_1\partial/\partial\eta_1 + 3\eta_2\partial/\partial\eta_2) \\ &+ \mathbb{C}\left(3e\eta_1 + 3\eta_1\eta_2\partial/\partial\eta_1 + \frac{1}{2}(3h - z)\eta_2\right) + \mathbb{C}\left(-3f\eta_2 + 3\eta_1\eta_2\partial/\partial\eta_2 + \frac{1}{2}(Z + 3h)\eta_1\right),\end{aligned}$$

where Z is the grading operator of $SHO(3,3)$ with respect to its grading of type $(2,2,2|1,1,1)$. Here sl_2 has generators $e + \eta_2\partial/\partial\eta_1$, $f + \eta_1\partial/\partial\eta_2$ and $h + \eta_2\partial/\partial\eta_2 - \eta_1\partial/\partial\eta_1$, where e, f, h is the Chevalley basis of the copy of sl_2 of outer derivations of $SHO(3,3)$ described in Remark 2.37.

Recall that the $\mathbb{Z}$-grading of type $(2,2,2|1,1,1)$ is an irreducible grading of $Der\,SHO(3,3)$ (cf. Theorem 2.48(iii)), therefore $Gr\,L$ is irreducible. It follows that L_0 is a maximal subalgebra of L.

Remark 10.9. Let $T' = \langle x_1\partial/\partial x_1 - x_2\partial/\partial x_2, x_2\partial/\partial x_2 - x_3\partial/\partial x_3\rangle$.

1. Let us consider the odd elements $x_i v_h$ for $i = 1,2,3$ and $h = 1,2$. Then:
   - $x_i v_h$ and $x_j v_k$ have the same weights with respect to T' if and only if $i = j$;
   - $x_i v_h$ and v_k have different T'-weights, for every i, h, k.
2. For every $i \neq j$, the T'-weight of $dx_i \wedge dx_j$:
   - is equal to the T'-weight of $dx_h \wedge dx_k$ if and only if $\{i, j\} = \{h, k\}$;
   - is different from the T'-weight of v_h and $x_k v_h$ for every h, k.

3. The T'-weight of the vector field $x_i\partial/\partial x_j$, for $i \neq j$:
   - is different from $(0,0)$;
   - is equal to the T'-weight of $x_h\partial/\partial x_k$ if and only if $(i,j)=(h,k)$;
   - is different from the T'-weight of the vector field $\partial/\partial x_k$, for every k;
   - is different from the T'-weight of any vector field X such that $X(0)=0$ of order 2;
   - is different from the T'-weight of any element $x_h a$ for any $a \in sl_2$.
4. The elements E, F and H have T'-weight $(0,0)$.

Theorem 10.10. *Let L_0 be a maximal open T'-invariant subalgebra of $L=E(3,8)$. Then L_0 is conjugate either to a graded subalgebra of type* $(1,1,1,-1)$*,* $(2,1,1,-2)$ *or* $(2,2,2,-3)$*, or to one of the non-graded subalgebras constructed in Examples* 10.3–10.8*. In particular L_0 is regular.*

Proof. We first notice that the even elements $\partial/\partial x_i + X + z$ such that $X \in W_3$, $X(0)=0$ and $z \in \Omega^0(3)\otimes sl_2 + d\Omega^1(3)$, cannot lie in L_0 since they are not exponentiable. Likewise, no non-zero linear combination of the vector fields $\partial/\partial x_i$ lies in L_0. Up to conjugation, we may distinguish the following three cases:

1. The elements $v_1 + fv_1 + gv_2 + \omega v_1 + \sigma v_2$ and $v_2 + fv_1 + gv_2 + \omega v_1 + \sigma v_2$ do not lie in L_0 for any $f,g \in \Omega^0(3)$ such that $f(0)=0=g(0)$, and any $\omega,\sigma \in \Omega^2(3)$ such that $\omega(0)=0=\sigma(0)$.
2. The elements $v_1 + fv_1 + gv_2 + \omega v_1 + \sigma v_2$ and $v_2 + f'v_1 + g'v_2 + \omega' v_1 + \sigma' v_2$ lie in L_0 for some $f,g,f',g' \in \Omega^0(3)$ such that $f(0)=f'(0)=0=g(0)=g'(0)$ and some $\omega,\sigma,\omega',\sigma' \in \Omega^2(3)$ such that $\omega(0)=\omega'(0)=0=\sigma(0)=\sigma'(0)$.
3. The element $v_1 + fv_1 + gv_2 + \omega v_1 + \sigma v_2$ lies in L_0 for some $f,g \in \Omega^0(3)$ such that $f(0)=0=g(0)$ and some $\omega,\sigma \in \Omega^2(3)$ such that $\omega(0)=0=\sigma(0)$, but the elements $v_2 + f'v_1 + g'v_2 + \omega' v_1 + \sigma' v_2$ do not lie in L_0 for any f',g' such that $f'(0)=0=g'(0)$ and any $\omega',\sigma' \in \Omega^2(3)$ such that $\omega'(0)=0=\sigma'(0)$.

Let us analyze case 1. Two possibilities may occur:

(1a) The elements $\alpha v_1 + \beta v_2 + fv_1 + gv_2 + \omega v_1 + \sigma v_2$ do not lie in L_0 for any $\alpha,\beta \in \mathbb{C}$ such that $(\alpha,\beta) \neq (0,0)$, any $f,g \in \Omega^0(3)$ such that $f(0)=0=g(0)$, and any $\omega,\sigma \in \Omega^2(3)$ such that $\omega(0)=0=\sigma(0)$.

(1b) The element $v_1 + \beta v_2 + fv_1 + gv_2 + \omega v_1 + \sigma v_2$ lies in L_0 for some $\beta \in \mathbb{C}$, $\beta \neq 0$, some $f,g \in \Omega^0(3)$ such that $f(0)=0=g(0)$ and some $\omega,\sigma \in \Omega^2(3)$ such that $\omega(0)=0=\sigma(0)$. It follows that $v_1 - \beta v_2 + f'v_1 + g'v_2 + \omega' v_1 + \sigma' v_2$ does not lie in L_0 for any $f',g' \in \Omega^0(3)$ such that $f'(0)=0=g'(0)$ and any $\sigma',\omega' \in \Omega^2(3)$ such that $\omega'(0)=0=\sigma'(0)$. Therefore, up to a change of basis of $\mathbb{C}^2$, this is equivalent to case 3, that we will analyze below.

Case 1 therefore reduces to case (1a). Then two possibilities may occur:

(1A) The elements $x_i v_1 + fv_1 + gv_2 + \omega v_1 + \sigma v_2$ and $x_i v_2 + fv_1 + gv_2 + \omega v_1 + \sigma v_2$ do not lie in L_0 for any i, any $f,g \in \Omega^0(3)$ such that $f(0)=0=g(0)$ of order greater than or equal to 2, and any $\omega,\sigma \in \Omega^2(3)$ such that $\omega(0)=0=\sigma(0)$.

(1B) The element $x_iv_k + f'v_1 + g'v_2 + \omega'v_1 + \sigma'v_2$ lies in L_0 for some i, k, some $f', g' \in \Omega^0(3)$ such that $f'(0) = 0 = g'(0)$ of order greater than or equal to 2, and some $\omega', \sigma' \in \Omega^2(3)$ such that $\omega'(0) = 0 = \sigma'(0)$. Up to conjugation, we can assume $i = 1$ and $k = 1$, i.e., $x_1v_1 + f'v_1 + g'v_2 + \omega'v_1 + \sigma'v_2 \in L_0$.

Let us first analyze case (1B). In this case the odd elements $x_2v_2 + f''v_1 + g''v_2 + \omega''v_1 + \sigma''v_2$ do not lie in L_0 for any f'', g'' such that $g''(0) = 0$ of order greater than or equal to 2 and $f''(0) = 0$, and any $\omega'', \sigma'' \in \Omega^2(3)$ such that $\omega''(0) = 0 = \sigma''(0)$. Indeed, if such an element lies in L_0, then L_0 contains the element $[x_1v_1 + f'v_1 + g'v_2 + \omega'v_1 + \sigma'v_2, x_2v_2 + f''v_1 + g''v_2 + \omega''v_1 + \sigma''v_2] = \partial/\partial x_3 + Y + z$ for some vector field Y such that $Y(0) = 0$ and some $z \in \Omega^0(3) \otimes sl_2 + d\Omega^1(3)$. But such an element cannot lie in L_0 since it is not exponentiable.

Likewise, $x_3v_2 + f''v_1 + g''v_2 + \omega''v_1 + \sigma''v_2$ does not lie in L_0 for any $f'', g'' \in \Omega^0(3)$ such that $g''(0) = 0$ of order greater than or equal to 2 and $f''(0) = 0$, and any $\omega'', \sigma'' \in \Omega^2(3)$ such that $\omega''(0) = 0 = \sigma''(0)$.

We distinguish two cases:

(1Bi) $x_1v_2 + \tilde{f}v_1 + \tilde{g}v_2 + \tilde{\omega}v_1 + \tilde{\sigma}v_2$ does not lie in L_0 for any $\tilde{f}, \tilde{g}$ such that $\tilde{f}(0) = 0 = \tilde{g}(0)$ of order greater than or equal to 2, and any $\tilde{\omega}, \tilde{\sigma} \in \Omega^2(3)$ such that $\tilde{\omega}(0) = 0 = \tilde{\sigma}(0)$.

It follows that $x_1v_2 + \beta x_1v_1 + \hat{f}v_1 + \hat{g}v_2 + \hat{\omega}v_1 + \hat{\sigma}v_2$ does not lie in L_0 for any $\beta \in \mathbb{C}$, any $\hat{f}, \hat{g}$ such that $\hat{f}(0) = 0 = \hat{g}(0)$ of order greater than or equal to 2, and any $\hat{\omega}, \hat{\sigma} \in \Omega^2(3)$ such that $\hat{\omega}(0) = 0 = \hat{\sigma}(0)$.

Suppose that the even element $F + fH + gE + X + Y + \check{\omega}$ lies in L_0 for some $f, g \in \Omega^0(3)$ such that either f and g lie in $\mathbb{C}$ or $f(0) = 0 = g(0)$ of order greater than or equal to 2, some $X \in W_3$ such that $X(0) = 0$ of order greater than or equal to 2, some $Y \in T$ and $\check{\omega} \in d\Omega^1(3)$. Then L_0 contains the element $[F + fH + gE + X + Y + \tilde{\omega}, x_1v_1 + f'v_1 + g'v_2 + \omega'v_1 + \sigma'v_2] = x_1v_2 + \beta x_1v_1 + \varphi v_1 + \psi v_2 + \tau v_1 + \rho v_2$ for some $\beta \in \mathbb{C}$, some φ, ψ such that $\varphi(0) = 0 = \psi(0)$, contradicting our hypotheses. By Remark 10.9, L_0 is contained in the maximal graded subalgebra of type $(1, 1, 1, -1)$, hence it coincides with it by maximality.

(1Bii) $x_1v_2 + \tilde{f}v_1 + \tilde{g}v_2 + \tilde{\omega}v_1 + \tilde{\sigma}v_2$ lies in L_0 for some $\tilde{f}, \tilde{g}$ such that $\tilde{f}(0) = 0 = \tilde{g}(0)$ of order greater than or equal to 2, and some $\tilde{\omega}, \tilde{\sigma} \in \Omega^2(3)$ such that $\tilde{\omega}(0) = 0 = \tilde{\sigma}(0)$.

As a consequence, the elements $x_2v_1 + f''v_1 + g''v_2 + \omega''v_1 + \sigma''v_2$ and $x_3v_1 + f''v_1 + g''v_2 + \omega''v_1 + \sigma''v_2$ do not lie in L_0 for any f'', g'' such that $f''(0) = 0$ of order greater than or equal to 2 and $g''(0) = 0$, and any ω'', σ'' such that $\omega''(0) = 0 = \sigma''(0)$.

Now consider the elements $x_i\partial/\partial x_1 + \sum_j f_jA_j + Y + \delta$ for $i \neq 1$, where $f_j \in \Omega^0(3)$, $f_j(0) = 0$ of order greater than or equal to 2, $A_j \in sl_2$, $\delta \in d\Omega^1(3)$, and Y is a vector field such that $Y(0) = 0$ of order greater than or equal to 3. If such an element lies in L_0, then the commutator $[x_i\partial/\partial x_1 + \sum f_jA_j + Y + \delta, x_1v_1 + f'v_1 + g'v_2 + \omega'v_1 + \sigma'v_2] = x_iv_1 + \varphi v_1 + \psi v_2 + \tau v_1 + \rho v_2$ lies in L_0, for some $\varphi, \psi \in \Omega^0(3)$ such that $\varphi(0) = 0$ of order greater than or equal to 2 and $\psi(0) = 0$, and some $\tau, \rho \in \Omega^2(3)$ such that $\tau(0) = \rho(0) = 0$, contradicting our hypotheses. By Remark 10.9, L_0 is contained in the graded subalgebra of L of type $(2, 1, 1, -2)$, thus coincides with it due to its maximality.

Let us now go back to case (1A). Again, we distinguish two possibilities:

(1Ai) L_0 does not contain any element of the form $\alpha x_iv_1 + \beta x_iv_2 + fv_1 + gv_2 + \omega v_1 + \sigma v_2$, for any i, any $\alpha, \beta \in \mathbb{C}$, any $f, g \in \Omega^0(3)$ such that $f(0) = 0 = g(0)$ of order greater than or equal to 2, and any $\omega, \sigma \in \Omega^2(3)$ such that $\omega(0) = 0 = \sigma(0)$.

Then, using arguments similar to those used above and Remark 10.9, one shows that L_0 is contained in the maximal graded subalgebra of type $(2,2,2,-3)$, thus coincides with it due to its maximality.

(1Aii) L_0 contains the element $\alpha x_i v_1 + \beta x_i v_2 + \tilde{f} v_1 + \tilde{g} v_2 + \tilde{\omega} v_1 + \tilde{\sigma} v_2$ for some i, some $\alpha, \beta \in \mathbb{C}$ such that $(\alpha, \beta) \neq (0,0)$, some $\tilde{f}, \tilde{g} \in \Omega^0(3)$ such that $\tilde{f}(0) = 0 = \tilde{g}(0)$ of order greater than or equal to 2, and some $\omega, \sigma \in \Omega^2(3)$ such that $\omega(0) = 0 = \sigma(0)$.

Up to conjugation, we can assume $i = 1$ and $\alpha \neq 0$, i.e., $x_1 v_1 + \beta x_1 v_2 + \tilde{f} v_1 + \tilde{g} v_2 + \tilde{\omega} v_1 + \tilde{\sigma} v_2 \in L_0$. It follows that $x_1 v_1 - \beta x_1 v_2 + f'' v_1 + g'' v_2 + \omega'' v_1 + \sigma'' v_2$ does not lie in L_0 for any f'', g'' such that $f''(0) = 0 = g''(0)$ of order greater than or equal to 2. Therefore, up to a change of basis, this case is equivalent to (1Bi).

Let us now consider case 2. Arguing as above, one shows that, up to conjugation, the following possibilities may occur:

(2a) The elements $x_i v_1 + \tilde{f}_i v_1 + \tilde{g}_i v_2 + \tilde{\omega}_i v_1 + \tilde{\sigma}_i v_2$ lie in L_0 for every $i = 1,2,3$, some $\tilde{f}_i, \tilde{g}_i \in \Omega^0(3)$ such that $\tilde{f}_i(0) = 0 = \tilde{g}_i(0)$ of order greater than or equal to 2, and some $\tilde{\omega}_i, \tilde{\sigma}_i \in \Omega^2(3)$ such that $\tilde{\omega}_i(0) = 0 = \tilde{\sigma}_i(0)$ and the elements $x_i v_2 + f'' v_1 + g'' v_2 + \omega'' v_1 + \sigma'' v_2$ do not lie in L_0 for any $i = 1,2,3$, any $f'', g'' \in \Omega^0(3)$ such that $g''(0) = 0$ of order greater than or equal to 2 and $f''(0) = 0$, and any $\omega'', \sigma'' \in \Omega^2(3)$ such that $\omega''(0) = 0 = \sigma''(0)$. Then L_0 is the non-graded Lie superalgebra constructed in Example 10.7.

(2b) The elements $x_1 v_1 + \bar{f} v_1 + \bar{g} v_2 + \bar{\omega} v_1 + \bar{\sigma} v_2$ and $x_1 v_2 + f'' v_1 + g'' v_2 + \omega'' v_1 + \sigma'' v_2$ lie in L_0 for some $\bar{f}, \bar{g}, f'', g'' \in \Omega^0(3)$ such that $\bar{f}(0) = f''(0) = 0 = \bar{g}(0) = g''(0)$ of order greater than or equal to 2, and some $\bar{\omega}, \bar{\sigma}, \omega'', \sigma'' \in \Omega^2(3)$ such that $\bar{\omega}(0) = \omega''(0) = 0 = \bar{\sigma}(0) = \sigma''(0)$ and the elements $x_i v_1 + \varphi v_1 + \psi v_2 + \tau v_1 + \rho v_2$, $x_i v_2 + \varphi' v_1 + \psi' v_2 + \tau v_1 + \rho v_2$ do not lie in L_0 for any $i = 2,3$, any $\varphi, \psi, \varphi', \psi' \in \Omega^0(3)$ such that $\varphi(0) = 0 = \psi'(0)$ of order greater than or equal to 2 and $\varphi'(0) = 0 = \psi(0)$, and any $\tau, \rho \in \Omega^2(3)$ such that $\tau(0) = 0 = \rho(0)$. Then L_0 is the non-graded subalgebra of L constructed in Example 10.6.

(2c) The elements $x_i v_1 + f'' v_1 + g'' v_2 + \omega'' v_1 + \sigma'' v_2$, $x_i v_2 + f'' v_1 + g'' v_2 + \omega'' v_1 + \sigma'' v_2$ do not lie in L_0 for any $i = 1,2,3$, any $f'', g'' \in \Omega^0(3)$ such that $f''(0) = 0 = g''(0)$ of order greater than or equal to 2, and any $\omega'', \sigma'' \in \Omega^2(3)$ such that $\omega''(0) = 0 = \sigma''(0)$. Then L_0 is the non-graded Lie subalgebra of L constructed in Example 10.8.

Likewise, in case 3, one shows that, up to conjugation, the following cases may occur:

(3a) The elements $x_i v_1 + \tilde{f}_i v_1 + \tilde{g}_i v_2 + \tilde{\omega}_i v_1 + \tilde{\sigma}_i v_2$ lie in L_0 for every $i = 1,2,3$, some $\tilde{f}_i, \tilde{g}_i \in \Omega^0(3)$ such that $\tilde{f}_i(0) = 0 = \tilde{g}_i(0)$ of order greater than or equal to 2, and some $\tilde{\omega}_i, \tilde{\sigma}_i \in \Omega^2(3)$ such that $\tilde{\omega}_i(0) = 0 = \tilde{\sigma}_i(0)$ and the elements $x_i v_2 + f'' v_1 + g'' v_2 + \omega'' v_1 + \sigma'' v_2$ do not lie in L_0 for any $i = 1,2,3$, any $f'', g'' \in \Omega^0(3)$ such that $g''(0) = 0$ of order greater than or equal to 2 and $f''(0) = 0$, and any $\omega'', \sigma'' \in \Omega^2(3)$ such that $\omega''(0) = 0 = \sigma''(0)$. Then L_0 is the non-graded Lie superalgebra constructed in Example 10.3.

(3b) The elements $x_1 v_1 + \bar{f} v_1 + \bar{g} v_2 + \bar{\omega} v_1 + \bar{\sigma} v_2$ and $x_1 v_2 + f'' v_1 + g'' v_2 + \omega'' v_1 + \sigma'' v_2$ lie in L_0 for some $\bar{f}, \bar{g}, f'', g'' \in \Omega^0(3)$ such that $\bar{f}(0) = f''(0) = 0 = \bar{g}(0) = g''(0)$ of order greater than or equal to 2, and some $\bar{\omega}, \bar{\sigma}, \omega'', \sigma'' \in \Omega^2(3)$ such that $\bar{\omega}(0) = \omega''(0) = 0 = \bar{\sigma}(0) = \sigma''(0)$ and the elements $x_i v_1 + \varphi v_1 + \psi v_2 + \tau v_1 + \rho v_2$, $x_i v_2 + \varphi' v_1 + \psi' v_2 + \tau v_1 + \rho v_2$ do not lie in L_0 for any $i = 2,3$, any $\varphi, \psi, \varphi', \psi' \in \Omega^0(3)$ such that $\varphi(0) = 0 = \psi'(0)$ of order greater than or equal to 2 and $\varphi'(0) = 0 = \psi(0)$, and any $\tau, \rho \in \Omega^2(3)$ such that $\tau(0) = 0 = \rho(0)$. Then L_0 is the non-graded subalgebra of L constructed in Example 10.4.

(3c) The elements $x_i v_1 + f'' v_1 + g'' v_2 + \omega'' v_1 + \sigma'' v_2$, $x_i v_2 + f'' v_1 + g'' v_2 + \omega'' v_1 + \sigma'' v_2$ do not lie in L_0 for any $i = 1, 2, 3$, any $f'', g'' \in \Omega^0(3)$ such that $f''(0) = 0 = g''(0)$ of order greater than or equal to 2, and any $\omega'', \sigma'' \in \Omega^2(3)$ such that $\omega''(0) = 0 = \sigma''(0)$. Then L_0 is the non-graded Lie subalgebra of L constructed in Example 10.5. □

Corollary 10.11. *All irreducible gradings of* $E(3, 8)$ *are, up to conjugation, the gradings of type* $(1, 1, 1, -1)$, $(2, 1, 1, -2)$ *and* $(2, 2, 2, -3)$.

Theorem 10.12. *All maximal open subalgebras of* $L = E(3, 8)$ *are, up to conjugation, the following*:

(i) *the graded subalgebras of type* $(1, 1, 1, -1)$, $(2, 1, 1, -2)$, $(2, 2, 2, -3)$;
(ii) *the non-graded regular subalgebras constructed in Examples* 10.3–10.8.

Proof. Let L_0 be a maximal open subalgebra of L and let $Gr\,L$ be the graded Lie superalgebra associated to the Weisfeiler filtration corresponding to L_0. Then $\overline{Gr\,L}$ has growth equal to 3 and size equal to 16, and, by Proposition 7.1, it is of the form (7.1). It follows, using Table 2, Remark 7.3 and Proposition 7.4, that $S = HO(3, 3)$, $SHO(3, 3)$, $SKO(3, 4; \beta)$, and $n = 1, 2, 1$, respectively, or $S = S(3, 2)$, $E(3, 8)$ and $n = 0$. Therefore $\overline{Gr\,L}$ necessarily contains a torus $\hat{T}$ of dimension greater than or equal to 2, thus L_0 contains a torus $\tilde{T}$ of dimension greater than or equal to 2 which is the lift of $\hat{T}$. In particular, the weights of $\tilde{T}$ on L/L_0 coincide with the weights of $\hat{T}$ on $Gr\,L/Gr_{\geqslant 0}\,L$. Since L is transitive, these weights determine the torus $\tilde{T}$ completely. Therefore we may assume, up to conjugation, that L_0 contains the standard torus T' of S_3. Now the statement follows from Theorem 10.10. □

We conclude this section with an immediate corollary of the work we have done in Sections 2–10. It is assumed here that $\Lambda(s)$, $\Lambda(\eta)$, etc, as well as $\mathfrak{a}$, $\mathfrak{b}$, etc, have zero degree.

Corollary 10.13. *The following is a complete list of infinite-dimensional linearly compact irreducible graded Lie superalgebras that admit a non-trivial simple filtered deformation* (*listed in the parentheses at the beginning of each item*):

($H(2k, n+s)$) $H(2k, n) \otimes \Lambda(s) + H(0, s)$ *with* $H(2k, n)$ *having gradings of type* $(1, \ldots, 1| 2, \ldots, 2, 1, \ldots, 1, 0, \ldots, 0)$ *with* t *zeros and* t *2's, for* $0 \leqslant t \leqslant [n/2]$;

($KO(n, n+1)$) $HO(n, n) \otimes \Lambda(\eta) + \mathfrak{a}$ *with* $HO(n, n)$ *having gradings of type* $(1, \ldots, 1|0, \ldots, 0)$ *and* $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0)$ *with* t *zeros and* t *2's, for* $0 \leqslant t \leqslant n-2$, *where* $\mathfrak{a} = \mathbb{C}\partial/\partial\eta + \mathbb{C}(E - 2 + 2\eta\partial/\partial\eta)$ *and* E *is the Euler operator*;

($SKO(n, n+1; \beta)$) $SHO(n, n) \otimes \Lambda(\eta) + \mathfrak{a}$ *for* $n \geqslant 3$, *with* $SHO(n, n)$ *having gradings of type* $(1, \ldots, 1|0, \ldots, 0)$ *and* $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0)$ *with* t *zeros and* t *2's, for* $0 \leqslant t \leqslant n-2$, *where* $\mathfrak{a} = \mathbb{C}\partial/\partial\eta + \mathbb{C}(E - 2 - \beta ad(\Phi) + 2\eta\partial/\partial\eta)$ *and* $\Phi = \sum x_i \xi_i$, *or* $\mathfrak{a} = \mathbb{C}\partial/\partial\eta + \mathbb{C}(E - 2 - \beta ad(\Phi) + 2\eta\partial/\partial\eta) + \mathbb{C}\xi_1 \ldots \xi_n$;

($SKO(2, 3; \beta)$, $\beta \neq 0$) $SHO(2, 2) \otimes \Lambda(\eta) + \mathfrak{a}$ *with* $SHO(2, 2)$ *having grading of type* $(1, 1|1, 1)$, *where* $\mathfrak{a} = \mathbb{C}\partial/\partial\eta + \mathbb{C}(E - 2 - \beta ad(\Phi) + 2\eta\partial/\partial\eta) + \mathbb{C}\xi_1\xi_2$;

($SHO^\sim(n, n)$) $SHO'(n, n)$ *with the gradings of type* $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0)$ *with* t *zeros and* t *2's, for* $0 \leqslant t \leqslant n-2$;

($SKO^\sim(n, n+1)$) $SKO'(n, n+1; (n+2)/n)$ *with the gradings of type* $(1, \ldots, 1| 0 \ldots, 0, 1)$ *and* $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0, 2)$ *with* t *zeros and* $t+1$ *2's, for* $0 \leqslant t \leqslant n-2$;

($SKO^{\sim}(n, n+1)$) $SHO(n,n) \otimes \Lambda(\eta) + \mathfrak{a}$ *with* $SHO(n,n)$ *having gradings of type* $(1, \dots, 1, 2, \dots, 2|1, \dots, 1, 0, \dots, 0)$ *with* t *zeros and* t *2's, for* $0 \leqslant t \leqslant n-2$, *where* $\mathfrak{a} = \mathbb{C}(\partial/\partial\eta - \xi_1 \dots \xi_n \otimes \eta) + \mathbb{C}\xi_1 \dots \xi_n + \mathbb{C}(E - 2 + \frac{n+2}{n} ad(\Phi) + 2\eta\partial/\partial\eta)$;

($E(4,4)$) $SHO(4,4) + \mathbb{C}E$, *where* E *is the Euler operator, with* $SHO(4,4)$ *having gradings of type* $(1,1,1,2|1,1,1,0)$, $(1,1,2,2|1,1,0,0)$, *and* $(1,1,1,1|\,0,0,0,0)$;

($E(3,8)$) $SKO(3,4;-1/3) \otimes \Lambda(\xi) + \mathfrak{a}$, *where* $\mathfrak{a} = \mathbb{C}\partial/\partial\xi + \mathbb{C}(Z + \xi\partial/\partial\xi)$ *and* Z *is the grading operator, with* $SKO(3,4;-1/3)$ *having gradings of type* $(1,1,1|1,1,1,2)$, $(2,1,1|0,1,1,2)$, $(1,1,1|0,0,0,1)$;

($E(3,8)$) $SHO(3,3) \otimes \Lambda(2) + \mathfrak{b}$ *with* $SHO(3,3)$ *having gradings of type* $(2,1,1|0,1,1)$, $(1,1,1|0,0,0)$, $(2,2,2|1,1,1)$, *where* $\mathfrak{b}$ *is the finite-dimensional subalgebra of* $Der(SHO(3,3) \otimes \Lambda(2))$ *described in Examples* 10.6, 10.7, 10.8.

11. Invariant maximal open subalgebras and the canonical invariant

Given a linearly compact Lie superalgebra L, we call *invariant* a subalgebra of L which is invariant with respect to all its inner automorphisms, or, equivalently, which contains all exponentiable elements of L.

In order to obtain all invariant maximal open subalgebras of all linearly compact infinite-dimensional simple Lie superalgebras L, we take the list of all maximal open subalgebras of L, up to conjugation by G (obtained in the previous sections), select those which contain all exponentiable elements of L, and then apply to each of them the subgroup of G of outer automorphisms. This leads to the following

Theorem 11.1. *The following is a complete list, up to conjugation by G, of invariant maximal open subalgebras in infinite-dimensional linearly compact simple Lie superalgebras L*:

(a) *the graded subalgebras of principal type in* $L \neq SKO(2,3;0)$, $SHO^{\sim}(n,n)$ *or* $SKO^{\sim}(n, n+1)$;
(b) *the non-graded subalgebra* $L_0(n)$ *in* $SHO^{\sim}(n,n)$ *and* $SKO^{\sim}(n, n+1)$, *constructed in Examples* 5.2 *and* 5.8 *respectively*;
(c) *the graded subalgebras of subprincipal type in* $W(m,1)$, $S(m,1)$, $H(m,2)$, $K(m,2)$, $KO(2,3)$, $SKO(2,3;\beta)$, $SKO(3,4;1/3)$;
(d) *the graded subalgebra of type* $(1,1|-1,-1,0)$ *in* $SKO(2,3;\beta)$ *for* $\beta \neq 1$;
(e) *the non-graded regular subalgebra* $L_0(0)$ *in* $H(m,1)$, *constructed in Example* 3.3;
(f) *the graded subalgebra of type* $(2,1,\dots,1|0,2)$ *in* $K(m,2)$ *and the graded subalgebra of type* $(1,\dots,1|0,2)$ *in* $H(m,2)$.

Next theorem follows from our classification of maximal open subalgebras and Theorem 11.1.

Theorem 11.2.

(a) *In all infinite-dimensional linearly compact simple Lie superalgebras* $L \neq SKO(3,4;1/3)$ *there is a unique, up to conjugation by automorphisms of* L, *subalgebra of minimal codimension. These are the subalgebras listed in Theorem* 11.1(a) *and* (b) *if* $L \neq KO(2,3)$, $SKO(2,3;\beta)$, *and the graded subalgebra of subprincipal type in* $KO(2,3)$ *and* $SKO(2,3;\beta)$.
(b) *If* $L \neq W(1,1)$, $S(1,2)$, $SHO(3,3)$ *and* $SKO(3,4;1/3)$, L *contains a unique subalgebra of minimal codimension. In* $L = W(1,1)$, $S(1,2)$ *and* $SHO(3,3)$, *subalgebras of minimal*

codimension are invariant with respect to inner automorphisms and are conjugate by outer automorphisms of L.

(c) $L = SKO(3,4;1/3)$ *contains infinitely many subalgebras of minimal codimension which are conjugate by an outer automorphism of L to the subalgebra of subprincipal type*; *besides, the subalgebra of principal type has minimal codimension and it is not conjugate to the previous ones.*

Remark 11.3. Let L be an infinite-dimensional linearly compact simple Lie superalgebra. If $L = W(1,1)$ the subalgebras of principal and subprincipal type, which are invariant with respect to inner automorphisms, are permuted by an outer automorphism of L. If $L = S(1,2)$ or $SHO(3,3)$, then L has infinitely many invariant subalgebras: these subalgebras have minimal codimension and are permuted by an SL_2-copy of outer automorphisms of L. If $L = SKO(2,3;1)$ then L has a unique invariant subalgebra of minimal codimension (the subalgebra of subprincipal type) and infinitely many invariant maximal open subalgebras of codimension $(2|3)$, which are permuted by an SL_2-copy of outer automorphisms of L. If $L = SKO(3,4;1/3)$, then there are infinitely many subalgebras of minimal codimension which are conjugate to the subalgebra of subprincipal type by the automorphisms $\exp(ad(t\xi_1\xi_2\xi_3))$ with $t \in \mathbb{C}$. If $L = K(m,2)$ (respectively $H(m,2)$) the subalgebra of subprincipal type, which is invariant with respect to inner automorphisms, is conjugate by an outer automorphism to the subalgebra of type $(2,1,\dots,1|0,2)$ (respectively $(1,\dots,1|0,2)$). In all other cases all invariant maximal open subalgebras of L, listed in Theorem 11.1, are invariant with respect to all automorphisms of L.

Let L be an infinite-dimensional linearly compact simple Lie superalgebra and let L_0 be a maximal open subalgebra of L. In the introduction we defined the subspace $\pi(L_0)$ of $V = L/S_0$, where S_0 is the canonical subalgebra, defined as the intersection of all subalgebras of minimal codimension. Since S_0 contains all exponentiable elements of L and all even elements of L_0 are exponentiable, we conclude that $\pi(L_0)$ is an abelian subspace of $V_{\bar{1}}$.

Denote by $\overline{G}$ the linear subgroup of $GL(V_{\bar{1}})$ induced by the action of G on L, and by Π the map from the set of conjugacy classes of open maximal subalgebras of L to the set of $\overline{G}$-orbits of abelian subspaces of $V_{\bar{1}}$. Recall that the $\overline{G}$-orbit of $\pi(L_0)$ is called the canonical invariant of L_0.

We list below in all cases the linear group $\overline{G}$, all its orbits of abelian subspaces of $V_{\bar{1}}$, and those of them which are canonical invariants of maximal open subalgebras. When $L = W(1,1)$, $S(1,2)$, $SHO(3,3)$ or $SKO(3,4;1/3)$, we will describe the canonical subalgebra of L. In all other cases, since L has a unique subalgebra of minimal codimension, this will be its canonical subalgebra.

(1) $L = W(1,1)$. L has two invariant subalgebras of minimal codimension: the graded subalgebras of principal and subprincipal type. It follows that the canonical subalgebra of L is its graded subalgebra of type $(2|1)$. Therefore $V_{\bar{1}} = \langle \partial/\partial\xi, \xi\partial/\partial x\rangle$ with the symmetric bilinear form $(\partial/\partial\xi, \partial/\partial\xi) = 0$, $(\xi\partial/\partial x, \xi\partial/\partial x) = 0$, $(\partial/\partial\xi, \xi\partial/\partial x) = 1$, and the abelian subspaces of $V_{\bar{1}}$ are its isotropic subspaces; $\overline{G} = \mathbb{C}^\times \times \mathbb{C}^\times$.

If L_0 is the graded subalgebra of L of type $(1|1)$ then $\pi(L_0) = \langle \xi\partial/\partial x\rangle$; if L_0 is the graded subalgebra of L of type $(1|0)$ then $\pi(L_0) = \langle \partial/\partial\xi\rangle$. It follows from Theorem 2.3 that the map Π is injective but it is not surjective since the orbit of the trivial subspace of $V_{\bar{1}}$ is not in the image of Π.

(2) $L = S(1,2)$. L has infinitely many invariant subalgebras of minimal codimension whose intersection is the graded subalgebra of type $(2|1,1)$ which is, therefore, the canonical subalgebra

of L (cf. Remark 2.12). It follows that $V_{\bar{1}} = \langle \partial/\partial\xi_1, \partial/\partial\xi_2, \xi_1\partial/\partial x, \xi_2\partial/\partial x\rangle$ with the symmetric bilinear form $(\partial/\partial\xi_i, \partial/\partial\xi_j) = 0$, $(\xi_i\partial/\partial x, \xi_j\partial/\partial x) = 0$, $(\partial/\partial\xi_i, \xi_j\partial/\partial x) = \delta_{ij}$; the abelian subspaces of $V_{\bar{1}}$ are its isotropic subspaces and $\overline{G} = \mathbb{C}^\times SO_4$. The orbit of an h-dimensional isotropic subspace of $V_{\bar{1}}$ is determined by h if $h < 2$; besides, there are two orbits of maximal isotropic subspaces: the orbit of the subspace $\langle \xi_1\partial/\partial x, \xi_2\partial/\partial x\rangle$ and the orbit of the subspace $\langle \partial/\partial\xi_2, \xi_1\partial/\partial x\rangle$.

If L_0 is the graded subalgebra of L of type $(1|1,1)$ then $\pi(L_0) = \langle \xi_1\partial/\partial x, \xi_2\partial/\partial x\rangle$; if L_0 is the graded subalgebra of L of type $(1|1,0)$ then $\pi(L_0) = \langle \partial/\partial\xi_2, \xi_1\partial/\partial x\rangle$. It follows from Theorem 2.13(b) that the map Π is injective, but it is not surjective: its image consists of the orbits of the maximal isotropic subspaces of $V_{\bar{1}}$.

(3) $L = W(m,n)$ with $(m,n) \neq (1,1)$, or $S(m,n)$ with $(m,n) \neq (1,2)$. $V_{\bar{1}} = \langle \partial/\partial\xi_1, \ldots, \partial/\partial\xi_n\rangle$, $\overline{G} = GL_n(\mathbb{C})$, any subspace of $V_{\bar{1}}$ is abelian and its $\overline{G}$-orbit is determined by the dimension.

If L_0 is the graded subalgebra of L of type $(1,\ldots,1|1,\ldots,1,0,\ldots,0)$ with k zeros, for some $k = 0,\ldots,n$, then $\pi(L_0) = \langle \partial/\partial\xi_{n-k+1}, \ldots, \partial/\partial\xi_n\rangle$. By Theorems 2.3 and 2.13(a), the map Π is bijective.

(4) $L = K(m,n)$: we identify $K(m,n)$ with $\Lambda(m,n)$. Therefore $V_{\bar{1}} = \langle \xi_1, \ldots, \xi_n\rangle$ with symmetric bilinear form $(\xi_i, \xi_j) = \delta_{i,n-j+1}$, the abelian subspaces of $V_{\bar{1}}$ are its isotropic subspaces, and $\overline{G} = \mathbb{C}^\times SO_n(\mathbb{C})$. The $\overline{G}$-orbit of any abelian subspace of $V_{\bar{1}}$ is determined by the dimension k of the subspace unless $n = 2h$ and $k = h$. If $n = 2h$ there are two distinct $\overline{G}$-orbits of h-dimensional isotropic subspaces.

Let $L = K(1,2h)$: if L_0 is the graded subalgebra of L of type $(1|1,\ldots,1,0,\ldots,0)$ with h zeros, then $\pi(L_0) = \langle \xi_1, \ldots, \xi_h\rangle$; if L_0 is the graded subalgebra of L of type $(1|1,\ldots,1,0,1,0,\ldots,0)$ with h zeros, then $\pi(L_0) = \langle \xi_1, \ldots, \xi_{h-1}, \xi_{h+1}\rangle$; if L_0 is the graded subalgebra of L of type $(2|2,\ldots,2,1,\ldots,1,0,\ldots,0)$ with $s+1$ 2's and s zeros, for some $s = 0,\ldots,h-2$, then $\pi(L_0) = \langle \xi_1, \ldots, \xi_s\rangle$. Therefore, by Theorem 2.31(i), all possible images of π are the isotropic subspaces of $V_{\bar{1}}$ except those of dimension $h-1$, and Π is injective.

Let $L = K(2k+1,n)$ where n is odd and $k = 0$, or n is arbitrary and $k > 0$: if L_0 is the graded subalgebra of L of type $(2,1,\ldots,1|2,\ldots,2,1,\ldots,1,0,\ldots,0)$ with $s+1$ 2's and s zeros, for some $s = 0,\ldots,[n/2]$, then $\pi(L_0) = \langle \xi_1, \ldots, \xi_s\rangle$. If $n = 2h$ the graded subalgebra of L of type $(2,1,\ldots,1|2,\ldots,2,0,2,0,\ldots,0)$, with h zeros and $h+1$ 2's, is not conjugate to the graded subalgebra of type $(2,1,\ldots,1|2,\ldots,2,0,\ldots,0)$ with h zeros and $h+1$ 2's, and its image through π is the subspace $\langle \xi_1, \ldots, \xi_{h-1}, \xi_{h+1}\rangle$. By Theorem 2.31(ii) and (iii), Π is bijective.

(5) $L = SHO(3,3)$: we identify L with the set of elements in $\{f \in \Lambda(3,3)/\mathbb{C}1 \mid \Delta(f) = 0\}$ not containing the monomial $\xi_1\xi_2\xi_3$, with reversed parity. L has infinitely many invariant subalgebras of minimal codimension whose intersection is the subalgebra of type $(2,2,2|1,1,1)$ which is, therefore, the canonical subalgebra of L (cf. Remark 2.38). It follows that $V_{\bar{1}} = \langle x_1, x_2, x_3, \xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3\rangle$ and $\overline{G} = SL_3 \times GL_2$. Consider the map $\psi : S^2V_{\bar{1}} \to \langle \xi_i \mid i = 1,2,3\rangle$ given by $\psi(x_j \otimes x_k) = 0$, $\psi(\xi_i\xi_j \otimes \xi_h\xi_k) = 0$, $\psi(x_i \otimes \xi_j\xi_k) = \delta_{ij}\xi_k - \delta_{ik}\xi_j$. A subspace of $V_{\bar{1}}$ is abelian if and only if $\psi(a \otimes b) = 0$ for any pair of elements a, b of this subspace. It follows that the $\overline{G}$-orbits of the non-trivial abelian subspaces of $V_{\bar{1}}$ are the orbits of the following subspaces: $\langle x_1\rangle$, $\langle x_1, x_2\rangle$, $\langle x_1, \xi_2\xi_3\rangle$, $\langle x_1, x_2, x_3\rangle$.

If L_0 is the graded subalgebra of L of type $(1,1,1|1,1,1)$, then $\pi(L_0) = \langle \xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3\rangle$; if L_0 is the graded subalgebra of L of type $(1,1,2|1,1,0)$, then $\pi(L_0) = \langle \xi_1\xi_2, x_3\rangle$. By Theorem 2.42(b), the map Π is injective but not surjective. Indeed its image does not contain the orbit of the trivial subspace, that of the one-dimensional subspaces and that of the subspace $\langle x_1, x_2\rangle$.

(6) $L = HO(n,n)$ (respectively $L = SHO(n,n)$ with $n > 3$): we identify $HO(n,n)$ with $\Lambda(n,n)/\mathbb{C}1$ with reversed parity, and $SHO(n,n)$ with the set of elements in $\{f \in \Lambda(n,n)/\mathbb{C}1 \mid \Delta(f) = 0\}$ not containing the monomial $\xi_1 \dots \xi_n$. Then $V_{\bar{1}} = \langle x_1, \dots, x_n\rangle$, $\overline{G} = GL_n(\mathbb{C})$, any subspace of $V_{\bar{1}}$ is abelian and its $\overline{G}$-orbit is determined by the dimension.

If L_0 is the graded subalgebra of L of type $(1,\dots,1|0,\dots,0)$, then $\pi(L_0) = \langle x_1, \dots, x_n\rangle$; if L_0 is the graded subalgebra of L of type $(1,\dots,1,2,\dots,2|1,\dots,1,0,\dots,0)$ with $n-s$ 2's and $n-s$ zeros, for some $s = 2,\dots,n$, then $\pi(L_0) = \langle x_{s+1}, \dots, x_n\rangle$. By Theorem 2.42(a), the image of π consists of all subspaces of $\langle x_1, \dots, x_n\rangle$ except those of codimension 1, and the map Π is injective.

(7) $L = H(2k,n)$: we identify L with $\Lambda(2k,n)/\mathbb{C}1$. Then $V_{\bar{1}} = \langle \xi_1, \dots, \xi_n\rangle$ with the bilinear form $(\xi_i, \xi_j) = \delta_{i,n-j+1}$ (cf. Example 3.3), $\overline{G} = \mathbb{C}^{\times} SO_n(\mathbb{C})$, and any subspace of $V_{\bar{1}}$ is abelian. Let S be a subspace of $V_{\bar{1}}$ and let $S = S^0 \oplus S^1$ where S^0 is the kernel of the restriction of the bilinear form $(\cdot,\cdot)$ to S. Let $s_i = \dim S^i$. Then the $\overline{G}$-orbit of S is determined by the pair (s_0, s_1) unless $s_1 = 0$, n is even and $s_0 = n/2$. If n is even then there are two distinct orbits of maximal isotropic subspaces of $V_{\bar{1}}$.

If $L_0 = L_0(U)$ is the maximal open subalgebra of L constructed in Example 3.3, then $\pi(L_0(U)) = U^0 + (U^1)'$. By Theorem 3.10 and Remark 3.4, the map Π is bijective.

(8) $L = KO(2,3)$: we identify L with $\Lambda(2,3)$ with reversed parity. The canonical subalgebra of L is its subalgebra of subprincipal type. Therefore $V_{\bar{1}} = \langle 1, \xi_1\xi_2\rangle$ and any subspace of $V_{\bar{1}}$ is abelian. $\overline{G}$ is the subgroup of $GL_2(\mathbb{C})$ consisting of upper triangular matrices, thus there are four $\overline{G}$-orbits of abelian subspaces in $V_{\bar{1}}$: the orbit of the zero-dimensional subspace, the orbit of the two-dimensional subspace, the orbit of the one-dimensional subspace $\langle 1\rangle$ and the orbit of the one-dimensional subspace $\langle \xi_1\xi_2\rangle$.

If L_0 is the subalgebra of L of principal type or the subalgebra of subprincipal type, then $\pi(L_0) = \langle \xi_1\xi_2\rangle$ or $\pi(L_0) = \langle 0\rangle$, respectively; if L_0 is the subalgebra constructed in Example 4.7, then $\pi(L_0) = \langle 1\rangle$; finally, if $L_0(2)$ is the subalgebra constructed in Example 4.8, then $\pi(L_0(2)) = \langle 1, \xi_1\xi_2\rangle$. By Theorem 4.12, the map Π is bijective.

(9) $L = SKO(2,3;\beta)$ with $\beta \neq 0, 1$. The canonical subalgebra of L is its subalgebra of subprincipal type. Therefore $V_{\bar{1}} = \langle 1, \xi_1\xi_2\rangle$, any subspace of $V_{\bar{1}}$ is abelian and $\overline{G}$ is the subgroup of $GL_2(\mathbb{C})$ consisting of diagonal matrices. It follows that there are five $\overline{G}$-orbits of abelian subspaces in $V_{\bar{1}}$: the orbit of the zero-dimensional subspace, the orbit of the two-dimensional subspace, the orbit of the one-dimensional subspace $\langle 1\rangle$, the orbit of the one-dimensional subspace $\langle \xi_1\xi_2\rangle$, and the orbit of the one-dimensional subspace $\langle 1 + \xi_1\xi_2\rangle$.

If L_0 is the subalgebra of L of type $(1,1|0,0,1)$, $(1,1|1,1,2)$, $(1,1|-1,-1,0)$, then $\pi(L_0) = \langle 0\rangle$, $\pi(L_0) = \langle \xi_1\xi_2\rangle$, $\pi(L_0) = \langle 1\rangle$, respectively; if $S_0(2)$ is the subalgebra of L constructed in Example 4.21, then $\pi(S_0(2)) = \langle 1, \xi_1\xi_2\rangle$. By Theorem 4.24(a), the map Π is injective but not surjective, since its image does not contain the orbit of the subspace $\langle 1 + \xi_1\xi_2\rangle$.

(10) $L = SKO(2,3;1)$. The canonical subalgebra of L is its subalgebra of subprincipal type. Therefore $V_{\bar{1}} = \langle 1, \xi_1\xi_2\rangle$, any subspace of $V_{\bar{1}}$ is abelian and $\overline{G} = GL_2$. It follows that the $\overline{G}$-orbit of an abelian subspace of $V_{\bar{1}}$ is determined by its dimension.

If L_0 is the subalgebra of L of type $(1,1|0,0,1)$ or $(1,1|1,1,2)$, then $\pi(L_0) = \langle 0\rangle$ or $\pi(L_0) = \langle \xi_1\xi_2\rangle$, respectively; if $S_0(2)$ is the subalgebra of L constructed in Example 4.21, then $\pi(S_0(2)) = \langle 1, \xi_1\xi_2\rangle$. By Theorem 4.24(b), the map Π is bijective.

(11) $L = SKO(2,3;0)$. The canonical subalgebra of L is its subalgebra of subprincipal type. $V_{\bar{1}} = \langle 1\rangle$ and any subspace of $V_{\bar{1}}$ is abelian; $\overline{G} = \mathbb{C}^\times$. It follows that there are two $\overline{G}$-orbits of abelian subspaces in $V_{\bar{1}}$: the orbit of the zero-dimensional subspace and the orbit of the one-dimensional subspace.

If L_0 is the subalgebra of type $(1,1|0,0,1)$, then $\pi(L_0) = \langle 0\rangle$; if L_0 is the subalgebra of type $(1,1|-1,-1,0)$, then $\pi(L_0)$ is $\langle 1\rangle$. By Theorem 4.24(c), Π is bijective.

(12) $L = SKO(3,4;1/3)$. L has, up to conjugation by G, 2 subalgebras of minimal codimension: the subalgebras of principal and subprincipal type. These subalgebras are not conjugate since the grading of principal type has depth 2 and the grading of subprincipal type has depth 1. The canonical subalgebra is the graded subalgebra of type $(2,2,2|1,1,1,3)$, therefore $V_{\bar{1}} = \langle 1, x_1, x_2, x_3, \xi_2\xi_3, \xi_3\xi_1, \xi_1\xi_2\rangle$ with the non-trivial filtration: $V_{\bar{1}} = V_{-3} \supset V_{-1}$ where $V_{-1} = \langle x_1, x_2, x_3, \xi_2\xi_3, \xi_3\xi_1, \xi_1\xi_2\rangle$. $\overline{G} = \mathbb{C}^\times G'$ where G' consists of matrices $\left(\begin{smallmatrix} a & c\\ 0 & 1\end{smallmatrix}\right)$ where c is an arbitrary 6×1 matrix and a belongs to the subgroup of $GL_6(\mathbb{C})$ consisting of matrices

$$\left\{\left(\begin{array}{c|c} A & 0\\ \hline \sigma A & A\end{array}\right)\right\}$$

such that $A \in SL_3(\mathbb{C})$ and $\sigma \in \mathbb{C}$. Here $\mathbb{C}^\times$ acts on $\mathfrak{g}_{-1} = V_{-1}$ by multiplication by a scalar λ and on $\mathfrak{g}_{-3} = V_{-3}/V_{-1}$ by multiplication by λ^3. Consider the map $\psi : S^2 V_{\bar{1}} \to \langle \xi_i \mid i = 1,2,3\rangle$ given by: $\psi(1\otimes a) = 0$ for $a \in V_{\bar{1}}$, $\psi(x_i \otimes x_j) = 0 = \psi(\xi_i\xi_j \otimes \xi_k\xi_h)$, $\psi(x_i \otimes \xi_j\xi_k) = \delta_{ij}\xi_k - \delta_{ik}\xi_j$. A subspace of $V_{\bar{1}}$ is abelian if and only if $\psi(a\otimes b) = 0$ for any pair of elements a, b of this subspace. It follows that the $\overline{G}$-orbits of the non-trivial abelian subspaces of $V_{\bar{1}}$ are the orbits of the following subspaces: $\langle 1\rangle$, $\langle x_1\rangle$, $\langle \xi_1\xi_2\rangle$, $\langle 1, x_1\rangle$, $\langle 1, \xi_1\xi_2\rangle$, $\langle x_3, \xi_1\xi_2\rangle$, $\langle x_1, x_2\rangle$, $\langle \xi_1\xi_2, \xi_1\xi_3\rangle$, $\langle 1, x_1, x_2\rangle$, $\langle 1, \xi_1\xi_2, \xi_1\xi_3\rangle$, $\langle x_1, x_2, x_3\rangle$, $\langle \xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3\rangle$, $\langle 1, \xi_1\xi_2, x_3\rangle$, $\langle 1, x_1, x_2, x_3\rangle$, $\langle 1, \xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3\rangle$.

If L_0 is the subalgebra of type $(1,1,1|0,0,0,1)$, $(1,1,1|1,1,1,2)$, $(1,1,2|1,1,0,2)$, then $\pi(L_0) = \langle x_1, x_2, x_3\rangle$, $\pi(L_0) = \langle \xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3\rangle$, $\pi(L_0) = \langle x_3, \xi_1\xi_2\rangle$, respectively; if S_0' is the subalgebra of L constructed in Example 4.20, then $\pi(S_0) = \langle 1, x_1, x_2, x_3\rangle$; if $S_0(2)$ and $S_0(3)$ are the subalgebras of L constructed in Example 4.21, then $\pi(S_0(2)) = \langle 1, \xi_1\xi_2, x_3\rangle$ and $\pi(S_0(3)) = \langle 1, \xi_1\xi_2, \xi_1\xi_3, \xi_2\xi_3\rangle$. By Theorem 4.24(d), the map Π is injective but not surjective.

(13) $L = KO(n, n+1)$ with $n > 2$ (respectively $L = SKO(n, n+1; \beta)$ with $n \geqslant 3$ and $\beta \neq 1/3$ if $n = 3$). $V_{\bar{1}} = \langle 1, x_1, \ldots, x_n\rangle$. In this case $V_{\bar{1}}$ has a non-trivial filtration: $V_{\bar{1}} = V_{-2} \supset V_{-1}$ where $V_{-1} = \langle x_i \mid i = 1, \ldots, n\rangle$; $\overline{G} = \mathbb{C}^\times G'$ where G' consists of matrices $\left(\begin{smallmatrix} a & c\\ 0 & 1\end{smallmatrix}\right)$ with $a \in GL_n(\mathbb{C})$, and where c is an arbitrary $n\times 1$ matrix. Here $\mathbb{C}^\times$ acts on $\mathfrak{g}_{-1} = V_{-1}$ by multiplication by a scalar λ (respectively $\sigma^{1-\beta}$) and on $\mathfrak{g}_{-2} = V_{-2}/V_{-1}$ by multiplication by λ^2 (respectively σ^2). Any subspace of $V_{\bar{1}}$ is abelian. For any $k \in \mathbb{N}$, $1 \leqslant k \leqslant n$, there are two $\overline{G}$-orbits of abelian subspaces of $V_{\bar{1}}$ of dimension k: one containing 1 and the other contained in $\langle x_1, \ldots, x_n\rangle$.

Let $L = KO(n, n+1)$ with $n > 2$: if L_0 is the graded subalgebra of type $(1,\ldots,1|0,\ldots,0,1)$ then $\pi(L_0) = \langle x_1, \ldots, x_n\rangle$; if L_0 is the graded subalgebra of type $(1,\ldots,1,2,\ldots,2|1,\ldots,1,0,\ldots,0,2)$ with $n-t+1$ 2's and $n-t$ zeros, for some $t = 2,\ldots,n$, then $\pi(L_0) = \langle x_{t+1}, \ldots, x_n\rangle$; if L_0 is the subalgebra of L constructed in Example 4.7, then $\pi(L_0) = \langle 1, x_1, \ldots, x_n\rangle$; if $L_0(t)$ is the subalgebra of L constructed in Example 4.8, for some $t = 2,\ldots,n$, then $\pi(L_0) = \langle 1, x_{t+1}, \ldots, x_n\rangle$.

By Theorem 4.12 the image of π consists of all subspaces of $\langle x_1, \ldots, x_n\rangle$ except those of codimension 1, and of all subspaces of $\langle 1, x_1, \ldots, x_n\rangle$ containing 1 except those of codimension 1. By Theorem 4.24 the same description of the image of π holds for $L = SKO(n, n+1; \beta)$ with $n > 2$. The map Π is therefore injective but not surjective.

(14) $L = SHO^{\sim}(n, n)$. $V_{\bar{1}} = \langle x_1, \dots, x_n \rangle$; $\overline{G} = SL_n$; any subspace of $V_{\bar{1}}$ is abelian and its $\overline{G}$-orbit is determined by the dimension.

If L_0 is the graded subalgebra of type $(1, \dots, 1|0, \dots, 0)$ then $\pi(L_0) = \langle x_1, \dots, x_n \rangle$; if $L_0(t)$ is the maximal open subalgebra of L constructed in Example 5.2, for some $t = 2, \dots, n$, then $\pi(L_0(t)) = \langle x_{t+1}, \dots, x_n \rangle$.

By Theorem 5.4 the image of π consists of all subspaces of $\langle x_1, \dots, x_n \rangle$ except those of codimension 1. Therefore the map Π is injective but not surjective.

(15) $L = SKO^{\sim}(n, n+1)$. $V_{\bar{1}} = \langle 1, x_1, \dots, x_n \rangle$. As in the case of $KO(n, n+1)$, $V_{\bar{1}}$ has a non-trivial filtration: $V_{\bar{1}} = V_{-2} \supset V_{-1}$ where $V_{-1} = \langle x_i \mid i = 1, \dots, n \rangle$; $\overline{G} = \mathbb{C}^{\times} G'$ where G' consists of matrices $\left(\begin{smallmatrix} a & c \\ 0 & 1 \end{smallmatrix}\right)$ with $a \in SL_n(\mathbb{C})$, and where c is an arbitrary $n \times 1$ matrix. Here $\mathbb{C}^{\times}$ acts on $\mathfrak{g}_{-1} = V_{-1}$ by multiplication by a scalar $\sigma^{-2/n}$, and on $\mathfrak{g}_{-2} = V_{-2}/V_{-1}$ by multiplication by σ^2. The description of the $\overline{G}$-orbits of the abelian subspaces of $V_{\bar{1}}$ is the same as for $SKO(n, n+1; (n+2)/n)$ with $n > 2$.

If L_0 is the subalgebra of L constructed in Example 5.7, then $\pi(L_0) = \langle x_1, \dots, x_n \rangle$; if $L_0(t)$ is the subalgebra of L constructed in Example 5.8, for some $t = 2, \dots, n$, then $\pi(L_0(t)) = \langle x_{t+1}, \dots, x_n \rangle$; if $S_0(t)$ is the subalgebra of L constructed in Example 5.9, for some $t = 2, \dots, n$, then $\pi(S_0(t)) = \langle 1, x_{t+1}, \dots, x_n \rangle$.

By Theorem 5.11 all possible images of π are all subspaces of $\langle x_1, \dots, x_n \rangle$ except those of codimension 1, and all subspaces of $\langle 1, x_1, \dots, x_n \rangle$ containing 1 except those of codimension 1 and 0. The map Π is therefore injective but not surjective.

(16) $L = E(1, 6)$. $V_{\bar{1}}$, $\overline{G}$ and the $\overline{G}$-orbits of the abelian subspaces of $V_{\bar{1}}$ are the same as for $K(1, 6)$.

If L_0 is the graded subalgebra of type $(2|1, 1, 1, 1, 1, 1)$, $(1|1, 1, 1, 0, 0, 0)$, $(1|1, 1, 0, 0, 0, 1)$, $(1|2, 1, 1, 0, 1, 1)$, then $\pi(L_0) = \langle 0 \rangle$, $\langle \xi_1, \xi_2, \xi_3 \rangle$, $\langle \xi_1, \xi_2, \eta_3 \rangle$, and $\langle \xi_1 \rangle$, respectively. Therefore, by Theorem 7.5, all possible images of π are (as for $L = K(1, 6)$) all isotropic subspaces of $V_{\bar{1}}$ except those of dimension 2. The map Π is therefore injective but not surjective.

(17) $L = E(3, 6)$. $V_{\bar{1}} = \langle a_{ij} := dx_i\, v_j \mid i = 1, 2, 3,\ j = 1, 2 \rangle$; $\overline{G} = GL_3(\mathbb{C}) \times SL_2(\mathbb{C})$ acting on $V_{\bar{1}} \simeq \mathbb{C}^3 \otimes \mathbb{C}^2$. Consider the map $\psi : S^2 V_{\bar{1}} \to \langle \partial/\partial x_i \mid i = 1, 2, 3 \rangle$, given by $\psi(a_{ij} \otimes a_{rs}) = \epsilon(irk)\epsilon(js)\partial/\partial x_k$, where ϵ is the sign of the permutation irk (respectively js) if all i, r, k (respectively j, s) are distinct and $\epsilon = 0$ otherwise. A subspace of $V_{\bar{1}}$ is abelian if and only if $\psi(a \otimes b) = 0$ for any pair of elements a, b of this subspace.

By Theorem 7.6, all maximal open subalgebras are graded, and they are, up to conjugation, the subalgebras of type $(2, 2, 2, 0)$, $(2, 1, 1, 0)$ and $(1, 1, 1, 1/2)$, so that the corresponding abelian subspaces are 0, $\langle a_{11}, a_{12} \rangle$ and $\langle a_{11}, a_{21}, a_{31} \rangle$, respectively. Therefore all possible non-zero images of π are given by all maximal abelian subspaces of $V_{\bar{1}}$. Thus, the map Π is injective, but not surjective, as the remaining two $\overline{G}$-orbits of abelian subspaces, that of $\langle a_{11} \rangle$ and $\langle a_{11}, a_{21} \rangle$, are missing.

(18) $L = E(5, 10)$. $V_{\bar{1}} = \langle q_{ij} := dx_i \wedge dx_j \mid i, j = 1, 2, 3, 4, 5 \rangle$, $\overline{G} = GL_5(\mathbb{C})$, acting on $V_{\bar{1}} \simeq \Lambda^2 \mathbb{C}^5$. Consider the map $\varphi : S^2 V_{\bar{1}} \to \langle \partial/\partial x_i \mid i = 1, \dots, 5 \rangle$, given by $\varphi(q_{ij} \otimes q_{rs}) = \epsilon(ijrsk)\partial/\partial x_k$, where as before, ϵ is the sign of the permutation $ijrsk$ if all i, j, r, s, k are distinct and $\epsilon = 0$ otherwise. A subspace of $V_{\bar{1}}$ is abelian if and only if $\varphi(a \otimes b) = 0$ for any pair of elements of this subspace.

By Theorem 8.5 all maximal open subalgebras are graded, of type $(2,2,2,2,2)$, $(3,3,2,2,2)$, $(2,2,2,1,1)$ and $(2,1,1,1,1)$, up to conjugation, so that the corresponding abelian subspaces of $V_{\bar{1}}$ are 0, $\langle q_{12}\rangle$, $\langle q_{12}, q_{13}, q_{23}\rangle$ and $\langle q_{1j} \mid j=2,3,4,5\rangle$, respectively. Thus the map Π is injective, but not surjective, as the remaining two $\overline{G}$-orbits of abelian subspaces, that of $\langle q_{12}, q_{13}\rangle$ and $\langle q_{12}, q_{13}, q_{14}\rangle$, are missing.

(19) $L = E(4,4)$. $V_{\bar{1}} = \langle dx_i \mid i=1,2,3,4\rangle$; $\overline{G} = GL_4(\mathbb{C})$ acting on $V_{\bar{1}} \cong \mathbb{C}^4$. Any subspace of $V_{\bar{1}}$ is abelian and its $\overline{G}$-orbit is determined by the dimension.

If L_0 is the graded subalgebra of L of type $(1,1,1,1)$, then $\pi(L_0) = \langle 0\rangle$; if L_0 is the maximal open subalgebra of L constructed in Examples 9.2, 9.3, and 9.4, then $\pi(L_0) = \langle dx_1\rangle$, $\pi(L_0) = \langle dx_1, dx_2\rangle$, and $\pi(L_0) = \langle dx_i \mid i=1,2,3,4\rangle$ respectively. By Theorem 9.9 all possible images of π are all subspaces of $V_{\bar{1}}$ except those of codimension 1. Therefore the map Π is injective but not surjective.

(20) $L = E(3,8)$. $V_{\bar{1}} = \langle v_1, v_2, x_i v_1, x_i v_2 \mid i=1,2,3\rangle$ has a non-trivial filtration: $V_{\bar{1}} = V_{-3} \supset V_{-1}$ where $V_{-1} = \langle q_{ij} := x_i v_j \mid i=1,2,3,\ j=1,2\rangle$. We can give the following description of abelian subspaces of $V_{\bar{1}}$: consider the map $\varphi : S^2 V_{-1} \to \langle \partial/\partial x_i \mid i=1,2,3\rangle$, given by $\varphi(q_{ij} \otimes q_{rs}) = \epsilon(irk)\epsilon(js)\partial/\partial x_k$, where, as for $L = E(3,6)$, ϵ is the sign of the permutation irk (respectively js) if all i, r, k (respectively j, s) are distinct and $\epsilon = 0$ otherwise. A subspace of $V_{\bar{1}}$ is abelian if and only if $\varphi(a \otimes b) = 0$ for any a, b from this subspace.

$\overline{G} = \mathbb{C}^\times(SL_3 \times SL_2)$ acts on $V_{\bar{1}}$ as follows: $\mathbb{C}^\times$ acts on $\mathfrak{g}_{-1} = V_{-1}$ by multiplication by a scalar λ and on $\mathfrak{g}_{-3} = V_{-3}/V_{-1}$ by multiplication by λ^3; SL_3 acts trivially on $\mathfrak{g}_{-3}$ and it acts on $\mathfrak{g}_{-1} = \mathbb{C}^3 \otimes \mathbb{C}^2$ as on the direct sum of two copies of the standard SL_3-module; finally, SL_2 acts on $\mathfrak{g}_{-3}$ as on the standard SL_2-module and it acts on $\mathfrak{g}_{-1}$ as on the direct sum of three copies of the standard SL_2-module.

If L_0 is the graded subalgebra of type $(2,2,2,-3)$, $(2,1,1,-2)$, or $(1,1,1,-1)$, then $\pi(L_0) = \langle 0\rangle$, $\langle x_1 v_1, x_1 v_2\rangle$, or $\langle x_i v_1 \mid i=1,2,3\rangle$, respectively; if L_0 is the maximal subalgebra of L constructed in Example 10.3, 10.4, 10.5, 10.6, 10.7, or 10.8, then $\pi(L_0) = \langle v_1, x_i v_1 \mid i = 1,2,3\rangle$, $\langle v_1, x_1 v_1, x_1 v_2\rangle$, $\langle v_1, x_i v_2 \mid i=1,2,3\rangle$, $\langle v_1, v_2, x_1 v_1, x_1 v_2\rangle$, $\langle v_1, v_2, x_i v_1 \mid i=1,2,3\rangle$, or $\langle v_1, v_2\rangle$, respectively.

Therefore, by Theorem 10.12, all possible images of π are the subspace $\langle v_1, v_2\rangle$ and every subspace S of $V_{\bar{1}}$ such that $S \cap V_{-1}$ is a maximal abelian subspace of V_{-1}. It follows that the map Π is injective but not surjective. The $\overline{G}$-orbits of the following abelian subspaces of $V_{\bar{1}}$ are missing: $\langle v_1\rangle$, $\langle x_1 v_1\rangle$, $\langle v_1, x_1 v_1\rangle$, $\langle v_1, x_1 v_2\rangle$, $\langle x_1 v_1, x_2 v_1\rangle$, $\langle v_1, v_2, x_1 v_1\rangle$, $\langle v_1, x_1 v_1, x_2 v_1\rangle$, $\langle v_2, x_1 v_1, x_2 v_1\rangle$, $\langle v_1, v_2, x_1 v_1, x_2 v_1\rangle$.

We conclude by listing the maximal among $\mathfrak{a}_0$-invariant open subalgebras of S which are not maximal and also those maximal open subalgebras of S, none of whose conjugates is $\mathfrak{a}_0$-invariant. The lists follow from Theorems 2.17, 2.18, 2.47, 2.48, 4.28, 4.30, and 4.31.

Theorem 11.4. *Let S be an infinite-dimensional linearly compact simple Lie superalgebra and let $\mathfrak{a}_0$ be a subalgebra of the subalgebra $\mathfrak{a}$ of outer derivations of S.*

(a) *A complete list of pairs $(S_0, \mathfrak{a}_0)$ where S_0 is an open, maximal among the $\mathfrak{a}_0$-invariant subalgebras of S, which is not maximal, is as follows*:
- $S = S(1,2)$, *S_0 is the canonical subalgebra,* $\mathfrak{a}_0 = \mathfrak{a} \cong sl_2$;
- $S = SHO(3,3)$, *S_0 is the canonical subalgebra and* $\mathfrak{a}_0 = sl_2$, *or* $\mathfrak{a}_0 = \mathfrak{a} \cong gl_2$;
- $S = SKO(2,3;0)$, *S_0 is the subalgebra of principal type or S_0 is the subalgebra $S_0(2)$ constructed in Example* 4.21, *and* $\mathfrak{a}_0 = \mathbb{C}\xi_1\xi_2$ *or* $\mathfrak{a}_0 = \mathfrak{a}$ $(\dim \mathfrak{a} = 2)$.

(b) *A complete list of pairs* $(\mathfrak{a}_0, S_0)$ *where* S_0 *is a maximal open subalgebra of* S*, none of whose conjugates is* $\mathfrak{a}_0$*-invariant, is as follows*:
- $S = S(1,2)$ *or* $S = SKO(2,3;1)$: $\mathfrak{a}_0 = \mathfrak{a} \cong sl_2$ *and* S_0 *is the graded subalgebra of* S *of principal type*;
- $S = SHO(3,3)$: $\mathfrak{a}_0 = sl_2$ *or* $\mathfrak{a}_0 = \mathfrak{a} \cong gl_2$ *and* S_0 *is the graded subalgebra of* S *of principal type*;
- $S = S(1,n)$ *with* $n \geqslant 3$: $\mathfrak{a}_0 = \mathbb{C}\xi_1 \ldots \xi_n \partial/\partial x_1$ *or* $\mathfrak{a}_0 = \mathfrak{a}$ $(\dim \mathfrak{a} = 2)$*, and* S_0 *is the graded subalgebra of* S *of type* $(1|0,\ldots,0)$;
- $S = SHO(n,n)$ *with* $n \geqslant 4$: $\mathfrak{a}_0 = \mathbb{C}\xi_1 \ldots \xi_n \rtimes \mathfrak{t}$ *where* $\mathfrak{t}$ *is a torus of* $\mathfrak{a}$ $(\dim \mathfrak{a} = 3)$*, and* S_0 *is the graded subalgebra of* S *of type* $(1,\ldots,1|0,\ldots,0)$;
- $S = SKO(2,3;0)$: $\mathfrak{a}_0 = \mathbb{C}\xi_1\xi_2$ *or* $\mathfrak{a}_0 = \mathfrak{a}$ $(\dim \mathfrak{a} = 2)$*, and* S_0 *is the subalgebra of type* $(1,1|0,0,1)$ *or the subalgebra of type* $(1,1|-1,-1,0)$;
- $S = SKO(n,n+1;(n-2)/n)$ *with* $n > 2$: $\mathfrak{a}_0 = \mathbb{C}\xi_1 \ldots \xi_n$ *or* $\mathfrak{a}_0 = \mathfrak{a}$ $(\dim \mathfrak{a} = 2)$*, and* S_0 *is the graded subalgebra of* S *of type* $(1,\ldots,1|0,\ldots,0,1)$ *or the subalgebra* S_0' *constructed in Example* 4.20;
- $S = SKO(n,n+1;1)$ *with* $n > 2$: $\mathfrak{a}_0 = \mathbb{C}\xi_1 \ldots \xi_n \tau$ *or* $\mathfrak{a}_0 = \mathfrak{a}$ $(\dim \mathfrak{a} = 2)$*, and* S_0 *is the subalgebra* S_0' *constructed in Example* 4.20.

Appendix A. The radical of an artinian linearly compact Lie superalgebra

Let L be a linearly compact Lie superalgebra and let $rad\, L$ denote the closure of the sum of all its solvable ideals. This is a closed ideal of L, which, in general, is not solvable, but we will show that this is the case if L is artinian.

Lemma A.1. *Let* S *be a simple linearly compact Lie superalgebra. Then*

$$rad\big(S \hat{\otimes} \Lambda(m,n)\big) = S \hat{\otimes} J,$$

where J *is the ideal of* $\Lambda(m,n)$ *generated by the generators* $\xi_1,\ldots,\xi_n$.

Proof. It is clear that the right-hand side is a solvable ideal. Since the quotient by this ideal is $S \hat{\otimes} \Lambda(m,0)$, we need to prove that any abelian ideal of the latter Lie superalgebra is zero. Suppose the contrary, let

$$a = \sum_{i \in \mathbb{Z}_+^m} a_i x^i$$

be a non-zero element of an abelian ideal I of $S \hat{\otimes} \Lambda(m,0)$, where $a_i \in S$. Let $i_0 \in \mathbb{Z}_+^m$ be the minimal in the lexicographical ordering index, such that $a_{i_0} \neq 0$. Since S is simple, we conclude that for any $b \in S$, I contains an element of the form $bx^{i_0} + \sum_{i > i_0} a_i x^i$. Hence I is not abelian, a contradiction. □

Theorem A.2. *Let* L *be an artinian linearly compact Lie superalgebra. Then the ideal* $rad\, L$ *is solvable.*

Proof. By [13, Theorem 7.1], L contains a sequence of closed ideals $L = I_0 \supsetneq I_1 \supsetneq \cdots \supsetneq I_k = 0$, such that each quotient I_j/I_{j+1} is either abelian, or else there are no closed ideals of L properly contained between I_j and I_{j+1}. (The proof given in [13] works verbatim in the "super" case).

We will prove the theorem by induction on k. Consider the Lie superalgebra $\bar{L} = L/I_{k-1}$. Since it is again artinian, by the inductive assumption, $rad\,\bar{L}$ is solvable. If I_{k-1} is an abelian ideal of L, we immediately conclude that $rad\,L$ is solvable. If I_{k-1} is not abelian, it is a non-abelian minimal closed ideal of L, hence by [13, Theorem 7.1] and [11, Corollary 2.8], I_{k-1} is isomorphic to $S \hat{\otimes} \Lambda(m,n)$, where S is a simple linearly compact Lie superalgebra. Hence, by Lemma A.1, $(rad\,L) \cap I_{k-1}$ is a solvable ideal of L, and, as in the previous case, we conclude that $rad\,L$ is solvable. □

Examples A.3. (a) Let $\mathfrak{g}_n$ be an infinite sequence of finite-dimensional solvable Lie algebras of increasing length and let $L = \prod_n \mathfrak{g}_n$. Then $rad\,L = L$ is not solvable.

(b) Let S be simple. Then, $L = S \hat{\otimes} \Lambda(m,n)$ is not artinian if $m > 0$, but $rad\,L$ is solvable by Lemma A.1. Note that L is noetherian.

(c) The linearly compact Lie algebra $\mathbb{C}[\![t]\!] \rtimes d/dt$ is artinian, but not noetherian.

Theorem A.4. *Let L be an artinian linearly compact Lie superalgebra and let T be a maximal torus of L.*

(a) *Any ad-diagonalizable element t of L can be conjugated to an element of T by an inner automorphism of L.*
(b) *Any maximal torus T_1 of L can be conjugated to T by an inner automorphism of L.*

Proof. Note that the properties (a) and (b) are equivalent. Indeed, it follows from the proof of Theorem 1.7 that $\dim T < \infty$ and T has at most a countable number of weights in L. Hence there exists $t_0 \in T$ such that $\lambda(t_0) \neq 0$ for all these weights λ. Hence (b) follows from (a). Including t in a maximal torus, we obtain that (a) follows from (b).

Since $rad\,L$ is solvable by Theorem A.2, it has a finite derived series $rad\,L = J_0 \supsetneq J_1 \supsetneq \cdots \supsetneq J_{k-1} \supsetneq J_k = 0$. We prove (a) by induction on k. If $k = 0$, L is semisimple, and (b) is Theorem 1.7, hence (a) holds by the above remark. Hence we may assume that $k > 0$.

By the inductive assumption, the image of t in L/J_{k-1} is conjugate to an element of the image of T. Hence we may assume that $t = t_1 + r$, where $t_1 \in T$, $r \in J_{k-1}$. We can write:

$$r = \sum_i r_i, \quad \text{where } [t, r_i] = \lambda_i r_i,\ \lambda_i \in \mathbb{C}.$$

If $\lambda_i \neq 0$, applying $\exp(-\lambda_i^{-1}\, ad\, r_i)$ to t, kills r_i and does not change r_j with $j \neq i$ (since J_{k-1} is abelian). Thus, we may assume that $[t, r] = 0$, hence $[t_1, r] = 0$. Hence $ad\,r$ is diagonalizable, and since $[r, L] \subset J_{k-1}$ and $[r, J_{k-1}] = 0$, we conclude that r is a central element of L, hence $r \in T$, and (a) is proved. □

Appendix B. Description of the non-graded maximal open subalgebras of non-exceptional Lie superalgebras via their embedding in $W(m,n)$

Let S be a non-exceptional simple infinite-dimensional linearly compact Lie superalgebra. Then every maximal open subalgebra of S in its defining embedding in $W(m,n)$, can be constructed as the intersection of S with a graded subalgebra of $W(m,n)$. Here we describe this construction for all non-graded maximal open subalgebras of S.

If $S = KO(n, n+1)$ or $S = SKO(n, n+1; \beta)$ with $n > 2$, then, by Theorems 4.12 and 4.24, S has, up to conjugation by G, n non-graded maximal open subalgebras. These are obtained by intersecting S with the subprincipal subalgebra of $W(n, n+1)$ and with the subalgebras of $W(n, n+1)$ of type $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0, 0)$ with $n-t$ 2's and $n-t+1$ zeros, for $t = 2, \ldots, n$.

If $S = SKO(2, 3; \beta)$ with $\beta \neq 0$, then, by Theorem 4.24, S has, up to conjugation by G, only one non-graded maximal open subalgebra. This is obtained by intersecting S with the subalgebra of $W(2, 3)$ of type $(1, 1|1, 1, 0)$.

If $S = SHO^{\sim}(n, n)$, then, by Theorem 5.4, S has, up to conjugation by G, $n-1$ non-graded maximal open subalgebras. These are obtained by intersecting S with the subalgebras of $W(n, n)$ of type $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0)$ with $n-t$ 2's and $n-t$ zeros, for $t = 2, \ldots, n$.

If $S = SKO^{\sim}(n, n+1)$, then, by Theorem 5.11, S has, up to conjugation by G, $2n-1$ non-graded maximal open subalgebras. These are obtained by intersecting S with the subalgebras of $W(n, n+1)$ of type $(1, \ldots, 1|0, \ldots, 0, 1)$, $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0, 2)$ with $n-t+1$ 2's and $n-t$ zeros, for $t = 2, \ldots, n$, and $(1, \ldots, 1, 2, \ldots, 2|1, \ldots, 1, 0, \ldots, 0, 0)$ with $n-t$ 2's and $n-t+1$ zeros, for $t = 2, \ldots, n$.

If $S = H(m, n)$ with $n = 2h+1$, then, by Theorem 3.10, all maximal open subalgebras of S are regular. The non-graded maximal open subalgebras of S are obtained, up to conjugation by G, by intersecting S with the subalgebras of $W(m, 2h+1)$ of type

$$(1, \ldots, 1|\underbrace{2, \ldots, 2}_{t}, 1, \ldots, 1, \underbrace{0, \ldots, 0}_{s}, \alpha, \underbrace{0, \ldots, 0}_{s}, 1, \ldots, 1, \underbrace{0, \ldots, 0}_{t})$$

with $\alpha = 0, 1$, for $s = 0, \ldots, h$ and $t = 0, \ldots, h-s$, $(\alpha, s) \neq (0, 0)$.

If $S = H(m, n)$ with $n = 2h$, then all regular non-graded maximal open subalgebras of S, up to conjugation by G, are obtained by intersecting S with the subalgebras of $W(m, 2h)$ of type

$$(1, \ldots, 1|\underbrace{2, \ldots, 2}_{t}, 1, \ldots, 1, \underbrace{0, \ldots, 0}_{2s}, 1, \ldots, 1, \underbrace{0, \ldots, 0}_{t})$$

for $s = 1, \ldots, h$ and $t = 0, \ldots, h-s$.

All non-regular maximal open subalgebras of $H(m, 2h)$, up to conjugation by G, can be obtained as the intersection of $H(m, 2h)$ with the graded subalgebras of $W(m, 2h)$ defined as follows:

- $\deg(\xi_1 + \xi_n) = 1$;
- $\deg(\xi_1 - \xi_n) = 0$;
- grading of type

$$(1, \ldots, 1|\underbrace{2, \ldots, 2}_{t}, 1, \ldots, 1, \underbrace{0, \ldots, 0}_{2s}, 1, \ldots, 1, \underbrace{0, \ldots, 0}_{t})$$

  on the subalgebra $W(m, 2h-2)$ of $W(m, 2h)$ consisting of vector fields in the indeterminates $x_1, \ldots, x_m, \xi_2, \ldots, \xi_{n-1}$, with $s = 0, \ldots, h-1$ and $t = 0, \ldots, h-s-1$.

It follows that the number of non-regular maximal open subalgebras of $H(m, 2h)$, up to conjugation by G, is $h(1+h)/2$.

References

[1] D. Alekseevsky, D. Leites, I. Shchepochkina, Examples of simple infinite-dimensional Lie superalgebras of vector fields, C. R. Acad. Bulgare Sci. 33 (9) (1980) 1187–1190.

[2] B. Bakalov, A. D'Andrea, V.G. Kac, Theory of finite pseudoalgebras, Adv. Math. 162 (2001) 1–140.

[3] R.J. Blattner, A theorem of Cartan and Guillemin, J. Differential Geom. 5 (1970) 295–305.

[4] R.E. Block, Determination of the differentiably simple rings with a minimal ideal, Ann. of Math. 90 (2) (1969) 433–459.

[5] N. Cantarini, V.G. Kac, Automorphisms and forms of simple infinite-dimensional linearly compact Lie superalgebras, math.QA/0601292.

[6] N. Cantarini, S.-J. Cheng, V.G. Kac, Errata: Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields, Transform. Groups 9 (2004) 399–400.

[7] E. Cartan, Les groupes des transformations continués, infinis, simples, Ann. Sci. École Norm. Sup. 26 (1909) 93–161.

[8] S.-J. Cheng, Differentiably simple Lie superalgebras and representations of semisimple Lie superalgebras, J. Algebra 173 (1995) 1–43.

[9] S.-J. Cheng, V.G. Kac, Generalized Spencer cohomology and filtered deformations of $\mathbb{Z}$-graded Lie superalgebras, Adv. Theor. Math. Phys. 2 (1998) 1141–1182; Addendum, Adv. Theor. Math. Phys. 8 (2004) 697–709.

[10] S.-J. Cheng, V.G. Kac, Structure of some $\mathbb{Z}$-graded Lie superalgebras of vector fields, Transform. Groups 4 (1999) 219–272.

[11] D. Fattori, V.G. Kac, Classification of finite simple Lie conformal superalgebras, J. Algebra 258 (2002) 23–59.

[12] D. Fattori, V.G. Kac, A. Retakh, Structure theory of finite Lie conformal superalgebras, in: H.D. Dobner, V.V. Dobrev (Eds.), Lie Theory and Its Applications to Physics, World Sci., 2004, pp. 27–63.

[13] V.W. Guillemin, A Jordan–Hölder decomposition for a certain class of infinite-dimensional Lie algebras, J. Differential Geom. 2 (1968) 313–345.

[14] V.W. Guillemin, Infinite-dimensional primitive Lie algebras, J. Differential Geom. 4 (1970) 257–282.

[15] V.G. Kac, Lie superalgebras, Adv. Math. 26 (1977) 8–96.

[16] V.G. Kac, Classification of simple $\mathbb{Z}$-graded Lie superalgebras and simple Jordan superalgebras, Comm. Algebra 5 (1977) 1375–1400.

[17] V.G. Kac, Classification of infinite-dimensional simple linearly compact Lie superalgebras, Adv. Math. 139 (1998) 1–55.

[18] V.G. Kac, Classification of infinite-dimensional simple groups of supersymmetries and quantum field theory, in: GAFA, Geom. Funct. Anal. (2000) 162–183 (special volume GAFA2000).

[19] Yu. Kotchetkoff, Déformation de superalgébras de Buttin et quantification, C. R. Acad. Sci. Paris I 299 (14) (1984) 643–645.

[20] A.N. Rudakov, Groups of automorphisms of infinite-dimensional simple Lie algebras, Math. USSR-Izv. 3 (4) (1969) 707–722.

[21] I. Shchepochkina, The five exceptional simple Lie superalgebras of vector fields and their fourteen regradings, Represent. Theory 3 (1999) 373–415.

[22] B.Y. Weisfeiler, Infinite-dimensional filtered Lie algebras and their connections with graded Lie algebras, Funct. Anal. Appl. 2 (1968) 88–89.