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Razumikhin-type theorems on exponential stability of stochastic functional differential equations

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Abstract

Although the Razumikhin-type theorems have been well developed for the stability of functional differential equations and they are very useful in applications, so far there is almost no result of Razumikhin type on the stability of stochastic functional differential equations. The main aim of this paper is to close this gap by establishing several Razumikhin-type theorems on the exponential stability for stochastic functional differential equations. By applying these new results to stochastic differential delay equations and stochastically perturbed equations we improve or generalize several known results, and this shows the powerfulness of our new results clearly.

Keywords: Lyapunov exponent; Razumikhin theorem; Brownian motion; Burkholder–Davis–Gundy's inequality; Borel–Cantelli lemma

1. Introduction

Stochastic modelling has come to play an important role in many branches of science and industry. An area of particular interest has been the automatic control of stochastic systems, with consequent emphasis on the analysis of stability in stochastic models (cf. Arnold, 1972; Friedman, 1976; Has'minskii, 1981; Mao, 1991). One of the most useful stochastic models which appear frequently in applications are the stochastic functional differential equations of the form

$$\mathrm{d}x(t) = f(t,x_t)\,\mathrm{d}t + g(t,x_t)\,\mathrm{d}w(t), \quad t \geqslant 0, \tag{1.1}$$

with initial data $x_0 = \xi$, where $x_t = \{x(t+\theta)\colon -\tau \leqslant \theta \leqslant 0\}$ is regarded as a $C([-\tau,0];R^n)$-valued stochastic process. The stability of Eq. (1.1) has been studied by many authors and we here mention Kolmanovskii and Myshkis (1992), Ladde and Lakshmikantham (1980), Mao (1994) and Mohammed (1986) among others. Especially, Kolmanovskii and Nosov (1986) established the following theorem.

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Theorem 1.1 (Kolmanovskii and Nosov, 1986, p. 169). *Let the standing hypothesis* (H1) *imposed in Section* 2 *below hold. Let* $p \geqslant 2$ *and* c_1–c_3 *be positive constants. Let* $x(t;\xi)$ *denote the solution of Eq.* (1.1) *and* $x_t(\xi) = \{x(t+\theta;\xi)\colon -\tau \leqslant \theta \leqslant 0\}$. *Assume that there is a continuous functional* $V\colon R_+ \times C([-\tau,0];R^n) \to R$ *such that*

$$c_1|\varphi(0)|^p \leqslant V(t,\varphi) \leqslant c_2\|\varphi\|^p, \quad (t,\varphi) \in R_+ \times C([-\tau,0];\ R^n) \tag{1.2}$$

and

$$EV(t_2,x_{t2}(\xi)) - EV(t_1,x_{t_1}(\xi)) \leqslant -c_3\int_{t_1}^{t_2} E|x(s;\xi)|^p\,\mathrm{d}s, \quad t_2 > t_1 \geqslant 0. \tag{1.3}$$

Then the trivial solution of Eq. (1.1) *is asymptotically pth moment stable.*

This theorem is of course a natural generalization of the Lyapunov direct method but is somewhat not very convenient in applications. This is not only because condition (1.3) is not related to the coefficients f and g of Eq. (1.1) explicitly but also because it appears to be more difficult to construct the Lyapunov functionals than the Lyapunov functions. It is in this spirit that we would like to explore the possibility of using the rate of change of a function on R^n to determine sufficient conditions for stability.

To explain the idea, let $V(t,x) \in C^{2,1}(R_+ \times R^n; R_+)$. Then the expectation of the derivative of V along the solution of Eq. (1.1) is given by

$$\begin{aligned} E\mathscr{L}V(t,x_t) &:= E(V_t(t,x(t)) + V_x(t,x(t))f(t,x_t) \\ &\quad + \tfrac{1}{2}\,\mathrm{trace}[g^{\mathrm{T}}(t,x_t)V_{xx}(t,x(t))g(t,x_t)]). \end{aligned} \tag{1.4}$$

In order for $E\mathscr{L}V(t,x_t)$ to be negative for all initial data and $t \geqslant 0$, one would be forced to impose very severe restrictions on the functions $f(t,\varphi)$ and $g(t,\varphi)$. In fact, the point $\varphi(0)$ must play a dominant role and, therefore, the results will apply only to equations that are very similar to stochastic differential equations. This seems to indicate that it is not good enough to use the Lyapunov functions. Fortunately, a few moments of reflection in the proper direction indicate that it is unnecessary to require that (1.4) be negative for all initial data in order to have asymptotic stability, and this is the basic idea exploited in this paper. This idea originated with Razumikhin for the (1956, 1960) ordinary differential delay equation and was developed by several people to more general functional differential equations (cf. Hale and Lunel (1993) and the references therein). The results in this direction are generally referred to as theorems of Razumikhin type. However, so far there is almost no result of Razumikhin type for stochastic functional differential equations. The aim of this paper is to establish some Razumikhin-type theorems on exponential stability of stochastic functional differential equations.

In this paper we shall first establish the Razumikhin-type theorems on pth moment and almost sure exponential stability for stochastic functional differential equations in Section 2. These general results will then be applied to stochastic differential delay equations and stochastically perturbed equations in Sections 3 and 4 in order to improve or generalize several known results, and this shows the powerfulness of our new results. Furthermore, several interesting examples will be given in Section 5 to illustrate the theory.

2. Main results

Throughout this paper, unless otherwise specified, we let $\tau > 0$ and $C([-\tau,0];R^n)$ denote the family of continuous functions φ from $[-\tau,0]$ to R^n with the norm $\|\varphi\| = \sup_{\tau\leqslant\theta\leqslant 0}|\varphi(\theta)|$, where $|\cdot|$ is the Euclidean norm in R^n. If A is a vector or matrix, its transpose is denoted by A^{T}. If A is a matrix, its norm $\|A\|$ is defined by $\|A\| = \sup\{|Ax|: |x| = 1\}$ (without any confusion with $\|\varphi\|$). Moreover, let $w(t) = (w_1(t),\ldots, w_m(t))^{\mathrm{T}}$ be an m-dimensional Brownian motion defined on a complete probability space $(\Omega,\mathscr{F},P)$ with a natural filtration $\{\mathscr{F}_t\}_{t\geqslant 0}$ (i.e. $\mathscr{F}_t = \sigma\{w(s)$: $0\leqslant s\leqslant t\}$). Denote by $C^{\mathrm{b}}_{\mathscr{F}_0}([-\tau,0];R^n)$ the family of all bounded, $\mathscr{F}_0$-measurable, $C([-\tau,0];R^n)$-valued random variables. For $p > 0$ and $t\geqslant 0$, denote by $L^p_{\mathscr{F}_t}([-\tau,0];R^n)$ the family of all $\mathscr{F}_t$-measurable $C([-\tau,0];R^n)$-valued random variables $\phi = \{\phi(\theta)$: $-\tau\leqslant\theta\leqslant 0\}$ such that $\sup_{-\tau\leqslant\theta\leqslant 0}E|\phi(\theta)|^p < \infty$.

Consider an n-dimensional stochastic functional differential equation

$$\begin{aligned} &\mathrm{d}x(t) = f(t,x_t)\,\mathrm{d}t + g(t,x_t)\,\mathrm{d}w(t), \quad t\geqslant 0, \\ &x_0 = \xi. \end{aligned} \tag{2.1}$$

Here $\xi \in C^b_{\mathscr{F}_0}([-\tau,0];R^n)$ and $x_t = \{x(t+\theta)$: $-\tau\leqslant\theta\leqslant 0\}$ which is regarded as a $C([-\tau,0];R^n)$-valued stochastic process. Moreover,

$$f: R_+ \times C([-\tau,0];R^n) \to R^n, \quad g: R_+ \times C([-\tau,0];R^n) \to R^{n\times m}.$$

For the existence and uniqueness of the solution we impose a standing hypothesis:

(H1) Both f and g satisfy the local Lipschitz condition and the linear growth condition. That is, for each $i = 1,2,\ldots,$ there is an $h_i > 0$ such that

$$|f(t,\varphi_1) - f(t,\varphi_2)| + \|g(t,\varphi_1) - g(t,\varphi_2)\| \leqslant h_i\|\varphi_1 - \varphi_2\|$$

for all $t\geqslant 0$ and those $\varphi_1,\varphi_2 \in C([-\tau,0];R^n)$ with $\|\varphi_1\| \vee \|\varphi_2\|\leqslant i$, and, moreover, there is an $h > 0$ such that

$$|f(t,\varphi) + \|g(t,\varphi)\| \leqslant h(1 + \|\varphi\|)$$

for all $t\geqslant 0$ and all $\varphi \in C([-\tau,0];R^n)$.

It is known (cf. Mao (1994) or Mohammed (1986)) that, under (H1), Eq. (2.1) has a unique global solution, which is denoted by $x(t;\xi)$ in this paper, and, moreover, $E(\sup_{0\leqslant s\leqslant t}|x(s;\xi)|^r) < \infty$ for all $t\geqslant 0$ and $r > 0$. For the purpose of stability in this paper we also assume that $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$. So Eq. (2.1) admits a zero solution or trivial solution $x(t;0) \equiv 0$.

Let $C^{2,1}([-\tau,\infty) \times R^n;R_+)$ denote the family of all nonnegative functions $V(t,x)$ on $[-\tau,\infty)\times R^n$ which are continuously twice differentiable in x and once differentiable in t. If $V\in C^{2,1}([-\tau,\infty)\times R^n;R_+)$, define an operator $\mathscr{L}V$ from $R_+\times C([-\tau,0];R^n)$ to R by

$$\mathscr{L}V(t,\varphi) = V_t(t,\varphi(0)) + V_x(t,\varphi(0))f(t,\varphi) + \tfrac{1}{2}\,\mathrm{trace}[g^{\mathrm{T}}(t,\varphi)V_{xx}(t,\varphi(0))g(t,\varphi)],$$

where

$$V_t(t,x)=\frac{\partial V(t,x)}{\partial t},\quad V_x(t,x)=\left(\frac{\partial V(t,x)}{\partial x_1},\dots,\frac{\partial V(t,x)}{\partial x_n}\right),$$

$$V_{xx}(t,x)=\left(\frac{\partial^2 V(t,x)}{\partial x_i\,\partial x_j}\right)_{n\times n}.$$

Let us now first establish a Razumikhin-type theorem on the pth moment exponential stability for the stochastic functional differential equation.

Theorem 2.1. *Let* (H1) *hold. Let* λ, p, c_1, c_2 *all be positive numbers and* $q>1$. *Assume that there exists a function* $V(t,x)\in C^{2,1}([-\tau,\infty)\times R^n;R_+)$ *such that*

$$c_1|x|^p\leqslant V(t,x)\leqslant c_2|x|^p \quad \textit{for all } (t,x)\in[-\tau,\infty)\times R^n, \tag{2.2}$$

and also for all $t\geqslant 0$

$$E\mathscr{L}V(t,\phi)\leqslant -\lambda EV(t,\phi(0)) \tag{2.3}$$

provided $\phi=\{\phi(\theta): -\tau\leqslant\theta\leqslant 0\}\in L^p_{\mathscr{F}_t}([-\tau,0];R^n)$ *satisfying*

$$EV(t+\theta,\phi(\theta))<qEV(t,\phi(0)) \quad \textit{for all } -\tau\leqslant\theta\leqslant 0.$$

Then for all $\xi\in C^{\mathrm{b}}_{\mathscr{F}_0}([-\tau,0];R^n)$

$$E|x(t;\xi)|^p\leqslant\frac{c_2}{c_1}E\|\xi\|^p e^{-\gamma t} \quad \textit{on } t\geqslant 0, \tag{2.4}$$

where $\gamma=\min\{\lambda,\log(q)/\tau\}$.

Proof. Fix the initial data $\xi\in C^{\mathrm{b}}_{\mathscr{F}_0}([-\tau,0];R^n)$ arbitrarily and write $x(t;\xi)=x(t)$ simply. Let $\varepsilon\in(0,\gamma)$ be arbitrary and set $\bar\gamma=\gamma-\varepsilon$. Define

$$U(t)=\max_{-\tau\leqslant\theta\leqslant 0}[e^{\bar\gamma(t+\theta)}EV(t+\theta,x(t+\theta))] \quad \text{for } t\geqslant 0.$$

Since $E(\sup_{0\leqslant s\leqslant t}|x(s)|^r)<\infty$ for all $r>0$ and both $x(t)$ and $V(x,t)$ are continuous, $EV(t,x(t))$ is continuous. Hence $U(t)$ is well defined and is continuous. We claim that

$$D_+U(t):=\limsup_{h\to 0+}\frac{U(t+h)-U(t)}{t}\leqslant 0 \quad \text{for all } t\geqslant 0. \tag{2.5}$$

To show this, for each $t\geqslant 0$ (fixed for the moment), define

$$\bar\theta=\max\{\theta\in[-\tau,0]:\ e^{\bar\gamma(t+\theta)}EV(t+\theta,x(t+\theta))=U(t)\}.$$

Obviously, $\bar\theta$ is well defined, $\bar\theta\in[-\tau,0]$ and

$$U(t)=e^{\bar\gamma(t+\bar\theta)}EV(t+\bar\theta,x(t+\bar\theta)).$$

If $\bar\theta<0$, then

$$e^{\bar\gamma(t+\theta)}EV(t+\theta,x(t+\theta))<e^{\bar\gamma(t+\bar\theta)}EV(t+\bar\theta,x(t+\bar\theta)) \quad \text{for all } \bar\theta<\theta\leqslant 0.$$

It is therefore easy to observe that for all $h > 0$ sufficiently small

$$e^{\bar{\gamma}(t+h)}EV(t+h,x(t+h)) \leqslant e^{\bar{\gamma}(t+\bar{\theta})}EV(t+\bar{\theta},x(t+\bar{\theta})),$$

hence

$$U(t+h) \leqslant U(t) \quad \text{and} \quad D_+U(t) \leqslant 0.$$

If $\bar{\theta} = 0$, then

$$e^{\bar{\gamma}(t+\theta)}EV(t+\theta,x(t+\theta)) \leqslant e^{\bar{\gamma}t}EV(t,x(t)) \quad \text{for all} \quad -\tau \leqslant \theta \leqslant 0.$$

So

$$EV(t+\theta,x(t+\theta)) \leqslant e^{-\bar{\gamma}\theta}EV(t,x(t))$$
$$\leqslant e^{\bar{\gamma}\tau}EV(t,x(t)) \quad \text{for all} \quad -\tau \leqslant \theta \leqslant 0. \tag{2.6}$$

Note that either $EV(t,x(t)) = 0$ or $EV(t,x(t)) > 0$. In the case $EV(t,x(t)) = 0$, (2.6) and (2.2) yield that $x(t+\theta) = 0$ a.s. for all $-\tau \leqslant \theta \leqslant 0$. Recalling the fact that $f(t,0) \equiv 0$ and $g(t,0) \equiv 0$, one sees that $x(t+h) = 0$ a.s. for all $h > 0$, hence $U(t+h) = 0$ and $D_+U(t) = 0$. On the other hand, in the case $EV(t,x(t)) > 0$, (2.6) implies

$$EV(t+\theta,x(t+\theta)) < qEV(t,x(t)) \quad \text{for all} \quad -\tau \leqslant \theta \leqslant 0$$

since $e^{\bar{\gamma}\tau} < q$. Thus, by condition (2.3),

$$E\mathscr{L}V(t,x_t) \leqslant -\lambda EV(t,x(t)).$$

However, by Itô's formula, one can derive that for all $h > 0$

$$e^{\bar{\gamma}(t+h)}EV(t+h,x(t+h)) - e^{\bar{\gamma}t}EV(t,x(t))$$
$$= \int_t^{t+h} e^{\bar{\gamma}s}\,[\bar{\gamma}EV(s,x(s)) + E\mathscr{L}V(s,x_s)]\,ds.$$

Note that

$$\bar{\gamma}EV(t,x(t)) + E\mathscr{L}V(t,x_t) \leqslant -(\lambda - \bar{\gamma})EV(t,x(t)) < 0.$$

One sees from the continuity of V etc. that for all $h > 0$ sufficiently small

$$\bar{\gamma}EV(s,x(s)) + E\mathscr{L}V(s,x_s) \leqslant 0 \quad \text{if } t \leqslant s \leqslant h,$$

and consequently

$$e^{\bar{\gamma}(t+h)}EV(t+h,x(t+h)) \leqslant e^{\bar{\gamma}t}EV(t,x(t)).$$

So it must hold that $U(t+h) = U(t)$ for all $h > 0$ sufficiently small, and hence $D_+U(t) = 0$. Inequality (2.5) has been proved. It now follow from (2.5) immediately that

$$U(t) \leqslant U(0) \quad \text{for all } t \geqslant 0.$$

By the definition of $U(t)$ and condition (2.2) one sees

$$E|x(t)|^p \leqslant \frac{c_2}{c_1} E\|\xi\|^p e^{-\bar{\gamma}t} = \frac{c_2}{c_1} E\|\xi\|^p e^{-(\gamma-\varepsilon)t}.$$

Since ε is arbitrary, the required (2.4) must hold. The proof is complete. □

In the sequel of this section we shall deal with the almost sure exponential stability for the stochastic functional differential equation.

Theorem 2.2. *Suppose all of the conditions of Theorem* 2.1 *are satisfied and in addition* $p \geqslant 2$. *If there is a constant* $K > 0$ *such that for all* $t \geqslant 0$ *and* $\phi \in L^p_{\mathscr{F}_t}([-\tau,0];R^n)$

$$E|f(t,\phi)|^p + E(\mathrm{trace}[g^{\mathrm{T}}(t,\phi)g(t,\phi)])^{p/2} \leqslant K \sup_{-\tau\leqslant\theta\leqslant 0} E|\phi(\theta)|^p, \tag{2.7}$$

then for all $\xi \in C^{\mathrm{b}}_{\mathscr{F}_0}([-\tau,0];R^n)$

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t;\xi)| \leqslant -\frac{\gamma}{p} \quad \text{a.s.} \tag{2.8}$$

where γ *is the same as defined in Theorem* 2.1, *i.e.* $\gamma = \min\{\lambda, \log(q)/\tau\}$.

Proof. Fix any $\xi \in C^{\mathrm{b}}_{\mathscr{F}_0}([-\tau,0];R^n)$ and write again $x(t;\xi) = x(t)$ simply. For $t \geqslant \tau$,

$$\begin{aligned} E\|x_{t+\tau}\|^p &= E\left(\sup_{0\leqslant h\leqslant\tau}|x(t+h)|^p\right) \\ &\leqslant 3^{p-1}\left(E|x(t)|^p + E\left[\int_t^{t+\tau}|f(s,x_s)|\,\mathrm{d}s\right]^p\right. \\ &\quad \left. + E\left[\sup_{0\leqslant h\leqslant\tau}\left|\int_t^{t+h} g(s,x_s)\,\mathrm{d}w(s)\right|^p\right]\right). \end{aligned} \tag{2.9}$$

But by Hölder's inequality, condition (2.7) and Theorem 2.1, one derives that

$$\begin{aligned} E\left[\int_t^{t+\tau}|f(s,x_s)|\,\mathrm{d}s\right]^p &\leqslant \tau^{p-1}\int_t^{t+\tau} E|f(s,x_s)|^p\,\mathrm{d}s \\ &\leqslant K\tau^{p-1}\int_t^{t+\tau}\sup_{-\tau\leqslant\theta\leqslant 0} E|x(s+\theta)|^p\,\mathrm{d}s \\ &\leqslant \frac{Kc_2\tau^{p-1}}{c_1} E\|\xi\|^p \int_t^{t+\tau} e^{-\gamma(s-\tau)}\,\mathrm{d}s \\ &\leqslant \frac{Kc_2\tau^p}{c_1} E\|\xi\|^p e^{-\gamma(t-\tau)}. \end{aligned} \tag{2.10}$$

Also by the Burkholder–Davis–Gundy inequality (cf. Karatzas and Shreve (1991) or Mao (1994))

$$E\left[\sup_{0\leqslant h\leqslant\tau}\left|\int_t^{t+h} g(s,x_s)\,\mathrm{d}w(s)\right|^p\right] \leqslant C_p E\left(\int_t^{t+\tau}\mathrm{trace}[g^{\mathrm{T}}(s,x_s)g(s,x_s)]\,\mathrm{d}s\right)^{p/2},$$

where C_p is a positive constant dependent on p only. One can then show in the same way as (2.10) that

$$E\left[\sup_{0\leqslant h\leqslant\tau}\left|\int_t^{t+h} g(s,x_s)\,\mathrm{d}w(s)\right|^p\right]\leqslant\frac{C_pKc_2\tau^{p/2}}{c_1}E\|\xi\|^p\mathrm{e}^{-\gamma(t-\tau)}. \tag{2.11}$$

Substituting (2.10), (2.11) and (2.4) into (2.9) yields

$$E\|x_{t+\tau}\|^p\leqslant K_1\mathrm{e}^{-\gamma t}\quad\text{for all } t\geqslant\tau, \tag{2.12}$$

where

$$K_1=\frac{3^{p-1}c_2}{c_1}E\|\xi\|^p[1+K\mathrm{e}^{\gamma\tau}(\tau^p+C_p\tau^{p/2})].$$

We shall now show that (2.12) implies the required (2.8). Let $\varepsilon\in(0,\gamma)$ be arbitrary and let $k=1,2,\ldots$ It follows from (2.12) that

$$P(\omega\colon\ \|x_{(k+1)\tau}\|>\mathrm{e}^{-(\gamma-\varepsilon)k\tau/p})\leqslant\mathrm{e}^{(\gamma-\varepsilon)k\tau}E\|x_{(k+1)\tau}\|^p\leqslant K_1\mathrm{e}^{-\varepsilon k\tau}.$$

In view of the well-known Borel–Cantelli lemma, one sees that for almost all $\omega\in\Omega$

$$\|x_{(k+1)\tau}\|\leqslant\mathrm{e}^{-(\gamma-\varepsilon)k\tau/p} \tag{2.13}$$

holds for all but finitely many k. Hence there exists a $k_0(\omega)$, for all $\omega\in\Omega$ excluding a P-null set, for which (2.13) holds whenever $k\geqslant k_0$. Consequently, for almost all $\omega\in\Omega$,

$$\frac{1}{t}\log|x(t)|\leqslant-\frac{\gamma-\varepsilon}{p}$$

if $k\tau\leqslant t\leqslant(k+1)\tau,\ k\geqslant k_0$. Therefore

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|\leqslant-\frac{\gamma-\varepsilon}{p}\quad\text{a.s.}$$

and the required (2.8) follows by letting $\varepsilon\to0$. The proof is complete. □

3. Exponential stability of stochastic differential delay equations

In this section we shall apply the general Razumikhin-type theorems established in the previous section to deal with the exponential stability of stochasitc differential delay equations.

Consider a delay equation of the form

$$\begin{aligned}\mathrm{d}x(t)=\;&F(t,x(t),x(t-\delta_1(t)),\ldots,x(t-\delta_k(t)))\,\mathrm{d}t\\&+G(t,x(t),x(t-\delta_1(t)),\ldots,x(t-\delta_k(t)))\,\mathrm{d}w(t)\end{aligned} \tag{3.1}$$

on $t\geqslant0$ with initial data $x_0=\xi\in C^{\mathrm{b}}_{\mathscr{F}_0}([-\tau,0];R^n)$, where $\delta_i:R_+\to[0,\tau]$, $1\leqslant i\leqslant k$, are all continuous, and

$$F\colon R_+\times R^n\times R^{n\times k}\to R^n\quad\text{and}\quad G\colon R_+\times R^n\times R^{n\times k}\to R^{n\times m}.$$

We also impose a standing hypothesis:

(H2) Both F and G satisfy the local Lipschitz condition and the linear growth condition. That is, for each $j = 1,2,\ldots$, there is an $h_j > 0$ such that

$$|f(t,x,y_1,\ldots,y_k) - f(t,\bar{x},\bar{y}_1,\ldots,\bar{y}_k)| + \|g(t,x,y_1,\ldots,y_k) - g(t,\bar{x},\bar{y}_1,\ldots,\bar{y}_k)\|$$

$$\leqslant h_i(|x-\bar{x}| + |y_1 - \bar{y}_1| + \cdots + |y_k - \bar{y}_k|)$$

for all $t \geqslant 0$ and those $x, y_i, \bar{x}, \bar{y}_i \in R^n$ with $|x| \vee |y_i| \vee |\bar{x}| \vee |\bar{y}_i| \leqslant j$, and there is moreover an $h > 0$ such that

$$|f(t,x,y_1,\ldots,y_k)| + \|g(t,x,y_1,\ldots,y_k)\| \leqslant h(1 + |x| + |y_1| + \cdots + |y_k|)$$

for all $t \geqslant 0$ and $x, y_i \in R^n$.

Under (H2), Eq. (3.1) has a unique global solution which is again denoted by $x(t;\xi)$. Besides, we also assume that $F(t,0,\ldots,0) \equiv 0$, $G(t,0,\ldots,0) \equiv 0$.

Theorem 3.1. *Let $\lambda, \lambda_1, \ldots, \lambda_k, p, c_1, c_2$ be all positive numbers. Assume that there exists a function $V(t,x) \in C^{2,1}([-\tau,\infty) \times R^n; R_+)$ such that*

$$c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p \quad \text{for all } (t,x) \in [-\tau,\infty) \times R^n, \tag{3.2}$$

and

$$V_t(t,x) + V_x(t,x)F(t,x,y_1,\ldots,y_k)$$
$$+ \tfrac{1}{2}\,\mathrm{trace}[G^{\mathrm{T}}(t,x,y_1,\ldots,y_k)V_{xx}(t,x)G(t,x,y_1,\ldots,y_k)]$$
$$\leqslant -\lambda V(t,x) + \sum_{i=1}^{k} \lambda_i V(t - \delta_i(t), y_i) \tag{3.3}$$

for all $(t,x,y_1,\ldots,y_k) \in R_+ \times R^n \times R^{n\times k}$. If $\lambda > \sum_{i=1}^k \lambda_i$, then the zero solution of Eq. (3.1) is pth moment exponentially stable and its pth moment Lyapunov exponent should not be greater than $-(\lambda - q\sum_{i=1}^k \lambda_i)$, where $q \in (1, \lambda/\sum_{i=1}^k \lambda_i)$ is the unique root of $\lambda - q\sum_{i=1}^k \lambda_i = \log(q)/\tau$. In addition, if $p \geqslant 2$ and there is a $K > 0$ such that

$$|F(t,x,y_1,\ldots,y_k)|^2 + \mathrm{trace}[G^{\mathrm{T}}(t,x,y_1,\ldots,y_k)G(t,x,y_1,\ldots,y_k)]$$
$$\leqslant K\left(|x|^2 + \sum_{i=1}^{k} |y_i|^2\right) \tag{3.4}$$

for all $(t,x,y_1,\ldots,y_k) \in R_+ \times R^n \times R^{n\times k}$, then the zero solution of Eq. (3.1) is also almost surely exponentially stable and its sample Lyapunov exponent should not be greater than $-(\lambda_1 - q\sum_{i=1}^k \lambda_i)/p$.

Proof. Define, for $t \geqslant 0$ and $\varphi \in C([-\tau,0];R^n)$,

$$f(t,\varphi) = F(t,\varphi(0),\varphi(-\delta_1(t)),\ldots,\varphi(-\delta_k(t)))$$

and

$$g(t,\varphi) = G(t,\varphi(0),\varphi(-\delta_1(t)),\ldots,\varphi(-\delta_k(t))).$$

Then Eq. (3.1) becomes Eq. (2.1). Moreover, the operator $\mathscr{L}V$ becomes

$$\begin{aligned}\mathscr{L}V(t,\varphi) = {} & V_t(t,\varphi(0)) + V_x(t,\varphi(0))F(t,\varphi(0),\varphi(-\delta_1(t)),\ldots,\varphi(-\delta_k(t))) \\ & + \tfrac{1}{2}\mathrm{trace}[G^T(t,\varphi(0),\varphi(-\delta_1(t)),\ldots,\varphi(-\delta_k(t))) \\ & \times V_{xx}(t,\varphi(0))G(t,\varphi(0),\varphi(-\delta_1(t)),\ldots,\varphi(-\delta_k(t)))].\end{aligned}$$

If $t \geqslant 0$ and $\phi \in L^p_{\mathscr{F}_t}([-\tau,0];R^n)$ satisfying

$$EV(t+\theta,\phi(\theta)) < qEV(t,\phi(0)) \quad \text{for all } -\tau \leqslant \theta \leqslant 0,$$

then by condition (3.3)

$$\begin{aligned} E\mathscr{L}V(t,\phi) &\leqslant -\lambda EV(t,\phi(0)) + \sum_{i=1}^{k} \lambda_i EV(t-\delta_i(t),\phi(-\delta_i(t))) \\ &\leqslant -\left(\lambda - q\sum_{i=1}^{k}\lambda_i\right) EV(t,\phi(0)). \end{aligned} \tag{3.5}$$

So, by Theorem 2.1, the zero solution of Eq. (3.1) is pth moment exponentially stable and, moreover, its pth moment Lyapunov exponent should not be greater than $-(\lambda - q\sum_{i=1}^{k}\lambda_i)$. If furthermore $p \geqslant 2$ and (3.4) holds, then for all $t \geqslant 0$ and $\phi \in L^p_{\mathscr{F}_t}([-\tau,0];R^n)$,

$$\begin{aligned} & E|f(t,\phi)|^p + E(\mathrm{trace}[g^{\mathrm{T}}(t,\phi)g(t,\phi)])^{p/2} \\ &\leqslant 2E\left(K\left[|\phi(0)|^2 + \sum_{i=1}^{k}|\phi(-\delta_i(t))|^2\right]\right)^{p/2} \\ &\leqslant 2K^{p/2}(1+k)^{(p-2)/2}E\left[|\phi(0)|^p + \sum_{i=1}^{k}|\phi(-\delta_i(t))|^p\right] \\ &\leqslant 2[K(1+k)]^{p/2}\sup_{-\tau\leqslant\theta\leqslant 0} E|\phi(\theta)|^p. \end{aligned} \tag{3.6}$$

Therefore, by Theorem 2.2, the zero solution of Eq. (3.1) is almost surely exponentially stable and its sample Lyapunov exponent should not be greater than $-(\lambda - q\sum_{i=1}^{k}\lambda_i)/p$. The proof of the theorem is complete. □

We now use Theorem 3.1 to establish a useful corollary.

Corollary 3.2. *Assume that there is a* $\lambda > 0$ *such that*

$$x^{\mathrm{T}}F(t,x,0,\ldots,0) \leqslant -\lambda|x|^2 \quad \textit{for all } (t,x) \in R_+ \times R^n. \tag{3.7}$$

Assume also that there are nonnegative numbers α_i, β_i, $0 \leqslant i \leqslant k$ *such that*

$$|F(t,x,0,\ldots,0) - F(t,\bar{x},y_1,\ldots,y_k)| \leqslant \alpha_0|x-\bar{x}| + \sum_{i=1}^{k} \alpha_i|y_i| \tag{3.8}$$

and

$$\operatorname{trace}[G^{\mathrm{T}}(t,x,y_1,\ldots,y_k)G(t,x,y_1,\ldots,y_k)] \leqslant \beta_0|x|^2 + \sum_{i=1}^{k} \beta_i|y_i|^2 \tag{3.9}$$

for all $t \geqslant 0$, $x, \bar{x}, y_1, \ldots, y_k \in R^n$. *If* $p \geqslant 2$ *and*

$$\lambda > \sum_{i=1}^{k} \alpha_i + \frac{p-1}{2} \sum_{i=0}^{k} \beta_i, \tag{3.10}$$

then the zero solution of Eq. (3.1) *is pth moment exponentially stable and is also almost surely exponentially stable.*

Proof. Note first that (3.4) follows from (3.8), (3.9) and $f(t,0,\ldots,0) \equiv 0$. To check (3.3), let $V(t,x) = |x|^p$. Then for all $(t,x,y_1,\ldots,y_k) \in R_+ \times R^n \times R^{n\times k}$,

$$\begin{aligned}
&V_t(t,x) + V_x(t,x)F(t,x,y_1,\ldots,y_k) \\
&\quad + \tfrac{1}{2}\operatorname{trace}[G^{\mathrm{T}}(t,x,y_1,\ldots,y_k)V_{xx}(t,x)G(t,x,y_1,\ldots,y_k)] \\
&\quad = p|x|^{p-2}x^{\mathrm{T}}F(t,x,0,\ldots,0) + p|x|^{p-2}x^{\mathrm{T}}[F(t,x,y_1,\ldots,y_k) - F(t,x,0,\ldots,0)] \\
&\qquad + \frac{p}{2}|x|^{p-2}\operatorname{trace}[G^{\mathrm{T}}(t,x,y_1,\ldots,y_k)G(t,x,y_1,\ldots,y_k)] \\
&\qquad + \frac{p(p-2)}{2}|x|^{p-4}|x^{\mathrm{T}}G(t,x,y_1,\ldots,y_k)|^2 \\
&\quad \leqslant -\left(p\lambda - \frac{p(p-1)\beta_0}{2}\right)|x|^p + p\sum_{i=1}^{k}\alpha_i|x|^{p-1}|y_i| \\
&\qquad + \frac{p(p-1)}{2}\sum_{i=1}^{k}\beta_i|x|^{p-2}|y_i|^2.
\end{aligned} \tag{3.11}$$

Note the elementary inequality

$$u^\alpha v^{1-\alpha} \leqslant \alpha u + (1-\alpha)v \quad \text{for } u,v \geqslant 0,\ 0 \leqslant \alpha < 1. \tag{3.12}$$

Thus

$$|x|^{p-1}|y_i| = (|x|^p)^{(p-1)/p}(|y_i|^p)^{1/p} \leqslant \frac{p-1}{p}|x|^p + \frac{1}{p}|y_i|^p,$$

and similarly

$$|x|^{p-2}|y_i|^2 \leqslant \frac{p-2}{p}|x|^p + \frac{2}{p}|y_i|^p.$$

Substituting these into (3.11) gives

$$(3.11) \leqslant -\left(p\lambda - \frac{p(p-1)}{2}\beta_0 - (p-1)\sum_{i=1}^{k}\alpha_i - \frac{(p-1)(p-2)}{2}\sum_{i=1}^{k}\beta_i\right)|x|^p$$
$$+\sum_{i=1}^{k}(\alpha_i + (p-1)\beta_i)|y_i|^p.$$

Now the conclusions follow from Theorem 3.1 immediately and the proof is complete. □

The above corollary is in fact a generalization of Theorem 3.1 of Mao (1992), where the time lags $\delta_i(t), 1\leqslant i\leqslant k$ were required to be nonincreasing and continuously differentiable but we here only assume they are nonnegative bounded continuous functions. We would also like to mention that Caraballo (1990) studied the exponential stability of stochastic differential delay equations in Hilbert space where the time lag was also assumed to be nonincreasing and continuously differentiable, and we believe his result can be improved as above by Razumikhin's arguments. One more point we need point out is that conditions of Corollary 3.2 are delay-independent and so the conclusions. However, (3.7) may not hold sometimes and, instead, one may have $x^{\mathrm{T}}F(t,x,x,\ldots,x)\leqslant -\lambda|x|^2$. For example, $F(t,x,y_1,\ldots,y_k) = ax - \sum_{i=1}^{k} b_i y_i$ with $0\leqslant a < \sum_{i=1}^{k} b_i$. In this case, the delay effect plays the main role in stabilizing the system. The following corollary deals with this case.

Corollary 3.3. *Assume that there is a* $\lambda > 0$ *such that*

$$x^{\mathrm{T}}F(t,x,x,\ldots,x)\leqslant -\lambda|x|^2 \quad \textit{for all}\,(t,x)\in R_+\times R^n. \tag{3.13}$$

Let $p\geqslant 2$ *and assume furthermore that there are nonnegative numbers* $\alpha_i,\beta_i,\ 0\leqslant i\leqslant k$ *such that*

$$|F(t,x,x,\ldots,x) - F(t,\bar{x},y_1,\ldots,y_k)|^p \leqslant \alpha_0|x-\bar{x}|^p + \sum_{i=1}^{k}\alpha_i|x-y_i|^p \tag{3.14}$$

and

$$(\mathrm{trace}[G^{\mathrm{T}}(t,x,y_1,\ldots,y_k)G(t,x,y_1,\ldots,y_k)])^{p/2} \leqslant \beta_0|x|^p + \sum_{i=1}^{k}\beta_i|y_i|^p \tag{3.15}$$

for all $t\geqslant 0$, $x,\bar{x},y_1,\ldots,y_k\in R^n$. *If*

$$\lambda > (K\hat{\alpha})^{1/p} + \tfrac{1}{2}(p-1)\hat{\beta}^{2/p}, \tag{3.16}$$

where

$$\bar{C}_p = \left[\frac{(p-1)^2}{2}\left(\frac{p}{p-1}\right)^{p-1}\right]^{p/2},$$

$K = 2^{p-1}[\tau^p(\alpha_0+\hat{\alpha}) + \bar{C}_p\tau^{p/2}\hat{\beta}]$, $\hat{\alpha} = \sum_{i=1}^{k}\alpha_i$ *and* $\hat{\beta} = \sum_{i=0}^{k}\beta_i$, *then the zero solution of Eq.* (3.1) *is pth moment exponentially stable and is also almost surely exponentially stable.*

Proof. Regard Eq. (3.1) as a delay equation on $t \geqslant \tau$ with initial data on $[-\tau, \tau]$, i.e. consider the delay interval of length 2τ instead of τ. By the well-known martingale moment inequality (cf. Karatzas and Shreve, 1991), Hölder's inequality and the assumptions, one can derive that

$$E|x(t) - x(t - \delta_i(t))|^p \leqslant K \sup_{-2\tau \leqslant \theta < 0} E|x(t+\theta)|^p \tag{3.17}$$

for $t \geqslant \tau$, $1 \leqslant i \leqslant k$, where K is defined above. One can also show that for $t \geqslant \tau$,

$$E\mathscr{L}|x_t|^p \leqslant -p\lambda E|x(t)|^p + \varepsilon_1(p-1)E|x(t)|^p + \frac{1}{\varepsilon_1^{p-1}} \sum_{i=1}^{k} \alpha_i E|x(t) - x(t - \delta_i(t))|^p$$

$$+ \tfrac{1}{2}\varepsilon_2(p-1)(p-2)E|x(t)|^p + \frac{(p-1)\hat{\beta}}{\varepsilon_2^{(p-2)/2}} \sup_{-\tau \leqslant \theta \leqslant 0} E|x(t+\theta)|^p, \tag{3.18}$$

where the elementary inequality (3.12) has been used, and $\varepsilon_1, \varepsilon_2$ are two positive parameters to be chosen. Substituting (3.17) into (3.18) and choosing $\varepsilon_1 = (K\hat{\alpha})^{1/p}$, $\varepsilon_2 = \hat{\beta}^{2/p}$ one then obtains

$$E\mathscr{L}|x_t|^p \leqslant -p\lambda E|x(t)|^p + \left(p(K\hat{\alpha})^{1/p} + \tfrac{1}{2}p(p-1)\hat{\beta}^{2/p}\right)$$

$$\times \sup_{-2\tau \leqslant \theta \leqslant 0} E|x(t+\theta)|^p. \tag{3.19}$$

By (3.16), one can choose $q > 1$ such that

$$\lambda > q\left((K\hat{\alpha})^{1/p} + \tfrac{1}{2}(p-1)\hat{\beta}^{2/p}\right).$$

Therefore, if $E|x(t+\theta)|^p < qE|x(t)|^p$ for $-2\tau \leqslant \theta \leqslant 0$, (3.19) implies

$$E\mathscr{L}|x_t|^p \leqslant -p\left(\lambda - q(K\hat{\alpha})^{1/p} - \tfrac{1}{2}q(p-1)\hat{\beta}^{2/p}\right)E|x(t)|^p.$$

So the conclusions follow from Theorems 2.1 and 2.2. The proof is complete. □

4. Exponential stability of stochastically perturbed equations

In this section we shall use the general theorems established in Section 2 to deal with the exponential stability of stochastically perturbed equations. Consider a stochastic equation of the form

$$dx(t) = [\psi(t, x(t)) + F(t, x_t)]\,dt + g(t, x_t)\,dw(t) \quad \text{on } t \geqslant 0 \tag{4.1}$$

with initial data $x_0 = \xi \in C^b_{\mathscr{F}_0}([-\tau, 0]; R^n)$, where g is the same as defined in Section 2, while $\psi: R_+ \times R^n \to R^n$ and $F: R_+ \times C([-\tau, 0]; R^n) \to R^n$. As before, assume that ψ, F, g satisfy the local Lipschitz condition and the linear growth condition (similar to (H1) and (H2)), and moreover $\psi(t, 0) = F(t, 0) \equiv 0$, $g(t, 0) \equiv 0$. Under these conditions Eq. (4.1) has a unique global solution. Eq. (4.1) can be regarded as the stochastically perturbed equation of the ordinary differential equation

$$\dot{x}(t) = \psi(t, x(t)). \tag{4.2}$$

To a certain degree it is known that if Eq. (4.2) is exponentially stable and the stochastic perturbation is sufficiently small, then the perturbed Eq. (4.1) will remain exponentially stable (cf. Mao, 1994, Theorem 6.5.1). The critical research in this direction is to give better bound for the stochastic perturbation. We shall now apply the Razumikhin-type theorems to establish some new results.

Theorem 4.1. *Let $\lambda, c_1, c_2, \beta_1, \ldots, \beta_4$ all be positive numbers and $p \geqslant 2$, $q > 1$. Assume that there exists a function $V(t,x) \in C^{2,1}([-\tau,\infty) \times R^n; R_+)$ such that*

$$c_1|x|^p \leqslant V(t,x) \leqslant c_2|x|^p \quad \textit{for all } (t,x) \in [-\tau,\infty) \times R^n,$$

and

$$V_t(t,x) + V_x(t,x)\psi(t,x) \leqslant -\lambda V(t,x),$$

$$|V_x(t,x)| \leqslant \beta_1[V(t,x)]^{(p-1)/p}, \quad \|V_{xx}(t,x)\| \leqslant \beta_2[V(t,x)]^{(p-2)/p}$$

for all $(t,x) \in R_+ \times R^n$. Assume also that

$$E|F(t,\phi)|^p \leqslant \beta_3 EV(t,\phi(0)) \quad \text{and} \quad E(\text{trace}[g^{\mathrm{T}}(t,\phi)g(t,\phi)])^{p/2} \leqslant \beta_4 EV(t,\phi(0))$$

for all $t \geqslant 0$ and those $\phi \in L^p_{\mathscr{F}_t}([-\tau,0];R^n)$ satisfying

$$EV(t+\theta,\phi(\theta)) < qEV(t,\phi(0)) \quad \textit{for all } -\tau \leqslant \theta \leqslant 0. \tag{4.3}$$

If

$$\lambda > \beta_1\beta_3^{1/p} + \tfrac{1}{2}\beta_2\beta_4^{2/p}, \tag{4.4}$$

then the zero solution of Eq. (4.1) is pth moment exponentially stable. In addition, if there is a constant $K > 0$ such that for all $t \geqslant 0$ and $\phi \in L^p_{\mathscr{F}_t}([-\tau,0];R^n)$

$$E|\psi(t,\phi(0))|^p + E|F(t,\phi)|^p + E(\text{trace}[g^{\mathrm{T}}(t,\phi)g(t,\phi)])^{p/2} \leqslant K \sup_{-\tau\leqslant\theta\leqslant 0} E|\phi(\theta)|^p,$$

then the zero solution of Eq. (4.1) is also almost surely exponentially stable.

Proof. Define $f(t,\varphi) = \psi(t,\varphi(0)) + F(t,\varphi)$ and Eq. (4.1) becomes Eq. (2.1). Moreover,

$$\begin{aligned}\mathscr{L}V(t,\varphi) &= V_t(t,\varphi(0)) + V_x(t,\varphi(0))[\psi(t,\varphi(0)) + F(t,\varphi)] \\ &\quad + \tfrac{1}{2}\text{trace}[g^{\mathrm{T}}(t,\varphi)V_{xx}(t,\varphi(0))g(t,\varphi)].\end{aligned}$$

Hence for $t \geqslant 0$ and those $\phi \in L^p_{\mathscr{F}_t}([-\tau,0];R^n)$ satisfying (4.3) one can derive from the assumptions that

$$\begin{aligned}E\mathscr{L}V(t,\phi) &\leqslant -\lambda EV(t,\phi(0)) + \beta_1 E([V(t,\phi(0))]^{(p-1)/p}|F(t,\phi)|) \\ &\quad + \frac{\beta_2}{2}E([V(t,\phi(0))]^{(p-2)/p}\text{trace}[g^{\mathrm{T}}(t,\phi)g(t,\phi)]).\end{aligned} \tag{4.5}$$

But for any $\varepsilon > 0$

$$E([V(t,\phi(0))]^{(p-1)/p}|F(t,\phi)|) = E\left[(\varepsilon V(t,\phi(0)))^{(p-1)/p}\left(\frac{|F(t,\phi)|^p}{\varepsilon^{p-1}}\right)^{1/p}\right]$$
$$\leqslant \frac{\varepsilon(p-1)}{p}EV(t,\phi(0)) + \frac{1}{p\varepsilon^{p-1}}E|F(t,\phi)|^p$$
$$\leqslant \left(\frac{\varepsilon(p-1)}{p} + \frac{\beta_3}{p\varepsilon^{p-1}}\right)EV(t,\phi(0)),$$

where the elementary inequality (3.12) has been used once again. In particular, if we choose $\varepsilon = \beta_3^{1/p}$, then

$$E([V(t,\phi(0))]^{(p-1)/p}|F(t,\phi)|) \leqslant \beta_3^{1/p}EV(t,\phi(0)).$$

Similarly, one can show

$$E([V(t,\phi(0))]^{(p-2)/p}\,\mathrm{trace}[g^{\mathrm{T}}(t,\phi)g(t,\phi)]) \leqslant \beta_4^{2/p}EV(t,\phi(0)).$$

Substituting these into (4.5) yields

$$E\mathscr{L}V(t,\phi) \leqslant -(\lambda - \beta_1\beta_3^{1/p} - \tfrac{1}{2}\beta_2\beta_4^{2/p})EV(t,\phi(0)).$$

Now the conclusions follow from Theorems 2.1 and 2.2 immediately. The proof is complete. □

Corollary 4.2. *Assume that there is a $\lambda > 0$ such that*

$$x^{\mathrm{T}}\psi(t,x) \leqslant -\lambda|x|^2 \quad \textit{for all } (t,x)\in R_+ \times R^n.$$

Assume also that there are two functions $\alpha_1(\cdot),\alpha_2(\cdot)\in C([-\tau,0];R_+)$ such that

$$|F(t,\varphi)| \leqslant \int_{-\tau}^{0} \alpha_1(\theta)|\varphi(\theta)|\,\mathrm{d}\theta,$$

$$\mathrm{trace}[g^{\mathrm{T}}(t,\varphi)g(t,\varphi)] \leqslant \int_{-\tau}^{0} \alpha_2(\theta)|\varphi(\theta)|^2\,\mathrm{d}\theta$$

for all $t\geqslant 0$ and $\varphi\in C([-\tau,0];R^n)$. If $p\geqslant 2$ and

$$\lambda > (\tau\bar{\alpha}_1)^{1/p} + \frac{p-1}{2}(\tau\bar{\alpha}_2)^{2/p}, \tag{4.6}$$

where

$$\bar{\alpha}_1 = \left(\int_{-\tau}^{0} |\alpha_1(\theta)|^{p/(p-1)}\,\mathrm{d}\theta\right)^{p-1},$$

$$\bar{\alpha}_2 = \begin{cases} \max_{-\tau\leqslant\theta\leqslant 0} a_2(\theta) & \text{if } p=2,\\ \left(\int_{-\tau}^{0} |\alpha_2(\theta)|^{p/(p-2)}\,\mathrm{d}\theta\right)^{(p-2)/p} & \text{if } p>2, \end{cases}$$

then the zero solution of Eq. (4.1) *is pth moment exponentially stable. In addition, if there is a* $K > 0$ *such that* $|\psi(t,x)| \leqslant K|x|$ *for all* $(t,x) \in R_+ \times R^n$, *then the zero solution of Eq.* (4.1) *is also almost surely exponentially stable.*

Proof. Let $V(t,x) = |x|^p$. Then

$$V_t(t,x) + V_x(t,x)\psi(t,x) \leqslant -p\lambda|x|^p,$$

$$V_x(t,x)| \leqslant p|x|^{p-1}, \quad \|V_{xx}(t,x)\| \leqslant p(p-1)|x|^{p-2}$$

for all $(t,x) \in R_+ \times R^n$. By (4.6) one can choose $q > 1$ such that

$$\lambda > (q\tau\bar{\alpha}_1)^{1/p} + \frac{p-1}{2}(q\tau\bar{\alpha}_2)^{2/p}. \tag{4.7}$$

Now for $t \geqslant 0$ and $\phi = \{\phi(\theta): -\tau \leqslant \theta \leqslant 0\} \in L^p_{\mathscr{F}_t}([-\tau,0];R^n)$ satisfying

$$E|\phi(\theta)|^p < qE|\phi(0)|^p \quad \text{for all } -\tau \leqslant \theta \leqslant 0,$$

one can easily show that

$$E|F(t,\phi)|^p \leqslant q\tau\bar{\alpha}_1 E|\phi(0)|^p$$

and

$$E(\text{trace}[g^{\mathrm{T}}(t,\phi)g(t,\phi)]))^{p/2} \leqslant q\tau\bar{\alpha}_2 E|\phi(0)|^p.$$

So the conclusions follow from Theorem 4.1 and the proof is complete. □

5. Examples

In this section we shall discuss two examples to illustrate our theory due to the page limit. In the following examples we shall omit mentioning the initial data which are always assumed to be in $C^{\mathrm{b}}_{\mathscr{F}_0}([-\tau,0];R^n)$ anyway.

Example 5.1. Consider a linear stochastic differential delay equation

$$\mathrm{d}x(t) = -[Ax(t) + Bx(t-\delta(t))]\,\mathrm{d}t + Cx(t-\delta(t))\,\mathrm{d}w(t), \tag{5.1}$$

where A, B, C are all $n \times n$ constant matrices, $w(t)$ is a one-dimensional Brownian motion and $\delta : R_+ \to [-\tau, 0]$ is continuous.

Case (i). Assume that $A+A^{\mathrm{T}}$ is positive definite and its smallest eigenvalue is denoted by $\lambda_{\min}(A + A^{\mathrm{T}})$. In this case, one can easily conclude by Corollary 3.2 that if $p \geqslant 2$ and

$$\tfrac{1}{2}\lambda_{\min}(A + A^{\mathrm{T}}) > \|B\| + \frac{p-1}{2}\|C\|^2, \tag{5.2}$$

then the zero solution of Eq. (5.1) is both pth moment and almost surely exponentially stable.

Case (ii). Assume that $A+A^T+B+B^T$ is positive definite and its smallest eigenvalue is denoted by $\lambda_{\min}(A+A^T+B+B^T)$. To apply Corollary 3.2, write Eq. (5.1) as

$$dx(t) = -[(A+B)x(t) + Bx(t-\delta(t)) - Bx(t-\delta_2(t))]\,dt + Cx(t-\delta(t))\,dw(t) \tag{5.3}$$

with $\delta_2(t) \equiv 0$. One then easily sees that if $p \geqslant 2$ and

$$\tfrac{1}{2}\lambda_{\min}(A+A^T+B+B^T) > 2\|B\| + \frac{p-1}{2}\|C\|^2, \tag{5.4}$$

then the zero solution of Eq. (5.3), i.e. (5.1), is pth moment as well as almost surely exponentially stable. Of course, in this case one may also apply Corollary 3.3 to obtain a delay-dependent result. For simplicity, choose $p = 2$. Note that for any $\rho > 0$

$$|Ax + By - A\bar{x} - B\bar{y}|^2 \leqslant (1+\rho^{-1})\|A\|^2|x-\bar{x}|^2 + (1+\rho)\|B\|^2|y-\bar{y}|^2.$$

One can then apply Corollary 3.3 (with $p = 2$) to conclude that if

$$\tfrac{1}{2}\lambda_{\min}(A+A^T+B+B^T) > \tfrac{1}{2}\|C\|^2 + \inf_{\rho>0}\{\|B\|[2(1+\rho)(\tau^2[(1+\rho^{-1})\|A\|^2 + (1+\rho)\|B\|^2] + \tau\|C\|^2)]^{1/2}\}, \tag{5.5}$$

then the zero solution of Eq. (5.1) is second moment as well as almost surely exponentially stable. As a special case, let us look at a one-dimensional linear delay equation

$$dx(t) = -bx(t-\delta(t))\,dt + cx(t-\delta(t))\,dw(t) \tag{5.6}$$

with $b > c^2/2$. In this case, criteria (5.2) and (5.4) do not work but (5.5) reduces to

$$b > \frac{c^2}{2} + b\sqrt{2(\tau^2 b^2 + \tau c^2)}.$$

Hence, if

$$\tau < \frac{1}{2b^2}\left(\sqrt{c^4 + \frac{1}{2}(2b-c^2)^2} - c^2\right)$$

then the zero solution of Eq. (5.6) is both second moment and almost surely exponentially stable.

Example 5.2. Consider a stochastic oscillator described by a semi-linear stochastic functional differential equation

$$\ddot{z}(t) + 3\dot{z}(t) + 2z(t) = \sigma_1(z_t, \dot{z}_t) + \sigma_2(z_t, \dot{z}_t)\dot{w}(t), \tag{5.7}$$

where $\dot{w}(t)$ is a one-dimensional white noise, i.e. $w(t)$ a Brownian motion, both $\sigma_1, \sigma_2 : C([-\tau, 0]; R^2) \to R$ are locally Lipschitz continuous and, moreover,

$$|\sigma_1(\varphi)| \vee |\sigma_2(\varphi)| \leqslant \int_{-\tau}^{0} |\varphi(\theta)|\,d\theta, \quad \varphi \in C([-\tau, 0]; R^2).$$

We claim that if

$$\tau < \frac{\sqrt{42}-\sqrt{14}}{14} \tag{5.8}$$

then the zero solution of Eq. (5.7) is second moment as well as almost exponentially stable. To show this, introduce a new variable $x = (z,\dot{z})^{\mathrm{T}}$ and write Eq. (5.7) as a two-dimensional stochastic functional differential equation

$$\mathrm{d}x(t) = [Ax(t) + F(x_t)]\,\mathrm{d}t + G(x_t)\,\mathrm{d}w(t), \tag{5.9}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -3 \end{pmatrix}, \quad F(\varphi) = \begin{pmatrix} 0 \\ \sigma_1(\varphi) \end{pmatrix}, \quad G(\varphi) = \begin{pmatrix} 0 \\ \sigma_2(\varphi) \end{pmatrix}.$$

It is easy to find

$$H = \begin{pmatrix} 1 & 1 \\ -1 & -2 \end{pmatrix}, \quad \text{and hence} \quad H^{-1} = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}.$$

such that

$$H^{-1}AH = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}.$$

Set

$$Q = (H^{-1})^{\mathrm{T}}H^{-1} = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}$$

and define $V(x) = x^{\mathrm{T}}Qx$ for $x \in R^2$. It is easy to verify

$$\tfrac{1}{7}|x|^2 \leqslant V(x) \leqslant 7|x|^2.$$

We further compute

$$\begin{aligned}
\mathscr{L}V(\varphi) &= 2\varphi^{\mathrm{T}}(0)Q[A\varphi(0) + F(\varphi)] + G^{\mathrm{T}}(\varphi)QG(\varphi) \\
&\leqslant -2V(\varphi(0)) + 2|\varphi^{\mathrm{T}}(0)(H^{-1})^{\mathrm{T}}|\,|H^{-1}F(\varphi)| + 2|\sigma_2(\varphi)|^2 \\
&\leqslant -2V(\varphi(0)) + \sqrt{14}\tau V(\varphi(0)) + \frac{2}{\sqrt{14}\tau}|\sigma_1(\varphi)|^2 + 2|\sigma_2(\varphi)|^2 \\
&\leqslant -(2-\sqrt{14}\tau)V(\varphi(0)) + (\sqrt{14} + 14\tau)\int_{-\tau}^{0} V(\varphi(\theta))\,\mathrm{d}\theta.
\end{aligned} \tag{5.10}$$

By condition (5.8) one can find $q > 1$ such that

$$2 - \sqrt{14}(1+q)\tau - 14q\tau^2 > 0.$$

Therefore, for any $\phi \in L^2_{\mathscr{F}_t}([-\tau,0];R^n)$ satisfying $EV(\phi(\theta)) < qEV(\phi(0))$ on $-\tau \leqslant \theta \leqslant 0$, (5.10) yields

$$E\mathscr{L}V(\phi) \leqslant -(2 - \sqrt{14}(1+q)\tau - 14q\tau^2)EV(\phi(0)).$$

Thus the conclusions follows from Theorems 2.1 and 2.2.

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