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Inverse System of a Symbolic Power II. The Waring Problem for Forms

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The classical Waring problem for forms is to determine the smallest length s of an additive decomposition of a general degree d homogeneous polynomial or form f in r variables as sum of s dth powers of linear forms. We show that its solution is implied by a result of J. Alexander and A. Hirschowitz, concerning the Hilbert functions of the ideal of functions vanishing to order two at a generic set of s points in $\mathbb{P}^{r-1}$. Using Macaulay's inverse systems, we show that the Alexander–Hirschowitz result is equivalent to determining the number of linear syzygies of s homogeneous forms in r variables that are dth powers of a given set of general linear forms. We also determine the dimension of the family of degree d forms that have additive decompositions of length s. We then study several notions of length for forms f, having to do with the kind of length-s, zero-dimensional schemes Z in $\mathbb{P}^{r-1}$ whose defining ideal $I(Z)$ annihilates the inverse system of f. When Z is to consist of distinct points, we obtain the above length of additive decomposition of f. When Z is smoothable we obtain the "smoothable length" of f; when Z is arbitrary, we obtain a "scheme length" of f. All these lengths are at least as large as the dimension of the vector space of all order-i partial derivates of f, for each i. The above-mentioned length functions are distinct. Using results about the existence of nonsmoothable Gorenstein point singularities in codimension 4, we show that when $r = 5$ there are forms f of scheme length s, which are not in the closure of the family of forms having additive decompositions of length s. Finally, we propose a new set of Waring problems for forms, using these lengths.

1. INTRODUCTION

Any degree-d homogeneous form f in r variables over an algebraically closed field k with char $k = 0$ or char $k = p > d$ has additive decompositions $f = L_1^d + \cdots + L_s^d$ as sums of powers of linear forms. The number of

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summands is the *length* of the decomposition. We let $g_k(f)$ denote the minimal length of any such decomposition of f. The Waring problem for degree-d forms in r variables over k is to determine the length $g_k(r,d)$ of a general enough form f of degree d. If $W = k^r$, $\mathrm{Sym}^d W$ is the d-fold symmetric product of W, and $\mathbb{P}(V)$ is the projective space on V, then

$$g_k(r,d) = \min\{s \mid \exists U_s \text{ open-dense in } \mathbb{P}(\mathrm{Sym}^d W) \\ \text{with } f \in U_s \Rightarrow g_k(f) = s\}.$$

The Waring problem has attracted interest for its naturality (see [B1], [B3], [C1], [EhR], [L], [Lu], [R2], [Re], [Wk]), and for its connections to other problems (see [R1] and [H1]).

Recently, A. Alexander and A. Hirschowitz determined the Hilbert function $H(r,s)$ of the ideal $K(P,2)$ of functions vanishing to order two at any general enough set P of s points in the projective space $\mathbb{P}^{r-1}(k)$, where k is an infinite field (see [A1], [AH1], [AH2]). We use Macaulay's inverse systems and their result to determine the number of linear syzygies of s homogeneous forms that are dth powers of s general enough linear forms (Theorem 1). It is well known that Theorem 1 implies the solution of the Waring problem for forms, when $k = \mathbb{C}$ (see [T2], [Wh], [EhR]). We show the implication also when $\mathrm{char}\, k = 0$, or when $\mathrm{char}\, k > d$ (Theorem 2). A third corollary of the Alexander–Hirschowitz theorem is a determination of the dimension $\dim(\mathrm{PS}(r,s,d))$ of the family of all degree-d homogeneous forms f that have additive decompositions of length s (Theorem 3).

Theorem 1 improves a result of M. Hochster and D. Laksov on the dimension of the vector space $\mathcal{R}_1 V = \langle\{X_1 v, \dots, X_r v, v \in V\}\rangle$, where V is a general enough s-dimensional vector subspace of the space $\mathcal{R}_d$ of homogeneous degree-d forms in a polynomial ring $\mathcal{R} = k[X_1, \dots, X_r]$ (see [HL]). The new result is that the expected generic dimensions are attained —with four exceptions—by vector spaces that can be written as $V = L^d = \langle L_1^d, \dots, L_s^d\rangle$.

The proof that Theorem 1 is a corollary of the Alexander–Hirschowitz theorem uses Theorem I of [EI2]. The proofs that Theorems 2 and 3 are corollaries of Theorem 1 when $k = \mathbb{C}$ requires also one of two classical results, the algebraic Lasker–Wakeford theorem ([L], [Wk]), or the geometric Terracini theorem ([T1], [T2], [B1]). For a different exposition of the algebraic route from the Alexander–Hirschowitz theorem to Theorem 2, when $k = \mathbb{C}$, see R. Ehrenborg and G.-C. Rota's article [EhR]; they state that the generalization of their results to fields of arbitrary characteristic is a "completely open problem of the utmost interest." Of course, the answer to the Waring problem changes over low characteristics, as when $d = p = \mathrm{char}\, k$, the pth powers of linear forms land in a proper subspace $\langle X_1^p, \dots, X_r^p\rangle$ of $\mathcal{R}_d$. There are two ways to avoid the crux of this problem. We choose one in handling the case char $k > d$. Another would be to replace $\mathscr{R}$ by the divided power algebra $\mathscr{D}$: Theorems 2 and 3 would then extend to all characteristics over $\mathscr{D}$. We have not treated the Waring problem for low characteristics char $k \leq d$ or for non-algebraically closed fields. For the real case see Bruce Reznick's memoir [R1].

In the final section, we compare several different notions of length of a form f, beginning with $g_k(f)$; the lengths are related to the kinds of annihilating length-s schemes of f in $\mathbb{P}^{r-1}$ (Lemma 4, Definitions 4A–4D). We compare these lengths and show that they are in general distinct (Theorem 5, Proposition 10). We show that for any of our notions of length, the vector space dimension of the space of ith order derivates of a length-s form f is at most s (Proposition 6). When $7 \leq r \leq 10$, $d \geq 5$, and $s = 2r$ there are families of degree-d forms f, having at most s linearly independent ith order derivates for each i, but not in the closure of $\mathrm{PS}(r, s, d)$ (Proposition 10, Example 11). We give similar examples for the pairs $(r, s) = (6, 42)$ and $(5{,}760)$ (Example 13). The examples use the existence of families of nonsmoothable length s punctual Gorenstein singularities of embedding dimension $r - 1 \geq 4$.

In a sequel the author will use inverse systems to give "Koszul" upper bounds for the Hilbert function of $K(P, \mathbf{a})$, $a > 2$ [I3]. In work joint with V. Kanev we will apply Theorem 3 and Section 3 to the study of determinantal loci of catalecticant matrices [IK].

2. Inverse Systems of Order Two Vanishing Ideals

We denote by R the polynomial ring $R = k[x_1, \ldots, x_r]$ in r variables over an infinite field k, and by R_d the subspace of forms of degree d. We denote by $K(P, \mathbf{a})$ the homogeneous ideal in R of functions vanishing to order at least a at each point of the set $P = (p_1, \ldots, p_s)$ of points in $\mathbb{P}^{r-1}$; and $K(p, \mathbf{a})_d$ is the vector space of degree-d homogeneous elements of $K(P, \mathbf{a})$. We recall the apolarity action from [EI2], for char $k = 0$ or char $k > d$. We denote by $\mathscr{R}$ the polynomial ring $\mathscr{R} = k[X_1, \ldots, X_r]$ upon which the polynomial ring $R = k[x_1, \ldots, x_r]$ acts as higher order partial differential operators: if $h \in R$, and $f \in \mathscr{R}$, then $h \cdot f = h(\partial/\partial X_1, \ldots, \partial/\partial X_r) \circ f$. The pairing $R_d \times \mathscr{R}_d \to k$ is exact. If I is an ideal of R, then the *inverse system* of I is the sequence of vector spaces

$$[I^{-1}]_d = \mathrm{Ann}(I) \cap \mathscr{R}_d = \langle I_d \rangle^{\perp} .$$

When $0 < \text{char } k \leq d$, we need to take $\mathscr{R} = \mathscr{D}$, the divided power ring, and the contraction action (see [EI2]).

We now recall the Alexander–Hirschowitz result.

VANISHING THEOREM (J. Alexander and A. Hirschowitz[1]). *Suppose that k is an infinite field, that $P = (p(1), \dots, p(s))$ is a sufficiently general set of points in $\mathbb{P}^{r-1}(k)$, and that the integer $d \geq 3$. Then $K(P, \mathbf{2})_d$ has codimension* $\min(sr, \dim_k R_d)$ *in R_d, except for the following four exceptional triples (r, s, d),*

$$(3,5,4), \qquad (4,9,4), \qquad (5,14,4), \qquad (5,7,3), \tag{1}$$

for which the codimension is $sr - 1$.

We next recall the result of [EI2] concerning the inverse system of the ideal $I = K(P, \mathbf{a})$. If $p = (p_1, \dots, p_r)$ is a k-valued point of affine r-space $\mathbb{A}^r(k)$ we let $L_p = p_1 X_1 + \cdots + p_r X_r$ be the corresponding linear form of $\mathscr{R}_1$. If $P = (p(1), \dots, p(s))$ is a set of s points of affine r-space $\mathbb{A}^r(k)$ we let $L_P^j = \langle L_{p(1)}^j, \dots, L_{p(s)}^j \rangle \subset \mathscr{R}_j$ denote the k-span of the jth powers of the corresponding linear forms. Since the vector space L_P^j depends only on the classes $\overline{L_{p(u)}}$ of the forms $L_{p(u)}$ up to nonzero constant multiple, it is well defined for a set $P = (\overline{p(1)}, \dots, \overline{p(s)})$ of s points in projective $r - 1$ space $\mathbb{P}^{r-1}$, where $\overline{p(i)} = p(i) \bmod k^*$ multiple. We will omit the bar and write P for the s-points of $\mathbb{P}^{r-1}$. If V is a vector subspace of $\mathscr{R}_v$, we let $\mathscr{R}_u V$ be the vector space span of $\{hv | h \in \mathscr{R}_u,\ v \in V\}$ in $\mathscr{R}_{u+v}$.

LEMMA (Theorem I of [EI2]). *When* $\operatorname{char} k = 0$ *or is larger than i, the annihilator $[I^{-1}]_i$ in $\mathscr{R}_i$ of the degree-i piece of the ideal $I = K(P, \mathbf{a})_i$ of R satisfies*

$$[I^{-1}]_i = \mathscr{R}_{a-1} L_P^{i+1-a}. \tag{2}$$

We can now show

THEOREM 1 (number of syzygies of homogeneous forms that are powers of linear forms). *Suppose that k is infinite and* $\operatorname{char} k = 0$, *or that* $\operatorname{char} k = p$ *and that $p > j + 1$. If $L_1, \dots, L_s$ are general enough linear forms, and $L^j = \langle L_1^j, \dots, L_s^j \rangle$ is the span of their jth powers, and $j \geq 2$, then the Alexander–Hirschowitz theorem implies that the dimension of $\mathscr{R}_1 L^j$ satisfies*

$$\dim_k \mathscr{R}_1 L^j = \min(rs, \dim_k \mathscr{R}_{j+1}), \tag{3}$$

except when $(r, s, d = j + 1)$ is in the list (1) of four exceptional triples, for which the dimension of $\mathscr{R}_1 L^j$ is $rs - 1$. When $j = 1$, and $s \leq r$, then the dimension of $\mathscr{R}_1 L$ satisfies

$$\dim_k \mathscr{R}_1 L = rs - \binom{s}{2} = s(2r + 1 - s)/2.$$

[1]See [A1] for degrees $d \geq 5$ and degrees 3 and 4 for $r \leq 5$ [AH1] for degree 4 and $r > 5$, and [AH2] for degree 3 when $r \geq 5$. In view of the scattered nature of the work, and the authors' report of improvement and simplification of their method in the later papers, a coherent exposition of the complete result would be useful!

If char $k = p$ *and is less or equal* $j + 1$, *then the above statements are true with* L^j *replaced by the space of divided powers* $L^{[j]}$, *and with* $\mathcal{R}$ *replaced by the divided power ring* $\mathcal{D}$.

Proof. The case char $k = 0$, or char $k = p > j + 1$ is immediate from (2) of the lemma (Theorem I of [EI2]) and the Alexander–Hirschowitz theorem. The case $j = 1$ is simply the Koszul resolution for complete intersections L generated by forms of degree one. The case char $k \leq j + 1$ comes from the exact duality between R and the divided power algebra $\mathcal{D}$, (see [EI2]) and the Alexander–Hirschowitz theorem.

Remark (comparison with a result of Hochster and Laksov). M. Hochster and D. Laksov in [HL] have shown that if V is a generically chosen s-dimensional vector subspace of $\mathcal{R}_j$, and $j \geq 3$, then the dimension of $\mathcal{R}_1 V = \langle \{X_i v | v \in V\rangle$ satisfies $\dim_k \mathcal{R}_1 V = \min(\dim_k \mathcal{R}_{j+1}, sr)$. Theorem 1 strengthens the result of Hochster and Laksov since here the vector space L^j has a special form. Since the dimension of $\mathcal{R}_1 V$ depends semicontinuously on V, Theorem 1 implies the Hochster–Laksov result—except for the list of four triples in (1).

How special is the condition $V = L^j$? The dimension of the family $\mathrm{PS}(r, s, j) \subset \mathbb{P}^N$, $N = \binom{r+j-1}{j} - 1$, parametrizing vector spaces L^j, $L = [L_1, \ldots, L_s]$, is no greater than $sr - 1$, as there are s choices of a form L_i in the space $\langle X_1, \ldots, X_r \rangle$. The dimension of the Grassmannian $\mathrm{Grass}(s, \mathcal{R}_j)$ parametrizing all s-dimensional subspaces V of $\mathcal{R}_j$ is $s(\dim_k \mathcal{R}_j - s)$, so the codimension of the family $\mathrm{PS}(r, s, j)$ in $\mathrm{Grass}(s, \mathcal{R}_j)$ satisfies

$$\operatorname{cod}(\mathrm{PS}(r,s,j) \geq s\big(\dim_k R_j - (s+r)\big) + 1, \tag{4}$$

a degree $r - 1$ polynomial in j if r, s are fixed.

Of course, the amount of effort expended in achieving this stronger result is substantially greater, as the Alexander–Hirschowitz result is over a hundred pages of journal articles!

We had shown Theorem 1, when Rob Lazarsfeld noted that the solution of the Waring problem is a consequence of the theorem of J. Alexander and A. Hirschowitz.[2] Bruce Reznick and Michael Johnson then respec-

[2]We received on December 23, 1992 an e-mail message from I. Dolgachev via V. Kanev, who informed us of R. Lazarsfeld's observation.

tively pointed out to us the algebraic route by the Lasker–Wakeford lemma, and the geometric route for obtaining Theorem 2 from Theorem 1. We extend these to the case char $k > d$. We are grateful to L. Avramov, who pointed out to us the principle that polynomial invariants extend from characteristic zero to characteristic p, used in the proof. Recall that $g(r, d)$ is the smallest integer s such that the general degree-d r-ary form in $\mathcal{R}$ over the field k can be written as the sum of s powers of linear forms. We let $\lceil a \rceil$ denote the smallest integer greater than a.

THEOREM 2 (Waring problem: representability of a general form f as a sum of powers). *Suppose that $d > 2$, the field k is algebraically closed, and k is of characteristic zero, or of characteristic greater than d. Then the Alexander–Hirschowitz theorem implies that*

$$g(r,d) = \left\lceil \frac{1}{r}\binom{r+d-1}{d} \right\rceil, \tag{5}$$

except for the pairs $(r, d) = (3,4), (4,4), (5,4)$, and $(5,3)$, respectively, for which (r, s, d) is in the list of exceptional triples of (2), *and for which $g(r, d) = s + 1 = 6,\ 10,\ 15$, and 8, respectively. When $d = 2$, we have $g(r, 2) = r$.*

Proof. The classical Lasker–Wakeford lemma for $k = \mathbb{C}$ states that $g(r, d)$ is the smallest integer t for which there are linear forms $L = L_1, \dots, L_t$ in $\mathcal{R}_1$ such that $\mathcal{R}_1 L^{d-1} = \mathcal{R}_d$. Thus, Theorem 1 implies Theorem 2 when $k = \mathbb{C}$. See [L], [Wk], and for modern treatments [R2] or [EhR]. Alternatively, in the classical case, Michael Johnson has pointed out to us that J. Bronowski, in [B1], [B2], uses the classical Terracini theorem [T1] to show the equivalence of Theorems 1 and 2 when $k = \mathbb{C}$. But Terracini himself had shown this [T2].

We now assume that k is an algebraically closed field k with char $k = 0$ or char $k > d$. The key Theorems 4.1 and 4.2 of [EhR], showing that the Waring problem is equivalent to suitable special cases of the Alexander–Hirschowitz result, are stated only for $k = \mathbb{C}$. By (2) above, and the Alexander–Hirschowitz theorem, we need only show

CLAIM. Under the above hypothesis on k, the dimension of $\mathrm{PS}(r, s, d)$ satisfies

$$\dim \mathrm{PS}(r,s,d) = \max_{L \mid |L| = s} \left(\dim_k \mathcal{R}_1 L^{d-1}\right) - 1. \tag{6}$$

Proof of Claim. In characteristic zero, taking $V = \mathcal{R}_1$, the map

$$\phi \colon \mathbb{A}^{rs} \to S^d(V); \qquad \phi[L_1, \dots, L_s] = L_1^d + \cdots + L_s^d,$$

is generically onto its image. The dimension of the image is the same as that of the tangent space to the image at a general point of the domain. The tangent space $\mathcal{T}_{\phi(L)}$ to $\phi(\mathbb{A}^{rs})$ at the point $\phi(p_L)$ satisfies

$$\mathcal{T}_{\phi(L)} = \mathscr{R}_1 L^{d-1},$$

as can be seen by viewing the map as the s-fold secant to the map $\varphi\colon \mathbb{A}^r \to S^d(V)$, $\varphi(v) = v^d$. The claim follows, when char $k = 0$, after converting from $S^d(V)$ to $\mathbb{P}^N = P(S^d(V))$.

In characteristic p, the dimension of the image is at least that of the tangent space, equal when the map is separable, less if the characteristic intervenes essentially. Thus, we have $\dim \mathrm{PS}(r, s, d)$ greater than or equal to the right hand side of (6). By the Alexander–Hirschowitz theorem, the tangent map $\mathcal{T}_\phi\colon T_{\mathbb{A}^{rs}, p_L} \to T_{S^d(V), \phi(L)}$ has maximal rank, aside from the four exceptions. Thus, the dimension of the LHS of (6) could be larger than the RHS only in the exceptional cases. For the three exceptional cases where $d = 4$, equality in (6) now follows from comparing two results:

(i) The second partials of a general quartic are known to span $\mathscr{R}_2$ when char k > 4 (see the proof of Prop. 3.3 of [I1], which is valid for char k $>$ d). The locus of f for which the second partials do not span $\mathscr{R}_2$ is the catalecticant hypersurface of $\mathbb{P}(\mathscr{R}_4)$.

(ii) The second partials of a sum of s powers of linear forms spans a vector space of dimension no greater than s.

For $d = 4$, and (r, s, d) in (1), it follows that PS (r, s, d) lies on the nonzero catalecticant hypersurface of $\mathbb{P}(\mathscr{R}_4)$, implying that the left side of (6) is no larger than dim $(\mathbb{P}(\mathscr{R}_4)) - 1$, which is the right side by the Alexander–Hirschowitz theorem. This shows equality in (6) in these three cases.

For the last exceptional case $(r, d) = (5, 3)$ we conclude equality in (6) by using a general principal that polynomial invariants with integer coefficients in characteristic zero extend to characteristic p (see [Bo, Chap. IV, Sect. 2, Scholium after Theorem 3, p. 29]). We include a proof for completeness. First, let K be an algebraic closure of $\mathbb{Q}$, and note that if $g = \sum_{|J|=3} c_J X^J$, $c_j \in K$, satisfies $g = L_1^3 + \cdots + L_7^3$ then it is a classical result (see (12) of [Ri], (6) of [Wk], or Corollary 4.5 of [EhR]) that there is a homogeneous polynomial f in the ring $K[C] = K[\{C_J\} \mid |J| = 3\}]$ that vanishes at $C_j = c_j$, as $\dim_K \mathrm{PS}(5, 7, 3)$ is 33, not $34 = \dim \mathbb{P}(\mathscr{R}_3)$. By taking norms and clearing fractions we may assume that the coefficients of f are in the integers $\mathbb{Z}$, and that their GCD is 1. Then, over $\mathbb{Z}$, the locus $\mathrm{PS}_{\mathbb{Z}}(5, 7, 3)$ lies on the hypersurface $f = 0$. Letting $\bar{f}$ denote the image of f in the field $\mathbb{Z}_p$, we see that $\bar{f} = 0$ on $\mathrm{PS}_{\mathbb{Z}_p}(5, 7, 3)$, hence $\bar{f} \circ \phi = 0$ on $(\mathbb{A}_{\mathbb{Z}_p})^{rs}$. It follows that $\bar{f} \circ \phi = 0$ on $(\mathbb{A}_k)^{rs}$ over any field k of characteristic p. Thus, if k is algebraically closed, the dimension of the subvariety $PS_k(5,7,3)$ of $\mathbb{P}^{34}$ is at most 33. Since its dimension is at least 33, there is equality in (6), and $(r, d) = (5,3)$ remains an exceptional case for Theorem 2 when char $k = p > 3$.

Remark. As can be seen, the difficult part of the solution of the Waring problem is the Alexander–Hirschowitz theorem, which uses algebraic-geometric methods. B. Reznick has also independently determined $g_{\mathbb{C}}(r, d)$ in a large number of cases, and has obtained reasonable upper bounds in all cases [R2]; his results are the most complete among those eschewing use of algebraic geometry. His work is also a good source of references to the classical literature and recent improvements over classical results. We note again that Terracini [T2], Wakeford [W], as well as Ehrenborg and Rota in Theorem 4.2 of [ER], restate the Waring problem for $\mathbb{C}$ in the language of $K(P,2)$. The first exceptional case $(r, d) = (3,4)$ of Theorem 2 is due to Clebsch [C1], and is shown differently by J. Luroth [Lu]. The exceptions $(4,4)$ and $(5,4)$ use the same principle (see Reye [Re], Sylvester [S], Bronowski [B1]). The exception $(5,3)$ appears in Richmond [Ri] and Wakeford [W].

We have actually shown more than the Waring problem. Recall that $PS(r, s, d)$ denotes the family of homogeneous forms f in $\mathscr{R}$ of degree d up to k^*-multiple, such that f can be written $f = L_1^d + \cdots + L_s^d$, for some choice of linear forms $L_1, \ldots, L_s$.

THEOREM 3 (dimension of the family of power sums). *If $d > 2$, and char $k = 0$ or is greater than d, then the Alexander–Hirschowitz theorem implies that the dimension of $PS(r, s, d)$ in the projective space $\mathbb{P}(\mathscr{R}_d)$ satisfies*

$$\dim(PS(r, s, d)) = \min(rs - 1, \dim_k \mathscr{R}_d - 1), \tag{7}$$

except for the triples (r, s, d) of (1), where the dimension is one less.

Proof. See the proof of (6) in Theorem 2.

Remark. Note that when $s < g(r, d)$ and either the product $r \cdot g(r, d) \neq g(r, d)$ or (r, d) is an exceptional pair, Theorem 2 does not imply Theorem 3. To prove their result when $s < g(r, d)$, J. Alexander and A. Hirschowitz add a set Q of $(\dim_k R_d - rs)$ ordinary points to the double locus at P (see [H1]).

3. The Length of a Homogeneous Polynomial

We now compare several notions of length related to generalized additive decompositions of forms. Our goal is, first, to suggest some problems and methods related to the Waring problem just solved. Second, we establish a connection between definitions of lengths of forms f and the structure of the punctual annihilating schemes of f. This allows us to relate known examples of nonsmoothable length-s Gorenstein zero-dimensional schemes Z to the existence of forms f having no more than s linearly independent derivates in each degree, but which are not in the closure of $\mathrm{PS}(r, s, d)$, the family of sums of dth powers of s linear forms. We suppose henceforth that k is an algebraically closed field of arbitrary characteristic. Recall that if f is a degree-d homogeneous element in $\mathscr{R}_d$, then $R_i \circ f$ is the vector space of ith order (degree $d - i$) derivates of f. As before, when $0 < \operatorname{char} k \leq d$ below, we need to take $\mathscr{R} = \mathscr{D}$, the divided power ring, and the contraction action (see [EI2]).

A crude measure of the length of a degree-d form f is

$$\operatorname{ldiff}(f) = \max_{0 \leq i \leq d}\{\dim_k\langle R_{d-i} \circ f\rangle\}, \tag{8}$$

the maximum number of linearly independent degree-i derivates of f, for any i. When $d = 2t$ or $2t + 1$, there are at most $\dim_k R_t$ linearly independent degree-i derivates, no matter what the choice of f. The crude measure $\operatorname{ldiff}_k(f)$ is a lower bound for the notions of length in Definitions 4A–4D (Proposition 6). So these "annihilating scheme" notions of length are interesting only when d is large enough compared to s: if $t > 0$ and if $s = \operatorname{length}(f)$ satisfies $\dim_k R_{t-1} \leq s \leq \dim_k R_t$ then we normally consider degrees $d \geq 2t + 2$ (see Example 14 for the contrary case $d = 2t$).

We first give an "annihilating scheme" version of the usual notion of length $g_k(f)$ of a form. Recall that $g_k(f)$ is the minimal length of an additive decomposition of f as a sum of powers of linear forms.

LEMMA 4. *We have*

$$g_k(f) = \min(s | \exists P = P_1, \ldots, P_s \in \mathbb{P}^{r-1} \text{ with } K(P, 1) \subset \operatorname{Ann}(f)). \tag{9}$$

Proof. By (2) in the case $a = 1$, we have $f \in \langle L_1^d, \ldots, L_s^d\rangle$ iff there are s points $P = (p(1), \ldots, p(s))$ of $\mathbb{P}^{r-1}$ with $L_i = L_{p(i)}$ and $K(P, 1)_d \subset \operatorname{Ann}(f)_d$. It is well known that if $f \in \mathscr{R}_d$ and I is any ideal of R, then $I_d \subset \operatorname{Ann}(f)_d$ iff $I \subset \operatorname{Ann}(f)$ (see [Mac] or [IK]).

DEFINITION 4A. We say that a degree-d homogeneous form f in $\mathscr{R}_d$ has a *generalized additive decomposition of length* s and *multiplier degrees* D into powers of $L_1, \ldots, L_t$, iff there are forms $h_1, \ldots, h_t$ of degrees $D = d_1, \ldots, d_t$ such that

$$f = \sum_{1 \leq u \leq t} h_u L_u^{d-d_u} \quad \text{and} \quad s = \sum(\dim_k \mathscr{R}_{d_u}). \tag{9a}$$

We let $g'_k(f)$ be the minimal length of a generalized additive decomposition of f.

If $N = (d_1 + 1, \dots, d_t + 1)$ we let $K(P, N) = m_{p_1}^{d_1+1} \cap \cdots \cap m_{p_t}^{p_t+1}$ in R. Then by a generalization of (2) (Theorem I of [EI2]) and the proof of Lemma 4, $g'_k(f)$ is the minimal colength $s = \dim_k(R/K(P, N))$ of an ideal $K(P, N)$ such that $K(P, N) \subset \mathrm{Ann}(f)$.

DEFINITION 4B. We say that f has *power sum* decomposition length $\mathrm{psl}_k(f) \leq s$ if it is in the Zariski closure of $\mathrm{PS}(r, s, d)$: there is a family f_t, $t \in T$, such that $f = f_0$ and $f_t | t \neq 0$ is in $\mathrm{PS}(r, s, d)$. We let

$$\mathrm{psl}_k(f) = \min(s | f \in \overline{\mathrm{PS}(r, s, d)}). \qquad (9b)$$

I. Dolgachev and V. Kanev construct the variety $X_s(f)$ of polar s-polyhedrons—length-s additive decompositions of f, and the variety $X_s(f)^*$ of possibly degenerate polar s-polyhedrons of f, in Section 4.1.1 of [DK]. This variety is roughly the collection of limits of s-polyhedrons—limits of families $f_t = L_1(t)^d + \cdots + L_s(t)^d$ of additive decompositions, such that $f = \lim_{t \to 0} f_{t^*}$. The polar–polyhedral length of f is

$$\mathrm{ppl}_k(f) = \min(s | X_s(f)^* \neq \emptyset). \qquad (9c)$$

DEFINITION 4C. We say that f has *smoothable* or *additive length* $\mathrm{al}_k(f) \leq s$ if the ideal $\mathrm{Ann}(f)$ in R contains an ideal I_Z, where Z is a *smoothable* length-s zero-dimensional subscheme of P_k^{r-1}: the point p_Z parametrizing Z in the punctual Hilbert scheme $\mathrm{Hilb}^s(\mathbb{P}^{r-1})$ is in the closure of the family $U(s)$ parametrizing s distinct points.

$$\mathrm{al}_k(f) = \min(\{\text{length } Z | I_Z \subset \mathrm{Ann}(f), Z \subset \mathbb{P}^{r-1}, \dim(Z) = 0, \\ Z \text{ smoothable}\}). \qquad (9d)$$

DEFINITION 4D. We say that f has *scheme length* $\mathrm{l}_k(f) \leq s$ if the ideal $\mathrm{Ann}(f)$ in R contains an ideal I_Z, where Z is a length-s zero-dimensional subscheme of $\mathbb{P}^{r-1}$:

$$\mathrm{l}_k(f) = \min(\{\text{length } Z | I_Z \subset \mathrm{Ann}(f), Z \subset \mathbb{P}^{r-1}, \dim(Z) = 0\}). \qquad (9e)$$

LEMMA. *We have*

$$\mathrm{psl}_k(f) \leq \mathrm{ppl}_k(f) \qquad \textit{and} \qquad \mathrm{al}_k(f) \leq \mathrm{ppl}_k(f). \qquad (10)$$

Proof. That $\mathrm{ppl}_k(f) = s$ implies not only that $f = f_0$ is in the closure of a family f_t, $t \in T$, $T \subset \mathrm{PS}(r, s, d)$ (so $\mathrm{psl}_k(f) \leq \mathrm{ppl}_k(f)$) but also that there is a family $f_t = \Sigma L_i(t)^d$ of additive decompositions of f_t, and hence a limit length-s scheme $Z_0 = \lim_{t \to 0}(p_{L_1(t)} \cup \cdots \cup p_{L_s(t)})$, where p_{L_i} is the point in $\mathbb{P}^{r-1}$ corresponding to L_i. The limit length-s scheme Z_0 is by definition smoothable, and $I_{Z_0} \subset \mathrm{Ann}(f)$, implying $\mathrm{al}_k(f) \leq \mathrm{ppl}_k(f)$.

DEFINITION 4E. For all the lengths defined above, we let

$$\operatorname{len}_k(r,d) = \min\{s \mid \exists U_s \textit{ open-dense in } \mathbb{P}(\operatorname{Sym}^d W) \mid f \in U_s \Rightarrow \operatorname{len}_k(f) = s\},$$

$$\operatorname{maxlen}_k(r,d) = \max_{f \in \mathscr{R}_d} \operatorname{length}_k(f),$$

$$\operatorname{Len}_k(r,s,d) = \textit{subset of } \mathbb{P}_k(\mathscr{R}_d) \textit{ parametrizing } f \textit{ for which } \operatorname{len}_k(f) = s.$$

We say a length function $\operatorname{len}_k(f)$ is *semicontinuous* in f if $\operatorname{len}_k(f) \le s$ defines a closed subset of $\mathbb{P}(\operatorname{Sym}^d W) = P(\mathscr{R}_d)$. We say a length function $\operatorname{len}_k(f)$ defined by a family $\mathscr{F}_{\text{len}}$ of annihilating schemes is *weakly semicontinuous* in f if when $\{(f_t, Z_t) \mid t \in T\}$ is a flat family of pairs with $\operatorname{len}_k(f_t) = s$ for $t \ne 0$ and $Z_t \mid t \ne 0$ a length-s annihilating scheme of f_t in $\mathscr{F}_{\text{len}}$, then $\operatorname{len}_k(f_0) \le s$.

THEOREM 5. *We have*

$$l_k(f) \le \operatorname{al}_k(f) \le \operatorname{ppl}_k(f) \le g'_k(f) \le g_k(f). \tag{11}$$

If $k = \mathbb{C}$ and $\deg(f) > s$, *then* $\operatorname{al}_{\mathbb{C}}(f) \le \operatorname{psl}_{\mathbb{C}}(f)$. *When $r = 3$, then* $l_k(f) = \operatorname{al}_k(f)$. *When $r = 2$,* $l_k(f) = g'_k(f) = \operatorname{psl}_k(f)$. *All the lengths we have defined but* $g_k(f), g'_k(f)$ *are weakly semicontinuous in* $f \in \mathscr{R}_d = \operatorname{Sym}^d V$. *They all satisfy*

$$\operatorname{len}_k(f+g) \le \operatorname{len}_k(f) + \operatorname{len}_k(g), \tag{12}$$

and

$$\operatorname{len}_k(L^u f) \le (\operatorname{maxlen}_k(2, d+u)) \cdot \operatorname{len}_k(f),$$

$$\le \left\lceil \frac{d+u+1}{2} \right\rceil \cdot \operatorname{len}_k(f) \qquad \textit{for all lengths but } g_k(f). \tag{13}$$

The length $\operatorname{psl}_k(f)$ *is semicontinuous. When $d \ge s$,* $l_k(f)$ *and* $\operatorname{al}_k(f)$ *are semicontinuous, so* $\mathscr{L}(s,r,d)$ *and* $\operatorname{Al}(s,r,d)$ *are locally closed in* $\mathbb{P}(\mathscr{R}_d)$. *Then the dimension of* $\operatorname{Al}(s,r,d)$ *satisfies*

$$\dim(\operatorname{Al}(s,r,d)) \le rs - 1.$$

Proof. We first show $\operatorname{ppl}_k(f) \le g'_k(f)$. It is well known that the ideal $K(P,N) = m_{p_1}^{d_1+1} \cap \cdots \cap m_{p_t}^{d_t+1}$ of Definition 4A is smoothable; the inequality follows.

We next show that when $k = \mathbb{C}$ and $\deg(f) > s$, then $\operatorname{al}_{\mathbb{C}}(f) \le \operatorname{psl}_{\mathbb{C}}(f)$. Suppose that $(f_n \mid n \in N)$, is a sequence of degree-d forms in $\operatorname{PS}(r,s,d)$, convergent in $\operatorname{Sym}^d(V_{\mathbb{C}})$ and let $f = \lim_{n\to\infty} f_n$. Let Z_n be a sequence of length-s subschemes $Z_n = \Sigma p_i(n)$, corresponding to the length-s additive decompositions of f_n. Since the closure $\overline{U(s)}$ of the "distinct points" open subscheme of $\operatorname{Hilb}^s \mathbb{P}^{r-1}$ is compact, the schemes Z_n have a limit scheme

Z_0. We need to show that $I_{Z_0} \subset \mathrm{Ann}(f)$. By Gotzmann's regularity result in [G], a length-s scheme is m-regular by degree $m = s$. Since $d = \deg(f) \geq s$, we have $\dim_k([I_{Z_n}]_d) = \dim_k([I_{Z_0}]_d) = \dim_k \mathscr{R}_d - s$ is fixed. It follows that $\lim_{n \to \infty}[I_{Z_n}^{-1}]_d = [I_{Z_0}^{-1}]_d$, and that f is in $[I_{Z_0}^{-1}]_d$. We conclude that $\mathrm{al}_{\mathbb{C}}(f) \leq s$, implying $\mathrm{al}_{\mathbb{C}}(f) \leq \mathrm{psl}_{\mathbb{C}}(f)$.

The remaining inequalities of (11) are tautological, or have been shown. That $\mathrm{l}_k(f) = \mathrm{al}_k(f)$ when $r = 3$ follows from the smoothability of punctual schemes in $\mathbb{P}^2$. That all the definitions except $g_k(f)$ coincide when $r = 2$ is the classical Jordan lemma (see Appendix III of [GY], Example 4.1.2 of [DK], or Section 2C of [I1]). The weak semicontinuity of the lengths (except for $g_k(f)$ and $g'_k(f)$) is immediate from their definitions. Likewise, the subadditivity of (12) is immediate.

The point of (13) is that $L^u \cdot L_i^d$ has length no greater than $\mathrm{maxlen}_k(2, d + u)$) for two variables, which is $\lceil (d + u + 1)/2 \rceil$ by the above-mentioned Jordan lemma, except for $g_k(f)$.

When $d \geq s$, that the subsets $\mathscr{S}(\leq s, r, d)$ and $\mathrm{Al}(\leq s, r, d)$ are closed is shown by an argument similar to the proof that $\mathrm{al}_{\mathbb{C}}(f) \leq \mathrm{psl}_{\mathbb{C}}(f)$. It follows that the subsets $\mathscr{S}(s, r, d)$ and $\mathrm{Al}(s, r, d)$ are locally closed in $\mathbb{P}(\mathscr{R}_d)$. Since the dimension of the smoothable length-s subschemes of $\mathbb{P}^{r-1}$ is no greater than $s(r - 1)$, and $\mathrm{Al}(r, s, d)$ is fibred over the ideal I_Z defining the point $p_Z \in \mathrm{Hilb}^n_{\mathrm{smoothable}}(\mathbb{P}^{r-1})$ by the projective space $\mathbb{P}^{s-1} = P(I^{-1}(Z)_d)$ of dimension $s - 1$, we conclude that $\dim(\mathrm{Al}_k(s, r, d)) \leq rs - 1$. This completes the proof of the theorem.

We now apply the concept of scheme length of f. The following proposition generalizes the elementary fact that if $f = L_1^d + \cdots L_s^d$, where the L_i are linear forms, then f has at most s linear independent degree-i derivates, because of the inclusion, $R_{d-i} \circ f \subset \langle L_1^i, \ldots, L_s^i \rangle$. We use a result of A. Geramita and P. Maroscia, that the graded ideal in R of a punctual subscheme Z of $\mathbb{P}^{r-1}$ has Hilbert function $H(R/I_Z)$ that is nondecreasing, and that attains $H(R/I_Z)_i = \mathrm{length}(Z)$ for $i \gg 0$ (Proposition 1.4 of [GM]).

PROPOSITION 6. *If* $\mathrm{l}_k(f) = s$, *then the dimension of the vector space* $R_i \circ f$ *satisfies* $\dim_k \langle R_i \circ f \rangle \leq s$. *Thus*, $\mathrm{ldiff}_k(f) \leq \mathrm{l}_k(f)$.

Proof. Suppose Z is a length-s zero-dimensional subscheme of $\mathbb{P}^{r-1}$ whose homogeneous vanishing ideal I_Z in R satisfies $I_Z \subset \mathrm{Ann}(f)$. Then Z is Cohen–Macaulay, as it is zero-dimensional. If $x = 0$ defines a hyperplane missing the support of Z, then the class $\bar{x}$ of x in R/I_Z is a regular element, so R/I_Z is Cohen–Macaulay. It follows that the Hilbert function $H(R/I_Z)$ is nondecreasing, and attains its stable value s. Since $\mathrm{Ann}(f) \supset I_Z$, the Hilbert function $H(R/\mathrm{Ann}(f)) \leq H(R/I_Z)$ termwise, so $H(R/\mathrm{Ann}(f))_{d-i} \leq s$. By a result of Macaulay (see [Mac], [El1], or [IK]), $(R/\mathrm{Ann}(f))_{d-i}$ is dual to the space $R_i \circ f$ of ith derivates of f; this implies that the space of ith order derivates of f has vector space dimension over k no greater than s.

How computable are these lengths? One problem is that there may not be a unique minimal length scheme Z with I_Z annihilating f. For example, if $f = X^2 + Y^2$ in $k[X, Y]$ then $f = [(aX - Y)^2 + (X + aY)^2]/(a^2 + 1)$, so f has a family of length-two annihilating schemes $xy = 0$ and $(ax - y)(x + ay) = 0$. However, we can show a partial converse to Proposition 6. If V is a vector subspace of R_i, we let $\mathrm{Sat}(V)$ denote its saturation in R. We say a symmetric sequence $H(0), \dots, H(d)$, $d = 2t$ or $2t + 1$ is *strongly unimodal* if there is an integer δ such that

$$H(0) < H(1) < \cdots < H(t-\delta) = H(t-\delta+1) = \cdots H(d-t+\delta) > \cdots > H(d).$$

PROPOSITION 7. *Suppose* $1 \le s \le t < d$. *If* $\dim_k(R_t \circ f) = \dim_k(R_{t+1} \circ f) = s$ *or, equivalently, if* $H_f = H(R/\mathrm{Ann}(f))$ *satisfies* $H_f(t) = H_f(t+1) = s$, *then*

(i) $\mathrm{l}_k(f) = \mathrm{ldiff}_k(f) = s$,

(ii) *f has a unique length-s annihilating scheme Z_f defined by the graded ideal* $\mathrm{Sat}(\mathrm{Ann}(f)_t)$.

(iii) *The Hilbert function H_f is strongly unimodal.*

Proof. The hypothesis implies that the ideal $I = (\mathrm{Ann}(f)_t)$, generated by the degree-t forms in R that annihilate f, is extremal in Macaulay's sense of having minimal growth: $\dim_k R_1 I_t$ is the minimum possible given r, t, and $\dim_k I_t$. Gotzmann's persistence and regularity theorems of [G] imply that the ideal $\mathrm{Sat}(I_t)$ defines a length-s scheme Z, so $\mathrm{Sat}(I_t) = I_Z$. As in the proof of Lemma 4, it follows that $I_Z \subset \mathrm{Ann}(f)$, and $\mathrm{l}_k(f) \le s$. By assumption $s \le \mathrm{ldiff}_k(f)$, so Proposition 6 implies $\mathrm{l}_k(f) = \mathrm{ldiff}_k(f) = s$. If Z' of length s satisfies $I_{Z'} \subset \mathrm{Ann}(f)$, then $I_{Z'}$ is regular in degree t, by Gotzmann's regularity theorem; hence $(I_{Z'})_t \subset \mathrm{Ann}(f)_t = (I_Z)_t$ have the same colength s in R_t, so $(I_{Z'})_t = (I_Z)_t$, implying $I_{Z'} = I_Z$. Without loss of generality, by the symmetry of H_f around $d/2$, we may assume that $t \ge d/2$. We have shown that $\mathrm{Ann}(f)_i = (I_Z)_i$ for $i \le t + 1$; the Geramita–Maroscia result now implies H_f is nondecreasing for $i \le t + 1$, and we have $H_f(t) = s$. The symmetry of H_f implies $H_f(i) = s$ for $d - t - 1 \le i \le t + 1$. If there is an $i < d/2$ with $H_f(i) = H_f(i+1) = s'$ $< s$, then an application of the above proof to $t' = d - i - 1$ yields a contradiction. We conclude that H_f is strongly unimodal.

We now prepare to show that the two length functions $\mathrm{al}_k(f)$, defined from smoothable schemes, and $\mathrm{l}_k(f)$, defined from arbitrary length-s schemes, are distinct functions when $r \ge 7$ (Proposition 10, mod Conjecture 9.0).

LEMMA 8. *Suppose that g is a homogeneous degree-d_0 element of the polynomial ring $\mathscr{R}'$ in $r-1$ variables $X_2,\dots,X_r$, let $\mathrm{Ann}(g)$ be the ideal of the ring $R' = k[x_2,\dots,x_r]$ annihilating g, let $T'_g = H(R/\mathrm{Ann}(g))$ be the Hilbert function, and let $T = \Sigma T'$ be the sum function of T':*

$$t_i = (t'_0 + \cdots + t'_i).$$

Suppose $t \geq d_0 - 1$, $d = 2t$ or $2t+1$, and let $f = gX_1^{d-d_0}$. The Hilbert function $H_f = H(R/\mathrm{Ann}(f))$ is symmetric around $d/2$ and satisfies

$$(H_f)_{\leq t} = T_{\leq t}. \tag{14}$$

Proof. The submodule $R_{d-i} \circ f$ of $\mathscr{R}_i$ satisfies

$$\bigoplus_u R'_{d_0-u} x_1^{d-i-(d_0-u)} \circ f = \bigoplus_{\max(0,\, i-(d-d_0)) \leq u \leq \min(d_0, i)} (R'_{d_0-u} \circ g) X_1^{i-u}$$

$$\subset \oplus \mathscr{R}'_u X_1^{i-u} = \mathscr{R}_i. \tag{15}$$

We have $t'_i = \dim_k R'_{d_0-i} \circ g$, and $H(R/\mathrm{Ann}(f))_i = \dim_k R_{d-i} \circ f$ by Macaulay's inverse systems (see [Mac], [EI1], [I2], or [IK]). It is not hard to see that (15) implies that

$$H(R/\mathrm{Ann}(f))_i = t'_{\max(0,\, i-(d-d_0))} + \cdots + t'_{\min(i, d_0)}, \tag{16}$$

so implies the lemma.

Recall that a Gorenstein graded Artin algebra A is *compressed* if it has the maximum Hilbert function possible, given the embedding dimension r and socle degree d (see [EI1]). It is "generic" if it lies on a single component of the punctual Hilbert scheme, and has no deformations (where A is the special "point" of a family $A(t)$) to algebras $A(t)$ with different discrete invariants. A form is "general" for property P if there is a dense open subvariety U_P of the projective space $\mathbb{P}(\mathscr{R}_d)$ such that $f \in U_P$ implies property P.

LEMMA 9. *When g is a general degree-d form in the polynomial ring $\mathscr{R}' = k[X_2,\dots,X_r]$ in $r' = r-1$ variables, then for the following cases, the Gorenstein algebra $A = R'/\mathrm{Ann}(g)$ is compressed, nonsmoothable, and generic: $(r', d) = (r', 3)$, $6 \leq r' \leq 10$; $(r', d) = (5, 5)$; $(r', d) = (4, 15)$.*

Proof. We have verified the nonexistence of negative deformations in each of these cases by computer calculation of I/I^2 using the "Macaulay" symbolic algebra program [BSE].[3] See [EI1] and Example 7 of [I2] for a discussion of the "small tangent space" method used.

[3]The case (4, 15) took 23 MB of RAM and more than 8 hours on an accelerated SE-30. When $r' = 6$, $R' = k[a,b,c,x,y,z]$ then $g = 2a^2z + 2ab^2 + 3abc + 2bc^2 + 3bcx + 2cx^2 + 3cxy + 2xy^2 + 3xyz + 2yz^2$ is general enough to be nonsmoothable (Example 7 of [I2]). Finding $(\mathrm{Ann}(g))^2$ when $r \geq 11$ ran up against degree bounds in "Macaulay."

Conjecture 9.0. The conclusion of Lemma 9 is valid for (r', d) when $r' \geq 6$ and $d \geq 3$ is odd; for $(5, d)$ when $d \geq 5$ is odd; and for $(4, d)$ when $d \geq 15$ is odd.

PROPOSITION 10. *When $7 \leq r \leq 11$, if g is a general enough degree-3 form of R'_3, if $d \geq 7$, and we take $f = gX_1^{d-3}$, then*

(i) $H(R/\mathrm{Ann}(f)) = (1, r, 2r-1, (2r)^{d-5}, 2r-1, r, 1)$.

(ii) *There is a unique scheme Z of length $s = 2r$ such that $I_Z \subset \mathrm{Ann}(f)$. The scheme Z is concentrated at the point $p = (1, 0, \dots, 0)$ in $\mathbb{P}^{r-1}$, and has Hilbert functions*

$$H(R/I_Z) = (1, r, 2r-1, 2r, 2r, \dots,),$$

and

$$H(\mathcal{O}_Z) = (1, r-1, r-1, 1).$$

(iii) *The lengths of f satisfy $s = 2r = \mathrm{l}_k(f) < \mathrm{al}_k(f)$.*

(iv) $\mathrm{ldiff}_k(f) = s = 2r$.

Proof. The statements (i) and (iv) follow from Lemmas 8 and 9 and Proposition 6. We take $\mathcal{O}_Z \cong R'/\mathrm{Ann}(g)$ concentrated at p. By construction, $I_Z \subset \mathrm{Ann}(f)$. The integer d is large enough that by (14) we have $\mathrm{Ann}(f)_2 \subset (x_2, \dots, x_r)$, so $\mathrm{rad}(\mathrm{Ann}(f)_2) \subset (x_2, \dots x_r)$, implying that any zero-dimensional annihilating scheme Y is concentrated at p. By construction, $(\mathrm{Ann}(f)_3 = (I_Z)_3$, and since I_Z is regular in degree 3, we have $Y \subset Z$; if the lengths are equal then $Y = Z$, as claimed.

If there is a family $f(t)|t \in T$ of forms having additive decompositions $f(t) = L_1(t)^d + \cdots + L_{2r}(t)^d$ of length $2r$ for $t \neq 0$, and approaching f, then we may further deform the family so that when $t \neq 0$, $f(t)$ has $2r$ linearly independent degree-3 derivates. Since Z is regular in degree 3, and the annihilator scheme $Y(t)$ of $\langle L_1(t)^3, \dots, L_{2r}(t)^3 \rangle$ consisting of $2r$ smooth points would then also be 3-regular, it follows from considering $(I_{Y(t)})_3 = \mathrm{Ann}(f(t))_3 \to (\mathrm{Ann}(f)_3) = (I_Z)_3$ that the scheme Z would be smoothable, contradicting its choice. This shows (iii) and completes the proof.

We let $\mathrm{Gor}(T) \subset \mathbb{P}(\mathcal{R}_d)$ parametrize the f, up to nonzero constant multiple, such that $H_f =_{\mathrm{def}} H(R/\mathrm{Ann}(f)) = T$.

EXAMPLE 11 (form of small length s not in the closure of sums of powers of s linear forms). When $(r, d) = (7, d)$, $d \geq 7$, $R' = k[u, \dots, w]$, $R = k[t, \dots, w]$, then the form $f = t^4 g$,

$$g = 2u^2z + 2uv^2 + 3uvw + 2vw^2 + 3vwx + 2wx^2 + 3wxy + 2xy^2 + 3xyz + 2yz^2,$$

is general enough to satisfy the hypotheses of Proposition 10, so $14 = \mathrm{ldiff}_k(f) = \mathrm{l}_k(f) < \mathrm{al}_k(f)$. Calculation gives $H_g = T' = (1, 6, 6, 1)$, $H(R'/(\mathrm{Ann}(g))^2) = (1, 6, 21, 56, 6)$ of length 90, and if $I = \mathrm{Ann}(f)$ we have $A_f = R/I$ has length 70, and satisfies

$$H_f = (1, 7, 13, 14, 14, 13, 7, 1),$$

and

$$H(R/I^2) = (1, 7, 28, 84, 90, 90, 90, 75, 20, 14, 13, 7, 1).$$

The number $75 = H(R/I^2)_d$ bounds the dimension of the tangent space $\mathcal{T}_f$ to the affine cone over $\mathrm{Gor}(T)$ at f, by [IK], so the dimension of the family of forms f with this Hilbert function is small compared to $\dim(\mathrm{PS}(7, 14, d)) = 97$. By taking g a general form of degree 3 in R', considering $f = L^4(g + m \circ g)$—adding to g multiples of its 13 linearly independent derivates of orders 1 to 3—and homogenizing to degree d by a general linear form L, we find that for the sequence $T = H_f$, there is a neighborhood $U_f = \mathrm{Gor}(T)_f$ of the point parametrizing f in $\mathrm{Gor}(T)$, such that

$$\dim(\mathrm{Gor}(T)_f) \geq \dim(55 + 13 + 6) = 74,$$

the same as $\dim_k \mathcal{T}_f - 1$. Thus, U_f is composed entirely of similarly constructed forms for which $14 = \mathrm{ldiff}_k(f) = \mathrm{l}_k(f) < \mathrm{al}_k(f)$!

The tangent space $\mathcal{T}_A$ to the punctual Hilbert scheme $\mathrm{Hilb}^{70}\mathbb{A}^7$ at $\mathrm{Spec}(A_f)$, satisfies $\mathcal{T}_A \cong I/I^2$; hence

$$\dim_k \mathcal{T}_A = \dim_k(I/I^2) = \dim_k(R/I^2) - \dim_k R/I = 520 - 70 = 450,$$

less than the 490 required for smoothable schemes. Thus, $A_f = R/\mathrm{Ann}(f)$ is also nonsmoothable.

EXAMPLE 12. By adding together several forms arrived at as in Proposition 10 or Example 11, we may create new examples where $\mathrm{ldiff}_k(f) = \mathrm{l}_k(f) < \mathrm{al}_k(f)$. For example, taking $f = t^4 g + u^4 g'$, where g' is $g(t, v, w, x, y, z)$, we find $H(R/\mathrm{Ann}(f)) = (1, 7, 20, 28, 28, 20, 7, 1)$, with $28 = \mathrm{ldiff}_k(f) = \mathrm{l}_k(f) < \mathrm{al}_k(f)$, and a tangent space $\mathcal{T}_f$ of dimension 131. Taking $f = t^6 g + u^6 g' + z^6 g''$ with $g'' = g(t, u, v, w, x, y)$, we find $H(R/\mathrm{Ann}(f)) = (1, 7, 24, 41, 42, 42, 41, 24, 7, 1)$, with $42 = \mathrm{ldiff}_k(f) = \mathrm{l}_k(f) < \mathrm{al}_k(f)$, and a tangent space $\mathcal{T}_f$ of dimension 196.[4]

[4]The Hilbert functions $H(R/(f))$ and the tangent space size $\dim_k \mathcal{T}_f = H(R/\mathrm{Ann}(f)^2)_d$ were found using the "Macaulay" symbolic algebra program [BSE].

EXAMPLE 13A. When $(r, d) = (6, d)$, $d \geq 11$, there are homogeneous degree-d polynomials f satisfying

(i) $H_f = (1, 6, 21, 36, 41, 42^{d-9}, 41, 36, 21, 6, 1)$ and $(\Delta H_f)_{\leq 5} = (1, 5, 15, 15, 5, 1)$.

(ii) There is a unique scheme Z of length $s = 42$ and concentrated at a single point of P^5, such that $I_Z \subset \text{Ann}(f)$. $H(R/I_Z) = (1, 6, 21, 36, 41, 42, 42, \dots)$.

(iii) $\text{ldiff}_k(f) = 42 = \text{l}_k(f) < \text{al}_k(f)$.

EXAMPLE 13B. When $(r, d) = (5, d)$, $d \geq 31$, there are homogeneous degree-d polynomials f satisfying

(i) $H_f = (1, 5, 15, 35, 70, 126, 210, 330, 450, 534, 690, 725, 745, 755, 759, 760^{d-29}, 759, \dots, 1)$ and $(\Delta H_f)_{\leq 15} = (1, 4, 10, 20, 35, 56, 84, 120, 120, 84, 56, 35, 20, 10, 4, 1)$.

(ii) There is a unique scheme Z of length $s = 760$ and concentrated at a single point of $\mathbb{P}^4$, such that $I_Z \subset \text{Ann}(f)$. $H(R/I_Z) = (1, 5, \dots, 755, 759, 760, 760, \dots)$.

(iii) $\text{ldiff}_k(f) = 760 = \text{l}_k(f) < \text{al}_k(f)$.

Proof. If F is a general degree 5 homogeneous polynomial in $R' = k[x_1, \dots, x_5]$ it defines a Gorenstein ideal $J_F = \text{Ann}(F)$ in R' that has no deformations except to ideals of the same Hilbert function H_F—by an argument of Emsalem and the author [EI1], verified by calculation in "Macaulay," that there are only the trivial negative degree tangents $\partial/\partial x_i$ in $\text{Hom}(J_F, R'/J_F)$. We take $f = FX_1^{d-5}$. The proof of Proposition 10 applies to show the assertions in Example 13A. Example 13B is constructed similarly from the general degree-15 form F in $R' = k[x_1, \dots, x_4]$.

EXAMPLE 14. It is not hard to show that for dimension reasons, if the length s is large compared to d, then there are forms f not in the closure of $\text{PS}(r, s, d)$, but satisfying $\text{ldiff}_k(f) \leq s$. The lowest degree example for $r = 3$ has $s = 14$ and $d = 8$. The projective dimension of $\text{Gor}(T)$, $T = (1, 3, 6, 10, 14, 10, 6, 3, 1)$, is 43, as $\text{Gor}(T)$ is the locus of the catalecticant hypersurface on $\mathbb{P}^{44} = \mathbb{P}(\mathscr{R}_8)$. Thus, $\dim(\text{Gor}(T)) > \dim(\text{PS}(3, 14, d)) = 41$. It follows that there are degree-8 forms f in $k[x, y, z]$ with $14 = \text{ldiff}_k(f) < \text{al}_k(f)$.

Remark. Let $H(3, s, d)$ be the "maximal Hilbert function H_f bounded by s,"

$$H(3, s, d)_i = \min(s, \dim_k R_i, \dim_k R_{d-i}), \qquad 0 \leq i \leq d. \tag{16}$$

If s is small enough, the author and V. Kanev show that $f \in \mathrm{Gor}(T)$, $T = H(3, s, d)$, implies that f is in the closure of $\mathrm{PS}(3, s, d)$.

THEOREM [IK]. *When* $r = 3$, $s \leq \dim_k R_{t-1}$, *and* $d \geq 2t$, *the family* $\mathrm{Gor}(3, s, d)$ *of all forms* $f \in \mathscr{R}_d$ *with* $H_f = H(3, s, d)$ *is an irreducible open-dense subvariety of the Zariski closure of* $\mathrm{PS}(3, s, d)$.

Proposition 10 and Examples 11 and 12 above are obstructions to extending this result to more general Hilbert functions bounded by s when $r \geq 7$. However, it does not so extend even when $r = 3$ (see [IK])!

S. J. Diesel has shown that when $r = 3$, and T is any permissible symmetric sequence, the family $\mathrm{Gor}(T)$ of forms $f \in \mathscr{R}_d$ such that $H_f = T$ is irreducible [D]. In [IK], we show that $\mathrm{Gor}(T)$ has several components when T is any of the Hilbert functions H_f from Proposition 10 or Example 11; we show similar examples for $r \geq 5$. Punctual Gorenstein singularities in $\mathbb{P}^3$ are smoothable, so when $r = 4$ the construction of Proposition 10 does not show $\mathrm{l}_k(f) < \mathrm{al}_k(f)$. When $r = 4$, the questions of whether $\mathrm{Gor}(T)$ is always irreducible and whether there are forms for which $\mathrm{l}_k(f) < \mathrm{al}_k(f)$ are still open.

We do not show that the Gorenstein algebras $A = R/\mathrm{Ann}(f)$ constructed in Proposition 10 are nonsmoothable; we expect that they are smoothable. A tangent space argument shows that A determines a component of $\mathrm{Gor}(T)$, when $r \geq 5$; in fact by an example of Geramita and Orecchia, one can show that there is a component parametrizing forms $f \in \mathrm{Gor}(T)$ such that the minimal length annihilating scheme Z with $I_Z \subset \mathrm{Ann}(f)$ is smooth (see [IK]). But there are negative weight elements of the tangent space $\mathrm{Hom}(I, A) \cong I/I^2$, $I = \mathrm{Ann}\, f$, to the punctual Hilbert scheme at A.

We propose several problems, related to these notions of length.

Problem A (Waring problems). What are the values of $\mathrm{len}_k(r, d)$ and $\mathrm{maxlen}_k(r, d)$ for the above length functions? What is the structure of the family of degree-d homogeneous polynomials f in R for which $\mathrm{len}_k(f) \leq s$?

B. Reznick has studied $\max g_k(r, d)$ in [R1], [R2], and he determines $\max g_{\mathbb{C}}(3, 3)$ in [R2].

The problem of decomposing t forms simultaneously as sums of powers of the same set of linear forms is classical (see [B2]).

Problem B (simultaneous decompositions). What are the values of the analogous constructs $\mathrm{len}_k^t(r, d)$ and $\mathrm{maxlen}_k^t(r, d)$ for simultaneous decompositions of t forms, or, equivalently, for t-dimensional vector spaces W of degree-d forms in the polynomial ring $\mathscr{R}$?

Problem C. What limit behavior is there for $\mathrm{length}(f^k)/k$?

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